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Four Classical Kleinian Group Problems In The Complex Setting

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"Which is why I am writing this book.
To think.
To understand.
It just happens to be the way I'm made.
I have to write things down to feel I fully comprehend them."
— Haruki Murakami (Norwegian Wood)

To my parents and all the people that was part of my life in the time I write this thesis; specially to Adrian, that made me realize that I could be a better human being.

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Introduction

The main interest in Kleinian groups theory arises from deep results such as the Uniformization theorems, Rigidity theorems, Geometrization theorems, and its applications on diverse areas (Theichmüller theory, Representation varieties, dynamical systems, etc.). Kleinian groups provide a bridge between projective, conformal and hyperbolic geometries. Unfortunately, in dimensions greater than two, this bridge is no longer available, so the generalization of the classical Kleinian group theory into higher dimensions takes various directions; quoting M. Kapovich ([92]), "There is a vast variety of Kleinian groups in higher dimensions: it appears that there is no hope for a comprehensive structure theory similar to the discrete groups of isometries of $\mathbb{H}^3$". In the direction of conformal geometry, a higher-dimensional Kleinian group is a discrete group of the group of conformal transformations of the $n-$dimensional sphere, which through Poincaré's extension is related with real hyperbolic geometry. However, there is not a direct connection with projective geometry. In [92], M. Kapovich state a general vision of the higher-dimensional Kleinian group, the relations between the topological and geometric properties of their limit sets and the topological and algebraic properties of the group. In the case of the projective geometry in higher dimensions, we have to look for J. Seade and A. Verjovsky ([145]) construction of *complex Kleinian groups* as a discrete subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$, *i.e.*, the biholomorphisms group of the $n-$dimensional complex projective space. Nowadays the complex Kleinian group theory is in its childhood in comparison with the classical theory. For example, little is known about the structure of the discontinuity region divided by the group action, how

the deformations of the group affect the geometry of the limit set and the topology of the representation varieties, the relation with the iteration of functions in higher dimensions. One expects that complex Kleinian groups in dimension two should be easier to understand than in higher dimensions, and in fact a lot more is known for dimension two. Yet, there are many points to be understood in dimensions greater or equal to two. A main point in this sense is the nature of the limit set. We refer to [122],[124], [16], [20],[21] as examples in dimension two. Despite the progress achieved so far, the known techniques are not enough to gain a deeper understanding. We need new tools to work with higher dimensional complex Kleinian groups.

The goal of this thesis is to contribute to set up the bases for the study of complex Kleinian groups in higher dimensions, and to broaden the research directions of this area, providing generalized versions of classical theorems, or conditions that prevent their generalization. We work with the following problems: the limit set in higher dimensions, higher-dimensional uniformization, Hausdorff dimension of limit sets, and constructions of representations in complex Kleinian groups. There are six chapters where these problems are studied individually and three appendices that complement the contents of the thesis.

In Chapter 0 we introduce the definitions, examples and some features needed to understand complex Kleinian groups. At the end of this Chapter we present a summary of results of complex Kleinian groups in dimension two.

It is key to the theory of complex Kleinian groups to understand and characterize their limit sets. The first approximation of this problem was due in [42]. L. Loeza in [107] introduces a description of the limit set of cyclic complex Kleinian group. This depends on the type of the generating element, that could be parabolic, elliptic and loxodromic (see Chapter 0 for details on this classification). The original proof of L. Loeza had a gap in the case of parabolic cyclic groups; in Chapter 1, we cover this gap and set new efficient tools that help us to understand the limit set in higher dimensions. The understanding of the limit set of cyclic groups is fundamental for the understanding of the limit set of other kind of groups. Additionally, we provide conditions that ensure the existence of a loxodromic element in a complex Kleinian group, a property that has significant implications on the geometry and dynamics of the limit set. We have to mention that in the classical case, the existence of loxodromic elements in a Kleinian group is a generic property (see [88]); even if our conditions do not guarantee that this property is generic, we are moving towards the generalizations of this classical result to higher dimension. There are three main results over this Chapter. The first, Theorem 1.1.7, provide a characterization of subspaces of $\mathbb{C}\mathrm{P}^n$ associated with a cyclic group that concentrate the dynamics as "repulsor" and "attractor" set. The relevance of this Theorem is the generalization of similar behavior obtained for the classical Kleinian groups and the dimensional two complex Kleinian groups. Theorem 1.1.17, deals with the computing of the Kulkarni limit set of the parabolic cyclic groups with the help of the previous theorem. Finally, Theorem 1.2.9 provides algebraic conditions under which a complex Kleinian group have a loxodromic element. The relevance of this last theorem is that the loxodromic elements play a key role in the dynamics of the groups and their limit sets.

Despite the information we know about the limit set for higher-dimensional cyclic groups, there is not much known about this set for other group families. We rely on the unique (up to conjugation) irreducible representation of $\mathrm{PSL}(2,\mathbb{C})$ into $\mathrm{PSL}(n+1,\mathbb{C})$, as a tool to construct complex Kleinian groups with controllable dynamics; such groups are in second order of simplicity if we compare them with the cyclic groups. In Chapter 2, we study the different notions of the limit set for the particular case of complex Kleinian groups obtained via the irreducible representation. This study is a natural step to have a better understanding of the limit set in higher dimensions. There are two main results over this Chapter. Theorem 2.2.2, computes the complement of the Equicontinuity region for Veronese groups generalizing similar results obtained for the dimension two complex Kleinian groups. Theorem 2.3.9, implies that for the Veronese groups there exists a closed invariant subset of $\mathbb{CP}^n$ that contains the Kulkarni limit set and the complement of the Equicontinuity region, in the complement of such closed set the action of the group is properly discontinuous. The last theorem provides examples of groups where the Kulkarni limit set is not the best candidate of the limit set in higher dimensions.

The uniformization of Riemann surfaces theorem is one of the most influential results for the Riemann surfaces' study, from their generalization (Koebe's retrosection theorem and Bers' simultaneous uniformization theorem), it started a close relationship between the Riemann surfaces and the Kleinian groups. From the principal role of the Kleinian groups in uniformization theorems, we wonder about the existence of complex Kleinian group versions of these uniformization theorems. It is not the first time that this question is asked, see [80], and nowadays there are many papers that look for answers, see [146]. In Chapter 3, we answer the previous question. We have to recall that the Complex manifold's geometry is vast and complicated, and even ask for minimum requirements over the manifolds, the answer to that question is that it does not exist such theorems. Over this Chapter, there are two main results. Theorem 3.2.3 implies that there is no version of the Bers' simultaneous uniformization theorem for complex Kleinian groups. Theorem 3.2.4 implies that there is no version of the Koebe's retrosection theorem for complex Kleinian groups either for Schottky groups or Free purely loxodromic groups. The previous theorems refute our expectations to obtain similar uniformization theory as in the case of Riemann surfaces.

The Hausdorff dimension is a metric invariant that allows us to understand how a set is, and for this reason, we are interested in the Hausdorff dimension of the limit sets of complex Kleinian groups acting on the $n-$dimensional complex projective space. D. Sullivan ([150]) showed that given a geometrically finite Kleinian group there exists an induced family of measures on the limit set which are intimately related to the Hausdorff dimension of the limit set in question. Late, C. McMullen ([116]) took advantage of this relation, and he developed an algorithm that approximates the Hausdorff dimension of the limit set. In Chapter 4, we feature a version of Sullivan's density at infinity theorem for complex Kleinian groups, which combined with a result of S. Hersonsky and F. Paulin ([84]), allowed us to generalize McMullen's algorithm for complex Kleinian groups (see Theorem 4.2.6 and Theorem 4.2.9); in the second place we developed a computational implementation of the generalized McMullen's algorithm.

The representation of surfaces' fundamental group is a way to construct examples of classical Kleinian groups, and the study of this representation is key to "map" the differences between geometric structures that the surface can admit. We start this Chapter 5 with the detailed construction of a real hyperbolic structure on the Borromean link complement, a reason to do this is to complete the literature and lack of computations, the relevance of the representations of the fundamental group of the complement of the Borromean links is that it is used by Kapovich ([94], [90], [93]) to prove that there is no version of the Sullivan's finiteness theorem for higher dimensional conformal Klenian groups. Aditionaly, we start the study of the representation variety of the fundamental group of the complement of the Borromean ring into $\mathrm{PU}(2,1)$, more precisely, E. Falbel in [62] proposed an algorithm to understand the representation variety of fundamental groups of finite volume 3-manifolds. Unfortunately, Falbel's algorithm had some constrains that need to take into consideration, for example: it is unable to guarantee the existence of the representations, it is only applicable succesfuly to few links and knots, and it does not parametrize the whole representation variety. Our main contribution in this chapter is that we impose constraints over Falbel's algorithm in order to optimize it in the particular case of the fundamental group of the Borromean link complement (see Proposition 5.2.10). Even more, Theorem 5.2.11 gives an upper bound for the (real) dimension of the Representation variety of the fundamental group of the Borromean link complement.

The reader is encouraged to treat each Chapter independently, however, as a collection the Chapters form an example of extensive research in higher-dimensional complex Kleinian groups, and it provides new directions to research the dimensional-two case.

Introducción

Los grupos kleinianos han sido de gran interés debido a la profundidad de sus resultados como los teoremas de Uniformización, Teoremas de rigidez, teoremas de Geometrización y su aplicación en diversas áreas (Teoría de Teichmüller, Sistemas Dinámicos, Variedades de Representación, etc) y son un puente que conecta entre las geometrías proyectiva, conforme e hiperbólica. Lamentablemente en dimensiónes mayores que dos no contamos con este puente, por lo que generalizar las ideas del caso clásico hacia dimensión superior ha tomado diferentes direcciones, bien lo menciona M. Kapovich en ([92]): "...hay una extensa diversidad de grupos kleinianos en varias dimensiones,... ...y no hay esperanza de tener una teoría que comprenda a estos de la manera que sucede con la teoría de grupos discretos de $\mathbb{H}^3$...". Utilizando el enfoque de la geometría conforme, se define un grupo kleiniano en varias dimensiones como un subgrupo discreto del grupo de transformaciones conformes de la esfera $n-$dimensional, que a través de la extensión de Poincaré podemos hablar de estos grupos desde el punto de vista de la geometría hiperbólica real. En [92], M. Kapovich presenta una visión general de grupos kleinianos en varias dimensiones, en la cual proporciona relaciones topologicas y geométricas del conjunto límite y las propiedades topológicas y algebraicas del grupo Kleiniano. Desafortunadamente esta dirección no se relaciona con la geometría proyectiva compleja, por lo que para el caso projectivo J. Seade y A. Verjovsky en [145] introdujeron el concepto de grupo kleiniano complejo como un subgrupo discreto de $\mathrm{PSL}(n+1,\mathbb{C})$, *i.e.*, el grupo transformaciones biholomorfas del espacio proyectivo complejo $n-$dimensional. Hasta el momento la teoría de grupos Kleinianos

complejos se encuentra en sus inicios y ha sido poco explorada en comparación con la teoría clásica. Por ejemplo, se sabe poco sobre la estructura de los cocientes de la discontinuidad y el grupo, cómo las deformaciones del grupo afectan la geometría del conjunto limite y la topología de las variedades de representación. Uno puede esperar que en dimensión dos sea más sencillos de entender los grupos kleinianos en comparación con los de dimensión superior, dado que es el caso de donde se conocen más resultados, pero hay varios puntos sin comprender para dimensiones mayores o iguales a dos. Un punto importante en este sentido es la naturaleza del conjunto limite, ver [122],[124], [16], [20],[21] para algunos ejemplos en dimension dos. A pesar de estos avances, para dimensiones mayores que dos, las técnicas que se han trabajado no son suficientes para entender más alla, de allí que se necesiten nuevas técnicas para trabajar en varias dimensiones.

La intención de esta tesis es contribuir a las bases para el estudio de los grupos kleinianos complejos en varias dimensiones así como expandir las direcciones de investigación de éstos, proporcionando versiones de resultados clásicos de grupos kleinianos o condiciones que impiden su generalización. En esta tesis trabajamos con las siguientes problemas: el conjunto límite en varias dimensiones, uniformización en dimensión superior, dimensión de Hausdorff de conjuntos límite, y construcción de representaciones en grupos kleinianos complejos. La tesis se encuentra organizada en seis capítulos que trabajan de manera independiente cada problemática y tres apéndices que complementan el contenido de la tesis.

En el Capítulo 0 nos encargamos de presentar las definiciones correspondientes para hablar de grupos kleinianos complejos, así como algunos ejemplos y propiedades que difieren de la teoría clásica de grupos kleinianos. Cabe mencionar que este capítulo presenta los resultados que formaron la base de el estudio de los grupos kleinianos complejos y los resultados más recientes para el caso de dimensión dos.

Entender y caracterizar el conjunto límite de grupos kleinianos en dimensión superior es un problema importante, desde su primera aproximación en [42] hasta ahora no se ha tenido un resultado de manera total. L. Loeza en [107] propone

una descripción del tipo de conjuntos límites que se obtienen para grupos cíclicos generados por elementos de $\mathrm{PSL}(n, \mathbb{C})$, que depende del tipo de elemento que se tenga. Lamentablemente la prueba original de L. Loeza presentaba una laguna en el caso de grupos ciclicos parabólicos. En el Capítulo 1 nos encargamos de rellenar esta laguna, además proporcionamos técnicas nuevas y más eficientes para estudiar el conjunto límite en dimensión superior. Como primer capítulo, el entender bien el comportamiento de los grupos cíclicos nos permitirá tener una mejor noción de otros grupos, además de que estos grupos son los más sencillos de estudiar en dimensión superior. Adicionalmente a los resultados antes mencionados, proporcionamos condiciones para que un grupo complejo kleiniano contenga un elemento loxodrómico. Cabe mencionar que en el caso clásico de grupos kleinianos esta propiedad es genérica ([88]), a pesar de que no se obtuvo la generalidad de esta propiedad en el caso complejo, es una primera aproximación hacia este resultado en varias variables. En este Capítulo se obtuvieron tres resultados principales. El primero, Teorema 1.1.7, proporciona una caracterización de subespacios de $\mathbb{CP}^n$ asociados a un grupo ciclicos que concentran la dinámica como conjuntos "atractor" y "repulsor". La relevancia de este teorema radica en la generalización de comportamientos similares obtenidos en grupos kleinianos clásicos y grupos kleinianos complejos de dimensión dos. Teorema 1.1.17, calcula el conjunto límite de Kulkarni para grupos cíclicos parabólicos con la ayuda del teorema anterior. Finalmente, Teorema 1.2.9 proporciona condiciones algebraicas sobre un group kleiniano complejo de tal manera que se garantice la existencia de un elemento loxodrómico. La relevancia de este último teorema radica en la importancia de los elementos loxodrómicos a nivel dinámicos y su relación con el conjunto límite.

A pesar de lo que se conoce sobre el conjunto límite de grupos kleinianos complejos cíclicos no se conoce mucho acerca de este conjunto para otro tipo de familias de grupos. Nos basamos en la única (salvo conjugación) representación irreducible de $\mathrm{PSL}(2, \mathbb{C})$ en $\mathrm{PSL}(n+1, \mathbb{C})$ como una herramienta para construir ejemplos de grupos Kleinianos complejos con una dinámica controlado. En el

Capítulo 2 estudiamos las diferentos nociones de conjunto límite para el caso particular de la familia de grupos en PSL$(n+1,\mathbb{C})$ que son imagen de un grupo kleiniano clásico a través de representación irreducible de PSL$(2,\mathbb{C})$ en PSL$(n+1,\mathbb{C})$, utilizando las herramientas introducidas en el Capítulo 1. El estudio de esta familia es un paso natural en el camino a comprender el conjunto límite en varias dimensiones, debido a la estrecha relación que mantienen esta familia de grupos con los subgrupos discretos de PSL$(2,\mathbb{C})$. El siguiente Capítulo concentra dos resultados principales. En el Teorema 2.2.2 se realiza el calculo del complemento de la región de Equicontinuidad para grupos de Veronese, del cual se obtiene una "generalización de los resultados obtenidos para grupos kleinianos complejos de dimensión dos. El Teorema 2.3.9 implica que para grupos de Veronese existe un cerrado invariante de $\mathbb{CP}^n$ que contiene al conjunto límite de Kulkarni y al complemento de la región de Equicontinuidad, y en su complemento la acción del grupo es propiamente discontinua. El último teorema proporciona ejemplos de grupos donde el conjunto límite de Kulkarni no es el mejor candidato a conjuto limite en dimensiones superiores.

El teorema de uniformización de superficies de Riemann es uno de los resultados que más influyeron en el estudio de estas superficies, a partir de sus generalizaciones (el teorema de Retrosección de Koebe y el teorema de uniformización simultanea de Bers) se obtuvo una relación íntima entre las superficies y los grupos kleinianos. Debido al papel que tienen los grupos kleinianos clásicos dentro de estos resultados, nos preguntamos: ¿es posible la existencia de teoremas de uniformización en el contexto de los grupos kleinianos complejos?, *i.e.*, ¿existen condiciones para una variedad compleja que asegure que se puede obtener como un cociente de un abierto de $\mathbb{CP}^n$ bajo un grupo kleiniano complejo?. Cabe mencionar que no es la primera vez que se pregunta acerca de la uniformización de variedades complejas en dimensión superior, podemos mencionar a [80], y es una pregunta que sigue generando investigación, podemos mencionar [146]. En el Capítulo 3 proporcionamos una respuesta a la pregunta anterior; cabe mencionar que la geometría de las variedades complejas es vasta y por si misma complicada, por ello pedimos requerimientos mínimos en las variedades y a pesar de eso la respuesta a esta pregunta es no, recomendamos al lector ver el Capítulo 3 para mejores enunciados sobre las respuestas a la pregunta. A lo largo de este capítulo se probaron dos resultados principales. El Teorema 3.2.3, implica que no hay una versión del Teorema de uniformización simultanea de Bers para el caso de grupos Kleinianos complejos. El Teorema 3.2.4 implica que no existe una versión del teorema de retrosección de Koebe para grupos kleinianos complejos tanto para grupos de Schottky o grupos libres puramente loxodromicos. Los teoremas anteriores contrastan nuestas esperanzas de obtener una teoría de uniformización similar a la de superficies de Riemann en el caso de varias dimensiones complejas.

La dimensión de Hausdorff es un invariante métrico que nos permite inferir como luce un espacio, de allí que nos interese estudiar la dimensión de Hausdorff para conjuntos limites de grupos kleinianos complejos. D. Sullivan en [150] demostró que para un grupo kleiniano geométricamente finito, existe una relación entre la dimensión de Hausdorff del conjunto límite y una familia de medidas asociadas al grupo kleiniano. Más tarde, C. McMullen ([116]) aprovechando esta relación desarrolló un algoritmo para realizar aproximaciones para la dimensión de Hausdorff del conjunto límite de grupos kleinianos geométricamente finitos. En el Capítulo 4 presentamos una versión del Teorema de las densidades al infinito de Sullivan para grupos kleinianos complejos que combinado con un resultado de S. Hersonsky and F. Paulin [84], nos permite generalizar el algoritmo de McMullen, ver Teoremas 4.2.6 y 4.2.9. Además del algoritmo, damos una implementación computacional en Python, así como ejemplos y simulaciones de la implementación.

La rerpresentación de grupos fundamental de superficies es una forma de obtener ejemplos de grupos kleinianos clásicos y el estudio de estas representaciones nos permite "mapear" las diferencias entre las estructuras geométricas que pueda admitir la superficie. Comenzamos el 5 con una construcción detallada de la estructura hiperbólica real del complemento de los anillos de Borromeo, debido a que en la literatura no se encuentra completa o carece de cálculos. La relevancia de incluir esta parte es debido a su gran importancia en la prueba de M. Kapovich sobre la no generalización del teorema de finitud de Sullivan ([94], [90], [93]). De manera adicional, iniciamos el estudio de la variedad de representación del grupo fundamental de complemento de los anillos de Borromeo en $\mathrm{PU}(2,1)$; de manera más precisa, E. Falbel en [64] presenta un algoritmo para entender las variedades de repsentación de grupos fundamentales de 3-variedades de volumen finito. Desafortunamente, el algoritmo tiene ciertas restricciones que necesitan ser tomadas en consideración, por ejemplo: no garantiza la existencia de representaciones, sólo puede aplicarse de manera satisfactoria a cierto número de enlaces y nudos, y no parametriza toda la variedad de representación. Nuestra principal contribución en esta Capítulo es que impusimos restricciónes al algoritmo en pro de optimizarlo para el caso particular del grupo fundamental del complemento los anillos de Borromeo (ver Proposición 5.2.10). Más aún, el Teorema 5.2.11) proporciona una cota superior para la dimensión (real) de la variedad de representación del grupo fundamental del complemento del enlace de Borromeo.

Cada capítulo de esta tesis es independiente de los demás, pero en conjunto forman un estudio extensivo de los grupos kleinianos complejos en varias dimensiones, que además proporciona nuevas direcciones en el caso de dimensión superior.

Contents

List of Figures

0 Preliminaries

Introduction

Over this Chapter will set the notations, definitions and classical results that inspire our work. Our start point will be a short guide to the classical Kleinian group theory. After, we will provide a brief introduction to the Projective space and the geometry of some subspaces. Finally, we will introduce the concept of a complex Kleinian group and the different notions of a limit set that can admit; as well as the main results in this setting that will help to understand the point where the theory is.

0.1 Classical Kleinian Group Theory

This section is based on [113] by its actual treatment of the Kleinian group theory. We will introduce the concept of a Kleinian group and its relation to the real hyperbolic space.

0.1.1 The Hyperbolic Space and Kleinian Groups

Consider the space $B^3 = \{x \in \mathbb{R}^3 : \|x\| < 1\}$ the unit ball in $\mathbb{R}^3$.

Definition 0.1.1. We call B^3 equipped with the metric

$$ds_B^2 = \frac{4\|dx\|^2}{(1-\|x\|^2)^2} \tag{1}$$

the *unit ball model* of the 3-dimensional hyperbolic space $\mathbb{H}^3$. The metric ds_B^2 is known as the *hyperbolic metric.*

The real hyperbolic space is one of the first examples of spaces that do not satisfy the Euclides' axioms. For this, it is known as an example of non-Euclidean geometry. We have to mention that $\mathbb{H}^3$ is a Riemannian manifold with sectional curvature -1.

In order to talk about the automorphism of $\mathbb{H}^3$, we have to consider $\hat{\mathbb{R}}^3$ denoting the one-point compactification of $\mathbb{R}^3$.

Definition 0.1.2. A *Möbius transformation* of $\hat{\mathbb{R}}^3$ is an orientation-preserving automorphism of $\hat{\mathbb{R}}^3$ obtained by a composition of a finite number of:

- Similarities: is a map defined by $S(x) = \lambda Ax + b$ when $x \in \mathbb{R}^3$ and $S(\infty) = \infty$. Where $\lambda > 0$, $A \in \mathrm{O}(3)$ and $b \in \mathbb{R}^3$.
- Fundamental reflections: is a map defined by $J(x) = \frac{x}{\|x\|^2}$ if $x \in \mathbb{R}^3 \setminus \{0\}$, $J(0) = \infty$ and $J(\infty) = 0$.

We will denote by $Mob(B^3)$ the set of all Möbius transformations that preserves B^3.

We have to mention that any Möbius transformation maps spheres and planes to spheres and planes.

Theorem 0.1.3. *Any element $T \in Mob(B^3)$ is an isometry with respect to the hyperbolic metric on B^3.*

We will denote by $Isom^+(\mathbb{H}^3)$ the group of all orientation-preserving automorphisms of $\mathbb{H}^3$ that are isometric with respect to the hyperbolic metric.

Theorem 0.1.4. *$Isom^+(\mathbb{H}^3)$ is identified with $Mob(B^3)$.*

We can transform the unit ball model into other model of the Hyperbolic space. If we take $e = (0, 0, 1) \in \mathbb{R}^3$ and the Möbius transformation Π defined by

$$\Pi(x) = e + 2J(J(x) - e).$$

We obtain that $\Pi(B^3) = H^3$, where $H^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$. The map Π is known as the *stereographic projection*.

Definition 0.1.5. The upper half space H^3 with the metric

$$ds_H^2 = \Pi_*(ds_B^2) = \frac{\|dx\|^2}{(x_3)^2} \tag{2}$$

is called the *upper half-space model* of the hyperbolic space $\mathbb{H}^3$.

The Möbius transformations of H^3 are conjugated to $Mob(B^3)$ via Π and therefore can be identified with $Isom^+(\mathbb{H}^3)$. The upper half-space model is important because $Mob(H^3)$ have a special representation.

Theorem 0.1.6. *The group $Mob(H^3)$ can be identified with the linear fractional transformation group*

$$Mob = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}$$

and therefore with

$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm Id\}.$$

In 0.2.2, we will see that there are similar spaces that generalize the hyperbolic space notion as a complex manifold.

The elements of Mob admit a classification via conjugation of its dynamics.

Definition 0.1.7. Any element $\gamma \in Mob \setminus \{id\}$, viewed as a transformation an element of $\mathrm{PSL}(2, \mathbb{C})$, can be transformed by conjugation into either

1. $z \mapsto z + 1$
2. $z \mapsto \lambda z$ $(\lambda \in \mathbb{C} \setminus \{0, 1\})$.

We call γ conjugate 1 are *parabolic,* to 2 with $|\lambda| = 1$ *elliptic*, and to 2 with $\lambda > 1$ *hyperbolic*. Moreover, we also call γ conjugate 2 with $|\lambda| \neq 1$ *loxodromic.*

This classification can also be stated in terms of fixed points.

Proposition 0.1.8. *An element $\gamma \in Isom^+(\mathbb{H}^3) \setminus \{id\}$ is parabolic, loxodromic or elliptic if and only if the number of its fixed points in $\overline{\mathbb{H}}^3$ is one, two of infinity respectively. Moreover it has a fixed point in $\mathbb{H}^3$ if and only if γ is elliptic.*

Where $\overline{\mathbb{H}}^3$ denotes the union of $\mathbb{H}^3$ and its boundary, which in the ball model is the 2-sphere and in the half-space model is $\hat{\mathbb{C}}$.

In Chapter 1, we will explain a similar classification of the isometries of the complex hyperbolic space that is inspired in the classification of the Möbius transformations.

For us, the following definition state the concept of a classical Kleinian group.

Definition 0.1.9. A subgroup Γ of $Isom^+(\mathbb{H}^3)$ is called a *Kleinian group* is Γ acts on $\mathbb{H}^3$ properly discontinuously.

Under the identifications of $Isom^{+}(\mathbb{H}^3)$ with $Mob(B^3), Mob(H^3), Mob$ and $\mathrm{PSL}(2,\mathbb{C})$ the subgroups with the properly discontinuously action are also known as Kleinian group.

Some examples of Kleinian groups are:

Example 0.1.10.

1. **Fuchsian Groups:** consider the group $\mathrm{PSL}(2,\mathbb{R})$ acting on the upper half-space, we have that it preserves the boundary (who is identified with the real plane $z = 0$) of H^3. A discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is Kleinian group, this kind of Kleinian groups are called *Fuchsian groups.*

2. **Schottky Groups:** Let $C_1, C_1', \ldots, C_k, C_k'$ be $2k$ disjoint disks on $\mathbb{R}^2$ with common exterior. For each j, let g_j be the Möbius map that send C_j into C_j' such that the interior of C_j is mapped into the exterior of C_j'. Consider the transformations g_j now on $\mathbb{H}^3$, via the Poincaré extension, the group generated $G = \langle g_1, \ldots g_k \rangle$ is a discrete and free group. The groups obtained in this way are called *Shottky groups.*

The following lemma is a characterization of the properly discontinuous actions for the classical Kleinian groups.

Lemma 0.1.11. *For a subgroup Γ of $Isom^{+}(\mathbb{H}^3)$, the following two conditions are equivalent.*

1. *Γ acts on $\mathbb{H}^3$ properly discontinuously and no other element than id has a fixed point.*

2. *For any point $p \in \mathbb{H}^3$, there exists a neighborhood U of p that satisfies $\gamma(U) \cap U = \varnothing$ for any $\gamma \in \Gamma \setminus \{id\}$. In particular, every element of Γ except id has no fixed point in $\mathbb{H}^3$.*

For the special identification of $Isom^{+}(\mathbb{H}^3)$ with $\mathrm{PSL}(2,\mathbb{C})$, the Kleinian groups is given by the following Theorem.

Theorem 0.1.12. *A Kleinian group is a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$.

The previous Theorem inspired us to study the discrete subgroups of $\mathrm{PSL}(n,\mathbb{C})$ as possible Kleinian groups in higher dimension, see Definition 0.3.1.

0.1.2 Properties of Kleinian Groups

The action of a Kleinian group on $\overline{\mathbb{H}}^3$ induce a partition of the space by two disjoint subset with interesting dynamical properties. In what follows, we introduce this subsets and their properties.

Let $\Gamma \subset Mob(B^3)$ be a Kleinian group, we will denote by $\Gamma(p)$ the orbit of a point $p \in B^3$ under the action of Γ.

Definition 0.1.13. For a point $p \in B^3$, we will say that $q \in \overline{B}^3$ is a *limit point* if q is an accumulation point of the $\Gamma-$orbit of p.

From the fact that Γ acts properly discontinuous on B^3, we have that the limit points belong to $\partial\mathbb{H}^3 = \mathbb{S}^2$.

Definition 0.1.14. Let $\Gamma \subset Mob(B^3)$ be a Kleinian group. We will denote by $\Lambda(\Gamma)$ the closure of the set of all limit points of the orbits of Γ. The set $\Lambda(\Gamma)$ is called the *limit set*.

The Limit set of a group does not depend on the choice of the reference point $p \in B^3$. Moreover, the cardinality of the Limit set could be 0,1,2, or infinite.

Definition 0.1.15. We say that a Kleinian group Γ is *elementary* if the limit set consists of at most two points. Otherwise, it is *non-elementary*.

The following Lemma gives a characterization of the Limit set in terms of loxodromic fixed points.

Lemma 0.1.16. *A non-elementary Kleinian group Γ satisfies the following:*

1. Γ *contains a loxodromic element.*
2. *The limit set $\Lambda(\Gamma)$ coincides with the closure of the loxodromic fixed points of Γ.*
3. *For any loxodromic element $\gamma \in \Gamma$, there is another loxodromic element g such that $Fix(\gamma) \cap Fix(g) = \varnothing$.*

Theorem 0.1.17. *The limit set $\Lambda(\Gamma)$ of a non-elementary Kleinian group Γ coincides with the following sets.*

1. *The closure of the set of all loxodromic fixed points for Γ.*
2. *The set of accumulation points of $\Gamma(\zeta)$ for any $\zeta \in \partial\mathbb{H}^3$.*
3. *The minimal non-empty closed set that is invariant under the action of Γ.*

In particular, $\Lambda(\Gamma)$ is a perfect set, and nowhere dense when $\Lambda(\Gamma) \neq \partial\mathbb{H}^3$.

We will call the last property the minimal property of the Limit set.

Definition 0.1.18. Let Γ be a Kleinian group, the set $\partial\mathbb{H}^3 \setminus \Lambda(\Gamma)$ will be denoted by $\Omega(\Gamma)$ and we will called it the *region of discontinuity* of Γ.

Proposition 0.1.19. *Let Γ be a Kleinian group such that $\Omega(\Gamma) \neq \varnothing$. Then Γ acts on $\Omega(\Gamma)$ properly discontinuously. Even more, $\Omega(\Gamma)$ is the maximal open subset of $\partial\mathbb{H}^3$ where Γ acts properly discontinuously.*

One of the main interest to study the Discontinuity region is because we can construct a manifold by identifying the points under the action of Γ.

Definition 0.1.20. For a torsion-free Kleinian group Γ (a group with no elements of finite order), the manifold $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ possibly with boundary is called a Kleinian manifold.

In general, if Γ is a Kleinian group the quotient $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is well defined and has an orbifold structure (see Appendix C).

0.1.3 Some Results about Kleinian groups

In what follows we present some inspiring results about Kleinian groups, the constructions that we will show inspire future chapters and further studies (Chapter 4) in the higher dimensional complex case.

Hausdorff dimensions

D. McMullen in [116], introduced an algorithm that approximates the dimension of a conformal density in a dynamical system via a Markov partition. In particular, this algorithm can be used to approximate the Hausdorff dimension of the limit set of certain Kleinian groups; this particularity occurs because such Kleinian groups had a conformal density whose dimension coincides with the Hausdorff dimension of its limit set. In what follows, we will give a brief introduction of how to obtain this conformal density for a Kleinian group ([132], [150]), and we will describe McMullen's algorithm.

Definition 0.1.21. A *conformal density of dimension* δ on a manifold V is a function which assigns a positive finite measure $\mu(\rho)$ to each element ρ in a non-empty collection of Riemann metrics on V. Such that:

- If $\rho = \psi\rho'$ with ψ a positive function. Then $\mu(\rho)$ and $\mu(\rho')$ belong to the same measure class.

- The Radon-Nikodym ratio $\frac{d\mu}{d\mu'}$ is equal to $\left(\frac{\rho}{\rho'}\right)^{\delta}$.

A way to obtain a Conformal density on $\mathbb{H}^{d+1}$ induced by a discrete group Γ is via the Poincaré Series.

Definition 0.1.22 (Poincaré Series)**.** For s a positive number consider the infinite series

$$g_s(x,y) = \sum_{g\in\Gamma} e^{-s(x,gy)},$$

where (x, gy) the distance in $\mathbb{H}^{d+1}$.

The previous series converges for $s > \delta$ and diverges for $s < \delta$, where $\delta = \overline{\lim_{k\to\infty}}(\log s_k)/k$.

The number δ is known as the *critical exponent* of the group Γ.

If we consider the family of measures

$$\mu_s(x) = \frac{1}{g_s(y,y)} \sum_{\Gamma} e^{-s(x,gy)} \delta(gy),$$

where $\delta(gy)$ is the unit Dirac mass at gy, and let $\mu(x)$ denotes the weak limit of the previous measures in the space of measures of $\overline{\mathbb{H}}^{d+1}$. The measure $\mu(x)$ is concentrated on the accumulation points of $\Gamma(y)$. Therefore, $\mu(x)$ is a measure on the limit set $\Lambda(\Gamma)$.

One can show that $x \mapsto \mu(x)$ is a conformal density.

Theorem 0.1.23 (Theorem 1, [150])**.** *There is a conformal density of dimension* $\delta(\Gamma)$ *on* $\Lambda(\Gamma)$ *which is invariant by* Γ.

In general, every Kleinian group admits the previous construction, but when the group is convex-cocompact the conformal density has interesting properties.

Remember that for $\mathbb{H}^{d+1}$, we can identify $\partial\mathbb{H}^{d+1}$ as a $d-$sphere bounding a ball of dimension $d+1$.

Definition 0.1.24. If $\Lambda \subset \mathbb{S}^d$ (as boundary of $\mathbb{H}^{d+1}$) is a closed set. The *convex hull* of Λ, denoted by $C(\Lambda)$, is the minimal convex set of $\mathbb{H}^d$ that contains all the points of all geodesic joining two different points in Λ.

In the particular case of $\Lambda = \Lambda(\Gamma)$ where Γ is a Kleinian group, the convex-hull of Λ is a $\Gamma-$invariant set.

Definition 0.1.25 (Convex Cocompact Group)**.** Let Γ a discrete subgroup of $Isom^+(\mathbb{H}^{d+1})$. We say that Γ is convex cocompact group if the action of Γ in $C(\Lambda(\Gamma))$ has a compact fundamental domain.

Examples of convex cocompact groups are: Schottky groups whose set of generating circles is disjoint, the Fuchsian groups whose admit a finite sided fundamental polyhedron.

Theorem 0.1.26 (Theorem 8, [150])**.** *For a convex cocompact group* Γ *there is on* $\Lambda(\Gamma)$ *one and only one* $\Gamma-$*invariant conformal density whose dimension coincides with the Hausdorff dimension of* $\Lambda(\Gamma)$.

Now we will explain McMullen's algorithm to approximates the dimension of a conformal density. Consider a dynamical system $\mathcal{F}$ and a density μ with dimension δ invariant by the dynamical system.

Definition 0.1.27. A *Markov* partition for $(\mathcal{F}, \mu)$ is a nonempty collection $\mathcal{P} = \{(P_i, f_i)\}$ of connected compact blocks $P_i \subset \mathbb{S}^n$, and maps $f_i \in \mathcal{F}$ defined on P_i, such that:

1. $f_i(P_i) \subset \bigcup i \mapsto jP_j$, where $i \mapsto j$ means $\mu(f(P_i) \cap P_j) > 0$.

2. f_i is a homeomorphism on a neighborhood of $P_i \cap f_i^{-1}(P_j)$, when $i \mapsto j$.

3. $\mu(P_i) > 0$.

4. $\mu(P_i \cap P_j) = 0$ if $i \neq j$

5. $\mu(f(P_i)) = \sum_{i \mapsto j} \mu(P_j)$.

In addition, if $|f_i'(x)|_\rho > \xi > 1$ for a conformal metric ρ in $\mathbb{S}^n$ and a constant ξ. Then we say that the Markov partition is expanding.

Take an expanding Markov partition $\mathcal{P} = \{(P_i, f_i)\}$ and sample points $x_i \in P_i$. The algorithm provide a sequence of approximations $\alpha(R^n(\mathcal{P}))$ to δ.

1. For each $i \mapsto j$, solve for $y_{ij} \in P_i$ such that $f_i(y_{ij}) = x_j$.

2. Compute the transition matrix

$$T_{ij} = \begin{cases} |f_i'(y_{ij})|^{-1} & \text{if } i \mapsto j \\ 0 & \text{otherwise} \end{cases}$$

3. Solve for $\alpha(\mathcal{P}) \geq 0$ such that the spectral radius satisfies $\lambda(T^\alpha) = 1$. Here $(T^\alpha)_{ij} = T_{ij}^\alpha$.

4. Output $\alpha(\mathcal{P})$ as an approximation to δ.

5. Replace $\mathcal{P}$ with its refinement $R(\mathcal{P}$, define new sample points $x_{ij} = y_{ij}$ and return to first step.

D. McMullen shows that for a finite configuration of disjoint circles in $\mathbb{S}^2$, the Schottky group generated by the circles (see item 2 in Example 0.1.10) had a expanding Markov partition.

Proposition 0.1.28 (Proposition 3.3 in [116]). *Let $\mathcal{C}$ a finite configuration of disjoint circles and let $\Gamma(\mathcal{C})$ be the convex cocompact Schottky group induced by $\mathcal{C}$. Then $\mathcal{P} = \{(D(C_i), \rho_i\}$ is an expanding Markov partition for $\Gamma(\mathcal{C})$ for the dynamical system composed by the group $\Gamma(\mathcal{C})$ and the density described in Theorem 0.1.26. Here $D(C_i)$ denotes the interior of the circles $C_i \in \mathcal{C}$ and ρ_i the fundamental inversion induced by C_i.*

Therefore by the Theorem 0.1.26, we can apply McMullen's algorithm to approximate the Hausdorff dimension of the limit set of $\Gamma(\mathcal{C})$.

0.2 Projective Geometry

Let $\mathbb{C}^{n+1}$ the $n+1-$dimensional complex vector space and denote by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ the non zero complex numbers. There is a natural action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ by scalar multiplication. We will denote by $\mathbb{C}\mathrm{P}^n$ the set of orbits of $\mathbb{C}^{n+1} \setminus \{0\}$ under the $\mathbb{C}^*-$action and $[\cdot] : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$ the quotient map. The space $\mathbb{C}\mathrm{P}^n$ is known as the *$n-$complex projective space,* it is a compact connected complex $n-$dimensional manifold, diffeomorphic to the orbit space $\mathbb{S}^{2n+1}/U(1)$ where $U(1)$ acts coordinate-wise on the unit sphere on $\mathbb{C}^{n+1}$.

Let $L \subset \mathbb{C}\mathrm{P}^n$ a non-empty subset, we will say that L is a $k-$projective subspace if there is a $k+1-$dimensional $\mathbb{C}-$linear subspace $\tilde{L} \subset \mathbb{C}^{n+1}$ such that $L = [\tilde{L} \setminus \{0\}.]$ Given a non-empty subset $H \subset \mathbb{C}\mathrm{P}^n$, we define $\langle H \rangle = \bigcap \{L \subset \mathbb{C}\mathrm{P}^n : H \subset L \text{ and } L \text{ is a projective subspace}\}$, it is clear that $\langle H \rangle$ is a projective subspace and we will call it the generated projective subspace. In particular if $H = \{p, q\}$ with p and q linear independent, there is a unique projective space passing through p and q, such subspace will be called a *complex line* and it will be denoted by $\overleftrightarrow{pq}$. We recall that this "line" is the image under $[\cdot]$ of a $2-$dimensional $\mathbb{C}-$space generated by p and q. If we restrict to dimension two, the intersection of any pair of complex lines consists in exactly one point.

If $\{e_0, \cdots, e_n\}$ denotes the standard basis of $\mathbb{C}^{n+1}$, and we will denote by the same symbols their images under $[\cdot]$. If $x \in \mathbb{C}\mathrm{P}^n$ and $\tilde{x} = (x_0, \cdots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ is a representative of the class $[x]$, we will say that $x = [x_0 : \cdots : x_n]$ are the homogeneous coordinates of x.

Let $\mathrm{SL}(n+1, \mathbb{C})$ be the group of invertible squared complex matrices with determinant 1, and it is called the *Special Linear Group.* Let $\mathbb{Z}_{n+1}$ denote the multiplicative group of $n - th$ roots of unity in $\mathbb{C}$, this group has an action on $\mathrm{SL}(n+1, \mathbb{C})$ by scalar multiplication. We will denote by $\mathrm{PSL}(n+1, \mathbb{C})$ the orbits of $\mathrm{SL}(n+1, \mathbb{C})$ under this action, the set of orbits has a group structure, and it is known as the *Projective Special Linear Group.* It is clear that every element of $\mathrm{SL}(n+1, \mathbb{C})$ induces an holomorphic automorphism of $\mathbb{C}\mathrm{P}^n$, even more, every holomorphic transformation of $\mathbb{C}\mathrm{P}^n$ arises as an element of $\mathrm{PSL}(n+1, \mathbb{C})$.

We will denote by $[\![\cdot]\!] : \mathrm{SL}(n+1, \mathbb{C}) \to \mathrm{PSL}(n+1, \mathbb{C})$ to the natural map. Given $\gamma \in \mathrm{PSL}(n+1, \mathbb{C})$, we say that $\tilde{\gamma} \in \mathrm{GL}(n+1, \mathbb{C})$ is a lift for γ if there is $r \in \mathbb{C}^*$ such that $r \cdot \tilde{\gamma} \in \mathrm{SL}(n+1, \mathbb{C})$ and $\gamma = [\![r\tilde{\gamma}]\!]$. Notice that $\mathrm{PSL}(n+1, \mathbb{C})$ acts transtively, effectively and biholomorphically on $\mathbb{C}\mathrm{P}^n$, even more the image of projective subspaces under an element is projective subspaces.

Theorem 0.2.1 (Theorem 2.1.1 [40])**.** *The group of projective automorphisms of* $\mathbb{C}\mathrm{P}^n$ *is*

$$\mathrm{PSL}(n+1, \mathbb{C}) := \mathrm{GL}(n+1, \mathbb{C})/(\mathbb{C}^*)^{n+1} \tag{3}$$

where $(\mathbb{C}^*)^{n+1}$ *is regarded as the subgroup of diagonal matrices with a single non-zero proper value.*

The elements of $\mathrm{PSL}(n+1, C)$ are classified into three classes, as in the classical case of elements of $\mathrm{PSL}(2, \mathbb{C})$.

Definition 0.2.2. (see [39]) An element $\gamma \in \mathrm{PSL}(n+1, \mathbb{C})$ is called:

1. *elliptic,* if every lift $\tilde{\gamma} \in \mathrm{SL}(n+1, \mathbb{C})$ is diagonalizable and all its proper

values are unitary complex numbers.

2. *parabolic,* if every lift $\tilde{\gamma} \in \mathrm{SL}(n+1, \mathbb{C})$ is non-diagonalizable and each of its proper values are unitary complex numbers.

3. *loxodromic,* if it has a lift $\tilde{\gamma} \in \mathrm{SL}(n+1, \mathbb{C})$ with at least one non-unitary proper value. Even more, if $\tilde{\gamma}$ is diagonalizable and if all its proper values are different, then we will say that γ is *purely strongly loxodromic.*

0.2.1 Grasmannians

The dynamics of $\mathrm{PSL}(n+1, \mathbb{C})$ on $\mathbb{C}\mathrm{P}^n$ need a better comprehension of the ambient space and it will not be enough just to look at the dynamics of points. For this reason we will use the *Grassmanians* of $\mathbb{C}\mathrm{P}^n$.

Let $0 \leq k < n$, the $k-$Grassmanian, denoted by $\mathrm{Gr}(k, n)$, be the set of all $k-$projective subspaces of $\mathbb{C}\mathrm{P}^n$ provided of the Hausdorff topology. This is a compact, connected complex manifold of dimension $k(n-k)$. We can perceive the Grassmanian $\mathrm{Gr}(k, n)$ through the *Plücker embedding* as a subvariety of the projectivization of the $(k+1)-$th exterior power of $\mathbb{C}^{n+1}$, *i.e.,* $\mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right)$. The Plücker embedding is given by

$$
\begin{aligned}
\iota : \mathrm{Gr}(k, n) &\to \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right) \\
V &\mapsto [v_1 \wedge \cdots \wedge v_{k+1}]
\end{aligned}
$$

where $V = Span(\{v_1, \cdots, v_{k+1}\})$. The previous map is a well-defined embedding. There is a natural action of $\mathrm{PSL}(n+1, \mathbb{C})$ into $\iota(\mathrm{Gr}(k, n))$, this action is given by the induced maps over $\bigwedge^{k+1}(\mathbb{C}^{n+1})$, *i.e.*, for some $\gamma \in \mathrm{PSL}(n+1, \mathbb{C})$, we take a lift $\tilde{\gamma} \in \mathrm{SL}(n+1, \mathbb{C})$ and let $\bigwedge_{k+1} \tilde{\gamma}$ the induced transformation of $\bigwedge^{k+1} \mathbb{C}^{n+1}$ given by

$$\bigwedge_{k+1} \tilde{\gamma}(v_1 \wedge \cdots \wedge v_k) = \tilde{\gamma}(v_1) \wedge \cdots \tilde{\gamma}(v_{k+1}).$$

We will denote by $\bigwedge_{k+1} \gamma$, the image under $[\![\cdot]\!]$ of $\bigwedge_{k+1} \tilde{\gamma}$ in the appropiate dimension. The previous action gives the Plücker embedding the equivariant property.

0.2.2 Complex Hyperbolic Space

The understanding of the complex hyperbolic geometry will be helpful to a better understanding of the complex Kleinian groups, due to the role that plays the real version for the classical Kleinian groups, but will not be our savior.

If we denote by $\mathbb{C}^{n,1}$ the $n+1-$dimensional complex vector space with the Hermitian form

$$\langle u, v \rangle = u_0\overline{v}_0 + \cdots + u_{n-1}\overline{v}_{n-1} - u_n\overline{v}_n \tag{4}$$

where $u = (u_0, \cdots, u_n)$ and $v = (v_0, \cdots, v_n)$. An easy computation demonstrates that the Hermitian form is induced by the Hermitian matrix $H = \begin{pmatrix} I & \\ & -1 \end{pmatrix}$ where I is the $n-$dimensional identity, for this reason the signature of the Hermitian form is $(n, 1)$, corresponding to n positive proper values and one negative.

The Hermitian norm (4) induces a partition of $\mathbb{C}^{n,1}$ into three non-empty disjoint sets, each one corresponding to the sign of the Hermitian form, *i.e.,* we have V_- where the form is negative definite, V_+ where the form is positive definite and V_0 where the form is null.

If we take the intersection of $V_- \cup V_0$ with the hyperplane $z_{n+1} = 1$ in $\mathbb{C}^{n,1}$, we obtain the closed ball in $\mathbb{C}^{n+1}$ bounded by the sphere $\mathbb{S} = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n,1} : |z_0|^2 + \cdots |z_n|^2 = 1\}$. The open ball equipped with the following metric:

$$\frac{4}{(1-\langle z,z\rangle)^2}\left(\langle z,dz\rangle\langle dz,z\rangle + (1-\langle z,z\rangle)\langle dz,dz\rangle\right) \tag{5}$$

known as the Bergman metric, serves as a model for the complex hyperbolic geometry, we refer to [73] for more details and [40] for a beautiful explanation about complex hyperbolic geometry.

For the sake of completeness of this thesis we will explain the projective model of the complex hyperbolic space. Notice that a one dimensional linear space of $\mathbb{C}^{n+1}$ is null if its intersection with the ball V_0 is exactly one point. Hence the projectivization of $V_0 \setminus \{0\}$ is diffeomorphic to the $(2n-1)-$sphere in $\mathbb{C}^n$. If we do a similar analysis of the one dimensional linear spaces that are negative, we have that the projectivization of $V_- \setminus \{0\}$ corresponds to the $2n-$ball bounded by the previous sphere. We will denote by $\mathbb{H}^n_{\mathbb{C}}$ to $[V_- \setminus \{0\}]$ and by $\partial\mathbb{H}^n_{\mathbb{C}}$ to $[V_0 \setminus \{0\}]$.

Given a point $z \in \mathbb{H}^n_{\mathbb{C}}$, one wishes to know how the geodesic passing through z look like, and more generally, if there are totally geodesics subspaces and how they look like? If we consider a complex line $L \subset \mathbb{C}\mathrm{P}^n$ passing through z, the intersection $L \cap \mathbb{H}^n_{\mathbb{C}}$ is an holomorphic submanifold isometric to $\mathbb{H}^1_{\mathbb{C}}$. This "2-plane" is a totally geodesic subspace of $\mathbb{H}^n_{\mathbb{C}}$. This type of spaces in $\mathbb{H}^n_{\mathbb{C}}$ are called *complex geodesics*, they have constant negative curvature in the Bergman metric. The space L intersects $\partial\mathbb{H}^n_{\mathbb{C}}$ too, and this subset is homeomorphic to a circle $\mathbb{S}^1$, this boundary circles on the sphere at infinity are called *chains.* We recall that for every pair of distinct points at $\mathbb{H}^n_{\mathbb{C}}$ there is a unique complex line that contains them, we can extend this assertion to every pair in $\overline{\mathbb{H}^n_{\mathbb{C}}}$. More generally, if $\mathcal{P}$ is a $k-$complex projective subspace of $\mathbb{C}\mathrm{P}^n$ whose intersection with $\mathbb{H}^n_{\mathbb{C}}$ is non-empty, then $\mathcal{P} \cap \mathbb{H}^n_{\mathbb{C}}$ is a complex holomorphic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, then the Theorem 3.1.10 of [73] asserts that $\mathcal{P} \cap \mathbb{H}^n_{\mathbb{C}}$ is a totally geodesic submanifold which is biholomorphically isometric to $\mathbb{H}^k_{\mathbb{C}}$. Such manifold is called a $\mathbb{C}^k-$plane and its boundary, $\mathcal{P} \cap \partial \mathbb{H}^n_{\mathbb{C}}$ is a $(2k-1)-$real sphere called $\mathbb{C}^k-$chain. Goldman also shows that besides the $\mathbb{C}^k-$planes there is only another type of totally geodesics spaces in $\mathbb{H}^n_{\mathbb{C}}$; if we have a linear real subspace of $\mathbb{C}^{n,1}$, $\tilde{R}^{k+1}$ of real dimension $k+1$ with negative vectors on it, we say that $\tilde{R}^{k+1}$ is totally real respect Q if $i \cdot \tilde{R}^{k+1}$ is $Q-$orthogonal to $\tilde{R}^{k+1}$. A totally real subspace of $\mathbb{H}^n_{\mathbb{C}}$ is the intersection of $\left[\tilde{R}^{k+1} \setminus \{0\}\right] \cap \mathbb{H}^n_{\mathbb{C}}$ and we will call those real subspaces $\mathbb{R}^k-$planes.

Theorem 0.2.3. *Every totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$ is either a $\mathbb{C}^k-$plane or a $\mathbb{R}^k-$plane. In particular $\mathbb{H}^n_{\mathbb{C}}$ has no totally geodesic real submanifolds of codimension 1 for $n > 1$.*

We recall that every $\mathbb{C}^k-$plane with the restricted Bergman metric is biholomorphically isometric to $\mathbb{H}^k_{\mathbb{C}}$ and a $\mathbb{R}^k-$plane with the restricted Bergman metric is isometric to $\mathbb{H}^k_{\mathbb{R}}$ with the Klein-Beltrami metric, see section 3.1 of [73] for a detailed description.

The isometries for the complex hyperbolic space are induced by the *pseudo-orthogonal unitary group* of signature $(n,1)$ denoted by $\mathrm{U}(n,1)$, *i. e.,* the subgroup of $\mathrm{GL}(n+1,\mathbb{C})$ whose elements preserve the Hermitian form that defines $\mathbb{H}^n_{\mathbb{C}}$ or equivalently

$$A \in \mathrm{U}(n,1) \Leftrightarrow A^* H A = H \tag{6}$$

where A^* denotes the conjugated transpose. One can easly check that $\mathrm{U}(n,1)$ acts transtively on the unitary ball of $\mathbb{C}^{n,1}$. As we did for the projective special group, the group that acts on $\mathbb{H}^n_{\mathbb{C}}$ is the image of $\mathrm{U}(n,1)$ under the map $[\![\cdot]\!]$, called the *projective pseudo-orthogonal unitary group.* In the [73] and [127], they show that this group is the group of holomorphic isometries for $\mathbb{H}^n_{\mathbb{C}}$ respect to the Bergman metric.

As in the $\mathrm{PSL}(n+1,\mathbb{C})$ case, the elements of $\mathrm{PU}(n,1)$ are also clasified into three classes: *parabolic, elliptic* and *loxodromic.* This classification agrees with the one of $\mathrm{PSL}(n+1,\mathbb{C})$ but there are dynamical features that the elements of $\mathrm{PSL}(n+1,\mathbb{C})$ do not have, for example:

- the parabolic elements have exactly one fixed point at the sphere at infinity, $\partial\mathbb{H}^n_{\mathbb{C}}$.
- the elliptic elements have a fixed point inside the ball $\mathbb{H}^n_{\mathbb{C}}$.
- the loxodromic elements have exactly two fixed points in the sphere at infinity, $\partial\mathbb{H}^n_{\mathbb{C}}$.

As one can notice, this dynamical features are similar as the dynamical classification of isometries of $\mathbb{H}^2_{\mathbb{R}}$ and $\mathbb{H}^3_{\mathbb{R}}$, see [112], [121], [22] and [96].

0.3 Complex Kleinian Groups

As we explain earlier, the principal motivation of this thesis are the *Kleinian groups.* The concept of *complex Kleinian groups* was introduced in [145], as

Definition 0.3.1. A discrete subgroup Γ of $\mathrm{PSL}(n+1,\mathbb{C})$ is a complex Kleinian group if there exists a non-empty open subset of $\mathbb{C}\mathrm{P}^n$ where the action of Γ is properly-discontinuous.

First of all, we will recall that a group of diffeomorphisms G acting on a smooth manifold M is *discontinuous* at $x \in M$ if there exists an open neighborhood U of x such that $\{g \in G : g(U) \cap U \neq \varnothing\}$ is finite, and if for every compact $K \subset M$ the set $\{g \in G : g(K) \cap K\}$ is finite the action of G is called *properly discontinuous.* It is clear that every properly discontinuous action is discontinuous, but the converse is not true in general. For the case of the Kleinian group the *domain of discontinuity,* where the action of the group is discontinuous, the action is also properly discontinuous, but in higher dimensional projective actions the discontinuity domain and the properly-discontinuity domain does not coincide in general. The following example illustrate why we are asking for a properly-discontinuous action and that there exists several subsets of $\mathbb{C}\mathrm{P}^2$ to study the action of the group.

Example 0.3.2. ([16], proposition 3.1.1 in [40])

Let $\gamma \in \mathrm{PSL}(3,\mathbb{C})$ whose lift at $\mathrm{SL}(3,\mathbb{C})$ is

$$\tilde{\gamma} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

where $|\lambda_1| > |\lambda_2| > |\lambda_3|$, *i. e.,* a purely strongly loxodromic element and consider the group $\Gamma = \langle \gamma \rangle$. One can show that $\{e_0, e_1, e_2\}$ are fixed points, even more the orbits of $x \in \mathbb{C}\mathrm{P}^2$ under Γ accumulates at e_0 and e_2. Therefore, the closure of the accumulation points of orbits of Γ or the *classical limit set* is equal to $\{e_0, e_1, e_2\}$. A similar example was studied in [154], and it can be shown that the $\Gamma-$action is not discontinuous on $\mathbb{C}\mathrm{P}^2 \setminus \{e_0, e_1, e_2\}$, so the classical definition of limit set does not guarantee an open set where action is "good".

Beside the previous, the lines $\overleftrightarrow{e_0e_1}$ and $\overleftrightarrow{e_1e_2}$ are invariant sets for the $\Gamma-$action, even more, outside e_2 the positive iterates of γ accumulates to $\overleftrightarrow{e_0e_1}$, and outside e_1 the positive iterates of γ^{-1} accumulates to $\overleftrightarrow{e_1e_2}$. Then, the action of Γ is discontinuous on $\mathbb{C}\mathrm{P}^2 \setminus (\overleftrightarrow{e_0e_1} \cup e_2)$, $\mathbb{C}\mathrm{P}^2 \setminus (\overleftrightarrow{e_1e_2} \cup e_0)$, and $\mathbb{C}\mathrm{P}^2 \setminus \overleftrightarrow{e_0e_1} \cup \overleftrightarrow{e_1e_2}$. The first two sets are maximal open sets where the $\Gamma-$action is properly-discontinuous and the last one is the largest open set where Γ is a normal family.

All four sets are closed invariant sets made of orbits of points, but every has a different property about the dynamics of Γ.

From the previous example, we can observe that there is more than one invariant closed subset of $\mathbb{C}\mathrm{P}^2$ such that the action on his complement has a "nice" property for the action as proper-discontinuity or normality, and in general these sets are not comparable by contention. Therefore, how we decide or adequate our classical definition of "limit set"?

0.3.1 The Limit Set

In the classic setting, the *limit set* is defined as the cluster points of orbits of one point of the space. Consider the following definition of *limit set.*

Definition 0.3.3. Let $\Gamma \subset \mathrm{PSL}(n+1,\mathbb{C})$ be a discrete subgroup. Then consider the subset $\mathrm{Lim}(\Gamma) \subset \mathbb{C}\mathrm{P}^n$ as the closure of the accumulation points of $\Gamma-$orbits of points in $\mathbb{C}\mathrm{P}^n$.

The previous definition is the most natural description of what we expect to be a *limit set*, but from the previous example we saw that there are groups where the complement of $\mathrm{Lim}(\cdot)$ the action is not discontinuous nor properly-discontinuous. Despite this, the closed set $\mathrm{Lim}(\cdot)$ is related to other notions of limit set, see Chapter **??**.

We recall that in the classic Kleinian group theory, the limit set of an infinite cyclic group coincides with the set of fixed points of the generating element. For the complex Kleinian groups, the set of fixed points of an element is given in terms of the proper vectors of the lifts, in order to generalize the classical result about limit set we need to introduce the notion of unitary decomposition.

Definition 0.3.4. Let $\gamma \in \mathrm{SL}(n+1,\mathbb{C})$ be a linear transformation, $\{V_j\}_{j=1}{}^k$ a set of linear subspaces of $\mathbb{C}^{n+1}$; $k \in \mathbb{N}$; $\gamma_j : V_j \to V_j$ $\mathbb{C}-$linear transformations and $\{r_j\}_{j=1}{}^k$ real numbers. The quadruple $(k, \{V_j\}_{j=1}{}^k, \{\gamma_j\}_{j=1}{}^k, \{r_j\}_{j=1}{}^k)$ will be called a *unitary decomposition* for γ if it satisfies that:

1. $\bigoplus_{j=1}{}^k V_j = \mathbb{C}^{n+1}$.

2. For each $1 \leq j \leq k$, the eignevalues of γ_j are unitary complex numbers.

3. $0 < r_j < r_{j+1}$ for $j = 1, \cdots, k-1$.

4. $\bigoplus_{j=1}{}^{k} r_j \gamma_j = \gamma$.

Theorem 0.3.5 (Theorem 1.6, [39]). *Let $\gamma \in \mathrm{PSL}(n+1, \mathbb{C})$ be a projective transformation with infinite order. If $\tilde{\gamma} \in \mathrm{SL}(n+1, \mathbb{C})$ is a lift of γ and $(k, \{V_j\}_{j=1}^k, \{\gamma_j\}_{j=1}^k, \{r_j\}_{j=1}^k)$ is a unitary decomposition of $\tilde{\gamma}$, then:*

$$\mathrm{Lim}(\langle\gamma\rangle) = \bigcup_{j=1}^{k} \langle\langle [x \in V_j : x \textit{ is an eigenvector of } \gamma_j] \rangle\rangle .$$

Remark 0.3.6. In the case of subgroups of $\mathrm{PU}(n, 1)$, the set $\mathrm{Lim}(\Gamma)$ is known as the *Chen-Greenberg limit set*, introduced by Chen and Greenberg in [45] and the definition was presented in [122] proving the following proposition:

Proposition 0.3.7. *Let $(\gamma_m)_{m\in\mathbb{N}} \subset G$ be a sequence of different elements of a discrete subgroup $G \subset \mathrm{PU}(2, 1)$. Then there exists $x, y \in \partial\mathbb{H}^2_{\mathbb{C}}$ and a subsequence, still denoted (γ_m), such that $\gamma_m \xrightarrow[m\to\infty]{} x$ uniformly on compact subsets of $\overline{\mathbb{H}^2_{\mathbb{C}}} \setminus \{y\}$. Even more, if there is a closed subset $X \subset \partial\mathbb{H}^2_{\mathbb{C}}$ containing more than one point where Γ acts discontinuously, then $\mathrm{Lim}(\Gamma) \subset X$.*

The previous Proposition implies that for discrete subgroups of $\mathrm{PU}(2, 1)$, whose Chen-Greenberg limit set contains more than two points, the limit set is minimal for the contention, even for groups in $\mathrm{PU}(n, 1)$ this theorem is valid but when we look for other discrete groups of $\mathrm{PSL}(n + 1, \mathbb{C})$ these properties does not follow directly, even for $n = 2$.

Another good property given by this limit set is one related to the existence of loxodromic elements, as we know in the classical setting the classical limit set can be described as the closure of fixed points of loxodromic elements of the group.

In the complex setting this does not follow trivially, for this reason the following Proposition is key:

Proposition 0.3.8 ([88]). *Let* $\Gamma \subset \mathrm{PU}(2,1)$ *discrete subgroup such that* $\mathrm{Lim}(\Gamma)$ *contains at least two points, then there existe one loxodromic element in* Γ.

Even for general groups in $\mathrm{PSL}(3,\mathbb{C})$ the problem of existence of loxodromic elements is not easy. In next Chapter we will work with this problem.

0.3.2 The Complement of Equicontinuity Set

The first key closed subset is one related with dynamics on manifolds. Let X be a manifold and G a subgroup of the endomorphisms of X.

Definition 0.3.9. We say that the action of G is equicontinuous at x if there is an open neighborhood of $x \in X$ such that the family $G|_U$ of the restriction of the elements to U, is a normal family. The subset of X of all equicontinuous points is called the Equicontinuity Set and we will denoted by $\mathrm{Eq}(G)$.

It is clear that the equicontinuity set is an open subset of X.

In order to understand the equicontinuity region we need to introduce the concept of *Pseudo-Projective transformations.*

Let $\widetilde{T} \in \mathrm{Mat}(n+1,\mathbb{C})$ a squared matrix with complex entries not necessary invertible and let $K = [\ker(\widetilde{T}) \setminus \{0\}] \in \mathbb{C}\mathrm{P}^n$. There is a map, induced by $\tilde{T}$, in the following way:

$$T : \mathbb{C}\mathrm{P}^n \setminus K \to \mathbb{C}\mathrm{P}^n \tag{7}$$

$$[x] \mapsto [\widetilde{T}(x)] \tag{8}$$

We will denote by $\ker(T) = [\ker(\widetilde{T}) \setminus \{0\}]$ and by $\mathrm{im}(T) = [\mathrm{im}(\widetilde{T} \setminus \{0\}]$.

Definition 0.3.10. We will call T the *pseudo-projective* map induced by $\widetilde{T}$. We will denote by $\mathrm{PsP}(n+1,\mathbb{C})$ the set of all pseudo-projective maps.

It is not hard to prove that $\mathrm{PSL}(n+1,\mathbb{C})$ is an open dense set in $\mathrm{PsP}(n+1,\mathbb{C})$, hence the set of pseudo-projective maps is a completation of $\mathrm{PSL}(n+1,\mathbb{C})$.

Proposition 0.3.11 (See Proposition 7.4.1, page 157 in [40]). *Let* $(\gamma_m)_{m\in\mathbb{N}} \subset \mathrm{PSL}(n+1,\mathbb{C})$ *be a sequence of distinct elements, then*

1. *There is a subsequence* $(\tau_m)_{m\in\mathbb{N}}$ *and* $\tau_0 \in \mathrm{PsP}(n+1,\mathbb{C})$ *such that* $\tau_m \xrightarrow[m\to\infty]{} \tau_0$ *as points in* $\mathrm{PsP}(n+1,\mathbb{C})$.

2. *If* $(\tau_m)_{m\in\mathbb{N}}$ *is the sequence given by the previous part of this lemma, then* $\tau_m \xrightarrow[m\to\infty]{} \tau_0$, *as functions, uniformly on compact sets of* $\mathbb{C}\mathrm{P}^n \setminus \ker(\tau_0)$.

The previous proposition is a wonderful tool to describe the equicontinuity region for subgroups in $\mathrm{PSL}(n+1,\mathbb{C})$.

Proposition 0.3.12 (Proposition 2.5, [37]). *Let* $\Gamma \subset \mathrm{PSL}(n+1,\mathbb{C})$ *be a group, we say that* $\gamma \in \mathrm{PsP}(n+1,\mathbb{C})$ *is a limit of* Γ, *in symbols* $\gamma \in \mathrm{Lim}(\Gamma)$, *if there is a sequence* $(\gamma_m)_{m\in\mathbb{N}} \subset \Gamma$ *of distinct elements satisfying* $\gamma_m \xrightarrow[m\to\infty]{} \gamma$. *Thus we have*

$$\mathrm{Eq}(\Gamma) = \mathbb{C}\mathrm{P}^n \setminus \overline{\bigcup_{\gamma\in\mathrm{Lim}(\Gamma)} \ker(\gamma)}. \tag{9}$$

Corollary 0.3.13 (Corollary 2.6 [37]). *Let* $\Gamma \subset \mathrm{PSL}(n+1,\mathbb{C})$ *be a discrete group, then* Γ *acts properly discontinuously on* $\mathrm{Eq}(\Gamma)$.

The previous Corollary implies that the elements of the family of discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$ are complex Kleinian. A particular example of complex Kleinian groups, was studied in [42], the authors provide a characterization of the Equicontinuity set as a complement of hyperspaces in $\mathbb{C}\mathrm{P}^n$, as follows.

Theorem 0.3.14 (Lemma 5.1, [42]). *Let* Γ *be a discrete subgroup of* $\mathrm{PU}(1,n)$, *then*

$$\mathrm{Eq}(\Gamma) = \mathbb{C}\mathrm{P}^n \setminus \bigcup_{p \in \mathrm{Lim}(\Gamma)} \{p\}^{\perp}$$

where $\{p\}^{\perp}$ *denotes the* H_2-*orthogonal space to* p.

0.3.3 The Kulkarni Limit Set

In [122] the author studied discrete groups of $\mathrm{PU}(2,1)$ in order to give some examples of complex Kleinian groups. He looks for a special limit set introduced by Kulkarni in [103]. We will call this limit set, the *Kulkarni Limit Set* and its defined by

Definition 0.3.15. Let $\Gamma \subset \mathrm{PSL}(n+1,\mathbb{C})$ a discrete subgroup and consider the following subsets of $\mathbb{C}\mathrm{P}^n$

1. $L_0(\Gamma)$ the closure of points whit infinite stabilizer.
2. $L_1(\Gamma)$ the closure of cluster points of orbits of points in $\mathbb{C}\mathrm{P}^n \setminus L_0(\Gamma)$.
3. $L_2(\Gamma)$ the closure of cluster points of orbits of compact subsets of $\mathbb{C}\mathrm{P}^n \setminus (L_0(\Gamma) \cup L_1(\Gamma))$.

The *Kulkarni limit set* for Γ is the union $L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma)$ and we will denoted by $\Lambda_{Kul}(\Gamma)$. We will denote by $\Omega_{Kul}(\Gamma)$ to the complement of the Kulkarni limit set and it will be called, the Kulkarni discontinuity region.

The Kulkarni limit set has the following properties:

Proposition 0.3.16. *Let* Γ *be a complex Kleinian group. Then:*

1. *The set* $\Lambda_{Kul}(\Gamma)$ *is a* $\Gamma-$*invariant closed set.*

2. *The group* Γ *acts properly-discontinuous on* $\Omega_{Kul}(\Gamma)$.

3. *Let* $\mathcal{C} \subset \mathbb{C}\mathrm{P}^n$ *a closed* $\Gamma-$*invariant set such that for every compact set* $K \subset \mathbb{C}\mathrm{P}^n \setminus \mathcal{C}$, *the set of cluster points of* ΓK *is contained in* $(L_0(\Gamma) \cup L_1(\Gamma)) \cap \mathcal{C}$. *Then* $\Lambda_{Kul}(\Gamma) \subset \mathcal{C}$.

4. *The equicontinuity set of* Γ *is contained in* $\Omega_{Kul}(\Gamma)$.

Navarrete proved that the non-elementary discrete subgroups of $\mathrm{PU}(2,1)$ are complex Kleinian groups whose Kulkarni domain of discontinuity is non-empty, and that it is maximal respect the contention, as it is explained in the following proposition.

Proposition 0.3.17 (Corollary 4.13, [122])**.** *If* $\Gamma \subset \mathrm{PU}(2,1)$ *non-elementary discrete subgroup and* $W \subset \mathbb{C}\mathrm{P}^n$ *is a* $\Gamma-$*invariant open set where* Γ *acts properly discontinuous, then* $W \subset \Omega_{Kul}(\Gamma)$.

Even if the Kulkarni limit set looks like it is the best definition of *limit set* for minimal property, for dimensions higher than 3 the computations of this limit set get more complicated, although in [37] the authors prove that

Theorem 0.3.18. *For a discrete non elementary subgroup of* $\mathrm{PSL}(3,\mathbb{C})$ *the complement of the Equicontinuity domain and the Kulkarni limit set, coincide. Even more, this set is the largest open subset of* $\mathbb{C}\mathrm{P}^2$ *where the* $\Gamma-$*action is properly discontinuous.*

As one can see the Kulkarni limit set seems to be the quickest way to obtain a domain where the action of the group is properly discontinuous, but its hard computability overshadows this practicality. Even in dimension two, there is a small set of examples of complex Kleinian groups with its Kulkarni limit set totally computed, see [122].

From the fact that the Kulkarni limit set is the first notion of limit set that achieves most of the properties that we are looking for, we will use it as a standard for complex Kleinian groups despite its hard computability.

0.4 The Dimension Two

The dimension two is the most explored dimension for the complex Kleinian groups, this dimension presents results that are the basis and breakthroughs in the area. The following section is compound with the most remarkable theorems related to the limit set of complex Kleinian group in dimension two, these theorems are an inspiration for the study of the limit set in higher dimensions, in particular Chapter 1 and 2; we have to mention that at this point there are no similar results in higher dimensions.

For classical Kleinian groups, the Domain of Discontinuity satisfy certain geometric properties, as we described formerly, for Complex Kleinian groups there is no standard notion of Domain of discontinuity. The use of Kulkarni domain of discontinuity put a start point where to look at and try to find properties similar to the classical ones. The following Theorem states that under certain conditions on the group, the domain of discontinuity presents a notion of hyperbolicity.

Theorem 0.4.1 (Theorem 1.1 in [124]). *If $G \subset \mathrm{PU}(2,1)$ is an infinite discrete group acting on $\mathbb{C}\mathrm{P}^2$ without invariant complex projective lines, then the connected component of the Kulkarni domain of discontinuity containing $\mathbb{H}^2_{\mathbb{C}}$ is $G-$invariant and complete Kobayashi hyperbolic.*

The idea of the proof of the previous theorem is to use the existence of configurations of four projective lines in general position in the Kulkarni limit set. The relevance of this theorem resides on the characterization of families of groups where the Kulkarni limit set have nice properties that can be translated into the hyperbolicity of its complement.

After this theorem, it was desired to get the right notion of "elementary" group. Recall that property of a classical Kleinian group of being "elementary" is in correspondence with the cardinality of the limit set. Since the Kulkarni limit set is composed by projective lines, the following theorem is a resume of several result about configurations of projective lines.

Theorem 0.4.2 (Theorem 6.2.1 in [40]). *Let Γ be an infinite discrete subgroup of* $\mathrm{PSL}(3,\mathbb{C})$ *and let $\Omega \subset \mathbb{C}\mathrm{P}^2$ be either its equicontinuity region, its Kulkarni region of discontinuity or a maximal region where the group acts properly discontinuously. Then:*

1. *The number of lines contained in $\mathbb{C}\mathrm{P}^2 \setminus \Omega$ is either 1, 2, 3 or infinite.*
2. *The number of lines contained in $\mathbb{C}\mathrm{P}^2 \setminus \Omega$ lying in general position is either 1, 2, 3, 4 or infinite.*

There are interesting examples of complex Kleinian groups having 1, 2, 3 of 4 complex lines in their limit set, and they are clearly classified according to this number, we can refer to [17],[19], [18] and [20] for more details and recent advances over this classification.

Finally but not least, the most recent advance on the two-dimensional Complex Kleinian groups is related to a duality property on the Kulkarni limit set.

Theorem 0.4.3 (Theorem 1 in [21]). *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be an infinite discrete subgroup acting on $\mathbb{C}\mathrm{P}^2$ without fixed points not invariant lines. Then*

$$\Lambda_{Kul}(G) = \bigcup_{\ell \in \hat{L}(G)} \ell.$$

Where $\hat{L}(G) \subset (\mathbb{C}\mathrm{P}^2)^*$ is an accumulation set of G^*.

The authors of the Theorem define $\hat{L}(\cdot)$ as an extension of Conze-Guivarc'h limit set, i.e., closure of proximal spaces. This theorem states a precise algebraic relation between this two types of limit set, in which it is easier to prove classical properties, we refer to [21] for the whole description of the properties.

1 First Steps Trough Higher Dimensional Setting

Introduction

Complex Kleinian groups where introduced in [145], their first study was in [122]. Since the first works in the area the higher dimensional groups were of particular interest, not only for the aim to look for more examples but their high complexity and apparently non-generalizable properties. One of the purposes of this chapter is to set the notations and proper tools to work in the higher dimensional setting in the particular case of cyclic groups.

This work was primarily addresed by L. Loeza in [107], where he studied and compared different notions of limit sets for the cyclic groups in higher dimensions and gave beautiful properties intrinsically related to the generator's class. Unfortunately, there was a gap in Loeza's formulations for the parabolic cyclic group class. In the later we fulfill the gap in Loeza's work and propose a new version of his theorem in terms of more adequate language for higher dimensions.

Although cyclic groups play a significant role in the understanding of higher dimensional groups, there exists a wider family of groups that is left unexplored. The existence of a loxodromic element in a group is a generic property for the classical Kleinian groups but in the complex case, it is only known for dimension two groups. We show that this property is also generic for higher dimensions, *i.e.,* if a group is Zarisky dense in $\mathrm{PSL}(n+1,\mathbb{C})$ then the group has a loxodromic element. The relevance of this theorem is related to the classical construction of the limit set by "attracting spaces" associated to loxodromic elements in $\mathrm{PSL}(2,\mathbb{C})$.

1.1 Parabolic Classes

L. Loeza in his Ph. D. thesis [107] worked with this comparison between limit sets of cyclic groups. He proved the following results:

Theorem 1.1.1 (The discontinuity set). *Let $\gamma \in \mathrm{PSL}(n+1,\mathbb{C})$ be a projective transformation. Then,*

1. *γ is elliptic if and only if the set of accumulation points of orbits of points in $\mathbb{C}\mathrm{P}^n$ under $\langle\gamma\rangle$ is either empty, if the element is of finite order, or the whole space $\mathbb{C}\mathrm{P}^n$, if the element if of infinite order.*
2. *γ is parabolic if and only if the set of accumulation points of orbits of points in $\mathbb{C}\mathrm{P}^n$ under $\langle\gamma\rangle$ is a single proper projective subspace of $\mathbb{C}\mathrm{P}^n$.*
3. *γ is loxodromic if and only if the set of accumulation points of orbits of points in $\mathbb{C}\mathrm{P}^n$ under $\langle\gamma\rangle$ is a finite union of at least two proper projective subspaces of $\mathbb{C}\mathrm{P}^n$.*

Theorem 1.1.2 (The equicontinuity region). *Let $\gamma \in \mathrm{PSL}(n+1,\mathbb{C})$ a projective transformation. Then*

1. *γ is elliptic if and only if the equicontinuity region of $\langle\gamma\rangle$ is the whole space $\mathbb{C}\mathrm{P}^n$.*
2. *γ is parabolic if and only if the complement of the equicontinuity region of $\langle\gamma\rangle$ is a single projective subspace.*

3. *γ is loxodromic if and only if the complement of the equicontinuity region of $\langle\gamma\rangle$ is the union of at least two proper projective subspaces.*

For the Kulkarni limit set, we need detailed study of the dynamical properties of the group and its blocks decomposition. In the following section we studied the Kulkarni limit set for the parabolic classes, the next theorem give a more detailed proof of the one given by L. Loeza in his Ph.D. thesis.

Theorem 1.1.3 (The Kulkarni limit set)**.** *Let $\gamma \in \mathrm{PSL}(n+1,\mathbb{C})$ be a projective transformation. Then,*

1. *γ is elliptic if and only if the Kulkarni limit set of $\langle\gamma\rangle$ is empty, if the element has finite order, or the whole space $\mathbb{C}\mathrm{P}^n$, if the element has infinite order. In this case, the Kulkarni discontinuity region of $\langle\gamma\rangle$ is the largest open set where the action of $\langle\gamma\rangle$ is properly discontinuous.*

2. *γ is parabolic if and only if the Kulkarni limit set of $\langle\gamma\rangle$ is a projective subspace. In this case, the Kulkarni discontinuity region might not be the largest open set where $\langle\gamma\rangle$ acts properly discontinuous.*

3. *γ is loxodromic if and only if the Kulkarni limit set of $\langle\gamma\rangle$ can be described as the union of two different proper projective subspaces. In this case, the Kulkarni discontinuity region might not be the largest open set where the action of $\langle\gamma\rangle$ is properly discontinuos.*

In the following we will proof the part 2 of the previous theorem, for the proof of the other parts we refer to [39].

1.1.1 The $\lambda-$Lemma

The dynamical version of the $\lambda-$lemma asserts that for a diffeomorphism of an open set of $\mathbb{R}^n$, the stable manifold can be approximated by an open set transverse to the unstable manifold, see [126].

The first adaption of the $\lambda-$lemma into the complex Kleinian setting was due by Navarrete is his Ph.D. thesis (see [122]). A major implication of this lemma asserts that for those groups that contain a loxodromic element its Kulkarni limit set contains a line.

The following part is inspired by the ideas in [68], in the literature we can find spacecial cases of the lemma like in [37] for complex hyperbolic groups, in [10] for discrete subgroups of $\mathrm{PU}(k, l)$ and for subgroups of $O(4, \mathbb{C})$ we have the Ph.D. thesis of Mayra Mendez [117]. The previous cited works use the KAK decomposition to understand the type of dynamics we could have in subgroups. We generalize the previous ideas and versions insight to use this tool for a better comprehension of the dynamics in higher dimensions.

The main tool to generalize the $\lambda-$lemma is the KAK decomposition from the Lie group and Lie algebra theory. The KAK decomposition, for matrix Lie groups, asserts that the group can be viewed as a product of a compact Lie group and a diagonal group.

First we will explain the KAK decomposition of $\mathrm{SL}(2, \mathbb{C})$ because it is simplier to prove. Let $g \in \mathrm{SL}(2, \mathbb{C})$ and consider gg^*, where g^* denotes the conjugated transpose, we know that gg^* is an Hermitian matrix of the form

$$gg^* = \begin{pmatrix} x & \overline{z} \\ z & y \end{pmatrix},$$

where $x + y - |z|^2 = 1$.

We can compute the proper values and vectors. The characteristic polynomial of gg^* is given by $p(\lambda) = \lambda^2 - (x+y)\lambda + 1$ whose roots are given by

$$2\lambda_j = x + y \pm \sqrt{(x+y)^2 - 4}$$

and the proper values associated to these proper values are

$$v_j = \begin{pmatrix} x - \lambda_j \\ -\overline{z} \end{pmatrix}$$

an easy computation shows that these vectors are orthogonal in the standar Hermitian product. If we take the normalization of the vectors we obtain an unitary matrix U such that

$$gg^* = U^* \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U.$$

We recall that $\lambda_1, \lambda_2 > 0$, so the matrix

$$S = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}$$

is well defined. From the equation $gg^* = U^*SSU$, we can obtain $g = U^*S(SU(g^*)^{-1})$ and it is easy to prove that $SU(g^*)^{-1} \in U(2)$.

From the previous paragraph, each $\mathrm{SL}(2,\mathbb{C})$ matrix can be expresed by a product of two matrices in $U(2)$ and a diagonal matrix whose entries are real positive, the last one is known as the Cartan projection of the original matrix.

The $KAK-$decomposition of an element of $\mathrm{SL}(n+1,\mathbb{C})$ can be obtained by a similar proccess, so we have that an elemento of $\mathrm{SL}(n+1,\mathbb{C})$ can be viewed as a product ua_+v where $u, v \in U(n)$ and a_+ is a diagonal matrix whose entries are in non-creasing order and all positive, more explicitely

$$a_+ = \begin{pmatrix} \lambda_1\mathbf{I}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \lambda_2\mathbf{I}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \lambda_3\mathbf{I}_3 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \lambda_s\mathbf{I}_s \end{pmatrix} \tag{1.1}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_s$, $\mathbf{I}_j$ is the identity matrix and $\sum_{j=1}^{s} rank(\mathbf{I}_j) = n+1$. We have to mention that the number of blocks and the dimensions of each block is not fixed for the whole elements, these depends on the element.

Have control on the number and dimensions of the blocks will be important at the time to prove the $\lambda-$lemma. The following norm depends on the Singular values of the matrix, this will be key to have control on the blocks.

Definition 1.1.4 (Ky-Fan norms). Let $M \in \mathrm{SL}(n+1,\mathbb{C})$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ the proper values of MM^*, we define the $(k,p)-$Ky-Fan norm, see [67], by:

$$|M|_{k,p} = \left(\sum_{j=1}^{k} \lambda_j^{\frac{p}{2}}\right)^{\frac{1}{p}}. \tag{1.2}$$

The following theorem implies a weakly domination on the singular values of a matrix difference with the singular values of each matrix.

Theorem 1.1.5 (Theorem 3.4.5 in [87]). *Let $A, B \in M_{m,n}(\mathbb{C})$, $\sigma_1(A) \geq \cdots \geq \sigma_q(A) \geq 0$, $\sigma_1(B) \geq \cdots \geq \sigma_q(B) \geq 0$, $\sigma_1(A-B) \geq \cdots \geq \sigma_q(A-B) \geq 0$, the singular values of A, B and $A-B$ respectively where $q = \min\{m,n\}$. Let $s_i(A-B) = |\sigma_i(A) - \sigma_i(B)|$ and let $s_{[1]}(A-B) \geq s_{[2]}(A-B) \geq \cdots \geq s_{[q]}(A-B)$ a non-decreasing ordering of the $s_i(A-B)$ values. Then for every $k = 1, \cdots, q$ we* *have*

$$\sum_{i=1}^{k} s_{[i]}(A-B) \leq \sum_{i=1}^{k} \sigma_i(A-B).$$

We have to recall that if we have a sequence that diverge in $\mathrm{PSL}(n+1,\mathbb{C})$, the ratios of the elements on the Cartan projection diverge as well, but with a divergent sequence on $\mathrm{PSL}(n+1,\mathbb{C})$ there exist a Pseudo-projective transformation as limit.

Proposition 1.1.6. *Let $(\gamma_m)_{m\in\mathbb{N}}, (\tau_m)_{m\in\mathbb{N}} \subset \mathrm{SL}(n,\mathbb{C})$ divergent sequences. Let $k, n_1, \cdots, n_k$ and $t, l_1, \cdots, l_t$ the indices for the Cartan projection,* i.e.*, the number of diagonal blocks and its respective ranks. Suppose that for some $M \in \mathbb{N}$, $|\gamma_m - \tau_m|_{n,1} < \varepsilon$ if $m \geq M$. Then $k = l$ and $n_i = l_i$.*

Proof. By the theorem 1.1.5, we know that $|a(\gamma_m) - a(\tau_m)| < \varepsilon$, where $a(\cdot)$ represents the Cartan projection. Let $(\alpha_{i,m})_{m\in\mathbb{N}}$ and $(\beta_{i,m})_{m\in\mathbb{N}}$ the sequences of real numbers for the Cartan projections of (γ_m) and (τ_m). It is sufficient to suppose that exists some n_i, l_{t_i}, l_{t_i+1} such that $n_i = l_{t_i} + l_{t_i+1}$, for $m \geq M$. We can check that

$$\beta_{t_i+1,m} - \beta_{t_i,m} < |\alpha_{i,m} - \beta_{t_i+1,m}| + |\alpha_{i,m} - \beta_{t_i,m}| < 2\varepsilon.$$

The previous implies that the sequence $(\beta_{i,m})_{m\in\mathbb{N}}$ is bounded, but we supposed that our sequences are divergent. So $k = t$ and $n_i = l_i$, for m large enough. □

Since the metrics in the space $\mathrm{SL}(n+1,\mathbb{C})$ are equivalent, the previous Proposition implies that for close sequences the Cartan projections are similar, where close sequences we are speaking of a specific metric, for example, the maximum norm metric. Even more, if we have a convergent sequence, the Cartan projection of the elements are similar close to the limit.

Theorem 1.1.7 ($\lambda-$lemma)**.** *Let $(g_k)_{k\in\mathbb{N}} \subset \mathrm{PSL}(n+1,\mathbb{C})$ a sequence that converge to a pseudo-projective transformation in $\mathrm{PsP}(n+1,\mathbb{C}) \setminus \mathrm{PSL}(n+1,\mathbb{C})$. There exists* *two partial flags* $(F_1, F_{12}, \cdots, F_{1\cdots l})$ *and* $(\tilde{F}_l, \tilde{F}_{l-1,l}, \cdots, \tilde{F}_{1\cdots l})$, *that depends on the sequence. If* $x \in \tilde{F}_{j\cdots l} \setminus \tilde{F}_{j+1\cdots l}$, *then* $g_k(x)$ *accumulates in* $F_{1\cdots j}$.

Proof. Since we have that the sequence converge to some $g \in \mathrm{PsP}(n+1,\mathbb{C}) \setminus \mathrm{PSL}(n+1,\mathbb{C})$, for some lifts $\tilde{g}_k$, $\tilde{g}$ in $\mathrm{Mat}(n+1,\mathbb{C})$ we have that $|\tilde{g}_{m+1} - \tilde{g}_{m+1}| < \varepsilon$ in some metric, but in particular $|\tilde{g}_{m+1} - \tilde{g}_m|_{n,1} < \varepsilon$. Then, by Proposition 1.1.6, there exists $M \in \mathbb{N}$ such that for $m \geq M$ the Cartan projection of the KAK-decomposition of g_m is of the form

$$a_m = \begin{pmatrix} \alpha_{1m} I_1 & & & \\ & \alpha_{2m} I_2 & & \\ & & \ddots & \\ & & & \alpha_{lm} I_l \end{pmatrix} \tag{1.3}$$

where the diagonal matrices $I_1, \cdots, I_l$ are fixed and $g_m = u_m a_m v_m$. Consider β_j a basis of the I_j-invariant subspace in $\mathbb{C}\mathrm{P}^n$ such that $\mathbb{C}\mathrm{P}^n = \left\langle\left\langle \bigcup_{j=1}^{l} \beta_j \right\rangle\right\rangle$. Lets take $F_{1\cdots j} = u\left(\left\langle\left\langle \bigcup_{k=1}^{j} \beta_k \right\rangle\right\rangle\right)$ and $\tilde{F}_{j\cdots l} = v^{-1}\left(\left\langle\left\langle \bigcup_{k=j}^{l} \beta_k \right\rangle\right\rangle\right)$ where u and v are the limits of $(u_m), (v_m)$ in $U(n+1)$.

Let $x = v^{-1}([x_0 : \cdots : x_l]) \in \tilde{F}_{j\cdots l} \setminus \tilde{F}_{j+1\cdots l}$ where $x_i \in \langle\langle \beta_i \rangle\rangle$, and for $t = 0, \cdots, j$ let $\kappa_{tm} = \Pi_{k=0 k\neq t}^{j} \alpha_{km}$. If we take the sequence (x_m) where

$$x_m = v_m^{-1}([\kappa_{1m}x_1 : \kappa_{2m}x_2 : \cdots : \kappa_{jm}x_j : \kappa_{jm}x_{j+1} : \cdots : \kappa_{jm}x_l])$$

we have that $x_m \underset{m\to\infty}{\longrightarrow} x$, even more

$$g_m(x_m) = u_m([\alpha_{1m}\kappa_{1m}x_1 : \alpha_{2m}\kappa_{2m}x_2 : \cdots : \alpha_{lm}\kappa_j x_l])$$

that converge to $u\left([x_0 : \cdots : x_j : 0 : \cdots : 0]\right) \in F_{1\cdots j+1}$ as $m \to \infty$. □

1.1.2 The Kulkarni Limit Set

The following Lemma, due by Loeza, summarizes the dynamical behavior of a parabolic element whose lift consists of a single Jordan block.

Lemma 1.1.8 (Lemma 3.5.9 in [107]). *Let $\gamma \in \mathrm{PSL}(n+1, \mathbb{C})$ such that γ has a lift $\tilde{\gamma}$ that is a $(n+1) \times (n+1)-$Jordan block with 1 as its unique proper value, then:*

1. *For each $n > k \geq 0$, the action of $[\wedge_k \tilde{\gamma}]$ in $\mathbb{P}(\bigwedge_{j=1}^{k} \mathbb{C}^n)$ has a unique fixed point in $\iota(Gr(k-1, n))$, the image of the Grassmanian under the Plücker embedding, namely*
$$[e_1 \wedge \cdots \wedge e_k].$$
2. *For each $\ell \in \mathrm{Gr}(k, n)$, we have that $\gamma^m(\ell) \to \langle\langle [e_1], \cdots, [e_{k+1}] \rangle\rangle$.*
3. *It is verified that $\mathbb{C}\mathrm{P}^n \setminus \mathrm{Eq}(\langle \gamma \rangle) = \langle\langle [e_1], \cdots, [e_n] \rangle\rangle$.*
4. *It is verified that $\Lambda_{Kul}(\langle \gamma \rangle) = \langle\langle [e_1], \cdots, [e_n] \rangle\rangle$.*

The following Lemma is an implication of the previous one and it completes the dynamical understanding of the parabolic elements whose block decomposition is made by a single Jordan block.

Lemma 1.1.9. *Let $\gamma \in \mathrm{PSL}(n+1, \mathbb{C})$ such that γ has a lift $\tilde{\gamma}$ that is a $(n+1) \times (n+1)-$Jordan block with 1 as its unique proper value. If γ is conjugated to an element of $\mathrm{PU}(k, l)$, then $k = \lfloor \frac{n+1}{2} \rfloor$ and $l = \lceil \frac{n+1}{2} \rceil$.*

Proof. Assume that γ is conjugated to some element in $\mathrm{PU}(k, l)$ for some $l, k \in \mathbb{N}$ and $l + k = n + 1$. Let $l - k \geq 2$ and $(\gamma_m)_{m \in \mathbb{N}} \subset \langle \gamma \rangle$ be a sequence of different elements. If we take the KAK-decomposition of each γ_m we get

$$\gamma_m = u_m \begin{bmatrix} D_m^1 & & \\ & I_{l-k} & \\ & & D_m^2 \end{bmatrix} v_m$$

where $D_m^1 = \text{Diag}(\alpha_{1m}, \cdots, \alpha_{km})$, $D_m^2 = \text{Diag}(\alpha_{km}^{-1}, \cdots, \alpha_{1m}^{-1})$ and I_{l-k} is the identity in $GL(l-k, \mathbb{C})$. The sequences $(u_m)_{m\in\mathbb{N}}, (v_m)_{m\in\mathbb{N}}$ are sequences of different unitary matrices. Without loose of generality we can assume that $\alpha_{im} \xrightarrow[m\to\infty]{} \infty$ and the sequences $(u_m)_{m\in\mathbb{N}}$ and $(v_m)_{m\in\mathbb{N}}$ converge to some u and v, respectively. Define the following subspaces

$$\begin{aligned} J_1 &:= u(\langle\langle [e_{k+1}], \cdots, [e_l] \rangle\rangle) \\ J_2 &:= v^{-1}([e_{k+1}], \cdots, [e_l] \rangle\rangle) \\ J_3 &:= u(\langle\langle [e_1], \cdots, [e_k] \rangle\rangle) \\ J_4 &:= v^{-1}(\langle\langle [e_1], \cdots, [e_k] \rangle\rangle) \end{aligned}$$

and the function

$$\begin{aligned} \phi : J_1 &\to J_2 \\ p &\mapsto uv(p) \end{aligned}$$

Take $\mathcal{L} \subset \mathbb{C}\mathrm{P}^n$ a $k-$dimensional projective subspace such that $J_4 \subset \mathcal{L} \subset \langle\langle J_2 \cup J_4 \rangle\rangle$. There is $p \in J_2$, such that $\mathcal{L} = \langle\langle J_4, p \rangle\rangle$ with $p = [\sum_{j=k+1}^{l} x_j e_j]$.

Define the following sequence

$$x_m = v^{-1}([\alpha_{1m}^{-2} x_1 : \alpha_{2m}^{-2} x_2 : \cdots : \alpha_{km}^{-2} x_k : x_{k+1}, \cdots, x_l, 0, \cdots, 0]), \tag{1.4}$$

clearly $(x_m)_{m\in\mathbb{N}} \subset \mathcal{L}$ and $\gamma_m(x_m) \xrightarrow[m\to\infty]{} \phi(p)$. Since $\gamma_m(\mathcal{L})$ converge to J_3, the compacity of the Grassmannian and the equation 1.4, we deduce that $\gamma_m(\mathcal{L}) \xrightarrow[m\to\infty]{} \langle\langle J_3, \phi(p)\rangle\rangle$ which contradicts the part 2 of the previous Lemma, so $0 < l - k < 2$. □

Example 1.1.10. Consider the block matrix $S = \text{Diag}(B, B)$ where

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

a $3 \times 3-$Jordan block. A straight computation give us that
$\Omega_{Kul}(\langle S\rangle) = \text{Eq}(\langle S\rangle)$ and

$$\Lambda_{Kul}(\langle S\rangle) = \langle\langle [e_1], [e_2], [e_4], [e_5]\rangle\rangle.$$

Example 1.1.11. Consider the block matrix $T = \text{Diag}(B, D)$ where B is as in the previous example and

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a $4 \times 4-$Jordan block. A straight computation give us that
$\Omega_{Kul}(\langle T\rangle) = \mathbb{C}\text{P}^6 \setminus \langle\langle [e_1], [e_2], [e_4], [e_5], [e_6]\rangle\rangle$, and $\text{Eq}(\langle T\rangle) = \mathbb{C}\text{P}^6 \setminus \langle\langle [e_1], \cdots, [e_6]\rangle\rangle$.

The previous examples tell us that the Kulkarni limit set depends on the characteristics of the Jordan blocks in the lift of a projective transformation.

Lemma 1.1.12. *Let $A \in \text{GL}(k, \mathbb{C})$ a diagonal matrix with each one of its proper values is a unitary complex number, and $B \in \text{GL}(l, \mathbb{C})$ a $l \times l-$Jordan block with 1* *as its unique proper value. Let* $\tilde{\gamma} \in \mathrm{GL}(l+k,\mathbb{C})$ *the matrix given by*

$$\tilde{\gamma} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If $\gamma = [\tilde{\gamma}] \in \mathrm{PSL}(l+k,\mathbb{C})$, *then*

$$\Omega_{Kul}(\langle\gamma\rangle) = \mathrm{Eq}(\langle\gamma\rangle) = \mathbb{C}\mathrm{P}^{l+k-1} \setminus \langle\langle [e_1], \cdots, [e_{l+k-1}] \rangle\rangle.$$

Proof. We will proceed by induction over l. For $l = 2$, the result follows from an easy computation. Let us assume that the claim follows for $1, \ldots, l$.

Set $\mathcal{L}_1 = \langle\langle [e_1], \cdots, [e_{k+l-1}] \rangle\rangle$, it will be enough to show that for $z \in \mathcal{L}_1 \setminus \langle\langle [e_1], \cdots, [e_k] \rangle\rangle$, it holds that $\langle\langle z, [e_{k+1}] \rangle\rangle \subset \Lambda_{Kul}(\langle\gamma\rangle)$. If we take γ restricted to $\mathcal{L}_2 = \langle\langle [e_1], \cdots, [e_{k+l}] \rangle\rangle$, by the inductive hypothesis, we have $\Lambda_{Kul}(\gamma|_{\mathcal{L}_2}) = \mathcal{L}_1$. There is a sequence $(z_m^1)_{m\in\mathbb{N}} \subset \mathcal{L}_2$ such that its cluster points lies in $\mathbb{C}\mathrm{P}^{l+k} \setminus \Lambda_{Kul}(\langle\gamma|_{\mathcal{L}}\rangle)$ and

$$\gamma^m(z_m^1) \xrightarrow[m\to\infty]{} z.$$

Since $\mathcal{L}_2$ is compact, we can assume $z_m^1 \xrightarrow[m\to\infty]{} z_1$, where $z_1 = [v_1, \cdots, v_{k+l} : 0]$ and $\sum_{j=1}^{l} |v_{k+j}|^2 \neq 0$. On the other hand, let $\mathcal{L}_3 = \langle\langle [e_{k+1}], \cdots, [e_{k+l+1}] \rangle\rangle$, and $\mathcal{M} \subset \mathcal{L}_3$ be a projective subspace of dimension $l-1$, such that $[e_{k+1}], [\sum_{j=1}^{l} v_{k+j} e_{k+j}] \notin \mathcal{M}$. Applying the Lemma 1.1.8 to $\gamma|_{\mathcal{L}_3}$ and $\mathcal{M}$ we conclude that there is a sequence $(z_m^2)_{m\in\mathbb{N}} \subset \mathcal{M}$ such that

$$\gamma^m(z_m^2) \xrightarrow[m\to\infty]{} [e_{k+l}] .$$

We can assume that $z_m^1 \neq z_m^2$ for each $m \in \mathbb{N}$. If we take $\ell_m = \langle\langle z_m^1, z_m^2 \rangle\rangle$, thus we get $\gamma^m(\ell_m) \xrightarrow[m\to\infty]{} \langle\langle z_1, [e_{k+l}] \rangle\rangle$. To finish the proof we have to observe that $\bigcup_{m\in\mathbb{N}} \ell_m$ does not intersect $\Lambda(\langle\gamma\rangle)$.

□

Lemma 1.1.13. *Let $B \in \mathrm{GL}(l,\mathbb{C})$ a $l \times l-$Jordan block such that 1 is its unique proper value and $\tilde{\gamma} \in \mathrm{GL}(2l,\mathbb{C})$ be given by*

$$\tilde{\gamma} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

If $\gamma = [\tilde{\gamma}]$, then

$$\Omega_{Kul}(\langle\gamma\rangle) = \mathrm{Eq}(\langle\gamma\rangle) = \mathbb{C}\mathrm{P}^{2l-1} \setminus \langle\langle [e_1], \cdots, [e_{l-1}], [e_{l+1}], \cdots, [e_{2l-1}] \rangle\rangle.$$

Proof. By Theorem 2.8 of [39] and Proposition 0.3.16, to conclude the claim it will be enough to show that

$$W = \langle\langle [e_1], \cdots, [e_{l-1}], [e_{l+1}], \cdots, [e_{2l-1}] \rangle\rangle \subset L_2(\langle\gamma\rangle).$$

Let $\ell_1 \subset \langle\langle [e_1], \cdots, [e_l] \rangle\rangle \setminus \{[e_1]\}$ and $\ell_2 \subset \langle\langle [e_{l+1}], \cdots, [e_{2l}] \rangle\rangle \setminus \{[e_{l+1}]\}$, be projective subspaces of dimension $l-2$. Define $\mathcal{L} = \langle\langle \ell_1 \cup \ell_2 \rangle\rangle$, clearly $\mathcal{L} \cap \Lambda(\langle\gamma\rangle) = \varnothing$. By lemma 1.1.8 part 2 we get that

$$\gamma^m(\mathcal{L}) \xrightarrow[m\to\infty]{} W.$$

□

The following lemmas are for the Ky-Fan norms on Definition 1.1.4, that will be key tools to compute the Kulkarni limit set for the parabolic classes.

Lemma 1.1.14 (Lemma 10.9 in [59])**.** *Let $A \in \mathrm{SL}(n,\mathbb{C})$, then*

$$|A|_{1,1}^2(|A|_{2,1} - |A|_{1,1})^2 = \max\{\det(V^*AA^*V) : V \in M(n\times 2,\mathbb{C}), VV^* = Id_2\} \quad (1.5)$$

where the maximum is taken over all matrices $V \in M(n \times 2,\mathbb{C})$.

Lemma 1.1.15. *Let $M \in \mathrm{SL}(n+1,\mathbb{C})$ a $(n+1)\times(n+1)-$Jordan block. If 1 is its unique proper value, then*

1. *There are $0 < s \leq r$, such that for every $m \in \mathbb{N}$,*

$$s\binom{m}{n} < |M^m|_{1,1} < r\binom{m}{n}.$$

2. *Also we have*

$$\frac{|M^m|_{2,1} - |M^m|_{1,1}}{m} \xrightarrow[m\to\infty]{} 0.$$

Proof. An inductive argument shows that

$$M^m = \begin{pmatrix} 1 & \binom{m}{1} & \binom{m}{2} & \dots & \binom{m}{n} \\ 0 & 1 & \binom{m}{1} & \dots & \binom{m}{n-1} \\ 0 & 0 & 1 & \dots & \binom{m}{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & 1 \end{pmatrix}.$$

Define $B_m = M^m(M^m)^*$, a simple computation shows that B_m is given by

$$\begin{pmatrix} \sum_{j=0}^{n}\binom{m}{j}^2 & \sum_{j=1}^{n}\binom{m}{j}\binom{m}{j-1} & \sum_{j=2}^{n}\binom{m}{j}\binom{m}{j-2} & \cdots & \sum_{j=n}^{n}\binom{m}{j}\binom{m}{j-n} \\ \sum_{j=1}^{n}\binom{m}{j}\binom{m}{j-1} & \sum_{j=0}^{n-1}\binom{m}{j}^2 & \sum_{j=1}^{n-1}\binom{m}{j}\binom{m}{j-1} & \cdots & \sum_{j=n-1}^{n-1}\binom{m}{j}\binom{m}{j-n+1} \\ \sum_{j=2}^{n}\binom{m}{j}\binom{m}{j-2} & \sum_{j=1}^{n-1}\binom{m}{j}\binom{m}{j-1} & \sum_{j=0}^{n-2}\binom{m}{j}^2 & \cdots & \sum_{j=n-2}^{n-2}\binom{m}{j}\binom{m}{j-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=n}^{n}\binom{m}{j}\binom{m}{j-n} & \sum_{j=n-1}^{n-1}\binom{m}{j}\binom{m}{j-n+1} & \sum_{j=n-2}^{n-2}\binom{m}{j}\binom{m}{j-n+2} & & \sum_{j=0}^{0}\binom{m}{j}^2 \end{pmatrix}.$$

Let us show 1. From the previous equation we can deduce that

$$\left[\binom{m}{n}^2 B_m\right] \xrightarrow[m\to\infty]{} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

thus there are $0 < s \leq r$ satisfying

$$s\binom{m}{n}^2 < |B_m|_{1,1} = |M^m|_{1,1}^2 < r\binom{m}{n}^2.$$

In order to conclude, let us show 2. By the previous lemma, there are some $V_m \in M((n+1)\times 2, \mathbb{C})$ such that

$$k_m = |M^m|_{1,1}^2(|M^m|_{2,1} - |M^m|_{1,1})^2 = \det(V_m^* M^m (M^m)^* V_m).$$

If we take

$$k_m \left(m\binom{m}{n}\right)^{-2} = \det\left(V_m \left(m\binom{m}{n}\right)^{-2} B_m V_m\right) \xrightarrow[m\to\infty]{} 0.$$

□

Lemma 1.1.16. *Let $A \in \mathrm{GL}(k,\mathbb{C})$ and $B \in \mathrm{GL}(l,\mathbb{C})$ be two Jordan blocks such that 1 is their unique proper value and $k > l$. If $\tilde{\gamma} \in \mathrm{GL}(k+l,\mathbb{C})$ is given by*

$$\tilde{\gamma} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

and $\gamma = [\tilde{\gamma}]$. Then

$$\Lambda_{Kul}(\langle\gamma\rangle) = \langle\langle [e_1], \cdots, [e_{k-1}], [e_{k+1}], \cdots, [e_{l+k-1}] \rangle\rangle.$$

Proof. By Lemma 1.1.8 and a result in [125], there are $\nu_1 \in \mathrm{SL}(k,\mathbb{C})$, $\nu_2 \in \mathrm{SL}(l,\mathbb{C})$, $H_1 \in \mathrm{GL}(k,\mathbb{C})$ and $H_2 \in \mathrm{GL}(l,\mathbb{C})$ Hermitian matrices of signature $(\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil)$, $(\lfloor \frac{l}{2} \rfloor, \lceil \frac{l}{2} \rceil)$, respectively. We know that $\hat{A} = \nu_1 A \nu_1^{-1} \in U(H_1)$ and $\hat{B} = \nu_2 B \nu_2^{-1} \in U(H_2)$, and there are sequences $(u_{1m})_{m\in\mathbb{N}}, (v_{1m})_{m\in\mathbb{N}} \subset U(k)$ and $(u_{2m})_{m\in\mathbb{N}}, (v_{2m})_{m\in\mathbb{N}} \subset U(l)$, such that

$$\begin{aligned} \hat{A}^m &= u_{1m}\mathrm{Diag}(\alpha_{1m}, \cdots, \alpha_{km}) v_{1m} \\ \hat{B}^m &= u_{2m}\mathrm{Diag}(\beta_{1m}, \cdots, \beta_{lm}) v_{2m} \end{aligned}$$

where $\alpha_{im} > \alpha_{i+1,m}$, $\beta_{im} > \beta_{i+1,m}$ and

$$\begin{aligned} \alpha_{\lfloor \frac{k}{2} \rfloor - i,m}\alpha_{\lceil \frac{k}{2} \rceil+1+i,m} &= 1 \qquad 0 \le i \le \left\lfloor \frac{k}{2} \right\rfloor - 1. \\ \beta_{\lfloor \frac{l}{2} \rfloor - i,m}\beta_{\lceil \frac{l}{2} \rceil+1+i,m} &= 1 \qquad 0 \le i \le \left\lfloor \frac{l}{2} \right\rfloor - 1. \end{aligned}$$

Since the dynamics of A, B and $\hat{A}, \hat{B}$ are conjugated by the Lemma 1.1.15, we can assure that

$$\frac{\alpha_{1m}}{\beta_{1m}} \xrightarrow[m\to\infty]{} \infty,$$

$$\frac{\beta_{1m}}{\max\{\alpha_{im}, \beta_{im} : i \ge 2\}} \xrightarrow[m\to\infty]{} \infty.$$

Up to conjugation, we can assume that

$$\hat{\gamma}^m = \hat{u}_m \begin{pmatrix} D_m & & \\ & \beta_{1m}^{-1} & \\ & & \alpha_{1m}^{-1} \end{pmatrix} \hat{v}_m.$$

where $\hat{u}_m, \hat{v}_m$ are unitary matrices and

$$D_m = \mathrm{Diag}(\alpha_{1m}, \cdots, \alpha_{k-1,m}, \beta_{1m}, \cdots, \beta_{l-1,m}).$$

Without loose of generality we can assume that $\hat{u}_m \xrightarrow[m\to\infty]{} \hat{u}$ and $\hat{v}_m \xrightarrow[m\to\infty]{} \hat{v}$. By the Theorem 1.1.7 we can assure that exist a sequence $(\zeta_m)_{m\in\mathbb{N}}$ such that

$$\zeta_m \xrightarrow[m\to\infty]{} p \in \hat{v}^{-1}(\langle\langle [e_{s+1}], [e_{s+2}]\rangle\rangle)$$

and

$$\hat{\gamma}^m(\zeta_m) \xrightarrow[m\to\infty]{} q \in \hat{u}(\langle\langle [e_1], \cdots, [e_s]\rangle\rangle).$$

From the previous discussions it is clear that

$$\hat{u}(\langle\langle [e_1], \cdots, [e_s]\rangle\rangle) = \langle\langle [e_1], \cdots, [e_s]\rangle\rangle,$$

$$\hat{v}^{-1}(\langle\langle [e_{s+1}], [e_{s+2}]\rangle\rangle) = \langle\langle [e_{s+1}], [e_{s+2}]\rangle\rangle.$$

Let $l \subset \langle\langle [e_1], \cdots, [e_{s+2}]\rangle\rangle \setminus \langle\langle [e_1]\rangle\rangle \cup \langle\langle [e_k]\rangle\rangle$ of dimension $l + k - 2$, which by Theorem 0.3.5 we have $l \cap \Lambda(\langle\hat{\gamma}\rangle) = \varnothing$. The sequence $(\zeta_m)_{m\in\mathbb{N}}$ implies that l accumulates in $\langle\langle [e_1], \cdots, [e_s\rangle\rangle$, *i. e.*, $\langle\langle [e_1], \cdots, [e_s\rangle\rangle \subset L_2(\langle\hat{\gamma}\rangle)$. So by Proposition 0.3.16, we have that $\Lambda_{Kul}(\langle\hat{\gamma}\rangle) = \langle\langle \Lambda_{Kul}(\langle\hat{A}\rangle) \cup \Lambda_{Kul}(\langle\hat{B}\rangle)\rangle\rangle$ and the result follows from the conjugation of the dynamics. □

Theorem 1.1.17. *Let $\gamma \in \mathrm{PSL}(n+1,\mathbb{C})$ a parabolic element, $\tilde{\gamma} \in \mathrm{SL}(n+1,\mathbb{C})$ a lift of γ and $(k, \{V_j\}_{j=1}^k, \{\beta_j\}_{j=1}^k, \{\lambda_j\}_{j=1}^k, \{\gamma_j\}_{j=1}^k)$ a block decomposition for $\tilde{\gamma}$. If $k \geq 2$ and for $j \geq 2$ it is verified that γ_j is a $l_j \times l_j-$Jordan block with unique proper value 1, then*

$$\Omega_{Kul}(\langle\gamma\rangle) = \mathbb{C}\mathrm{P}^n \setminus \left\langle\left\langle V_1 \cup \bigcup_{j=2}^{k} \Lambda_{Kul}(\langle\gamma_j\rangle)\right\rangle\right\rangle.$$

Proof. By the previous lemmas the interesting case is where the ranks of the blocks in the Jordan blocks in the block decomposition are different. Note that the set $C := \left\langle\left\langle V_1 \cup \bigcup_{j=2}^{k} \Lambda_{Kul}(\langle \gamma_j \rangle) \right\rangle\right\rangle$ is a $\gamma-$invariant closed set, even more by the lemmas 1.1.12 and 1.1.13 one can deduce that

$$C \subset \Lambda_{Kul}(\langle \gamma \rangle).$$

So it will be enough to prove the other contention. By the tools used in lemma 1.1.16, after a conjugation and a rearrangement, we have that

$$\tilde{\gamma}^{n_m} = u_{n_m} \begin{pmatrix} D_{n_m} & & & \\ & |\tilde{\gamma}_k^{n_m}|_{l_k,1} - |\tilde{\gamma}_k^{n_m}|_{l_k-1,1} & & \\ & & \ddots & \\ & & & |\tilde{\gamma}_2^{n_m}|_{l_2,1} - |\tilde{\gamma}_2^{n_m}|_{l_2-1,1} \end{pmatrix} v_{n_m} \tag{1.6}$$

where the elements of the diagonal D_{n_m} are greater than or equal to the

$$\max_{2\leq j\leq k}\{|\tilde{\gamma}_j^{n_m}|_{l_j,1} - |\tilde{\gamma}_j^{n_m}|_{l_j-1,1}\}$$

and if ρ_{n_m} is a element of the diagonal D_{n_m} we have that

$$\frac{\rho_{n_m}}{\max\limits_{2\leq j\leq k}\{|\tilde{\gamma}_j^{n_m}|_{l_j,1} - |\tilde{\gamma}_j^{n_m}|_{l_j-1,1}\}} \xrightarrow[m\to\infty]{} \infty \tag{1.7}$$

and

$$\frac{\max\limits_{2\leq j\leq k}\{|\tilde{\gamma}_j^{n_m}|_{l_j,1} - |\tilde{\gamma}_j^{n_m}|_{l_j-1,1}\}}{|\tilde{\gamma}_i^{n_m}|_{l_i,1} - |\tilde{\gamma}_i^{n_m}|_{l_i-1,1}} \xrightarrow[m\to\infty]{} \infty \tag{1.8}$$

for every $i = 2, \cdots, k$ different from the index where the maximum is obtained. So taking a subespace in $\mathbb{P}^n_{\mathbb{C}} \setminus \Lambda(\langle\gamma\rangle)$ we can assure that the cluster points of the orbit is contained in C, that finish the proof. □

1.2 Existence of Loxodromic Elements

Loxodromic elements play an important role in the dynamic of Kleinian groups, in this part we will show that in the case Zariski dense groups we can assure the existence of loxodromic elements.

Lemma 1.2.1. *Let* $(\gamma_m)_{m\in\mathbb{N}} \subset \mathrm{PSL}(n+1,\mathbb{C})$ *and* $\gamma \in \mathrm{PsP}(n+1,\mathbb{C})\setminus\mathrm{PSL}(n+1,\mathbb{C})$, *such that* $\gamma_m \xrightarrow[m\to\infty]{} \gamma$ *as pseudo-projective transformations. If* $\ker(\gamma)\cap\mathrm{im}(\gamma)=\varnothing$, *then for* m *large we get that* γ_m *is loxodromic.*

Proof. We have that $\gamma_m \xrightarrow[m\to\infty]{} \gamma$ uniformly on compact sets of $\mathbb{C}\mathrm{P}^n \setminus \ker(\gamma)$. Let W an open neighborhood of $\mathrm{im}(\gamma)$ such that $\overline{W} \subset \mathbb{C}\mathrm{P}^n \setminus \ker(\gamma)$, for m large we get that $\gamma_m(\overline{W}) \subset W$, that is γ_m is loxodromic. □

Lemma 1.2.2. *Let* $(M_m)_{m\in\mathbb{N}}, (N_m)_{m\in\mathbb{N}} \subset M(n,\mathbb{C})$ *be sequences of matrices such that* $M_m \xrightarrow[m\to\infty]{} M$ *and* $N_m \xrightarrow[m\to\infty]{} N$ *point-wise. If* $\mathrm{im}(N) \nsubseteq \ker(M)$ *that*

$$[M_m N_m] \xrightarrow[m\to\infty]{} [MN]$$

in the sense of pseudo-projective transformations.

Proof. By the continuity of the matrices, we deduce that $M_m N_m \xrightarrow[m\to\infty]{} MN$, therefore in order to conclude the proof we need to show that $MN \neq 0$, which is trivial since $\mathrm{im}(N) \nsubseteq \ker(M)$. □

From linear algebra we know that for every linear transformation $T:\mathbb{C}^n \to \mathbb{C}^n$ and every $1 \le k \le n$, we got a linear transformation $\bigwedge_k T : \bigwedge_{j=1}^k \mathbb{C}^n \to \bigwedge_{j=1}^k \mathbb{C}^n$ which is induced by

$$\bigwedge_k T(v_1 \wedge \cdots \wedge v_k) = T(v_1)\wedge\cdots\wedge T(v_k).$$

Definition 1.2.3. If we have a subgroup $\Gamma \subset \mathrm{PSL}(n, \mathbb{C})$, we define

$$\bigwedge_k \Gamma = \left\{ \left[\bigwedge_k \gamma \right] : [\gamma] \in \Gamma \right\}.$$

Lemma 1.2.4. *Let $(M_m)_{m\in\mathbb{N}} \subset M(n, \mathbb{C})$ a sequence of matrices, such that $M_m \xrightarrow[m\to\infty]{} M$ point-wise.*

1. *If $\dim(\mathrm{im}(M)) \geq k$, then $[\bigwedge_k M_m] \xrightarrow[m\to\infty]{} [\bigwedge_k M]$ in the sense of pseudo-projective transformations.*

2. *If $\dim(\mathrm{im}(M)) = k$, then:*

$$\mathrm{im}\left(\left[\bigwedge_k M \right] \right) = \iota_{k-1,n-1}(\mathrm{im}(M)).$$

Moreover,

$$\mathrm{im}\left(\left[\bigwedge_k M \right] \right) \subset \ker\left(\left[\bigwedge_k M \right] \right)$$

if and only if $\ker([M]) \cap \mathrm{im}([M]) \neq \varnothing$.

Proof. Let us show 1. By the continuity of the wedge product we have that

$$\bigwedge_k M_m \xrightarrow[m\to\infty]{} \bigwedge_k M,$$

therefore in order to conclude the proof, we need to show that $\bigwedge_k M \neq 0$. Let $v_1, \cdots, v_k \in \mathrm{im}(M)$ be linearly independent vectors and $\tilde{v}_1, \cdots, \tilde{v}_k$ be vectors such that $M(\tilde{v}_i) = v_i$, then

$$\left(\bigwedge_k M_m \right) (\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k) \xrightarrow[m\to\infty]{} \left(\bigwedge M \right) (\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_k) = v_1 \wedge \cdots \wedge v_k \neq 0.$$

The proof of part 2 goes as follows. Let $W \in \mathrm{Gr}(k-1, n-1)$, then we can choose $w_1, \cdots, w_k \in W$ so $\iota_{k-1,n-1}(W) = [w_1 \wedge \cdots \wedge w_k]$, in consequence

$$\left[\bigwedge_k M\right][w_1 \wedge \cdots \wedge w_k] = \begin{cases} \iota_{k-1,n-1}(\mathrm{im}(M)) & \text{if } W \cap \ker(M) = \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

□

Definition 1.2.5. Let Γ a subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$, we will say that Γ is *Plücker irreducible* if $\bigwedge^k \Gamma$ is irreducibl for every $k \leq n+1$e.

Lemma 1.2.6. *Let $\Gamma \subset \mathrm{SL}(n+1,\mathbb{C})$ a Zariski dense subgroup and W a subspace of $\mathbb{C}^{n+1}$. For $w \in W$ fixed, there exists $\gamma \in \Gamma$ such that $\gamma w \notin W$.*

Proof. The space W is given as a set of solutions of a linear system of polynomials. So, for $g \in \mathrm{SL}(n+1,\mathbb{C})$, the vector $gw \in W$ if the coordinates of gw satisfy the polynomial system. Then for every polynomial p, defining W, can be seen as a polynomial

$$P : \mathrm{SL}(n+1,\mathbb{C}) \to \mathbb{C}$$

given by $P(g) = p(gw)$. Since each of those polynomials is non zero in $\mathrm{SL}(n+1,\mathbb{C})$, we can assure that it is non zero in Γ, so there exists $\gamma \in \Gamma$ such that $\gamma w \notin W$. □

Theorem 1.2.7. *Let $\Gamma \subset \mathrm{PSL}(n+1,\mathbb{C})$ a Zariski dense subgroup, then Γ is Plücker irreducible.*

Proof. Assume that there exists $0 < k \leq n$ such that $\bigwedge^k \Gamma$ is not irreducible, therefore there is a linear subspace $W \subset \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ who is invariant under the action of $\bigwedge^k \Gamma$. Since W is defined as zeros of some linear system, we can obtain a linear system on $\mathrm{PSL}(n+1,\mathbb{C})$ and by the previous Lemma, we obtain that there exists $\gamma \in \Gamma$ such that the linear system is not zero, and then W is not fixed by $\bigwedge^k \gamma$. □

Lemma 1.2.8. *Let $\mathcal{L} \subset \mathrm{Gr}(n-1,n)$ be a infinite collection whose elements are in general position and $\mu : \mathcal{L} \to \mathbb{C}\mathrm{P}^n$ be such that $\mu(\ell) \in \ell$ for every $\ell \in \mathcal{L}$, then there are $\ell_1, \ell_2 \in \mathcal{L}$ such that $\mu(\ell_2) \notin \ell_1$ and $\mu(\ell_1) \notin \ell_2$.*

Proof. We will proceed by induction on n. For $n = 2$, see [18]. Now, assume that the result is valid for n but is not for $n+1$. Then there are $\mathcal{L} \subset \mathrm{Gr}(n, n+1)$ and $\mu : \mathcal{L} \to \mathbb{C}\mathrm{P}^n$ as in the hypothesis, such that for $\ell_1, \ell_2 \in \mathcal{L}$ we have either $\mu(\ell_1) \in \ell_2$ or $\mu(\ell_2) \in \ell_1$.

Since the elements in $\mathcal{L}$ are in general position we can assume that μ is injective. Take $\ell_0 \in \mathcal{L}$ fixed, there is $\mathcal{L}' \subset \mathcal{L}$ a infinite subset such that

$$\mu(\ell_0) \notin \ell \qquad \text{for any } \ell \in \mathcal{L}'.$$

Thus $\mathcal{L}' \cap \ell_0 = \{\ell \cap \ell_0 : \ell \in \mathcal{L}'\}$ is an infinite set of projective subspaces with codimension one in ℓ_0 and in general position. Moreover, if

$$\mu'(\ell \cap \ell_0) = \mu(\ell)$$

for every $\ell \in \mathcal{L}'$, we get that $\mu'(\ell \cap \ell_0) \in \ell \cap \ell_0$. By the inductive hypothesis, there are $\ell_1, \ell_2 \in \mathcal{L}'$ such that

$$\begin{aligned} \mu'(\ell_1 \cap \ell_0) &\notin \ell_0 \cap \ell_2 \\ \mu'(\ell_2 \cap \ell_0) &\notin \ell_0 \cap \ell_1 \end{aligned}$$

therefore $\mu(\ell_1) \notin \ell_2$ and $\mu(\ell_2) \notin \ell_1$ which is a contradiction. □

Theorem 1.2.9. *Let $\Gamma \subset \mathrm{PSL}(n+1, \mathbb{C})$ be a discrete Zariski dense group, then Γ contains a loxodromic element.*

Proof. Since Γ is discrete we conclude that there is $\gamma \in \Gamma$ with infinite order, see

Lemma 6.6 in [42]. Let us assume that γ is parabolic, then there is a strictly increasing sequence $(n_m) \in \mathbb{N}$ and $\rho_0 \in \mathrm{PsP}(n+1,\mathbb{C}) \setminus \mathrm{PSL}(n+1,\mathbb{C})$ such that

$$\rho_m = \gamma^{n_m} \xrightarrow[m\to\infty]{} \rho_0$$

as pseudo-projective transformations and $\mathrm{im}(\rho_0) \subset \ker(\rho_0)$. Let $\tilde{\rho}_m \in M(n+1,\mathbb{C})$ be a lift of ρ_m such that $\tilde{\rho}_m \xrightarrow[m\to\infty]{} \tilde{\rho}_0$ point-wise. Set $k = \dim(\mathrm{im}\rho_0) + 1$, by lemma 1.2.4

$$\bigwedge_k \rho_m = \left[\bigwedge_k \tilde{\rho}_m\right] \xrightarrow[m\to\infty]{} \bigwedge_k \rho_0 = \left[\bigwedge_k \tilde{\rho}_0\right]$$

and $\mathrm{im}\left(\bigwedge_k \rho_0\right)$ is a single point contained in $\ker\left(\bigwedge_k \rho_0\right)$. Since the action of $\bigwedge_k \Gamma$ is irreducible by Theorem 1.2.7, we deduce that there is a sequence of different elements $(\tau_m) \subset \bigwedge_k \Gamma$ such that $\mathcal{L} = \{\tau_m (\ker \bigwedge_k \rho_0) : m \in \mathbb{N}\}$ is a family of hyperplanes in general position.

Applying lemma 1.2.8 with $\mathcal{L}$ and $\mu : \mathcal{L} \to \mathbb{P}\left(\bigwedge_{j=1}^k \mathbb{C}^{n+1}\right)$ given by

$$\mu\left(\tau_m\left(\ker\left(\bigwedge_k \rho_0\right)\right)\right) = \tau_m\left(\mathrm{im}\left(\bigwedge_k \rho_0\right)\right).$$

We deduce that there are $j_0, i_0 \in \{1, \cdots, \dim(\bigwedge_{j=1}^k \mathbb{C}^{n+1})\}$ satisfying

$$\tau_{i_0}\left(\mathrm{im}\left(\bigwedge_k \rho_0\right)\right) \notin \tau_{j_0}\left(\ker\left(\bigwedge_k \rho_0\right)\right),$$
$$\tau_{j_0}\left(\mathrm{im}\left(\bigwedge_k \rho_0\right)\right) \notin \tau_{i_0}\left(\ker\left(\bigwedge_k \rho_0\right)\right),$$

If $\tilde{\tau}_{i_0}, \tilde{\tau}_{j_0} \in \mathrm{End}(\bigwedge_{j=1}^k \mathbb{C}^{n+1})$ are lifts of τ_{j_0} and τ_{i_0}, respectively, by lemma 1.2.2, we conclude that

$$\eta_m = \tau_{i_0}^{-1}\tau_{j_0}\left(\bigwedge_k \rho_m\right)\tau_{j_0}^{-1}\tau_{i_0}\left(\bigwedge_k \rho_m\right) \xrightarrow[m\to\infty]{} \eta_0 = \left[\tilde{\tau}_{i_0}^{-1}\tilde{\tau}_{j_0}\left(\bigwedge_k \tilde{\rho}_0\right)\tilde{\tau}_{j_0}^{-1}\tilde{\tau}_{i_0}\left(\bigwedge_k \tilde{\rho}_0\right)\right]. \tag{1.9}$$

Observe that

$$\ker(\eta_0) = \left[\left(\bigwedge_k \tilde{\rho}_0\right)^{-1}\left(\tilde{\tau}_{i_0}^{-1}\left(\tilde{\tau}_{j_0}\left(\ker\left(\widehat{\bigwedge_k}\right)\right)\right)\right)\setminus\{0\}\right]. \tag{1.10}$$

Since $\mathrm{im}(\eta_0) \notin \tau_{i_0}^{-1}\tau_{j_0}\left(\ker\left(\bigwedge_k \rho_0\right)\right)$ by the equation 1.10, we deduce

$$\begin{aligned}\ker(\eta_0) &= \ker\left(\bigwedge_k \rho_0\right),\\ \mathrm{im}(\eta_0) &= \tau_{i_0}^{-1}\tau_{j_0}\left(\bigwedge_k \rho_0\right).\end{aligned}$$

Thus $\mathrm{im}(\eta_0)$ is a point not contained in $\ker(\eta_0)$, by lemma 1.2.1 for m large, n_m is loxodromic. If $\eta \in \Gamma$ satisfies that $\bigwedge_k \eta = \eta_m$ we conclude that η is loxodromic.

□

Since the Zariski dense property is a generic, we can assure that the existence of Loxodromic elements is generic.

2 Comparison of Limit Sets in the case of Veronese Groups

Introduction

The notion of the limit set for higher dimensions is not unique and there are many definitions for this set that satisfy different properties, in contrast with the dimension two case. At first, Kulkarni's notion of the limit set was preferred and more studied, but for dimensions greater than two we do not know that this concept generically works with the full properties that we require for a limit set. Even if this problem is open, there few papers with the computations of limit sets in higher dimensions because it is extremely hard to compute; instead of relying on just one definition, the different notions of limit sets are been used for specific cases where the type of the group make the computations become easier, we refer to [42], [37] and [10] for examples of limit set computing in higher dimensions. Understand the relations between the different notions of a limit set is key in the understanding of the complex Kleinian groups.

From the Lie group structure of $\mathrm{SL}(\cdot, \mathbb{C})$ there exist a particular group morphism from $\mathrm{SL}(2, \mathbb{C})$ to $\mathrm{SL}(n+1, \mathbb{C})$ known as the irreducible representation, representation that can be projected into $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{PSL}(n+1, \mathbb{C})$. If we consider this group morphism, we have a tool to construct examples of groups in higher dimensions.

The images of discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ under this homomorphism are called *Veronese groups*, and after cyclic groups, they are the simplest groups in higher dimensions. In this Chapter, we studied the irreducible representation of $\mathrm{PSL}(2,\mathbb{C})$ into $\mathrm{PSL}(n+1,\mathbb{C})$ and the Veronese groups, we gave properties of this groups and how are related the different notions of limit set for this family of groups.

This is a second phase to totally understand how the different notions of limit sets are related for generic groups in higher dimensions.

2.1 The $\mathrm{PSL}(2,\mathbb{C})$ irreducible representation

Let H_n be the vector space of homogeneous polynomials in $\mathbb{C}[z,w]$ of degree n, we can endorse this vector space with a natural basis $\{e_j(z,w)\}$, where $e_j(z,w) = z^{n-j}w^j$. We will denote by $P(H_n)$ the projectivization of H_n, i. e., the set of equivalence classes under the natural action of $\mathbb{C}^*$ in $H_n \setminus \{0\}$.

The projective special linear group $\mathrm{PSL}(2,\mathbb{C})$, has a natural action on $P(H_n)$ as a change of variables in a representative of a polynomial class.

$$\begin{array}{rccc} \rho: & \mathrm{PSL}(2,\mathbb{C}) \times P(H_n) & \rightarrow & P(H_n) \\ & ([A],[p(z,w)]) & \mapsto & [p\left((z,w)\cdot A\right)] \end{array} \tag{2.1}$$

Remark 2.1.1.

1. In morphism (2.1), the reason that the multiplication of A is by the right, relies that in the matricial version the matrix product be the standard. In the case of take the left multiplication, in the matricial version we have to adjust the image by a transpose in order to obtain the standard matrix product.

2. There are morphisms that identify $P(H_n)$ with the $n-$symmetric power of $\mathbb{CP}^1$, to know $(\mathbb{CP}^1)^n/S_n$, and this one with $\mathbb{CP}^n$. So we can think that the ρ action is a $\mathrm{PSL}(2,\mathbb{C})$ action on $\mathbb{CP}^n$.

We can translate the ρ action to a matrix representation, if we look in how a matrix A acts on the elements of the basis $\{e_j(z,w)\}$, computing

$$\rho(A,[e_m]) = (az+cw)^{n-m}(bz+dw)^m \quad \text{with} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

With a straight computation we can deduce

$$\rho(A,[e_m(z,w)]) = \sum_{j=0}^{n} \left[\sum_{k=\delta_{j,m}}^{\Delta_{j,m}} \binom{n-m}{k}\binom{m}{j-k} a^{n-m-k}c^k b^{m-j+k} d^{j-k} \right] z^{n-j}w^j \tag{2.2}$$

where $\delta_{j,m} = \max\{j-m,n\}$ and $\Delta_{j,m} = \min\{j,n-m\}$, and we can do this for every $m = 0,\cdots,n$. So we obtain a map between PSL$(2,\mathbb{C})$ and PSL$(n+1,\mathbb{C})$, that we will still denote ρ.

Lemma 2.1.2. *The map* $\rho : \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PSL}(n+1,\mathbb{C})$ *described above, is a well defined group morphism; even more, it is injective.*

Proof. The group morphism property follows from the action ρ. The injectivity follows from the form of $\rho([A])$ for a class $[A]$, to know

$$\rho(A) = \begin{bmatrix} a^n & \cdots & b^n \\ \vdots & \ddots & \vdots \\ c^n & \cdots & d^n \end{bmatrix} = [a_{ij}],$$

if $\rho([A]) = [Id_{n+1}]$, then $b^n = c^n = 0$, so $b = c = 0$, and the

$$a_{jj} = a^{n-j}d^j = 1,$$

for every $j = 0,\cdots,n$. In particular, $a^{n-1}d = 1$, and with this we have $a = d$, do $[A] = [Id_2]$. □

Remark 2.1.3. The morphism ρ is known as the irreducible representation of $\mathrm{PSL}(2,\mathbb{C})$ in $\mathrm{PSL}(n+1,\mathbb{C})$.

In what follows we will describe some properties related to the morphism and the elements of $\mathrm{PSL}(\cdot,\mathbb{C})$.

Proposition 2.1.4. *The representation ρ is type preserving, i. e., sends parabolic in parabolic, loxodromic in (strongly) loxodromic and elliptic in elliptic elements. Even more, if $G < \mathrm{PSL}(2,\mathbb{C})$ is a discrete subgroup purely loxodromic, then $\rho(G)$ is discrete and purely strongly loxodromic.*

Proof. In the sense that ϱ is a representation, it will be sufficient to show the assertion in the matrix of the form

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

First we calculate $\varrho(A)$, in the formula (2.2)

- If $j = m$, then $\delta_{j,m} = 0$ and

$$\begin{aligned} \sum_{k=0}^{\Delta_{j,j}} \binom{n-j}{k}\binom{j}{j-k} a^{n-m-k}c^k b^k d^{j-k} \\ = \binom{n-j}{0}\binom{j}{j} a^{n-j}b^0c^0d^j \\ = \lambda^{n-j}\lambda^{-j}. \end{aligned}$$

- If $j < m$, then in every term of (2.2) appears a b^{m-j+k} where $m-j+k > 0$ because $k = \delta_{j,m}, \cdots, \Delta_{j,m}$ and $\delta_{j,m} = 0$, but in A, $b = 0$, so the $j \times m$ element is zero.

- If $j > m$, in a similar way, in every term of (2.2) appears c^k where $k = \delta_{j,m}, \cdots, \Delta_{j,m}$ and $\delta_{j,m} = j - m > 0$, and for A, we have that $c = 0$, therefore the $j \times m$ element is zero.

This implies that $\varrho(A) = Diag(\lambda^{n-2m})$. Now let us calculate $\varrho(B)$,

- If $j > m$, then in every term of (2.2) appears c^k where for $k \geq \delta_{j,m} = j - m$, setting this sum equal to zero because $c = 0$.
- If $j \leq m$, then

$$\begin{aligned}\sum_{k=0}^{\Delta_{j,m}} \binom{n-m}{k}\binom{m}{j-k} a^{n-m-k}c^k b^{m-j+k} d^{j-k} \\ = \binom{n-m}{0}\binom{m}{j} a^{n-m} b^{m-j} d^{j-k} \\ = \binom{m}{j}\end{aligned}$$

Therefore

$$\rho(B) = \begin{bmatrix} 1 & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{n}{0} \\ 0 & 1 & \binom{2}{1} & \cdots & \binom{n}{1} \\ 0 & 0 & 1 & \cdots & \binom{n}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

in a inductive way we can prove that for $n > 1$, an easy computation shows that the previous matrix has a Jordan block decomposition of the form

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

To finish the proof, let us assume that $\rho(G)$ is not discrete. Let $(A_m) \subset G$ a sequence, such that

$$A_m = \begin{bmatrix} a_m & b_m \\ c_m & d_m \end{bmatrix}$$

and that $\rho(A_m) \to I_{n+1}$, where I_{n+1} the identity class in $\mathrm{PSL}(n+1,\mathbb{C})$. The form of the $\rho(A_m)$ matrices we can assure that $b_m^n = c_m^n \to 0$, and $a_m^{n-j} b_m^j \to 1$ for every $j = 0, \cdots, n$. We can conclude that $a_m = d_m \to 1$ and this is a contradiction, because G is discrete, therefore $\rho(G)$ is discrete. □

The previous Lemma provides explicitly the form of the elements in $\rho(\mathrm{PSL}(2,\mathbb{C}))$. Even more, with this in mind is not difficult to show that the KAK-decomposition is preserved by ϱ.

2.1.1 The Veronese Curve

Let

$$\begin{array}{rccc} \psi: & \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^n \\ & [z:w] & \mapsto & \left[z^n : \cdots : \binom{n}{m} z^{n-m} w^m : \cdots : w^n\right] \end{array} \tag{2.3}$$

this map is an embedding of $\mathbb{C}\mathrm{P}^1$ into $\mathbb{C}\mathrm{P}^n$ known as the *Veronese embedding.* We will denote by $\mathscr{C}_n$ the image of $\mathbb{C}\mathrm{P}^1$ under ψ, $\mathscr{C}_n$ is a normal rational algebraic curve called the *Veronese curve.*

The following Lemmas give geometric properties of the Veronese curve.

Lemma 2.1.5. *Every subset of $n+1$ different points in $\mathscr{C}_n$ is linearly independent.*

Proof. Let $\{p_j = \psi([1:t_j])\}_{j=0}^n$ with $t_j \neq t_k$ if $j \neq k$. Let $\sum_{j=0}^n a_j p_j = 0$ a linear combination, or equivalently

$$\begin{pmatrix} 1 & \binom{n}{1}t_0 & \binom{n}{2}t_0^2 & \cdots & t_0^n \\ 1 & \binom{n}{1}t_1 & \binom{n}{2}t_1^2 & \cdots & t_1^n \\ 1 & \binom{n}{1}t_2 & \binom{n}{2}t_2^2 & \cdots & t_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{n}{1}t_n & \binom{n}{2}t_n^2 & \cdots & t_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

the square matrix is a Van der Monde's matrix whose determinant is non-zero if and only if the $t_j \neq t_k$ for every $j = 0, \cdots, n$. Therefore $\{p_j\}_{j=0}^n$ is linearly independent. □

Proposition 2.1.6. *Every subset of $m > n+1$ different points of $\mathscr{C}_n$ is in general position.*

Proof. The proof follows from the previous lemma. We know that every $n+1$ different points in $\mathscr{C}_n$ are linearly independent, i. e., don't belong to the same hyperplane. Therefore every subset of $n+1$ points of the first m are in general position and then the subset of the m points is in general position. □

The following proposition characterize the set of all the projective automorphisms of the Veronese curve.

Proposition 2.1.7. *The group of projective automorphisms of $\mathscr{C}_n$ is $\rho(\mathrm{PSL}(2,\mathbb{C}))$.*

Proof. Let $p = [z : w] \in \mathbb{C}\mathrm{P}^1$ and $A \in \mathrm{PSL}(2,\mathbb{C})$, with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; ad - bc \neq 0.$$

The elements of $\rho(A)\cdot\psi(p)$ have the form

$$\sum_{m=0}^{n}\left[\binom{n}{j}z^{n-j}w^{j}\sum_{k=\delta_{j,m}}^{\Delta_{j,m}}\binom{n-m}{k}\binom{m}{j-k}a^{n-m-k}c^{k}b^{m-j+k}d^{j-k}\right] \tag{2.4}$$

After a rearrangement of (2.4), we obtain

$$\binom{n}{j}c^{j}z^{s}(az+bw)^{n-j}+\binom{n}{j}\binom{j}{1}c^{j-1}dz^{j-1}w(az+bw)^{n-s}+\cdots+\binom{n}{j}d^{j}w^{j}(az+bw)^{n-j}.$$

That is equal to $\binom{n}{j}(az+bw)^{n-j}(cz+dw)^{j}$, and therefore $\rho(A)\cdot\psi(p)=\psi(A\cdot p)\in\mathscr{C}_n$. We can conclude that $\rho(\mathrm{PSL}(2,\mathbb{C}))$ leaves invariant the curve $\mathscr{C}_n$. Assume that there is an element $B\in\mathrm{PSL}(n+1,\mathbb{C})$ such that $B(\mathscr{C}_n)=\mathscr{C}_n$. Denote by $\tilde{B}$ the map from $\mathbb{C}\mathrm{P}^1$ into $\mathbb{C}\mathrm{P}^1$ given by $\tilde{B}([z:w])=\psi^{-1}B\psi([z:w])$. The map $\tilde{B}$ is an holomorphic map, so belongs to $\mathrm{PSL}(2,\mathbb{C})$; even more the following diagram commutes,

$$\begin{array}{ccc} \mathbb{C}\mathrm{P}^1 & \xrightarrow{\tilde{B}} & \mathbb{C}\mathrm{P}^1 \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \mathscr{C}_n & \xrightarrow[B]{} & \mathscr{C}_n \end{array}$$

We can assure that $B|_{\mathscr{C}_n}=\rho(\tilde{B})|_{\mathscr{C}_n}$. Let us take $n+2$ different points of $\mathscr{C}_n$ in general position, the transformation $B\rho(\tilde{B})^{-1}$ fix the $n+2$ points, this implies that $B\rho(\tilde{B})^{-1}=Id_{n+1}$ in $\mathbb{C}\mathrm{P}^n$. So, the projective automorphisms of $\mathscr{C}_n$ is $\rho(\mathrm{PSL}(2,\mathbb{C}))$.

□

Corollary 2.1.8. *The Veronese embedding ψ is* $\mathrm{PSL}(2,\mathbb{C})-$*equivariant. In particular, for every $\Gamma\subset\mathrm{PSL}(2,\mathbb{C})$ group, ψ is $\Gamma-$equivariant.*

Remark 2.1.9. The previous Corollary implies that the action of subgroups of $\rho(\mathrm{PSL}(2,\mathbb{C}))$ on $\mathscr{C}$ is essentially the well know action of subgroups of $\mathrm{PSL}(2,\mathbb{C})$ in $\mathbb{C}\mathrm{P}^1$.

Definition 2.1.10. A Veronese group is the image under ρ of a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, i. e., a Kleinian group.

As we know, for complex Kleinian groups the limit sets are made of projective subspaces of $\mathbb{C}\mathrm{P}^n$. For complex Kleinian groups of $\mathrm{PU}(n,1)$ this projective subspaces are hyperplanes tangent to the ball in $\mathbb{C}\mathrm{P}^n$. Since the Veronese curve is of codimension $n-1$, intuitively there tangent space to the curve has to be of codimention $n-1$. From the algebraic structure of the Veronese group we can associate an hyperplane to every point of the curve and such point belonging to the hyperplane. The previous hyperplane will be usefull in what follows.

Definition 2.1.11 ([82])**.** Let $\phi : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ a smooth curve defined by

$$\phi([z:w]) = [v_0(z,w) : \cdots : v_n(z,w)]\,.$$

The osculating hyperplane of $\phi(\mathbb{C}\mathrm{P}^1)$ at $p = \phi([1:k])$ is the space spanned by the rows of the matrix

$$\left.\begin{pmatrix} v_0 & v_1 & \cdots & v_{n-1} & v_n \\ v_0^{(1)} & v_1^{(1)} & \cdots & v_{n-1}^{(1)} & v_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_0^{(n-1)} & v_1^{(n-1)} & \cdots & v_{n-1}^{(n-1)} & v_n^{(n-1)} \end{pmatrix}\right|_{[1:k]} \tag{2.5}$$

Remark 2.1.12. We recall the following facts

i. The osculating hyperplane of a curve $\phi(\mathbb{C}\mathrm{P}^1)$ is an hyperplane in $\mathbb{C}\mathrm{P}^n$ that intersects the curve in just one point.

ii. There is a mapping $\Pi_n : \phi(\mathbb{C}\mathrm{P}^1) \to (\mathbb{C}\mathrm{P}^n)^*$, where $(\mathbb{C}\mathrm{P}^n)^*$ is the dual space of $\mathbb{C}\mathrm{P}^n$, that identifies a point with its osculating hyperplane and it is known as a polarity, we refer to [71] and [55] for more information.

Lemma 2.1.13. *The osculating hyperplane of $\mathscr{C}$ at $p = \psi([1:t])$ is given by the equation*

$$\mathscr{L}_p := \sum_{j=0}^{n} (-1)^j t^j z_{n-j} = 0 \tag{2.6}$$

where $[z_0 : \cdots : z_n]$ are the homogeneous coordinates of $\mathbb{C}\mathrm{P}^n$.

Proof. A parametrization for the Veronese curve is given by $v_j([1:t])\binom{n}{j}t^j$, a straight computation shows that $v_j^{(k)}([1,t]) = \binom{n}{j}\frac{n!}{(j-k)!}t^{j-k}$, for $j > k$ and $v_j^{(k)}([1:t]) = 0$ other way. The matrix in (2.5) is of the form

$$F = \begin{pmatrix} 1 & \binom{n}{1}t & \binom{n}{2}t^2 & \cdots & \binom{n}{n-1}t^{n-1} & t^n \\ 0 & \binom{n}{1} & \binom{n}{2}2t & \cdots & \binom{n}{n-1}(n-1)t^{n-2} & nt^{n-1} \\ 0 & 0 & \binom{n}{2}2 & \cdots & \binom{n}{n-1}\frac{(n-1)!}{(n-3)!}t^{n-3} & n(n-1)t^{n-2} \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\frac{(n-1)!}{(n-4)!}t^{n-4} & \frac{n!}{(n-3)!}t^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}(n-1)! & n!t \end{pmatrix}.$$

It will be sufficient to compute the null space of F. A vector $p = (a_0, \cdots, a_n) \in \mathbb{C}\mathrm{P}^n$ is in the null space of F if satisfies the following equations

$$\frac{n!}{j!}a_{n-j} + \sum_{k=1}^{j} \frac{n!}{(j-k)!k!}t^k a_{n-j+k} = 0,\ j = 1, \cdots, n. \tag{2.7}$$

We claim that $a_{n-j} = (-t)^j a_n$. We will proceed by induction, for $j = 1$, we have that $n!a_{n-1} + n!ta_n = 0$, let us suppose that it's valid for $j = 1, \cdots, j-1$. By hypothesis the equation (2.7) is equivalent to

$$\frac{n!}{j!}a_{n-j} + \frac{n!}{j!}\left(\sum_{s=1}^{j}\binom{j}{j-s}(-1)^{j-s}\right)t^j a_n.$$

If j is even, then

$$\begin{aligned}\sum_{s=1}^{j}\binom{j}{j-s}(-1)^{j-s} &= \binom{j}{j-1}-\binom{j}{j-2}+\binom{j}{j-3}-\cdots-\binom{j}{3}+\binom{j}{2}-\binom{j}{1}+\binom{j}{0}\\ &= \binom{j}{0}.\end{aligned}$$

and $a_{n-j} = (-t)^j a_n$. If j is odd, then

$$\begin{aligned}\sum_{s=1}^{j}\binom{j}{j-s}(-1)^{j-s} &= 2\sum_{s=1}^{\frac{j}{2}-1}(-1)^s\binom{j}{s}+(-1)^{\frac{j}{2}}\binom{j}{j/2}+\binom{j}{0}\\ &= 2\sum_{s=1}^{\frac{j}{2}-1}(-1)^s\left[\binom{j-1}{s-1}+\binom{j-1}{s}\right]+2(-1)^{\frac{j}{2}}\binom{j-1}{\frac{j}{2}-1}+\binom{j}{0}\\ &= -2\binom{j-1}{0}+\binom{j}{0}\\ &= -1.\end{aligned}$$

and $a_{n-j} = (-t)^j a_n$.

By the previous the null space of F is spanned by $\alpha = [(-t)^n : (-t)^{n-1} : \cdots : -t : 1]$, so the row space is the hyperplane orthogonal to α, and from this we obtain the equation for the osculating hyperplane. □

2.2 The Equicontinuity region

The following ideas are inspired by [42] and the main theorem of [122].

In order to talk about the Equicontinuity region we have to look for the Pseudo-projective set of matrices.

It is not difficult to show that we can extend the irreducible representation $\rho : \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PSL}(n+1,\mathbb{C})$ to the Pseudo-projective completation. This is because the morphism ρ is continuous and the Pseudo-projective completation is made with sequences of matrices.

Proposition 2.2.1 ([42]). *Let $(\gamma_m)_{m\in\mathbb{N}} \subset \mathrm{PSL}(n+1,\mathbb{C})$ a sequence of different elements, then*

i. There is a subsequence $(\tau_m)_{m\in\mathbb{N}} \subset (\gamma_m)_{m\in\mathbb{N}}$ and $\tau_0 \in M(n+1,\mathbb{C}) \setminus \{0\}$, such that $\tau_m \to [[\tau_0]]$ as $m \to \infty$ as points in $\mathrm{PsP}(n+1,\mathbb{C})$.

ii. If $(\tau_m)_{m\in\mathbb{N}}$ is the sequence given by the previous part, then $\tau_m \to [[\tau_0]]$, as functions, uniformly in compact sets of $\mathbb{C}\mathrm{P}^n \setminus \ker[[\tau_0]]$.

Theorem 2.2.2. *Let Γ a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and $G = \rho(\Gamma)$ the correspondent Veronese group, then*

$$\mathbb{C}\mathrm{P}^n \setminus \mathrm{Eq}(G) = \bigcup_{z\in\Lambda(\Gamma)} T_{\psi(z)}\mathscr{C}. \tag{2.8}$$

where $\Lambda(\Gamma)$ is the limit set of Γ for its action on $\mathbb{C}\mathrm{P}^1$ and $T_z\mathscr{C}$ is the osculating hyperplane to $\mathscr{C}$ in z.

Proof. We can extend by continuity the morphism ρ to a map from $\mathrm{PsP}(2,\mathbb{C})$ into $\mathrm{PsP}(n+1,\mathbb{C})$, if $\gamma = \lim\limits_{m\to\infty} \gamma_m \in \mathrm{PsP}(2,\mathbb{C})$, then $\rho(\gamma) = \lim\limits_{m\to\infty} \rho(\gamma)$.

Let us assume that after a conjugation $[1:0], [0:1]$ do not belong to $\Lambda(\Gamma)$. Let $[1:k] \in \Lambda(\Gamma)$, following the ideas of [42] there is a sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$, such that $\gamma_n \to \gamma \in \mathrm{PsP}(2,\mathbb{C})$ and $\ker\gamma = [1:k]$, even more we can assure that

$$\begin{bmatrix} -bk & b \\ -dk & d \end{bmatrix}$$

and by the previous paragraph, under the map ρ we have a sequence in $\mathrm{PSL}(n+$

$1, \mathbb{C})$ and its limit in $\mathrm{PsP}(n+1, \mathbb{C})$, where the limit is of the form

$$\rho(\gamma) = \begin{bmatrix} (-k)^n b^n & (-k)^{n-1} b^n & \cdots & -kb^n & b^n \\ \binom{n}{1}(-k)^n b^{n-1} d & \binom{n}{1}(-k)^{n-1} b^{n-1} d & \cdots & \binom{n}{1}(-k) b^{n-1} d & \binom{n}{1} b^{n-1} d \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n}{n-1}(-k)^n b d^{n-1} & \binom{n}{n-1}(-k)^{n-1} b d^{n-1} & \cdots & \binom{n}{n-1}(-k) b d^{n-1} & \binom{n}{n-1} b d^{n-1} \\ (-k)^n d^n & (-k)^n d^n & \cdots & (-k) d^n & d^n \end{bmatrix}$$

A straight computation give us that the kernel of $\rho(\gamma)$ is spanned by $\beta = \{e_1 - (-k)^m e_{m+1}\}$, where $\{e_i\}_{i=1}^{m+1}$ is the standard basis of $\mathbb{C}^{n+1}$. By the lemma 2.1.13 the hyperplane spanned by β is the osculating hyperplane of $\mathscr{C}$ at $\psi([1:k])$, and by proposition 2.2.1 we obtain the theorem. □

The previous Theorem is in someway a generalization of the main theorem in [42]

Theorem 2.2.3. *Lemma 5.1 in [42] Given a discrete subgroup* $G \subset \mathrm{PU}(1,n)$. *The equicontinuity region of* G *is given by* $\mathrm{Eq}(G) = \mathbb{CP}^n \setminus \mathcal{C}(G)$, *where* $\mathcal{C}(G)$ *is defined as*

$$\bigcup_{p \in \mathrm{Lim}(G)} p^{\perp}$$

and $p^{\perp}$ *is the orthogonal complement of* p *in the Hermitian product preserved by* $\mathrm{PU}(1,n)$.

In both results, the equicontinuity region is in the same spirit as the computed in [122] where the author states that the complement of the Equicontinuity region is made as dual spaces of limit points in a sphere. We would expect the set described in the previous Theorem to be the Kulkarni limit set.

2.2.1 The Irreducible Action of Veronese Groups

The following Lemmas are known for the Veronese groups, but the proofs that we present are geometrical.

Lemma 2.2.4. *Let* $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ *a non-elementary Kleinian group. Let* $\{p_0, \cdots, p_n\} \subset \Lambda(\Gamma)$ *a set of* $n+1$ *different points. Then the set of osculating hyperplanes* $T_{\psi(p_i)}\mathscr{C}$ *is in general position.*

Proof. We can assume that after conjugation $[1:0], [0:1] \notin \Lambda(\Gamma)$. Let $p_i = [1:t_i]$, by lemma 2.1.13 we know that the osculating hyperplane to $\mathscr{C}$ in $\psi(p_i)$ satisfies the equation

$$\sum_{j=0}^{n} (-1)^j t_i^j z_{n-j} = 0.$$

A linear combination of points in these hyperplanes has to satisfy all the equations, i. e., belong to the null space of the matrix

$$\begin{pmatrix} (-t_0)^n & (-t_0)^{n-1} & (-t_0)^{n-2} & \cdots & 1 \\ (-t_1)^n & (-t_1)^{n-1} & (-t_1)^{n-2} & \cdots & 1 \\ (-t_2)^n & (-t_2)^{n-1} & (-t_2)^{n-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-t_n)^n & (-t_n)^{n-1} & (-t_n)^{n-2} & \cdots & 1 \end{pmatrix}$$

which is a Van derMonde matrix whose by hypothesis is invertible and the lemma follows. □

Proposition 2.2.5. *Let* $G < \rho(\mathrm{PSL}(2,\mathbb{C}))$ *a non-elementary Veronese group, then the action of* G *in* $\mathbb{C}\mathrm{P}^n$ *is irreducible.*

Proof. Let us assume that there is a proper vector subspace of $\mathbb{C}\mathrm{P}^n$, L, that is $G-$invariant. By the Bézout's theorem (see [82]), L intersects $\mathscr{C}$ in a finite number of points. By hypothesis L is $G-$invariant, so every element of G permutes the intersection points. The previous statement implies that the elements of G are of finite order, but this is a contradiction because G is the image of a non-elementary Kleinian group, in particular whose elements are of infinite order. $\square$

2.3 Kulkarni Limit Set of Veronese Groups

The following section will be dedicated to the comprehension of the Kulkarni limit set, for that reason we will deal with the description of the parts of the Kulkarni limit set for a Veronese Group. The reason to look for the Kulkarni limit set is because its complement in $\mathbb{CP}^n$ is a set where the action of the group is properly-discontinous. Since we look in the previous Chapter, there are examples of groups where the complement of the Equicontinuity region and the Kulkarni limit set does not coincide.

Lemma 2.3.1. *Let* $\rho(\Gamma) < \mathrm{PSL}(n+1,\mathbb{C})$ *a Veronese group with* Γ *infinite, then* $L_0(\rho(\Gamma))$ *(see Definition 0.3.15) is contained in a finite union of algebraic curves.*

Proof. Let $x \in \mathbb{CP}^n$ a point with infinite isotropy group for $\rho(\Gamma)$. Therefore x has to be either a fixed point of a parabolic element γ_1 or a fixed point of a loxodromic element γ_2.

If x is a fixed point of γ_1, we know that after conjugation this parabolic element is a $(n+1) \times (n+1)$ Jordan block (proposition 2.1.4), whose unique point with infinite isotropy group is e_1, so $x \in \overline{\rho(\Gamma) \cdot e_1}$.

If x is a fixed point of γ_2, we know that after a conjugation this loxodromic element is a diagonal matrix whose fixed points are $e_1, \cdots, e_{n+1}$, so either $x = \gamma_2 e_i$ for some $i = \underline{1, \cdots, n+1}$ or exist $g_m(e_{i_m})$ such that $g_m(e_{i_m}) \to x$ as $m \to \infty$, in any case $x \in \bigcup_{i=1}^{n+1} \rho(\Gamma) \cdot e_i$.

The part of the algebraic curves follows from the fact that $\mathrm{PSL}(2,\mathbb{C})$ is an algebraic group. □

The problem to describe the set $L_1(\cdot)$ of a Veronese group is way complicated in comparison with $L_0(\cdot)$. The main obstruction is the existence of *parabolic sequences,* these are sequence of transformations not necessarily parabolic transformations, that have the behavior of a parabolic transformation sequence. We recall that a parabolic transformation sequence the image of the pseudo-projective limit is contained in the kernel of this limit.

Example 2.3.2. Suppose that

$$\gamma_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \text{ and } \gamma_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and that $[e_1]$, $[e_2] \in \mathbb{C}P^1 \setminus \mathrm{Fix}(\gamma_2)$, and both transformations loxodromic.

If we look for the pseudo-projective limit of $(g_n)_{n\in\mathbb{N}}$ we have that

$$g_n = \begin{bmatrix} a & b\lambda^{2n} \\ c\lambda^{-2n} & d \end{bmatrix} = \begin{bmatrix} \frac{a}{\lambda^{2n}} & b \\ \frac{c}{\lambda^{4n}} & \frac{d}{\lambda^{2n}} \end{bmatrix}$$

whose limit has kernel $[e_1]$. Therefore the sequence $(g_n)_{n\in\mathbb{N}}$ converges to $[e_1]$ in compact sets of $\mathbb{C}\mathrm{P}^n \setminus \{[e_1]\}$. It is easy to prove that $[g_n(e_1)]$ converges to $[e_1]$.

The previous shows a sequence of loxodromic elements that have the parabolic sequence behavior.

The existence of these type of sequences imply that we cannot generalize the following property: for a Pseudo-projective limit the image of the limit is not contained in its kernel. For that reason we look for the following ideas. Since the Kulkarni limit set is hardly to compute we will look if there exists a set that satisfies the Properly-discontinuous action property and we compare it with the Kulkarni limit set.

2.3.1 Extending Proximal Spaces

As we saw in the Chapter 1 the KAK-decomposition is a usefull tool. As we commented earlier, the KAK-decomposition is mapped by the ρ morphism into a KAK-decomposition.

Corollary 2.3.3. *For an element $\gamma \in \mathrm{PSL}(2,\mathbb{C})$ whose KAK-decomposition is equal to k_1ak_2, then $\rho(k_1)\rho(a)\rho(k_2)$ is a KAK-decomposition of $\rho(\gamma)$.*

If we have a sequence $(\gamma_m)_{m\in\mathbb{N}} \subset \mathrm{PSL}(n+1,\mathbb{C})$ of distinct elements and γ is its limit Pseudo-Projective transformation. If $\gamma_m = u_{1,m}a_nu_{2,n}$ is the KAK-decomposition, then $\ker(\gamma) = u_2^{-1}(\ker(a))$ and $\mathrm{im}(\gamma) = u_1(\mathrm{im}(a))$ where $u_{2,m} \xrightarrow[m\to\infty]{} u_2$ and $u_{1,m} \xrightarrow[m\to\infty]{} u_1$ in $O(n+1)$ and $a_m \xrightarrow[m\to\infty]{} a$ in Pseudo-projective convergence.

In particular case of $\mathrm{PSL}(2,\mathbb{C})$ the previous paragraph implies that the for a sequence $(\gamma_m)_{m\in\mathbb{N}}$ of different elements whose KAK-decomposition is given by $(u_{1,m}a_mu_{2,m})_{m\in\mathbb{N}}$ and $a_m = \mathrm{Diag}(\lambda_1, \lambda_1^{-1})$. If γ correspond to the limit Pseudo-Projective transformation then $\ker(\gamma) = [u_2^{-1}(e_2)]$ and $\mathrm{im}(\gamma) = u_1(e_1)$ where e_1 and e_2 correspond to the standard basis of $\mathbb{CP}^1$ and u_i correspond to the limit of the sequences $u_{i,m}$ in $\mathrm{O}(2)$.

If we translate the previous paragraph for the case of a sequence in the Veronese automorphism group, *i.e.,* we take the $\rho-$image of the previous paragraph, then for $\rho(\gamma)$ we have that $\ker(\rho(\gamma)) = \rho(u_2^{-1})(\langle\langle u_1, \cdots, u_n\rangle\rangle)$ and $\mathrm{im}(\rho(\gamma)) = \varrho(u_1)([e_0])$, where $\{e_0, e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{CP}^n$. The previous ker and im description provide the following Lemma, that it is a special case of the $\lambda-$lemma (Theorem 1.1.7).

Corollary 2.3.4. *Assume that $(\gamma_m)_{m\in\mathbb{N}} \in \mathrm{PSL}(2,\mathbb{C})$ whose KAK-decomposition* *are of the form* $\gamma_m = u_m a_m v_m$ *and* $\gamma = uav$ *is the Pseudo-Projective Limit with* u *and* v *the limits of* $(u_m), (v_m)$*, respectively, and* a *the Pseudo-Projective limit of* (a_m)*. Then there exists two full flags in* $\mathbb{C}\mathrm{P}^n$*, let say*

$$(F_0, F_{01}, \cdots, F_{012\cdots n}) \qquad F_{1\cdots j} = \mathbb{P}\left(\varrho(u)(\langle\langle e_0, \cdots, e_j\rangle\rangle)\right) \tag{2.9}$$

$$\left(\tilde{F}_n, \tilde{F}_{n-1,n}, \cdots, \tilde{F}_{01\cdots n}\right) \qquad \tilde{F}_{j\cdots n} = \mathbb{P}\left(\varrho(v)^{-1}(\langle\langle e_j, \cdots, e_n\rangle\rangle)\right) \tag{2.10}$$

such that the set of accumulation points of $g_m\left(\tilde{F}_{j\cdots n} \setminus \tilde{F}_{j+1\cdots n}\right)$ *for* $g_m = \varrho(\gamma_m)$*, is* $F_{1\cdots j+1}$.

The following definition was introduced in [1].

Definition 2.3.5. We say that a matrix in $\mathrm{SL}(n+1, \mathbb{C})$ is proximal if it has a maximal norm eigenvalue. In the case of $\mathrm{PSL}(n+1, \mathbb{C})$ we say that an element is proximal if some lift is proximal.

The proximal property in an element γ implies that the eigenspace is one dimensional (thinking that $\gamma \in \mathrm{SL}(n+1, \mathbb{C})$ and its complement is an hyperplane, even more, the transformation acts as a contraction whose limit point is the proper vector space. In sight of this property and the Lemma 2.3.4 we can define the following *limit set*

Definition 2.3.6. Let $G \subset \rho(\mathrm{PSL}(2, \mathbb{C}))$ be a discrete subgroup such that $G = \varrho(\Gamma)$ for some subgroup $\Gamma \in \mathrm{PSL}(2, \mathbb{C})$. Let $(\gamma_m)_{m\in\mathbb{N}} \subset \mathrm{PSL}(2, \mathbb{C})$ a sequence of different elements and $\gamma \in \mathrm{PsP}(2, \mathbb{C})$ the Pseudo-Projective limit and $u_{1,\gamma} a_\gamma u_{2,\gamma}$ the limit of the KAK-decompisition of γ_m. The projective space $\varrho(u_{1,\gamma})(\langle\langle e_0, \cdots, e_{\lfloor \frac{n}{2} \rfloor}\rangle\rangle)$ is a *extended proximal* space for $\varrho(\gamma)$ associated to the point $p = \mathrm{im}(\gamma) \in \phi(\mathbb{C}\mathrm{P}^1) \cap \mathrm{Lim}(\varrho(\Gamma))$.

We recall that the Proposition 2.1.4, the form of $\varrho(A(n+1))$ has $\lfloor \frac{n+1}{2} \rfloor$ proper vectors whose proper values are "maximal" in comparison of the rest. The definition of the previous space is that has the attraction property (see Corollary 2.3.4). In [47] they introduced a notion of limit set conformed by the proximal vector spaces of elements of the group.

Definition 2.3.7. We define the *extended Conze-Guivarc'h limit set* of a discrete subgroup $\varrho(\Gamma)$, denoted by $L^{Ext}_{CG}(\varrho(\Gamma))$ as

$$L^{Ext}_{CG}(\varrho(\Gamma)) = \overline{\bigcup_{p \in \mathrm{Lim}(\Gamma)} H_{\phi(p)}} \tag{2.11}$$

where $H_{\phi(p)}$ denotes the extended proximal space associated to p.

Clearly the previous set is a closed set whose is invariant under the $\varrho(\Gamma)$, and by its definition it is contained in $\mathbb{C}\mathrm{P}^n \setminus \mathrm{Eq}(\varrho(\Gamma))$.

Proposition 2.3.8. *Let* $\Gamma \in \mathrm{PSL}(2,\mathbb{C})$ *a discrete group and* G *the associated Veronese group. Then* $\varrho(\Gamma)$ *acts properly discontinuous on* $\mathbb{C}\mathrm{P}^n \setminus L^{Ext}_{CG}(G)$.

The previous Proposition follows from the Lemma 2.3.4.

Theorem 2.3.9. *Let* $\Gamma \in \mathrm{PSL}(2,\mathbb{C})$ *a discrete group and* G *the associated Veronese group. Then* $L^{Ext}_{CG}(G)$ *is a minimal* $G-$*invariant closed set where the action is properly discontinuous.*

Proof. Assume that $W \subset \mathbb{C}\mathrm{P}^n$ is a $G-$invariant closed set where G acts properly discontinuous on its complement. For these reason we will assume that there exists $p \in \mathscr{C}$ such that $x \in T_p\mathscr{C}$ and we can assume that p is proximal for some $g \in G$. We can assume that $x \in k(p)(e_0, \cdots, e_{\lfloor \frac{n}{2} \rfloor})$ where $k(p)$ is the limit orthogonal element associated to a sequence convergent to p in Pseudo-projective sense. By the 2.3.4 we see that there is a sequence $(g_m)_{m\in\mathbb{N}} \subset G$ such that $\lim_{m\to\infty} g_m(x)$ belong to $L^{Ext}_{CG}(G)$ and this imply that $L^{Ext}_{CG}(G) \subset W$. □

The previous Theorem has big implications about our expectations over the Kulkarni limit set. The following Corollary implies that in higher dimensional setting the Kulkarni limit set is not the best option to look forward as a limit set and not just for the higher complication of computing but also for its big composition of elements.

Corollary 2.3.10. *For a discrete subgroup* $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ *and its associated Veronese group* G*. The following are satisfied*

$$Eq(G) \subset \mathbb{C}\mathrm{P}^n \setminus L_{CG}^{Ext}(G) \tag{2.12}$$

$$\Omega_{Kul}(G) \subset \mathbb{C}\mathrm{P}^n \setminus L_{CG}^{Ext}(G) \tag{2.13}$$

Since we do not compute the Kulkarni limit set for the Veronese groups, we prove that there exist a set, which is more easy to compute, where the action of the group in its complement is properly discontinuous.

3 Uniformization Theorems in Higher Dimensions

Introduction

Following [102], we understand that $A \subset \mathbb{C}^m$ is uniformizable if there exists $f = (f_1, \ldots, f_m)$ meromorphic functions and $D \subset \mathbb{C}^n$ such that $f : D_0 \to A_0$ is a holomorphic covering and where $A_0 \subset A$ and $D_0 \subset D$ are dense subsets, the covering has discrete fibers on which a discrete group (of automorphisms of D) acts transitively. A partially open problem, that has motivad numerous mathematicians for many years is to understand under which conditions a subset of $\mathbb{C}^m$ ($\mathbb{C}\mathrm{P}^m$) can be uniformized following the previous definition. A particular case of this problem is when the subset is a Riemannian manifold, in this scenario, the previous definition is equivalent to ask that the Riemannian manifold be biholomorphic to a quotient of a universal cover by a discrete group of automorphisms.

The most well known result about this problem is the Uniformization theorem, proved by H. Poincaré [134]; this theorem asserts that every simply connected Riemann surface is holomorphically equivalent to one of this: the Sphere $\hat{\mathbb{C}}$, the complex space $\mathbb{C}$ or the Disk $\mathbb{D}$. F. Klein extended this result by proving that each closed Riemann surface is of the form $\hat{S}/\Gamma$, where $\hat{S}$ is one of the spaces proposed by H. Poincaré and Γ is a discrete group of Mobius transformations of $\hat{S}$ acting freely.

In 1910, using a different approach, P. Koebe ([100]) proved a similar result to the Klein-Poincaré theorem known as the Koebe's retrosection theorem. Koebe's theorem asserts.

Theorem 3.0.1 (Retrosection Theorem [100], Theorem 2 in [85])**.** *Let S be a closed Riemann surface of genus $p \geq 1$. Then exists a Schottky group G of p generators, such that $\Omega(G)/G$ is holomorphically equivalent to S.*

B. Maskit in [111] gave an algebraic description of Schottky groups as finitely generated, purely loxodromic, free Kleinian groups. Therefore we can restate Koebe's Retrosection Theorem in a algebraic version as follows.

Theorem 3.0.2 (Algebraic Version of Koebe's retrosection Theorem)**.** *Let S be a closed Riemann surface of genus $p \geq 1$. Then exists a purely loxodromic, free Kleinian group of p generators, such that $\Omega(G)/G$ is holomorphically equivalent to S.*

Fifty years later, L. Bers ([28]) provided the Simultaneous uniformization theorem that generalizes the uniformization results for two Riemann surfaces. The Bers' simultaneous uniformization theorem asserts.

Theorem 3.0.3 (Simultaneous Uniformization [28])**.** *Let S and S' be two closed Riemann surfaces of the same genus. Let $f : S \to S'$ be a quasi-conformal map with quasi-conformal maximal dilatation $k(f)$ bounded above, that reverse orientation. Then there exists a quasi-Fuchsian group G, such that $S = I(\gamma_G)/G$, $S' = E(\gamma_G)/G$ where $E(\gamma_G)$ and $I(\gamma_G)$ denotes the exterior and interior of the oriented Jordan curve γ_G fixed by G. The map f induces an isomorphism between the fundamental groups of S and S'.*

As we can see, Kleinian groups are intimately related to the uniformization of Riemann surfaces. Therefore, it is natural to ask about the uniformization of higher-dimensional complex manifolds using complex Kleinian groups. Even if this question is natural, we have to take into consideration many obstructions in the process of generalization. In higher dimensions, the geometry of complex manifolds is much more diverse. For example, Krantz in [101] shows that the complex two-ball and the complex bi-disk are not biholomorphically equivalent, yet they are diffeomorphically equivalent. Therefore, there are higher-dimensional manifolds that admit non-equivalent geometric structures. The following facts show what kind of constraints we have to prevent to talk about uniformization in higher dimensions.

- The complex manifold $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ is a simply-connected complex manifold that does not admit a complex projective structure. Hence, if we want to ask for uniformization in higher dimensions in terms of complex Kleinian groups, we must require that the manifolds admit a complex projective structure.
- In [60] there is an exposition due to Smilie of a torus with a complex projective structure that is not complete, *i. e.,* the manifold cannot be realized as a quotient of an open set of the complex projective line and a discrete subgroup acting discontinuously on it. Once again, when we ask for uniformization result, we must assure that the manifolds have a complete complex projective structure.
- We have to recall that we understand a uniformization of a complex manifold as a quotient of a simply connected domain under a discrete group of automorphisms. For higher dimensional setting there are several simply-connected domains of $\mathbb{C}\mathrm{P}^n$ which are not biholomorphically equivalent; therefore our expectations of a higher dimensional version of the Uniformization theorem are null. Moreover, some of these domains arise as connected components in the equicontinuity region of non-trivial complex Kleinian groups, see [41].

Over this Chapter, we show that there are no higher dimensional versions of

Koebe's and Bers' theorems (Theorems 3.2.3 and 3.2.4) in the setting of complex Kleinian groups.

3.1 Schottky-like groups and some previous lemmas

The following lemmas give us dynamical properties of loxodromic and parabolic elements that we will use in the later. The next lemma is a particular case of the Proposition 3.6 in [37], we omit the proof because is similar to the cited Proposition.

Lemma 3.1.1. *Let $\gamma \in \mathrm{PU}(1,n)$ be a loxodromic element and $a, r \in \partial \mathbb{H}^n_{\mathbb{C}}$ be the atracting and repelling fixed points of γ, respectively. Then*

1. $\gamma^m \xrightarrow[m\to\infty]{} a$ *uniformly in compact sets of* $\mathbb{C}\mathrm{P}^n \setminus r^{\perp}$.

2. $\gamma^{-m} \xrightarrow[m\to\infty]{} r$ *uniformly in compact sets of* $\mathbb{C}\mathrm{P}^n \setminus a^{\perp}$.

3. *the transformation γ restricted to $a^{\perp} \cap r^{\perp}$ is conjugated to an element of* $\mathrm{PU}(n-1)$ *acting on* $\mathbb{C}\mathrm{P}^{n-1}$.

Lemma 3.1.2. *Let us consider the cyclic group $\Gamma \subset \mathrm{PSL}(n+1,\mathbb{C})$ generated by the element*

$$\gamma = \begin{bmatrix} A & \\ & B \end{bmatrix}$$

where A is a $k \times k$ unitary diagonal matrix and B is a $(n+1-k)\times(n+1-k)$ Jordan block, such that $\det(A)\det(B) = 1$. *Then*

1. *If $x \in \mathbb{C}\mathrm{P}^n \setminus \langle\langle e_1, \cdots, e_k \rangle\rangle$, then the set of accumulation points of Γx is e_1.*

2. *If $x \in \langle\langle e_1, \cdots, e_k \rangle\rangle$, the x belongs to the set of accumulation points of Γx.*

Definition 3.1.3. Let $\Sigma \subset \mathrm{PSL}(n+1, \mathbb{C})$ be a finite set which is symmetric (*i. e.,* $a^{-1} \in \Sigma$ for all $a \in \Sigma$) and $A_\sigma = \{A_\alpha\}_{\alpha\in\Sigma}$ a family of compact non-empty pairwise disjoint subsets of $\mathbb{C}\mathrm{P}^n$ such that for each $a \in \Sigma$ we have

$$\bigcup_{b\in\Sigma\setminus\{a^{-1}\}} a(A_b) \subset A_a.$$

The group generated by Σ, denoted by Γ_Σ, is called a *Schottky-like* group.

Definition 3.1.4. Given a Schottky-like group Γ defined by (Σ, A_σ) we define the following set

$$\Lambda_{\Sigma,\sigma,A_\sigma}(\Gamma) = \overline{\Big\{y \in \mathbb{C}\mathrm{P}^n : \exists(\phi_m) \subset \Sigma, (y_m) \subset A_{\phi_0}, \phi_{j+1}\phi_j \neq Id, \phi_m \cdots \phi_1(y_m) \xrightarrow[m\to\infty]{} y\Big\}}. \tag{3.1}$$

Remark 3.1.5. The previous set is a "kind" of limit set for the Schottky-like groups. The definition above depends on the choice of the set A_σ.

Remark 3.1.6. We recall that every Schottky-like group is free, finitely-generated and discrete.

Example 3.1.7. The following are examples of Schottky-like groups:

- Every Schottky group of $\mathrm{PSL}(2, \mathbb{C})$ acting on $\mathbb{C}\mathrm{P}^1$ is a Schottky-like group.
- The Schottky groups, appearing in Chapter 4, acting on $\mathbb{C}\mathrm{P}^2$ are Schottky-like groups.

Lemma 3.1.8. *Let $\Gamma \subset \mathrm{PU}(1, n)$ be a Schottky-like group. Then*

1. *The group is purely loxodromic.*
2. *We have $\Lambda_{CG}(\Gamma) \subset \Lambda_{\Sigma,\sigma,A_\sigma}(\Gamma)$.*
3. *The set $\Lambda_{CG}(\Gamma)$ is disconnected.*

Proof. Let us show 1. It will be enough to show that every generator is loxodromic. Let $\gamma \in \Gamma$ and assume that γ is a parabolic generator; since

$$\begin{aligned} \gamma(A_\gamma) &\subset A_\gamma, \\ \gamma^{-1}(A_{\gamma^{-1}}) &\subset A_{\gamma^{-1}} \end{aligned}$$

for some pairwise disjoint non-empty subsets of $\mathbb{C}\mathrm{P}^n$, we deduce that γ has at least two fixed points.

Since γ is parabolic, we can assure that γ has a lift $\tilde{\gamma} \in \mathrm{SL}(n+1, \mathbb{C})$ whose normal Jordan form is

$$\tilde{\gamma} = \begin{pmatrix} A & \\ & B \end{pmatrix}$$

where A is a $k \times k$ diagonal matrix whose proper values have modulus one and B is a $(n+1-k) \times (n+1-k)$ Jordan block, with $k \geq 2$ and $\det(A)\det(B) = 1$. Let $x \in A_\gamma$ and $y \in A_{\gamma^{-1}}$ be fixed points of γ. By the Lemma 3.1.2, we have that $x = y = e_{k+1}$, which is a contradiction.

For the proof of 2, we can deduce that Γ is non-elementary. Therefore, it will be enough to show that for every generator

$$\mathrm{Fix}(\gamma) \cap \partial \mathbb{H}^n_{\mathbb{C}} \cap \Lambda_{\Sigma,\sigma,A_\sigma}(\Gamma) \neq \varnothing. \tag{3.2}$$

On the contrary, let us assume that the previous intersection is empty for every generator γ. Let $\gamma_1, \gamma_2 \in \Gamma$ two generators satisfying $\gamma_1\gamma_2 \neq Id$. Let A_{γ_i} be the generating set of γ_i, $i = 1, 2$; since γ_1 is loxodromic, we can consider that $a, r \in \partial \mathbb{H}^n_{\mathbb{C}}$ are the attracting and repelling fixed points of γ_1, respectively. Therefore $A_{\gamma_2} \subset a^\perp \cap r^\perp$, in consequence $A_{\gamma_2} \subset A_{\gamma_1}$, by Lemma 3.1.1 we know that γ_1 restricted to $a^\perp \cap r^\perp$ is elliptic, which is a contradiction.

The part 3 follows from the previous part. □

3.2 The Higher dimensional version of Uniformization

Definition 3.2.1. A *complex hyperbolic manifold* is the quotient of an open subset of $\mathbb{H}^n_{\mathbb{C}}$ and a discrete subgroup of $\mathrm{PU}(1,n)$.

Mok-Yeung and Klingler proved, independently, that for a complex hyperbolic manifold with finite volume there is a unique complex projective structure compatible with the complex one.

Theorem 3.2.2 ([98], [119]). *Let $\Gamma \subset \mathrm{PU}(1,n)$ a discrete subgroup such that $M = \mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a manifold with finite volume. Then M has only one complex projective structure compatible with its complex structure.*

Theorem 3.2.3. *There is no analogue of Bers' simultaneous uniformization theorem for groups in $\mathrm{PSL}(n+1,\mathbb{C})$ acting on $\mathbb{C}\mathrm{P}^n$, where $n \geq 2$.*

Proof. Let M be a compact complex manifold such that $M = \mathbb{H}^n_{\mathbb{C}}/\Gamma$ where $n \geq 2$ and $\Gamma \subset \mathrm{PU}(1,n)$ is a discrete group. Let us consider the manifold $M \sqcup M$ and let us assume that there is a group $G \subset \mathrm{PSL}(n+1,\mathbb{C})$ and a $G-$invariant open set $U \subset \mathbb{C}\mathrm{P}^n$ satisfying $M \sqcup M = U/G$, (see Appendix C). By Theorem 3.2.2, we deduce that $G = \Gamma$ up to projective conjugation. On the other hand, by the main theorem in [37], we know that $\mathbb{H}^n_{\mathbb{C}}$ is the largest open set of $\mathbb{C}\mathrm{P}^n$ on which Γ acts properly discontinuously, which is a contradiction. $\square$

The following result shows that the *algebraic* and *geometric* higher dimensional analogs of the Köebe's retro-section theorem are false, here *algebraic* means that the fundamental group of a manifold has a representation as a purely loxodromic free discrete group and *geometric* means that the manifold can be realized as a quotient by a Schottky-like group.

Theorem 3.2.4. *The geometric and algebraic version of Köebe's retro-section theorem is no longer true for groups of* $\mathrm{PSL}(n+1,\mathbb{C})$ *acting on* $\mathbb{C}\mathrm{P}^n$, *where* $n \geq 2$.

Proof. For the proof of the geometric version, let M be a compact complex manifold such that $M = \mathbb{H}^n_{\mathbb{C}}/\Gamma$ where $n \geq 2$ and $\Gamma \subset \mathrm{PU}(1,n)$ is a discrete group. Let us assume that $G \subset \mathrm{PSL}(n+1,\mathbb{C})$ is a Schottky-like group such that there is a $G-$invariant open set $U \subset \mathbb{C}\mathrm{P}^n$ with $M = U/G$. By theorem 3.2.2, we deduce that $G = \Gamma$ up to projective conjugation. We have to notice that $\Lambda_{CG}(\Gamma) = \partial\mathbb{H}^n_{\mathbb{C}}$ since M is compact. However, this contradicts part 3 of Lemma 3.1.8.

For the algebraic version. Let M be a compact complex manifold such that $M = \mathbb{H}^n_{\mathbb{C}}/\Gamma$ where $n \geq 2$ and $\Gamma \subset \mathrm{PU}(1,n)$ is a discrete group. Let us assume that there is a discrete, purely loxodromic free group $G \subset \mathrm{PSL}(n+1,\mathbb{C})$ and a $G-$invariant open set $U \subset \mathbb{C}\mathrm{P}^n$ satisfying $M = U/G$. By theorem 3.2.2, we deduce that $G = \Gamma$ up to projective conjugation. Since M is compact, the Cayley graph $\Delta(\Gamma)$ of Γ is quasi-isometric to $\mathbb{H}^n_{\mathbb{C}}$, see Appendix B. Consequently the Gromov boundaries $\partial\Delta(\Gamma)$ and $\partial\mathbb{H}^n_{\mathbb{C}}$ are homeomorphic. This is a contradiction, since $\partial\Delta(\Gamma)$ is a Cantor set while $\partial\mathbb{H}^n_{\mathbb{C}}$ is a $2n-1$ sphere. □

4 Hausdorff Dimension and Complex Hyperbolic Groups

Introduction

The Patterson-Sullivan theorems are part of the classical theory of Kleinian, this theorems assign a family of measures to a Kleinian group, such measures have as support the limit set of the Kleinian group. The construction was first developed by Patterson and generalized by Sullivan, and the Kleinian groups involved where geometrically finite subgroups of the isometries group of the real hyperbolic space. We have to mention, that the dimension of the Patterson-Sullivan measures coincide with the Hausdorff dimension of the limit set of the associated group, (see Theorem 1 in [149]). Using this last quality of the Patterson-Sullivan measures, in addition with a Markov partition of the space induced by the group, McMullen developed an algorithm to approximate the Hausdorff dimension of the limit set of a geometrically finite Kleinian group.

It turns out that the Patterson-Sullivan measures are not exclusive of the real hyperbolic geometry, indeed these constructions could be generalized in several scenarios and geometries, in particular for the case of the complex hyperbolic geometry. In the present Chapter, we show that there is a connecting bridge between the Patterson-Sullivan theory in complex hyperbolic geometry and the

Heisenberg geometry at the boundary of the complex hyperbolic space that leads us to a generalization of the McMullen algorithm in the complex Kleinian setting, Theorems 4.2.8 and 4.2.9; also, we give a computational implementation of the algorithm.

4.1 Schottky groups

The following section pursuit the classical construction of Schottky groups, on the real hyperbolic 3-space, as a group generated by reflections and present the version of a Patterson-Sullivan measure associated to the group.

Let $\mathfrak{H}$ denote the space $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty\}$ with the Heisenberg geometry on it. We recall that there is a map that identifies $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty\}$ and $\mathbb{C} \times \mathbb{R}$ (see Section A.1 of Appendix A), and the Heisenberg structure on the boundary follows from that map, and the following product on $\mathbb{C} \times \mathbb{R}$,

$$(\zeta, v) * (\xi, t) = (\zeta + \xi, v + t + 2\,\mathrm{Im}(\bar{\xi}\zeta).$$

We have to recall that the space $\mathfrak{H}$ corresponds to the finite part of the boundary of $\mathbb{H}^2_{\mathbb{C}}$, in a similar way that $\mathbb{C}$ is the finite part of $\mathbb{H}^3_{\mathbb{R}}$. Therefore the boundary of the important subspaces in $\mathbb{H}^2_{\mathbb{C}}$, such as complex lines and Lagrangian spaces (Subsection 0.2.2), are studied and characterized. Those boundaries are known as complex chains and real circles for the complex lines and Lagrangian subspaces respectively, for more details of the description of those spaces see Section A.2 of Appendix A.

The following definition will be of help to understand how it looks like $\partial\mathbb{H}^2_{\mathbb{C}}$ close to ∞ and to define what will be our maps to construct the groups we will study.

Definition 4.1.1. The *Koranyi inversion* ι is a map on $\mathfrak{H} \setminus \{(0,0)\}$ into itself given by

$$(\zeta, v) \mapsto \left(\frac{-\zeta}{|\zeta|^2 - iv}, \frac{-v}{|\zeta|^4 + v^2} \right).$$

This map can be extended to whole $\partial \mathbb{H}^2_{\mathbb{C}}$ sending $\iota(0,0) = \infty$ and viceversa.

The Koranyi inversion is the analog of the inversion of the unit circle for the Möbius maps, and all inversions induced by finite chains are conjugated to this. We have to recall that the automorphism group of $\mathfrak{H}$ are called Heisenberg similarities, and it is conformed by translations, dilatations and inversions; we refer to Section A.1.2 of Appendix A.

Definition 4.1.2. Let $C \subset \mathfrak{H}$ be a finite chain there exist a complex reflection on $\mathbb{H}^2_{\mathbb{C}}$ induced by

$$\iota_C = D_\lambda T_{(\zeta,t)} \iota T_{(-\zeta,-t)} D_{\lambda^{-1}} \tag{4.1}$$

for some Heisenberg dilatation D_λ, $\lambda \in \mathbb{C}^*$, and Heisenberg translation by $(\zeta, t) \in \mathfrak{H}$ and ι the Koranyi inversion.

The space $\mathfrak{H}$ can be endowed with a metric space structure. The Cygan metric on $\mathfrak{H}$ (Section A.1.2) can be extended to $\overline{\mathbb{H}^2_{\mathbb{C}}} \setminus \{\infty\}$ as an incomplete metric (see Section 4 of [127]). The existence of the Cygan metric let us define an analog of the isometric circle for the Möbius transformations, but now for the complex hyperbolic transformation, that does not fix infinity, see [128], [73], [127].

Lemma 4.1.3. *The isometric Cygan sphere of a general complex reflection is given by*

$$S(\iota_C) = \{(\zeta, t) \in \mathfrak{H} : \rho_0\left((\zeta, t), (\xi, s)\right) = |\lambda|^2\}. \tag{4.2}$$

where ρ_0 *is the Cygan metric on* $\mathfrak{H}$.

Definition 4.1.4. Let $\mathscr{C} = \{C_j\}_{j=1}^{k}$, for $k \geq 3$, be a finite family of finite chains and $\{\iota_j\}$ the induced complex reflections. Assume that the associated isometric spheres $\{S(\iota_k)\}$ are mutually disjoint. Let $\Gamma(\mathscr{C})$ denotes the generated group by $\{\iota_k\}$. $\Gamma(\mathscr{C})$ is called *Schottky group* (see [121]).

Remark 4.1.5. We call this groups Schottky because restricted to the closure of the complex hyperbolic plane they have the Ping-Pong dynamics, but if we consider the action of the group to the whole projective space we have to call this group a *Schottky-like* group (see [47]). For the special case of this groups, generated by three complex reflections, we call them *triangular groups* (see [75], [144]).

We have to emphasize that these groups actions are only in the complex hyperbolic setting, in [10] the authors prove that in general not always it is possible to have a Schottky action on the projective space.

The complex Hyperbolic space is a Gromov hyperbolic space (see B) and the Gromov boundary coincide with the boundary of the space, i.e., we can endorse to $\partial\mathbb{H}^2_{\mathbb{C}}$ with the Gromov topology on the boundary. The Gromov hyperbolic structure will be helpful in the understanding of the Patterson-Sullivan theory on the complex hyperbolic space.

Definition 4.1.6 ([89], [31])**.** Let X be a proper Gromov hyperbolic space and Γ a discrete subgroup of isometries. We Say that Γ is quasi-convex co-compact if:

(i) Γ acts properly discontinuous on X.

(ii) Γ does not have a global fixed point on ∂X.

(iii) The quotient of $\Gamma-$invariant quasi-convex subset A of X by the group is compact.

The following Proposition of [89] gives an equivalence of a quasi-convex co-compact group.

Proposition 4.1.7. *Let Γ a infinite group of isometries acting properly discontinuous on a proper hyperbolic space (X, d) so that there is no point in ∂X fixed by Γ. Then the following are equivalent:*

1. *Γ is quasi-convex co-compact*

2. *Let $C(\Gamma)$ the convex hull of the limit set $\Lambda(\Gamma)$,* i.e., *the union bi-infinite geodesics in X with both endpoints in $\Lambda(\Gamma)$. Then $C(\Gamma)/\Gamma$ has a finite diameter.*

where $\Lambda(\cdot)$ is closure of the accumulation points of the $\Gamma-$orbits.

The following result is a corollary of the Proposition, and it implies that the Schottky groups on Definition 4.1.4 are quasi-convex co-compact as in the classical case.

Corollary 4.1.8. *Let Γ be a Schottky group as previously defined. Then Γ is quasi-convex co-compact for $\mathbb{H}^2_{\mathbb{C}}$.*

In the classical setting, for a convex-co-compact subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$, there exists a density μ associated with Γ called the Patterson-Sullivan measure whose support is the limit set of the group, see [132], [149], [150], [148] and [40].

The Patterson-Sullivan construction can be reinterpreted to different symmetric or Gromov hyperbolic spaces, see [32], [48], [49], [136], [89]. For the aims of this Chapter we will introduce the Gromov hyperbolic version of a Patterson-Sullivan density.

The following conventions will be useful for the definition of the Patterson-Sullivan measures for Gromov hyperbolic spaces. Let (X,d) be a metric space:

1. A Borel measure μ on X is called regular if for every measurable set A,

$$\mu(A) = \inf\{\mu(B) : A \subseteq B, \text{ where } B \text{ is a Borel subset}\}.$$

2. A measure μ' is absolutely continuous with respect to the measure μ, if $\mu(A) = 0$ implies $\mu'(A) = 0$ for any $\mu-$measurable set A.

3. Assume that μ' is absolutely continuous with respect to μ. The *Radon-Nikodym derivative,* $\dfrac{d\mu'}{d\mu}$, is a $\mu-$integrable function $g \in L^1(\mu)$, such that for any $f \in L^1(\mu')$ we have
$$\int_X f d\mu' = \int_X fg d\mu$$

We will assume that (X, d) is a proper hyperbolic space and all measures on ∂X to be Borel and regular.

Definition 4.1.9 (Busemann function)**.** Let $x_0 \in X$ a base point and $r : [0, \infty) \to X$ be a geodesic ray. The *Busemann function* associated to r is the function $h : X \to \mathbb{R}$ defined as:

$$h(x) := \lim_{t\to\infty} (d(x, r(t)) - t)$$

for every $x \in X$.

For every $p \in \partial X$ we choose a geodesic ray r_p representing p and denote by h_p the Busemann function of r_p. Let $a > 1$ and for any $p \in \partial X$ and any isometry γ of X we put

$$j_\gamma(p) : a^{\Delta(p)}$$

where $\Delta(p) = h_p(x_0) - h_p(\gamma^{-1}x_0)$.

The Busemann function helps us to control the boundedness of the Radom-Nikodym derivative between the original measure and its translated by an element of the group.

Definition 4.1.10. Let Γ a group of isometries of X, $D \geq 0$ and μ be a nonzero measure on ∂X with finite total mass. The measure μ is said to be $\Gamma-$*quasi-conformal of dimension* D if the measures $\gamma^*\mu$, with $\gamma \in \Gamma$, are absolutely continuous with respect to each other and if there exists $K \geq 1$ such that for every $\gamma \in \Gamma$,

$$\frac{1}{K} j_\gamma^D \leq \frac{d(\gamma^*\mu)}{d\mu} \leq K j_\gamma^D, \quad \mu - \text{almost-everywhere.}$$

Example 4.1.11. Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ a convex-co-compact group acting on $\mathbb{H}^3$. The Patterson-Sullivan measure is a $\Gamma-$quasi-conformal measure whose dimension correspond to the Hausdorff dimension of the limit set.

Example 4.1.12. Another example comes from the Hausdorff measure of a metric space. Let (E, d) be a metric space. Denote by $|U| = \sup\{d(x, y) : x, y \in U\}$ diameter of $U \subset E$. For a subset $A \subset E$ and $\varepsilon > 0$, we will say that $\{U_i\}_{i \in I}$ is a $\varepsilon-$covering of A if all the elements are subsets of E, $A \subset \bigcup_i U_i$ and $|U_i| \leq \varepsilon$ for every i.

Definition 4.1.13. Let A be a subset of E. Let $\mathscr{H}_\varepsilon^D(A)$ denotes

$$\inf \left\{ \sum_i |U_i|^D : \{U_i\}_{i \in I} \text{ is an } \varepsilon-\text{covering of } A \right\}.$$

The $D-$*Hausdorff measure of* A is given by

$$\mathscr{H}^D(A) = \lim_{\varepsilon \to \infty} \mathscr{H}_\varepsilon^D(A). \tag{4.3}$$

Definition 4.1.14. The *Hausdorff dimension* of a set A is

$$\dim_H(A) = \sup\{D : \mathscr{H}^D = \infty\} = \inf\{D : \mathscr{H}^D = 0\}. \tag{4.4}$$

Proposition 4.1.15 (Proposition 4.3 in [48]). *Let $E \subset \partial X$ a $\Gamma-$invariant Borel subset such that the $D-$Hausdorff measure is of total mass and non-zero. The*

$D-$Hausdorff measure at E defines a $\Gamma-$quasi-conformal measure of dimension D at ∂X.

The following theorem is an extension of the Sullivan's theorem ([150]) that relates the dimension of the Patterson-Sullivan measure with the Hausdorff dimension of the limit set of the group that induces the previous measure.

Theorem 4.1.16 ([48], Theorem 15.8 of [89])**.** *Let (X, d) a proper hyperbolic space and Γ a group acting on X by isometries properly discontinuous and quasi-convex co-compactly. Let d_a the visual metric on ∂X and $\Lambda(\Gamma)$ the limit set of Γ. Put $D = \delta_a(\Gamma)$ the exponent of growth of Γ and $H = H^D$ the $D-$Hausdorff measure on $\Lambda(\Gamma)$ with respect to d_a. Then*

1. *$\delta_a(\Gamma)$ is the Hausdorff dimension of $\Lambda(\Gamma)$.*

2. *H on $\Lambda(\Gamma)$ is a $\Gamma-$conformal density of dimension $\delta_a(\Gamma)$.*

3. *If μ is a $\Gamma-$conformal measure of dimension D' with support $\Lambda(\Gamma)$ then $D' = D = \delta_a(\Gamma)$ and μ and H are equivariant.*

Corollary 4.1.17. *Consider $\mathbb{H}^2_{\mathbb{C}}$ as a Gromov hyperbolic space with a visual metric. Let $\mathscr{C}$ a finite family of finite chains and $\Gamma(\mathscr{C})$ the generated Schottky group. Then, there exists a unique $\Gamma(\mathscr{C})-$conformal measure whose dimension coincide with the Hausdorff dimension of $(\Lambda_{CG}(\Gamma(\mathscr{C}))$ with the restriction of the visual metric on $\mathbb{H}^2_{\mathbb{C}}$.*

Proof. The proof is direct consequence of the Theorem 4.1.16 since every Schottky group in $\mathbb{H}^2_{\mathbb{C}}$ is quasi-convex co-compact. □

4.2 The McMullen Algorithm

The McMullen Algorithm to compute approximations of the Hausdorff dimension of a limit set relies on the existence of a Markov partition related to the dynamical system given by the group and its Patterson-Sullivan measure. Roughly speaking, a Markov partition for a dynamical system, is a partition that allowed us to translate the system into a symbolic dynamical system, see [95].

Following the ideas of [11], we will construct a Markov partition associated to a Schottky group.

Let $\Gamma < PU(2,1)$ be discrete, $\{P_i\}_{i=1}^k$ a finite collection of domains in $\mathfrak{H}$ such that $int(P_i) \cap int(P_j) = \varnothing$ for $i \neq j$, and let $P_0 = \overline{\mathfrak{H} \setminus \bigcup_{i=1}^k P_i}$ and has finitely many components, it is easy to note that $\mathfrak{H} = P_0 \cup \cdots \cup P_k$.

(M0) P_0 contains the closure of a fundamental domain for Γ.

(M1) $\partial P_j \cap \Lambda(\Gamma)$ is finite for every $j = 1, \cdots, k$.

There is a map $f : \mathfrak{H} \to \mathfrak{H}$ such that:

(M2) There are some $\gamma_j \in \Gamma$ such that $f|_{P_j} = \gamma_j|_{P_j}$ for $1 \leq j \leq k$ and $f|_{P_0} = id$.

(M3) $f(P_i) = P_{j_1} \cup \cdots \cup P_{j_n}$ for some $j_1, \cdots, j_n \in \{0, \cdots, k\}$.

For $x \in \mathfrak{H}$, let $(j_0, j_1, \cdots)$ such that $x \in P_{j_0}$, $f(x) \in P_{j_1}, \cdots$. A finite sequence $(j_0, \cdots, j_m)$ is called *admissible* if $f(P_{j_l}) \supseteq P_{j_{l+1}}$ for every $0 \leq l \leq m-1$, and define $P(j_0, \cdots, j_m) = \bigcap_{i=1}^m f^{-1}(P_{j_i})$. Consider the following conditions:

(M4) If for every sequence $(j_0, \cdots, j_l, \cdots)$, then

$$|(P(j_0, \cdots, j_n))|_{Cyg} \xrightarrow[n \to \infty]{} 0,$$

where $|\cdot|_{Cyg}$ denotes the diameter in the Cygan metric (see Section A.1.2) at $\mathfrak{H}$.

(M5) If there exists $N \in \mathbb{N}$ and $\beta > 1$ such that

$$\left|\det\left(\frac{\partial^N f}{\partial^N(\zeta, v)}(x)\right)\right| > \beta,$$

for every $x \in P(j_0, \cdots, j_l)$ where $(j_0, \cdots, j_l)$ is admissible.

Definition 4.2.1. Let $\Gamma \subset \mathrm{PU}(2,1)$ be a quasi-convex co-compact complex (Definition 4.1.6) Kleinian group, $\mathscr{P} = \{P_j\}_{j=0}^k$ a partition of $\mathfrak{H}$ and $f : \mathfrak{H} \to \mathfrak{H}$ a map. We say that the triplet $(\Gamma, \mathscr{P}, f)$ has the expanding Markov property if it satisfies the conditions (M0)-(M5).

The aim of this section is to prove that there exists a partition of $\mathfrak{H}$ and a map $f : \mathfrak{H} \to \mathfrak{H}$ associated to a Schottky group, such that the triplet has the expanding Markov property.

In the real hyperbolic geometry setting; for a Möbius map, let say $g \in \mathrm{PSL}(2, \mathbb{C})$ of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the image under g of a small circle centered at a point $z \in \hat{\mathbb{C}}$ is "distorted" by a factor near to

$$\frac{1}{|cz+d|^2} = |f'(z)|,$$

where the accuracy of the approximation depend on the radius of the taken circle. This property, permit us to approximate the radii of images of circles.

Lemma 4.2.2. *Let C a finite chain of center (ξ, t) and radius $|\lambda|$, and let ι_C the* *induced complex reflection. For* $(\zeta_0, v_0) \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \{(\xi, t), \infty\}$*, we have that*

$$\left|\det\left(\left.\frac{\partial \iota_C}{\partial(\zeta, v)}\right|_{(\zeta_0,v_0)}\right)\right| = \frac{|\lambda|^4}{(\rho_0((\zeta_0, v_0), (\xi, t)))^4}, \tag{4.5}$$

where $\frac{\partial \iota_C}{\partial(\zeta,v)}$ *denotes the Jacobian matrix of* ι_C *viewed as a real valued function.*

Proof. First, we will prove it for the map ι in Definition 4.1.1, who in real variables is of the form

$$\iota(x, y, z) = \left(\frac{x(x^2 + y^2) + yz}{(x^2 + y^2)^2 + z^2}, \frac{-xz - y(x^2 + y^2)}{(x^2 + y^2)^2 + z^2}, \frac{-z}{(x^2 + y^2)^2 + z^2}\right), \tag{4.6}$$

a straight computation gives that

$$\det\left(\left.\frac{\partial \iota}{\partial(x, y, z)}\right|_{(x_0,y_0,z_0)}\right) = \frac{1}{(x_0^2 + y_0^2)^2 + z_0^2}$$

and the claim is true. For the general case, we have to note that a general complex reflection is given by $\iota_C = T_{(\xi,t)}D_\lambda \iota D_{\lambda^{-1}}T_{(-\xi,-t)}$. Straights computations shows that

$$\det\left(\frac{\partial T_{(\xi,t)}}{\partial(\zeta, v}\right) = 1 \quad \det\left(\frac{\partial D_\lambda}{\partial(\zeta, v}\right) = |\lambda|^2$$

and these determinants does not depend on the evaluation point. By the chain rule, we have the claim. □

Lemma 4.2.3 ([73])**.** *Let* $g \in PU(2, 1)$ *that does not fix* ∞*. Then there exists a positive number* r_g *depending only in* g *such that for all* $z, w \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty, g^{-1}(\infty)\}$*, we have:*

1. $\rho_0(g(z), g(w)) = \frac{r_g^2 \rho_0(z,w)}{\rho_0(z,g^{-1}(\infty))\rho_0(w,g^{-1}(\infty))}$,
2. $\rho_0(g(z), g(\infty)) = \frac{r_g^2}{\rho_0(z,g^{-1}(\infty))}$.

Proposition 4.2.4 ([128]). *Let h be an element of $PU(2,1)$ not fixing ∞. Then the Cygan sphere of radius r and center $h^{-1}(\infty)$ is mapped by h into the Cygan sphere of radius r_h^2/r and centered at $h(\infty)$.*

Lemma 4.2.5. *Let $\iota_C \in PU(2,1)$ a complex reflection and $z \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty, \iota_C(\infty)\}$. A small Cygan sphere centered in z is distorted by ι_C a factor approximate*

$$\sqrt{\left|\det\left(\frac{\partial \iota_C}{\partial(\zeta, v)}\bigg|_z\right)\right|}. \tag{4.7}$$

The accuracy of the approximation depends on the radius of the initial circle.

Proof. It will be sufficient to prove it for the map ι of Definition 4.1.1, the general case is a consequence of the lemma 4.2.2 and the chain rule.

Let $(\zeta_0, v_0) \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \{0, \infty\}$ and let $S_r((\zeta_0, v_0))$ the Cygan sphere of radius r and center (ζ_0, v_0). Let us take $(\zeta, v) \in S_r(\zeta_0, v_0)$, by lemma 4.2.3, we know that for $(\zeta, v), (\zeta_0, v_0)$ is satisfied

$$\rho_0(\iota(\zeta_0, v_0), \iota(\zeta, v) = \frac{r}{\rho_0((\zeta_0, v_0), (0,0))\rho_0((\zeta, v), (0,0))},$$

when (ζ, v) is close enough (ζ_0, v_0) the previous equation is close to what we are claiming. □

Theorem 4.2.6. *Let $\mathscr{S} = \{C_j\}_{j=1}^d$ be a finite collection of finite chains such that $\Gamma(\mathscr{S})$ is a Schottky group. Let $\mathscr{D} = \{D_i\}_{i=0}^d$ such that $D_i = int(S_i)$ where S_i is the isometric sphere of the complex reflection induced by C_i and $D_0 = \overline{\mathfrak{H} \setminus \bigcup_{i=1}^d D_i}$, and $f : \mathfrak{H} \to \mathfrak{H}$ given by $f|_{D_j} = \iota_{C_j}|_{D_j}$ for $j = 1, \cdots, d$ and $f|_{D_0} = id$. Then $(\Gamma(\mathscr{S}), \mathscr{D}, f)$ has the expansive Markov property.*

Proof. By construction of the Schottky group $\Gamma(\mathscr{S})$, we can assure that P_0 contains a fundamental domain, so $\Gamma(\mathscr{D})$ satisfies (M0), and by construction $\partial D_j \cap \mathrm{Lim}(\Gamma(\mathscr{D}))$

has a finite number of points, then $\Gamma(\mathscr{D})$ satisfies (M1). By hypothesis, it's satisfied (M2). Since, every ι_j satisfies that $Int(S_j)$ is mapped to $Ext(S_j)$, then $\Gamma(\mathscr{D})$ satisfies (M3). By proposition 4.2.4 and 4.2.5, we can assure that (M4) holds, and finally by Proposition 4.2.5 we have that f has the expansive property for every point inside the isometric sphere. So we can conclude that $(\Gamma(\mathscr{S}), \mathscr{D}, f)$ has the expansive Markov property. □

The McMullen algorithm (see [114]) uses a Markov partition to approximate the dimension of the measure on a dynamical system. In the classical setting, given a Schottky group acting on the real hyperbolic space, the Markov partition and the Patterson-Sullivan measure are defined using the spherical metric on $\mathbb{H}$, for this reason there is no constraints to apply directly this algorithm.

At this far we have a Markov partition, induced by the complex reflections in the Cygan metric, and a $\Gamma-$conformal measure defined for the Gromov structure of $\mathbb{H}^2_{\mathbb{C}}$; both objects are defined using different metrics on $\mathfrak{H}$. At first, there is no clear relation between these two metrics, but the following Lemma implies that locally these metrics are equivalent. This equivalence will be helpful, because it will exist a partition of $\mathfrak{H}$ for a visual metric whose will be contained in the Markov partition of the Theorem 4.2.6.

Let ξ^+ the positive end of the geodesic that pass trough $o \in \mathbb{H}^2_{\mathbb{C}}$ and ∞, since the action of $\mathfrak{H}$ on $\partial\mathbb{H}^2_{\mathbb{C}}$ is transitive outside ∞, we can obtain a map $\phi : \mathfrak{H} \to \partial\mathbb{H}^2_{\mathbb{C}}$ given by $\phi(s) = s(\xi^+)$.

Lemma 4.2.7 ([83], [56])**.** *Let $\mathfrak{H}$ be endowed with the Cygan metric and $\partial\mathbb{H}^2_{\mathbb{C}}$ endowed with a visual metric (see Section B.2), then the map $\phi : \mathfrak{H} \to \partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty\}$, defined previously, is locally bi-Lipschitz.*

Theorem 4.2.8. *Let $\Gamma \subset \mathrm{PU}(2,1)$ a Schottky group. There exists a partition contained in the Markov partition of Γ such that it is Markov for a visual metric on $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty\}$.*

Proof. This is a direct consequence of the previous Lemma. □

The following theorem is similar to the Corollary 3.4 in [114].

Theorem 4.2.9. *For a disjoint family of finite chains, $dim_H(\Lambda_{CG}(\Gamma(\mathscr{C})))$ can be computing by applying the eigenvalue algorithm to the Markov partition given by the isometric spheres of the generating elements.*

The following theorem due to McMullen in [114] implies that for a given dynamical system the order of approximation of the digits of the measure dimension is lineal depending on the step of refinement in the Markov partition.

Theorem 4.2.10 (Theorem 2.2 in [114])**.** *Let $\mathscr{P}$ a expanding Markov partition for a conformal dynamical sistem $\mathscr{F}$ with invariant density μ of dimension δ. Then*

$$\alpha(\mathcal{R}^n(\mathscr{P})) \to \delta$$

as $n \to \infty$, where $\mathcal{R}^n(\mathscr{P})$ denotes the $n^{th}-$refinement of $\mathscr{P}$. At most $O(N)$ refinements are required to compute δ to N digits of accuracy.

Since we have that there exists a Bi-Lipzchits map between the Boundary with the Gromov metric and the Cygan metric, the Hausdorff dimensions are equal. So the previous theorem is valid and direct.

4.3 The Computational Algorithm

The following section intentions are to explain the computation implementation of the McMullen Algorithm in Python. This implementation is a combination of the eigenvalue algorithm (see [114]) and the Newton Method (see [72]).

Given a finite chain of $\mathfrak{H}$ whose center is c and radio $|r|$. The following function provides the Heisenberg coordinates of a given point under the complex reflection defined by a finite chain.

Algorithm 1: Complex reflection defined by a complex chain in Heisenberg coordinates

---

function Inversion(c, r, ζ);

Input : $c-$ array of the center of the complex chain.

$r-$ multiplier of the reflection, $|r|$ is the radius of the complex chain.

$\zeta-$ array of a point in Heisenberg different from ∞.

Output : $z-$ array of a point Heisenberg coordinates.

$z[1] = c[1] - \frac{|r|^2(\zeta[1]-c[1])(|\zeta[1]-c[1]|^2+i(\zeta[2]-c[2]-2Im(\overline{\zeta[1]}c[1])))}{|\zeta[1]-c[1]|^4+(\zeta[2]-c[2]-2Im(\overline{\zeta[1]}c[1]))^2}$;

$z[2] = c[2] -$

$\frac{|r|^4(\zeta[2]-c[2]-2Im(\overline{\zeta[1]}c[1]))+2|r|^2 Im(\overline{-(\zeta[1]-c[1])(|\zeta[1]-c[1]|^2+i(\zeta[2]-c[2]-2Im(\overline{\zeta[1]}c[1])))\zeta[1]})}{|\zeta[1]-c[1]|^4+(\zeta[2]-c[2]-2Im(\overline{\zeta[1]}c[1]))^2}$;

return z

The next routine allows us to compute the Perron-Frobenius pair of a squared matrix, i.e., the eigenvalue with maximal absolute value and its associated eigenvector. The routine starts with an initial vector and iterates the matrix product until the approximates the eigenvector.

Algorithm 2: Compute the Perron-Frobenious pair of a matrix

```
function PerronFrobenious(A, x);
Input  : A− square matrix.
         x− initial vector.
Output : alpha− eigenvalue of maximal absolute value.
         y− eigenvector associated to α.
cnt = 1;
M = ||x||;
x = x(1/M);
while cnt < 10000 do
    y = A * y;
    L = ||y||;
    y = y * (1/L);
    if ||x − y|| < 1e^{−15} then
        BREAK
    end
    else
        x = y;
        cnt = +1
    end
end
α = y^T Ay/||y||^2;
return α, y
```

The eigenvalue algorithm consists on given a set of example points, on the Markov partition, we associate a matrix that encrypts the dynamics on the Markov partition. For this reason, given the Markov partition of a Schottky group, we need to compute example points contained in every set of the refined initial partition.

Once we have the points for a big enough refinement, we can compute the transition matrix. The transition matrix is given by: if there exists an element of the group that sends P_i to P_j, elements of the partition, then the matrix entry (i, j) is given by the radii norm of the Jacobian's group element that relates them and zeroes in another case.

The Tagpoints and Words (Algorithm 3) routine is an adaptation of the Depth First Search algorithm, presented on [121]. This routine generates a list of points, which are contained in every refinement of the partition, and a list of words, which are group elements used to obtain the points.

Algorithm 3: Tagpoints and Words (see [121])

```
function words(k, m, reflex);
Input  : k− maximal depth of the Word tree, non-negative integer; m− number of reflections,
         non-negative integer; reflex− array of m points in 𝔥 × (ℂ \ {0}).
Output : w− array of tag-points; wordsN− array of characters of words in the Tree of length k − 1.
N = m ∗ (m − 1)^(k−1);
for i = 1, ··· , m do
    tagpoints[i] = reflex[i][1] − ϵ ;                                  // ϵ small
    tag[i] = i;
    words[i] = "i";
end
inv = array[1, 2, ··· , m];
num[1] = 1;
num[2] = m + 1;
for lev = 2, ··· , k + 2 do
    inew = num[lev];
    for j = 1, ··· , m do
        for iold = num[lev − 1], ··· , num[lev] do
            if j = inv[tag[iold]] then
                CONTINUE
            end
            else
                tagpoints[inew] = Inversion(reflex[j], tagpoints[iold]);
                words[inew − 1] = words[j] + words[iold];
                tag[inew] = j;
                inew = +1;
            end
        end
        num[lev] = inew;
    end
end
for i = 1, ··· , num[k] − num[k − 1] do
    w[i] = tagpoints[num[k]] + i]
end
for i = 1, ··· , num[k − 1] − num[k − 2] do
    wordsN[i] = words[num[k] + i]
end
return w, wordsN
```

The final routine is an adaptation of the Newton method; using an estimate of the Hausdorff measure of the Chen-Greenberg limit set. The Newton method is used in combination with the PerronFrobenious routine, applied to the matrix given by powered up each entry in the estimate of the Hausdorff dimension.

Algorithm 4: The Newton algorithm for the approximation of Hausdorff dimension

```
function NewtonHausdorff(d, T, ε);
Input  : d− estimated value for the Hausdorff dimension (usual valor 1).
         T− a transition matrix.
         ε− desired error.
Output : dHauss− approximated value for the Hausdorff dimension.
N = RankT;
aNew = d;
x = ones[N]/N;
x0 = x;
while cont < 350 do
    Td = T^aNew;
    a, x = PerronFrobenious(N, Td, x);
    if |a − 1| < ε then
        BREAK;
    end
    else
        d0 = d + 0.1;
        TdE = T^d0;
        aE, xE = PerronFrobenious(N, TdE, x0);
        Der = (aE − a)/0.01;
        aNew = d + (1 − a)/Der;
        d = aNew;
        if |d − 1| < ε then
            BREAK
        end
        else
            cont = +1
        end
    end
end
return d
```

We have to recall that the previous computational implementation of the McMullen algorithm does not provide an improvement to the one proposed by McMullen. The algorithm is not hard but leads us to an exponential computational process, i.e., if we want to compute the Hausdorff dimension of the limit set where the group is generated by a large set of inversions, the computational processes to compute the word tree grow exponentially at every depth of the tree.

Our proposal for an improvement in the algorithm relies on:

1. The Depth search algorithm could be done in parallel, and we expect that this reduce the time of the computational process.

2. The transition matrix has a big number of zero entries, we propose to improve the algorithm related to find the Perron-Frobenius pair insight to adapt these to the type of matrix that we have.

3. Find a better storage format for the transition matrix insight of its characteristics.

4.4 $\theta-$Schottky Groups

C. McMullen in [114] used his algorithm to approximate the Hausdorff dimension of a *Symmetric Pair of Pants.* The way to obtain this pair of pants is as follows: let $0 < \theta \leq 2\pi/3$ be fixed, and take three circles in $\hat{\mathbb{C}}$ whose intersection to $\mathbb{S}^1$ be orthogonal an of arc length θ. The group generated by the inversions on the three circles is a Fuchsian group of isometries of the real hyperbolic plane in the circle model. The limit set of this group is a Cantor set when $\theta < 2\pi/3$ and the whole $\mathbb{S}^1$ when $\theta = 2\pi/3$. The orientable double cover of the quotient of $\mathbb{H}$ by the group is a Pair of pants with $\mathbb{Z}/3-$symmetry and in the case of $\theta = 2\pi/3$ is the finite volume thrice-punctured sphere. In this section we will construct similar Schottky groups whose limit set is a Cantor set in $\mathfrak{H}$ and we will show that the Hausdorff dimension of the limit set has a similar behavior as the showed by McMullen.

4.4.1 $\mathbb{C}-$chain centers

Let $0 < \theta < \pi/3$, and let $\mathscr{S}$ denotes the configuration of three chains in $\mathfrak{H}$ whose centers belong to $\mathbb{C} \times \{0\}$ and symmetric under a rotation of $\pi/3$ around the vertical axis. These chains are parametrized by the following maps:

$$
\begin{aligned}
t &\mapsto \left(\frac{1}{\cos(\theta)} + \tan(\theta)e^{it}, \frac{2\sin(t)}{\cos(\theta)}\right)\\
t &\mapsto \left(\frac{w_1}{\cos(\theta)} + \tan(\theta)e^{it}, \frac{\sqrt{3}\cos(t) - \sin(t)}{\cos(\theta)}\right)\\
t &\mapsto \left(\frac{w_2}{\cos(\theta)} + \tan(\theta)e^{it}, \frac{-\sqrt{3}\cos(t) - \sin(t)}{\cos(\theta)}\right)
\end{aligned}
$$

where $\{1, w_1, w_2\}$ are the cubic roots of the unity.

Let $\iota_0, \iota_1, \iota_2$ the complex reflections induced by these complex chains and S_0, S_1, S_2 the isometric spheres of the complex reflections respectively. Notice that for the assumptions over $\mathscr{S}$ we have that the isometric spheres are mutually disjoint. The group $\Gamma(\mathscr{S})$ is Schottky and since the isometric spheres are disjoint the Chen-Greenberg limit set (See Remark 0.3.6) is a Cantor set in $\mathfrak{H}$.

We have to mention that these complex reflections doesn't leave invariant the standard chain, so the Chen-Greenberg limit set of this groups doesn't accumulate on the standard chain, the following picture shows how the points accumulate in $\mathfrak{H}$ under the group.

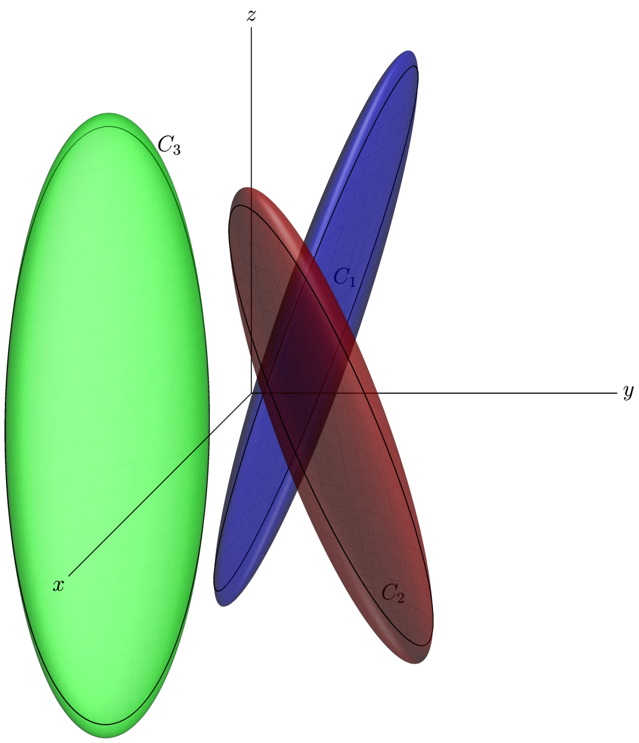


Figure 4.1 – Isometric spheres of the ι_i reflections.

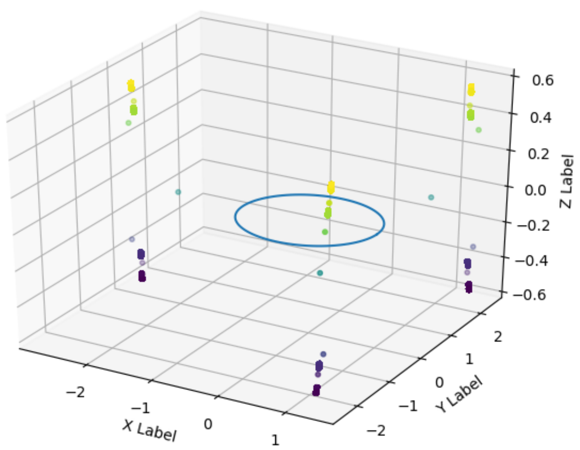


Figure 4.2 – Accumulation of points under the group Γ_θ

We can approximate the Hausdorff dimension of $\mathrm{Lim}_{CG}(\Gamma(\mathscr{S}))$ at the first digit with explicit computations for θ small or close to 0.

By Lemma 4.2.2, we have that for every generator of $\Gamma(\mathscr{S})$ it holds:

$$\det\left(\frac{\partial \iota_i}{\partial(\zeta, v)}(t)\right) = \frac{|\sin(\theta)|^4}{12}.$$

The transition matrix is given by

$$T = \frac{1}{12}\begin{pmatrix} 0 & |\sin(\theta)|^4 & |\sin(\theta)|^4 \\ |\sin(\theta)|^4 & 0 & |\sin(\theta)|^4 \\ |\sin(\theta)|^4 & |\sin(\theta)|^4 & 0 \end{pmatrix}$$

The eigenvalue of T^α have to satisfy $2(\frac{|\sin(\theta)|^4}{12})^\alpha = 1$, so

$$\alpha = \frac{\log(2)}{\log(12) - 4\log(|\sin(\theta)|)}.$$

4.4.2 Applying the Computational Algorithm

We start with a sample of 15 equally distributed angles on $(0, \pi/3)$, and for each angle, we constructed the $\theta-$Schottky group; for each group, we approximated the Hausdorff dimension of its Chen-Greenberg limit set with the previous algorithm. These approximations are displayed on the following image.

If we compare the previous graph with the one of the Pair of pants groups in [114], we can notice the similarity on the behavior in the Hausdorff dimensions.

4.4.3 $\mathbb{R}-$circle centers

We can put the generating chains' center in any part of the Heisenberg space; if we place these centers close to points in the standard finite $\mathbb{R}-$circle, more precisely on $\sec(\theta)(0,1), \sec(\theta)(0,-1)$ and $\sec(\theta)(-i,0)$, we can construct a $\theta-$Schottky group but instead of the angle belongs to $(0, \pi/3)$, we have to ask that the angle belongs

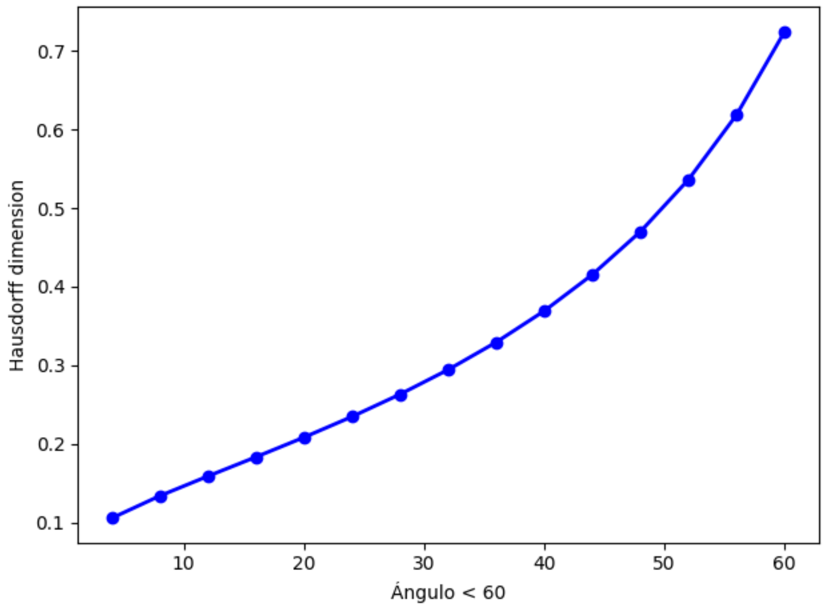


Figure 4.3 – The computed Hausdorff dimension for $\Lambda_{CG}(\Gamma(\mathscr{S}))$ varying θ.

to $(0, 9\pi/40)$, this is to guarantee the quasi-convex co-compact property on the group.

Since the standard finite $\mathbb{R}-$circle is a space circle whose planar projection is a lemniscate, we loose the symmetry on the chains and the centers.

The three generating chains are of parametrized by

$$\begin{aligned} t &\mapsto \big(\tan(\theta)e^{it}, \sec^2(\theta)\big) \\ t &\mapsto \big(\tan(\theta)e^{it}, -\sec^2(\theta)\big) \\ t &\mapsto \big(\tan(\theta)e^{it} - i\sec(\theta), -2\sec(\theta)\tan(\theta)\cos(t)\big) \end{aligned}$$

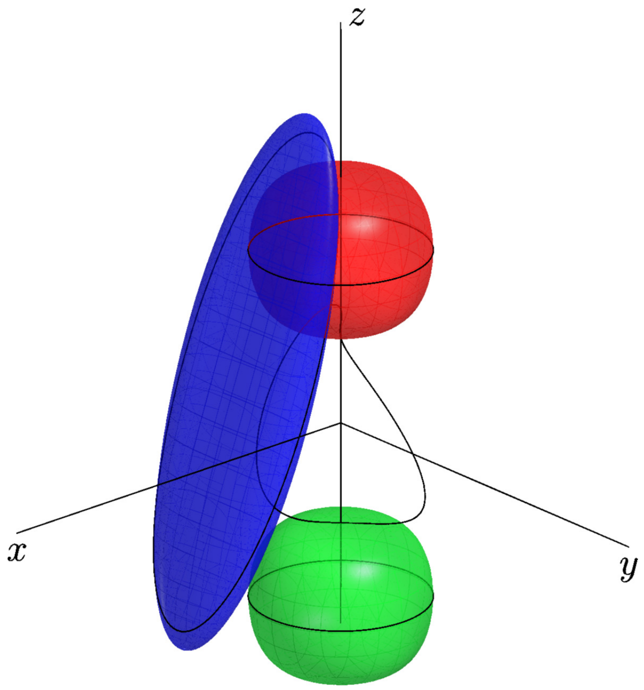
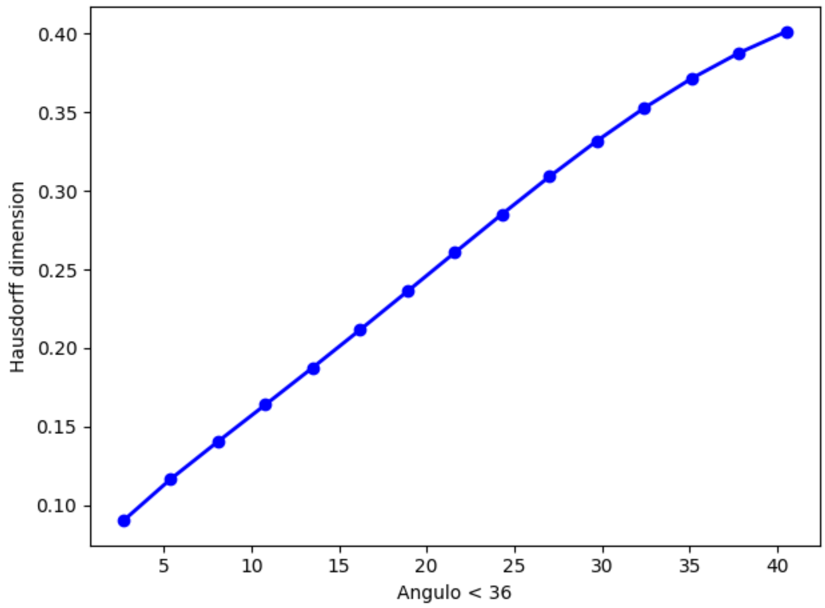


Figure 4.4 – Configuration of three finite chains that generate a $\theta-$Schottky group.

We denote by $\hat{\iota}_i$ the complex reflections generated by the previous chains, and a straight computations shows that the analysis done for the previous groups holds for these groups. For this reason if we took a set of uniformly distributed angles on $(0, 9\pi/40)$ the Hausdorff dimension of its Chen-Greenberg limit set behave similarly as you can see in the following figure.


0.40
0.35
0.30
0.25
0.20
0.15
0.10
Hausdorff dimension
5
10
15
20
25
30
35
40
Angulo < 36

5 Towards the Representation Variety of the Borromean Link Complement.

Introduction

We can understand the different geometric structures that a manifold M can admit if we study the different morphism of its fundamental group into a general Lie group, see [74]. Even more, if we fix the Lie group and we take the conjugation classes of the morphisms, we can explore the differences between the "same" structures. The space of classes $Hom(\pi_1(M), G)/G$ is known as the representation variety of $\pi_1(M)$ and it is denoted by $Rep(\pi_1(M), G)$, see [106]. Naively speaking, the Representation variety is a map that allows us to study the different group morphisms of a particular group of interest. In what follows, we are interested in the representation variety when $G = \mathrm{PU}(2,1)$ and M is the complement of the Borromean link.

W. Thurston, in [154], provides one of the first clear examples of representation varieties of fundamental groups of three-manifolds. The most remarkable in Thurston's work is that he passes through the hyperbolic structures of the figure-eight knot complement. Thurston provide an algorithm that uses a "triangulation" of the manifold where the face-pairing transformations give a representation of the fundamental group into $\mathrm{PSL}(2,\mathbb{C})$. Actually, by triangulation, he meant a decomposition of the manifold into ideal tetrahedra in $\mathbb{H}^3$.

M. Takahashi ([151], [152]) and I. Aitchison, E. Lumsden, J. Rubinstein ([6], [7]) works extended the set of links that can admit a process similar to Thurston's, and the conditions on the link that allowed this process only depend on combinatorial properties of the planar graph induced by the link. In particular, the Borromean link planar graph fulfilled this conditions.

The study of representation varieties of fundamental groups of 3-manifolds is at its first steps and there is a lot to study about.

In the higher-dimensional complex setting, E. Falbel ([64]) used Thurston's process in order to find representations of the fundamental group of the figure eight-knot complement in $\mathrm{PU}(2,1)$, but the triangulations are made of ideal tetrahedra on the complex hyperbolic space. It turns out that some of the representations found by E. Falbel provide a CR-structure on the figure-eight knot complement. Later, E. Falbel, A. Guilloux, P. Koseleff, F. Rouillier, and M. Thistlethwaite, in a series of works ([27], [54],[62], [63]), provide explicit computations of representation into $\mathrm{PSL}(3,\mathbb{C})$ for 3-manifolds that admit "few" tetrahedrons on its triangulation. Those computations provide a parametrization of subsets of the $\mathrm{PSL}(3,\mathbb{C})-$representation variety of such manifolds.

The Chapter is composed of two parts. In the first part, we present the construction of the real hyperbolic structure on the Borromean link complement. Our intentions are to provide a clear, precise and complete exposition of this well-known fact; we refer to [154], [137] and [120] for previous expositions of this work. This part is relevant for the later because it provides the explicit hyperbolic polyhedron that helps in the construction of the $\mathrm{PSL}(2,\mathbb{C})-$representation of this fundamental group.

In the second part, we provide an estimation of the dimension of the $\mathrm{PU}(2,1)-$representation variety of the fundamental group of the Borromean link complement (Theorem 5.2.11). We applied a modified version of Falbel's algorithm that instead of looking for a triangulation, we take the fundamental polyhedron described in the previous part; in order to do this, we had to describe what a complex hyperbolic ideal octahedron means ideas inspired by Thurston's and Fabel's ideal tetrahedra. Unfortunately, we do not compute examples of $\mathrm{PU}(2,1)-$representations of $\pi_1(M)$, but we provide a system of equations whose possible solutions will induce $\mathrm{PU}(2,1)-$representations of the fundamental group of the Borromean link complement (Proposition 5.2.10).

The results achieved in this Chapter conform the first attempts in the direction to prove the (non-) generalization of Sullivan's finiteness theorem for the higher-dimensional complex setting, see [94],[90], [93].

5.1 A Real Hyperbolic Structure

In this section, we deal with the construction of a real hyperbolic structure in the Borromean link complement on the three-dimensional sphere using the algorithm of Thurston ([154]). We have to mention that this algorithm was implemented for the figure Eight Knot complement, the Whitehead link complement, and others (see [137]); but the precise computations for the Borromean link aren't in the literature. As a key part of the forthcoming work, we will present the computations in the case of the Borromean link complement. This section is composed of two parts, the first one is a combinatorial version of Thurston's ideas and the second one are the exact computations to obtain the real hyperbolic structure.

5.1.1 The Combinatorial Realization

The combinatorial realization of a link complement (or knot complement) is an algorithm to obtain a manifold homeomorphic to the sphere complement of the link using a combinatorial object intrinsically related to the link, the algorithm is presented in [6], but also we can refer [151] and [155]. The combinatorial realization generalizes the Thurston hyperbolization algorithm for link complements since was proved for a more bigger family of links.

Assume we have the Borromean rings embedded in $\mathbb{R}^3$ and we took a planar projection of this link into the plane $z = 0$, denote by Γ this projection.

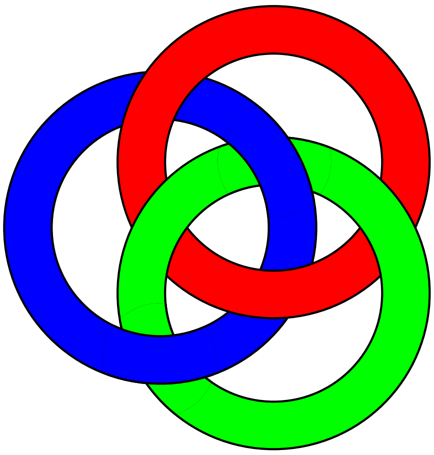


Figure 5.1 – Borromean Link, $\mathscr{B}$.

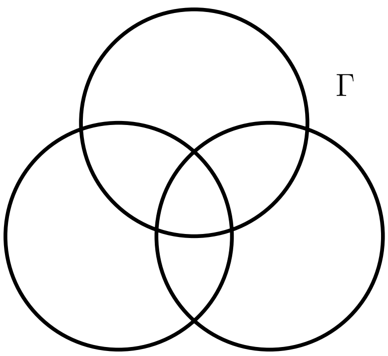


Figure 5.2 – Plannar projection of $\mathscr{B}$ in $z = 0$.

Notice that Γ is a planar connected degree four graph that gives a partition of $z=0$, even more this partition can be signed in the following way:

1. Assign $+$ to the unbound region.
2. Assign $-$ to the regions which have an adjacent edge to a $+$-signed region.
3. Assign $+$ to the regions which have an adjacent edge to a $-$-signed region.

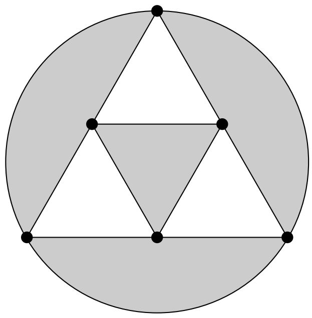


Figure 5.3 – The signed partition of $z = 0$ induced by Γ.

If we took the one-point compactification of $z \geq 0$ we obtain a 2-ball whose boundary sphere has a copy of Γ on it, notice that this is a CW-complex whose 1-skeleton is Γ; even more, there is a cellular map whose identifies this CW-complex with the Octahedron complex. We will denote the previous CW-complex by Π_Γ^+. If we took the dual sign of the partition we can construct a copy of the CW-complex and we will denote this by Π_Γ^-.

It is well known that we can construct $\mathbb{S}^3$ as a union of two copies of $\mathbb{D}^2$ identified by the boundaries; we can took the two closed balls Π_Γ^+ and Π_Γ^- for this process, but the most relevant part is the actual gluing process in comparison with the gluing pieces.

The gluing process is as follows: for every 3-cell ϕ_j of Π_Γ^+ with sign σ_j, we take the corresponding 3-cell $\phi_j' \subset \Pi_\Gamma^-$ and identify this cells with a rotation of $\sigma_j \frac{2\pi}{3}$ where $+$ denotes a clockwise rotation.

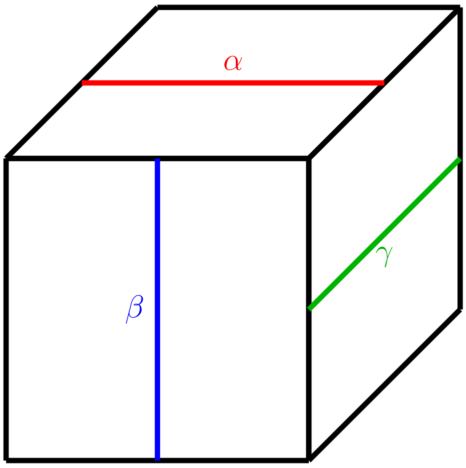
We will denoted by $\overline{M}_\Gamma$ the space obtained after gluing and by M_Γ if we remove the 0-skeleton of Γ in $\overline{M}_\Gamma$.

The following theorem due to Aitchison, describes the relation between the gluing process result and the complement of a link (associated to a graph). In particular, it is valid for the Borromean rings complement explained in the previous paragraphs.

Theorem 5.1.1 ([6]). *If $\mathscr{L}_\Gamma$ denotes the link associated to Γ, then M_Γ is canonically homeomorphic to $\mathbb{S}^3 \setminus \mathscr{L}_\Gamma$. Each edge of M_Γ is of degree* 4.

Since the previous combinatorial description doesn't show intuitively that the classes of the vertices are the components of the link, we will give a more naively description of how, with a similar gluing process, we construct the Borromean link.

Let's take a cube whose each opposite pair of faces has a middle line as the figure, where different pairs of faces have a different color of lines.



We can think the previous cube with extra edges as a degenerated dodecahedron. We will glue this "dodecahedron" over its faces, insight to obtain the Borromean link and its complement.

First, we glue the faces with the α edges with a reflection over this edge; we obtain a cylinder with an ellipsis and two pairs of lines in it. After, we glue the faces of the cylinder with a reflection over the β edges; we obtain a sphere whit two ellipses in it. Finally, we glue the faces of the sphere and the γ edges, we obtain the Borromean link.

We have to recall that the final of the previous process is the Borromean link and its complement. If we want just the complement we have to identify the extra edges as points at infinity.

The previous process gave us a Euclidean realization of the complement of the

Borromean link.

5.1.2 Thurston's Algorithm

Roughly speaking, Thurston's algorithm to obtain the hyperbolic structure of the figure Eight Knot complement relies on the construction of an ideal hyperbolic tetrahedron and compute the face-gluing transformations.

From the combinatorial construction done previously, we know that the right pieces for the Borromean link complement are two octahedrons; for this reason, we will need an ideal hyperbolic octahedron description.

Let's take a set of six points in $\partial\mathbb{H}^3(=\hat{\mathbb{S}})$ and that 0, 1 and ∞ belong to this set. To each pair of different points attach the bi-infinite geodesic in $\mathbb{H}^3$. The ideal octahedron will be the set delimited by the hyperplanes containing three different points and geodesics.

First, we need to compute the other points in the ideal octahedron; Thurston associated a system of equations involving invariant numbers given by the vertices. Let us consider our octahedron as a union of four tetrahedrons as we can see in the figure.

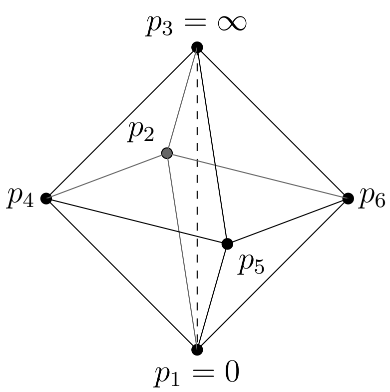


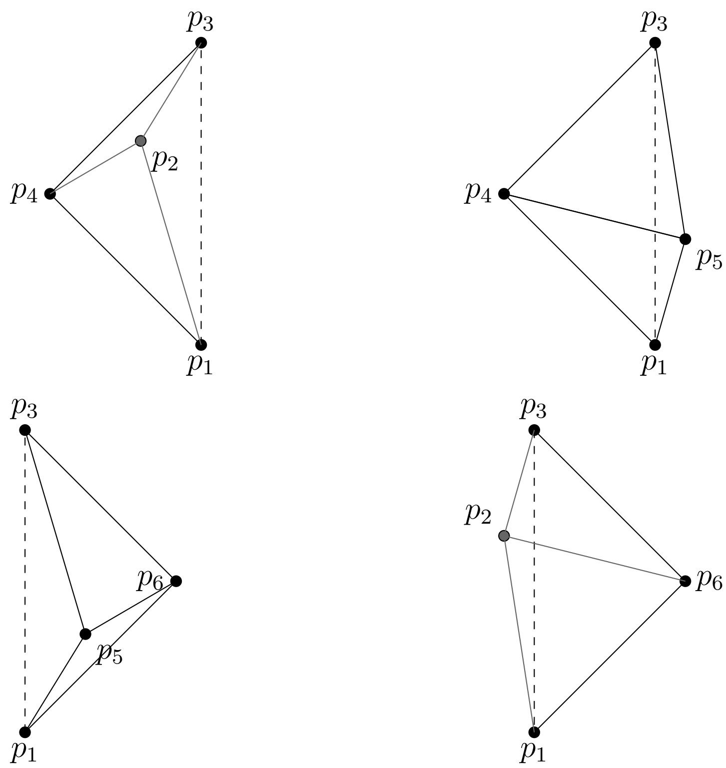
Figure 5.4 – Ideal octahedron in $\mathbb{H}^2_{\mathbb{R}}$.

We recall that for each of the tetrahedron has three different invariants, let say z_i, v_i and w_i. These numbers are obtained by a cross-ratio combination of the vertices and they are related by the following equations:

$$v_i = \frac{1}{1 - z_i} \tag{5.1}$$

$$w_i = \frac{z_i - 1}{z_i} \tag{5.2}$$

The following pictures shows the four tetrahedra that compose the octahedron.



Since these tetrahedra are glued in pairs by a face and share a common face, we need to guarantee that the gluing give us a well defined hyperbolic polyhedron. In [61], the authors provide conditions that guarantee our aims, naively speaking these conditions relate to the glued faces and edges. The following system of equation summarized these conditions.

$$z_1z_2z_3z_4 = 1 \tag{5.3}$$

$$w_1v_4z_1z_2w_2v_3 = 1 \tag{5.4}$$

$$v_1w_2v_3w_4w_3v_4w_1v_2 = 1 \tag{5.5}$$

$$v_2w_3z_3z_4v_1w_4 = 1 \tag{5.6}$$

The previous system is equivalent to

$$\frac{(z_1-1)(z_2-1)}{z_3-1} = \frac{1-z_1z_2z_3}{z_1z_2z_3}. \tag{5.7}$$

Since we started with the assumption that $p_1 = 0$, $p_3 = \infty$ and $p_2 = 1$. The previous equation implies that the other three points have to satisfy

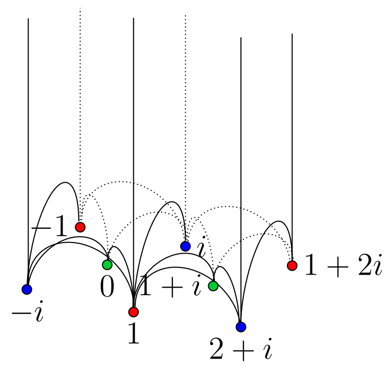
$$\frac{p_5}{p_4} \in \mathbb{C}^*$$
$$\frac{p_6}{p_5} \in \mathbb{C}^*$$

These conditions together with the equation 5.7, imply that the possible vertex points are in a four-dimensional real variety that contains the points: $p_4 = it$, $p_5 = -1$ and $p_6 = -it$, with $t > 0$.

For simplicity we will choose $t = 1$, so the points are $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$, $p_4 = i$, $p_5 = -1$ and $p_6 = -i$.

5.1.3 The Fundamental Region

Following our combinatorial model, the right pieces to obtain a fundamental region are two octahedra. Let's take a copy of the octahedra described in the previous paragraph applying the transformation $T(z) = z + 1 + i$, these two octahedra share a common face, let say $(1, \infty, i)$. The following picture shows the previous construction in the upper half-space model.



In order to obtain a representation into $\mathrm{PSL}(2, \mathbb{C})$, of Borromean link complement's fundamental group, we have to compute the face-gluing transformations.

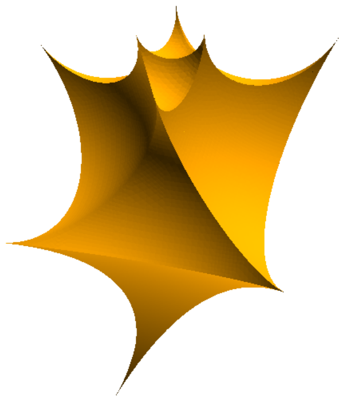


Figure 5.5 – Fundamental Region viewed in the Ball model of $\mathbb{H}^3$.

We have to recall that a face of the polyhedron is characterized by a triplet of points and since in $\hat{\mathbb{C}}$ always exists a transformation that sends a triplet into $(0, 1, \infty)$, the face-gluing transformations are totally computable; we just need to follow the instructions of the combinatorial gluing-process.

The following picture describes more precisely the combinatorial gluing-process and the faces of the polyhedron.

The following lemma resume the computations of the face-gluing transformations.

Lemma 5.1.2. *The face-gluing transformations are given by the extension to* $\mathbb{H}^3$ *of the Möbius transformations induced by them following matrices.*

- $\gamma_a = \begin{bmatrix} 2+i & -2+i \\ 1 & -1 \end{bmatrix}$
- $\gamma_b = \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}$

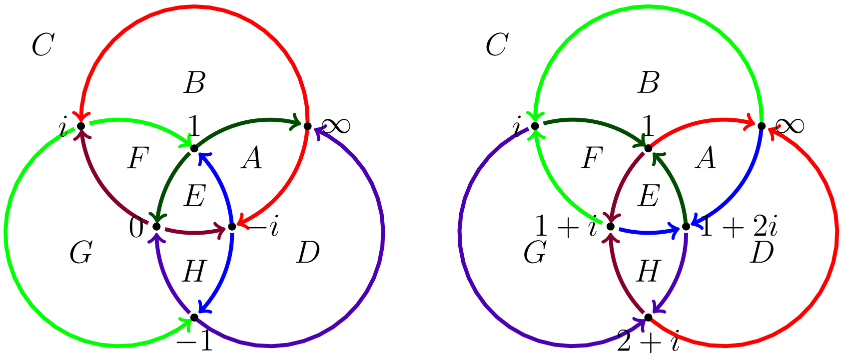


Figure 5.6 – Face and Edge Gluing instructions.

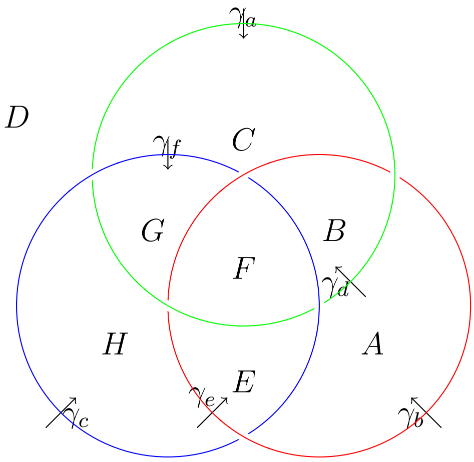
- $\gamma_c = \begin{bmatrix} i & 3i \\ 1 & 1 \end{bmatrix}$
- $\gamma_d = \begin{bmatrix} 2+i & 2+3i \\ 1 & 1 \end{bmatrix}$
- $\gamma_e = \begin{bmatrix} 3i & -2-i \\ 1+2i & -1 \end{bmatrix}$
- $\gamma_f = \begin{bmatrix} 2-i & -i \\ 1-2i & -1 \end{bmatrix}$
- $\gamma_g = \begin{bmatrix} -2+3i & i \\ 1+2i & 1 \end{bmatrix}$
- $\gamma_h = \begin{bmatrix} 4-i & 2+i \\ 1-2i & 1 \end{bmatrix}$

For Link complement's fundamental groups there exists a series of relations that the generators have to fulfilled, known as Wirtinger relations, see [86], [97], [141]. The Wirtinger relation are given by the type of crossings of the link. In the following picture is show the crossings for the Borromean link and how a loop behave when we apply a face-gluing transformation.

From the crossings we obtain six relations, but the $\pi_1(\mathbb{S}^3 \setminus \mathscr{B})$ have two relations, so we need to reduce the Wirtinger relations.

$$\begin{aligned} &R_1 : \gamma_d\gamma_b^{-1}\gamma_a\gamma_b = 1 && R_4 : \gamma_d\gamma_b^{-1}\gamma_a\gamma_b = 1 \\ &R_2 : \gamma_f\gamma_a^{-1}\gamma_c^{-1}\gamma_a = 1 && R_5 : \gamma_e\gamma_f^{-1}\gamma_b^{-1}\gamma_f = 1 \\ &R_3 : \gamma_e\gamma_c^{-1}\gamma_b^{-1}\gamma_c = 1 && R_6 : \gamma_c^{-1}\gamma_d\gamma_f\gamma_d^{-1} = 1 \end{aligned}$$

The reduction of these relations is made in [118]. With this we guarantee that



the group generated by the face-gluing transformation with the Wirtinger relations, is a representation of $\pi(\mathbb{S}^3 \setminus \mathscr{B})$ into $\mathrm{PSL}(2,\mathbb{C})$.

5.2 The Complex Hyperbolic Setting

The following ideas and constructions are inspired by the ones done in [62], [64], [65]. In the series of articles, previously mentioned, the authors provide an algorithm to provide Complex Hyperbolic structures using tetrahedra to the figure Eight Knot complement.

We will take advantage of the previous section in the sense that we already know that the right way to obtain a representation of $\pi_1(\mathbb{S}^3 \setminus \mathscr{B})$ is by face-pairing transformations of two octahedra.

The main difference between our ideas and the ones done by Falbel ([64],[65],[62], [54]) are that we do not look for a tetrahedral "triangulation" of the fundamental polyhedron obtained in the real hyperbolic structure, instead, we deal with the known polyhedron that work as a fundamental domain in the real hyperbolic space. The previous provide us a reduction on the number of equations that constrain our possible representation.

5.2.1 Ideal Complex Hyperbolic Octahedra

First, we need to define what will be an ideal octahedron in the complex hyperbolic setting and look if we can replicate the associate numeric invariants from the real hyperbolic case.

As Falbel mention in [62], there is no notion of a two-dimensional complex hyperbolic geodesic subset that works as the face of a polyhedron. For this reason, we will leave aside the face for our proposes.

A *generic ideal complex hyperbolic octahedron* will be a configuration of six points of $\partial\mathbb{H}^2_{\mathbb{C}}$ in general position as points in $\mathbb{C}\mathrm{P}^2$. A *normalized ideal complex hyperbolic octahedron* is an ideal octahedron whose set of points contain $\infty, (0,0)$ and $(1,0)$. Notice that the previous points coordinates are given in Heisenberg coordinates because this will be the coordinates in where we will give our computations and constraints. If we consider that the following points describe a normalized ideal octahedron:

$$p_0 = \infty \qquad q_0 = (0,0)$$

$$p_1 = (1,t) \qquad q_1 = (\zeta_1, s_1|\zeta_1|^2)$$

$$p_2 = (\zeta_0, s_0|\zeta_0|^2) \qquad q_2 = (\zeta_2, s_2|\zeta_2|^2)$$

We recall that in the previous Section, we compute the points of an octahedron from equations involving numeric invariants associated to a generic configuration of points. In [64], the author uses an invariant similar to the real hyperbolic cross-ratio, we have to mention that these numbers have no geometric interpretation in the complex hyperbolic geometry, but for the algorithm described in [64] it worked. The following definition describes these invariants in the case of a normalized ideal complex hyperbolic octahedron.

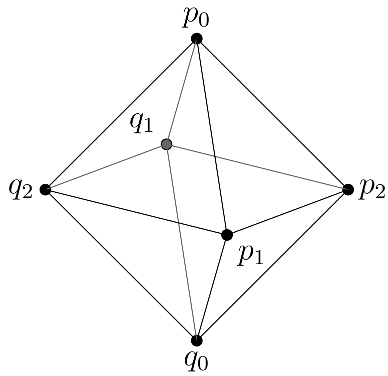
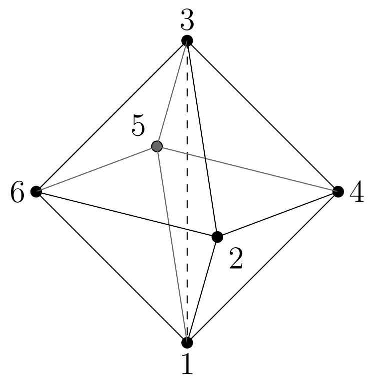


Figure 5.7 – Normalized Ideal Complex Hyperbolic Octahedron.

Definition 5.2.1. Let P an normalized ideal complex hyperbolic octahedron in $\partial\mathbb{H}^2_{\mathbb{C}}$, whose points are labeled as in the following figure:



We recall that we will always assume that 3 correspond to ∞. The invariant z_{ij} is defined as follows:

Look in P the tetrahedron that contains $3, 1, i, j$ and send the point i to 3 by a Heisenberg translation composed with the Koranyi inversion, denoted by ψ_{i3}, see figure 5.8.

$$z_{ij} = \frac{\mathscr{Z}(\psi_{i3}(1)) - \mathscr{Z}(\psi_{i3}(j))}{\mathscr{Z}(\psi_{i3}(3)) - \mathscr{Z}(\psi_{i3}(j))}. \tag{5.8}$$

Where $\mathscr{Z}(p)$ of $p \in \partial\mathbb{H}^2_{\mathbb{C}}$ denotes the complex part of the Heisenberg coordinates

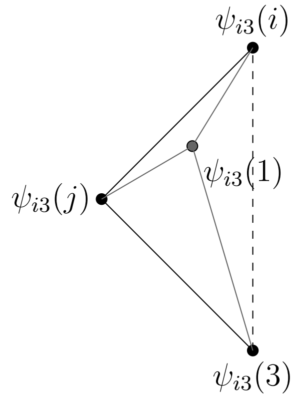


Figure 5.8 – Image under ψ_{i3} of the Tetrahedra containing $3, 1, i, j$.

of the point. We recall that we are assuming that $\psi_{i3}(1)$ is the right-most point from $\psi_{i3}(j)$.

From the way these numeric invariants are defined, it is not difficult to show the relations between them in the following Lemma.

Lemma 5.2.2. *For some vertex i on the generic octahedron, with the vertex belonging to the tetrahedron $3, i, j, 1$. Then:*

$$z_{3i} = \frac{1}{1 - z_{31}} \tag{5.9}$$

$$z_{3j} = 1 - \frac{1}{z_{31}} \tag{5.10}$$

The previous Lemma is similar to Definition 4.4 on [64] because an octahedron can be triangulated by four ideal tetrahedrons.

5.2.2 The Pairing Transformation

In the real hyperbolic representation, the generators are face-pairings of the fundamental polyhedra. These face-pairings are transformations that send an ordered triplet of points into another one. In the complex hyperbolic setting, the triplets of ordered points are determined up to $\mathrm{PU}(2,1)$ in terms of the *Cartan angular invariant.*

Definition 5.2.3. Let (x_1, x_2, x_3) a triplet of points each one belonging to $\partial\mathbb{H}^2_{\mathbb{C}}$. The *Cartan angular invariant*, denoted by $\mathbb{A}(x_1, x_2, x_3)$, is defined as

$$\mathbb{A}(x_1, x_2, x_3) = \arg(-\langle\tilde{x}_1, \tilde{x}_2\rangle\langle\tilde{x}_2, \tilde{x}_3\rangle\langle\tilde{x}_3, \tilde{x}_1\rangle) \tag{5.11}$$

where $\langle\cdot,\cdot\rangle$ is the Hermitian product with signature (2,1) (see Appendix A).

It is not difficult to show that the previous argument does not depends on the election of lifts that we take for every point in the triplet.

Proposition 5.2.4 ([44],[73])**.** *The Cartan angular invariant classifies ordered triplets of points up to* $\mathrm{PU}(2,1)$.

In the real hyperbolic case, the ideas to compute the transformations that pair two triplets of points rely on the property that every triplet can be sent to $(\infty, 0, 1)$. In [64], the author made similar computations to describe a $PU(2,1)$ pairing between a pair of triplets with the same Cartan angular invariant. We will replicate these computations for sake of completeness in our treatment about the representation of the fundamental group of the Borromean link complement.

First, consider two ordered triplets of points, let say $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$; for our proposes, we will assume that each points of these triplets don't belong to the same $\mathbb{C}-$circle since our assumptions on the points of an ideal complex hyperbolic octahedron satisfy these. For simplicity assume that $x_1 = \infty$, $x_2 = (0,0)$ and $x_3 = (1,t)$, and $y_1 = (z_1, t_1)$, $y_2 = (z_2, t_2)$ and $y_3 = (z_3, t_3)$. If we assume that $y_1 \neq \infty$ and $y_2 \neq \infty$, from the lift formula of Heisenberg coordinates, the matrix representative of the class $g_y \in \mathrm{PU}(2,1)$ is of the form

$$g_y = \begin{pmatrix} \frac{-|z_1|^2+it_1}{2} & b & \lambda \cdot \frac{-|z_2|^2+it_2}{2} \\ z_1 & e & \lambda z_2 \\ 1 & h & \lambda \end{pmatrix} \tag{5.12}$$

Where $g_{\mathbf{y}}$ denotes the transformation that pairs the triplets $\mathbf{x}$ and $\mathbf{y}$. To compute the remaining entries and λ, we have to remember that for an element in $\mathrm{PU}(2,1)$ each representative have to satisfy

$$g_{\mathbf{y}}^* J g_{\mathbf{y}} = J$$

where $g_{\mathbf{y}}^*$ is the conjugated transpose and J is the Hermitian matrix that defines the Hyperbolic space. From this we can obtain the following cases:

- If $z_1 \neq z_2$, then

$$e = \frac{h}{2} \cdot \frac{|z_1|^2 - |z_2|^2 + i(t_1 - t_2)}{\overline{z}_1 - \overline{z}_2}, \tag{5.13}$$
$$b = -\frac{h}{2} \cdot \frac{\overline{z}_2\left(|z_1|^2 + it_1\right) - \overline{z}_1\left(|z_2|^2 + it_2\right)}{\overline{z}_1 - \overline{z}_2} \tag{5.14}$$

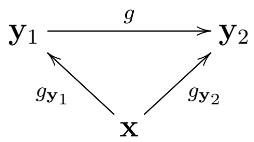
- If $z_1 = z_2$, then $h = 0$ and $b = -\overline{z}_1 e$.

Since $g_{\mathbf{y}}$ have to satisfy

$$g_{\mathbf{y}} * \begin{bmatrix} \frac{-1+it}{2} \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-|z_3|^2+it_3}{2} \\ z_3 \\ 1 \end{bmatrix}$$

we obtain the final equations needed; such that in the case $z_1 \neq z_2$, solving these we obtain h and λ and in the case $z_1 = z_2$, we obtain b and λ.

The $PU(2,1)$ transformation that pairs two triplets of ordered points will be the one that make the following diagram commutes.



From the previous paragraphs, we saw that to be able to talk about a pairing transformation we need to compute the Cartan angular invariants for triplets of points in a normalized ideal complex hyperbolic octahedron. In the following, we will show the values of Cartan angular invariant for different choices of three points in the octahedron, but we will omit several of the computations.

Consider the normalized ideal complex hyperbolic octahedron from the figure 5.7 and denoted by $\mathcal{O}$, since we can see this as a union of four tetrahedra, the next lemma is inspired in the Proposition 4.6 in [64].

Lemma 5.2.5. *For an ordered triplet of points in* $\mathcal{O}$, *it Cartan angular invariant* *are:*

$$
\begin{aligned}
\tan \mathbb{A}(q_0, p_0, p_1) &= t \\
\tan \mathbb{A}(q_0, p_0, p_2) &= s_0 \\
\tan \mathbb{A}(q_0, p_0, q_1) &= s_2 \\
\tan \mathbb{A}(q_0, p_0, q_2) &= s_3 \\
\tan \mathbb{A}(q_0, p_1, p_2) &= \frac{s_0|\zeta_0|^2 + 2\mathrm{im}(\zeta_0) - t}{|\zeta_0 - 1|^2} \\
\tan \mathbb{A}(q_0, p_1, q_1) &= \frac{s_1|\zeta_1|^2 + 2\mathrm{im}(\zeta_1) - t}{|\zeta_1 - 1|^2} \\
\tan \mathbb{A}(q_0, p_1, q_2) &= \frac{s_2|\zeta_2|^2 + 2\mathrm{im}(\zeta_2) - t}{|\zeta_2 - 1|^2} \\
\tan \mathbb{A}(p_0, p_1, p_2) &= \frac{2(s_0 - t)\,\mathrm{Re}(\zeta_0) + 2(1 + ts_0)\,\mathrm{Im}(\zeta_0) + t(1 + s_0^2)|\zeta_0|^2 - s_0(1 + t^2)}{|(s_0 - i)\zeta_0 + i - t|^2} \\
\tan \mathbb{A}(p_0, p_1, q_1) &= \frac{2(s_1 - t)\,\mathrm{Re}(\zeta_1) + 2(1 + ts_1)\,\mathrm{Im}(\zeta_1) + t(1 + s_1^2)|\zeta_1|^2 - s_1(1 + t^2)}{|(s_1 - i)\zeta_1 + i - t|^2} \\
\tan \mathbb{A}(p_0, p_1, q_2) &= \frac{2(s_2 - t)\,\mathrm{Re}(\zeta_2) + 2(1 + ts_2)\,\mathrm{Im}(\zeta_2) + t(1 + s_2^2)|\zeta_2|^2 - s_2(1 + t^2)}{|(s_2 - i)\zeta_2 + i - t|^2} \\
\tan \mathbb{A}(q_0, p_2, q_1) &= \frac{|\zeta_0|^2 s_0 + |\zeta_1|^2 s_1 + 2\,\mathrm{Re}\,\zeta_1\,\mathrm{Im}\,\zeta_0 - 2\,\mathrm{Re}\,\zeta_0\,\mathrm{Im}\,\zeta_1}{|\zeta_0 + \zeta_1|^2} \\
\tan \mathbb{A}(q_0, p_2, q_2) &= \frac{|\zeta_0|^2 s_0 + |\zeta_2|^2 s_2 + 2\,\mathrm{Re}\,\zeta_2\,\mathrm{Im}\,\zeta_0 - 2\,\mathrm{Re}\,\zeta_0\,\mathrm{Im}\,\zeta_2}{|\zeta_0 + \zeta_2|^2} \\
\tan \mathbb{A}(q_0, q_1, q_2) &= \frac{|\zeta_1|^2 s_1 + |\zeta_2|^2 s_2 + 2\,\mathrm{Re}\,\zeta_1\,\mathrm{Im}\,\zeta_2 - 2\,\mathrm{Re}\,\zeta_2\,\mathrm{Im}\,\zeta_1}{|\zeta_1 + \zeta_2|^2}
\end{aligned}
$$

$$
\begin{aligned}
\tan \mathrm{A}(p_0, p_2, q_2) &= \frac{(s_2 - s_0)|\zeta_0 + \zeta_2|^2 + (1 + s_0 s_2)\left(|\zeta_0|^2 s_0 + |\zeta_3|^2 s_3 + 2\,\mathrm{Re}\,\zeta_3\,\mathrm{Im}\,\zeta_1 - 2\,\mathrm{Re}\,\zeta_1\,\mathrm{Im}\,\zeta_3\right)}{(1 + s_0 s_3)|\zeta_0 + \zeta_2|^2 - (s_2 - s_0)\left(|\zeta_0|^2 s_0 + |\zeta_2|^2 s_2 + 2\,\mathrm{Re}\,\zeta_2\,\mathrm{Im}\,\zeta_0 - 2\,\mathrm{Re}\,\zeta_0\,\mathrm{Im}\,\zeta_2\right)} \\
\tan \mathrm{A}(p_0, q_1, q_2) &= \frac{(s_2 - s_0)|\zeta_0 + \zeta_2|^2 + (1 + s_0 s_2)\left(|\zeta_0|^2 s_0 + |\zeta_3|^2 s_3 + 2\,\mathrm{Re}\,\zeta_3\,\mathrm{Im}\,\zeta_1 - 2\,\mathrm{Re}\,\zeta_1\,\mathrm{Im}\,\zeta_3\right)}{(1 + s_0 s_3)|\zeta_0 + \zeta_2|^2 - (s_2 - s_0)\left(|\zeta_0|^2 s_0 + |\zeta_2|^2 s_2 + 2\,\mathrm{Re}\,\zeta_2\,\mathrm{Im}\,\zeta_0 - 2\,\mathrm{Re}\,\zeta_0\,\mathrm{Im}\,\zeta_2\right)}
\end{aligned}
$$

Proof. It follows from direct computations. □

5.2.3 The ideal complex hyperbolic polyhedron

We should recall that for the Real hyperbolic structure on Borromean link's complement, Poincaré polyhedron theorem plays a key role to obtain a representation of its fundamental group into $\mathrm{PSL}(2, \mathbb{C})$. This theorem allows us to guarantee the discreteness of a group of face-identification of a polyhedron under certain conditions. Let $P \subset \mathbb{H}^3$ be a polyhedron and $\iota : P_2 \to P_2$ an involution on the set of faces of P such that for every $f \in P_2$ there exists a Möbius transformation γ_f

whose $\gamma_f : f \to \iota(f)$ is an isometry, $\gamma_f \gamma_{\iota(f)} = id$ and γ_f maps the interior of P to the exterior.

Theorem 5.2.6 (Poincaré Polyhedron Theorem, [112]). *Let $X := P/\sim$ the quotient space under the equivalence $x \sim \gamma_f(x)$ for $x \in f$. If X is complete with the quotient metric from P, the map $F : P \to X$ is finite-to-one, and if for every edge class $\overline{e} \in P_1/\sim$ there exists an integer $N_{\overline{e}}$ such that the sum of the measure dihedral angles incidents to $\overline{e}$ is $\frac{2\pi}{N_{\overline{e}}}$. Then the group $G \subset \mathrm{PSL}(2, \mathbb{C})$ generated by the set $\{\gamma_f : f \in P_2\}$ is discrete, P is a fundamental domain of G and $\mathbb{H}^3/G$ is homeomorphic to X.*

In the complex hyperbolic setting, there is no a unique version of Poincaré's polyhedron theorem, the different version theorems depend on the authors' requirements and problem characteristics, we can refer to [81], [58], [61]. Since we don't have this tool, we need to be careful if the possible group that we obtain from a "face"-pairing transformation set satisfies the properties that we are looking forward.

In the earlier, we said that the algorithm, used by Falbel, can lead us to a system of equations with many equations or variables and with no security of existence of a solution. From the previous, we won't proceed to glue several tetrahedra (as Falbel); instead, we will use the previous polyhedron, from the Real hyperbolic case, we know that two glued octahedra make a fundamental domain. Take two normalized ideal octahedra in $\partial\mathbb{H}^2_{\mathbb{C}}$ that share the "face" (p_0, p_1, q_0).

Let assume that two octahedra sharing a common face, described previously, is a configuration of nine different points in $\partial\mathbb{H}^2_{\mathbb{C}}$ with the following coordinates, and will be denoted by $\mathscr{P}$:

$$p_0 = \infty \qquad q_0 = (0,0) \qquad q'_0 = (\xi_0, k_0|\xi_0|^2)$$

$$p_1 = (1,t) \qquad q_1 = (\zeta_1, s_1|\zeta_1|^2) \qquad q'_1 = (\xi_1, k_1|\xi_1|^2)$$

$$p_2 = (\zeta_0, s_0|\zeta_0|^2) \qquad q_2 = (\zeta_2, s_2|\zeta_2|^2) \qquad q'_2 = (\xi_2, k_2|\xi_2|^2)$$

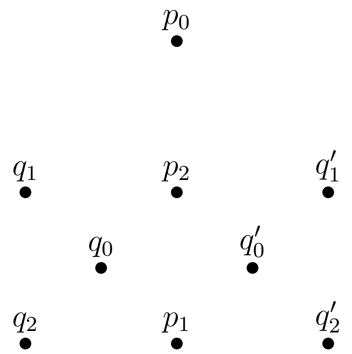


Figure 5.9 – Configuration of nine points of our complex hyperbolic polyhedron.

To avoid the using of a preferred version of the Poincaré polyhedron for the complex hyperbolic setting, we will impose conditions on the "face" and "edge" identifications in order to obtain a "well" glued polyhedron, where we will understand "well" as Epstein and Petronio in [61].

To properly describe the equations we need to define what a *face* and *edge* means for us in our ideal complex hyperbolic polyhedra.

Definition 5.2.7. Let (p,q,r) be a triplet of points in $\mathscr{P}$. The triplet will be called a face of $\mathscr{P}$, if bound a triangular region in the figure 5.10

Definition 5.2.8. Let (p,q) be a pair of different points in $\mathscr{P}$. We will say that the pair is an edge of $\mathscr{P}$, if there is a face of $\mathscr{P}$ whose points contain the pair.

Definition 5.2.9. Let (p_0,q,r,s) a quadruple of different point in $\mathscr{P}$. We will say that (p_0,q,r,s) is an admissible tetrahedron in $\mathscr{P}$ if

1. $s = q_0$ or $s = q'_0$.

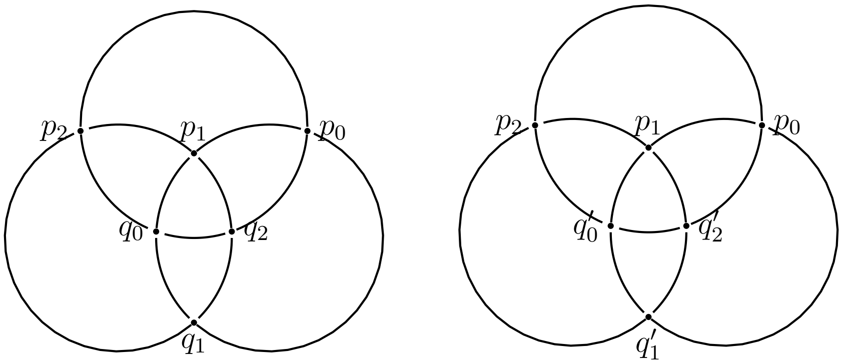


Figure 5.10 – Combinatorial Polyhedra

2. (p_0, q, r) and (q, r, s) are faces of $\mathscr{P}$.

The equations that we will constraint our polyhedron are determined by face and edge identifications, identifications that was studied in the combinatorial model in 5.1.1, we refer to Section 4.1 on [62] for the case of manifolds with "few" tetrahedrons on its triangulation.

1. **Face Equations:** For a pair of identified faces (p_i, p_j, p_k) and (q_i, q_j, q_k) (oriented) we need that

$$z_{il}z_{jl}z_{kl}w_{il}w_{jl}w_{kl} = 1. \tag{5.15}$$

   Where l denotes the unique point such that (l, p_i, p_j, p_k) and (l, q_i, q_j, q_k) (or (or $((p_i, p_j, p_k, l), (q_i, q_j, q_k, l))$ are admissible tetrahedrons and the complex numbers z_* and w_* are the invariants associated to each tetrahedra (see Definition 5.2.1).

2. **Edge Equations:** For an edge e in the polyhedra and let $v_1, \cdots, v_{n(e)}$ the other edges that should be identified in the two octahedrons. Then

$$z(v_1)z(v_2)\cdots z(v_{n(e)}) = 1 \tag{5.16}$$

where $z(v_i)$ correspond to the invariants in an admissible tetrahedra (as in Definition 5.2.1).

From the face identifications rules described in Aitchinson's theorem (see Theorem 5.1.1) we can deduce the following equations system, we have to recall that this identifications are similar to the ones used to compute the Real hyperbolic structure (see Figure 5.6).

Edge Equations

$$z_{23}z_{21}w_{42}w_{62} = 1 \qquad z_{34}z_{36}w_{23}w_{53} = 1$$
$$z_{62}z_{65}w_{36}w_{16} = 1 \qquad z_{53}z_{51}w_{65}w_{45} = 1$$
$$z_{14}z_{16}w_{21}w_{51} = 1 \qquad z_{45}z_{42}w_{14}w_{34} = 1$$

Face Equations:

$$z_{21}z_{31}z_{61}w_{21}w_{31}w_{61} = 1 \qquad z_{13}z_{63}z_{23}w_{13}w_{63}w_{23} = 1$$
$$z_{31}z_{41}z_{21}w_{31}w_{41}w_{21} = 1 \qquad z_{13}z_{43}z_{23}w_{13}w_{43}w_{23} = 1$$
$$z_{31}z_{51}z_{41}w_{31}w_{51}w_{41} = 1 \qquad z_{13}z_{43}z_{53}w_{13}w_{43}w_{53} = 1$$
$$z_{31}z_{51}z_{61}w_{31}w_{51}w_{61} = 1 \qquad z_{13}z_{53}z_{63}w_{13}w_{53}w_{63} = 1$$

The previous system are in terms of the numeric invariants associated to our complex hyperbolic polyhedron, but with the coordinates of the points given by the figure 5.9 we can obtain the following system:

Proposition 5.2.10. *The gluing data of the Complex Polyhedron (5.9) in order to obtain a homomorphism between $\pi_1(\mathbb{S}^3 \setminus \mathscr{B})$ and* PU(2, 1) *is contained in the following equation system:*

- $z_{23}z_{21}w_{42}w_{62} = 1$ *is equivalent to*

$$-\left(-(s_0-i)|\zeta_0|^2+t-2i\zeta_0+i\right)\left(i|\zeta_2|^2-s_2|\zeta_2|^2+\zeta_2(t-i)\right)\left((k_0-i)(\xi_2-1)|\xi_0|^2-(k_2-i)(\xi_0-1)|\xi_2|^2\right.$$

$$+(t-i)(\xi_0-\xi_2))\left((k_0-i)|\xi_0|^2(\zeta_0-1)-(s_0-i)(\xi_0-1)|\zeta_0|^2+(t-i)(\xi_0-\zeta_0)\right)(t+i)^{-1}(\xi_2-1)^{-1}(\zeta_0-1)^{-2}$$

$$\left((i-s_2)|\zeta_2|^2+t-2i\zeta_2+i\right)^{-1}\left(\overline{\xi}_2((k_2+i)\xi_2-2i\xi_0)-(k_0-i)|\xi_0|^2\right)^{-1}\left(\overline{\zeta}_0((s_0+i)\zeta_0-2i\xi_0)-(k_0-i)|\xi_0|^2\right)^{-1}=1$$

- $z_{62}z_{65}w_{36}w_{16}=1$ *is equivalent to*

$$\frac{(\xi_0-\xi_2)^2(\zeta_1-\zeta_2)\left(\overline{\xi}_0(-ik_0\xi_0+\xi_0-2)+it+1\right)\left(is_2\overline{\zeta}_2+\overline{\zeta}_2-it-1\right)}{(\xi_2-1)\zeta_1(\zeta_2-1)\left((s_1-i)\overline{\zeta}_1-(s_2-i)\overline{\zeta}_2\right)\left((k_0-i)(\xi_2-1)|\xi_0|^2-(k_2-i)(\xi_0-1)|\xi_2|^2+(t-i)(\xi_0-\xi_2)\right)}=1$$

- $z_{14}z_{16}w_{21}w_{51}=1$ *is equivalent to*

$$\frac{(t-i)^2\zeta_0^2\left((s_0-i)\overline{\zeta}_0-(k_1-i)\overline{\xi}_1\right)}{(-(s_0-i)|\zeta_0|^2+t-2i\zeta_0+i)\left(-(s_2-i)\overline{\zeta}_2+t-i\right)\left(-(k_1+i)|\xi_1|^2+(s_0-i)|\zeta_0|^2+2i\zeta_0\overline{\xi}_1\right)}=1$$

- $z_{34}z_{36}w_{23}w_{53}=1$ *is equivalent to*

$$\frac{(\xi_0-1)\zeta_0\zeta_2(\xi_0-\xi_1)\left((s_0-i)|\zeta_0|^2-t+2i\zeta_0-i\right)\left(\overline{\xi}_1((k_1+i)\xi_1-2i\zeta_0)-(s_0-i)|\zeta_0|^2\right)}{(\zeta_0-1)^2(\zeta_2-1)(\xi_1-\zeta_0)\left(-(k_0-i)|\xi_0|^2+t-2i\xi_0+i\right)\left(\overline{\xi}_1((k_1+i)\xi_1-2i\xi_0)-(k_0-i)|\xi_0|^2\right)}=1$$

- $z_{53}z_{51}w_{65}w_{45}=1$ *is equivalent to*

$$\zeta_0\left((s_0-i)\overline{\zeta}_0-(s_1-i)\overline{\zeta}_1\right)\left((k_0-i)|\xi_0|^2(\xi_1-\xi_2)-(k_1-i)|\xi_1|^2(\xi_0-\xi_2)+(k_2-i)|\xi_2|^2(\xi_0-\xi_1)\right)$$

$$\left(-(k_0-i)|\xi_0|^2(\xi_1-\zeta_0)+(k_1-i)|\xi_1|^2(\xi_0-\zeta_0)+(s_0-i)|\zeta_0|^2(\xi_1-\xi_0)\right)(s_1+i)^{-1}\overline{\zeta}_1^{-1}(\xi_1-\xi_2)^{-1}$$

$$(\xi_1-\zeta_0)^{-1}(\zeta_0-\zeta_1)^{-1}\left(\overline{\xi}_2((k_2+i)\xi_2-2i\xi_0)-(k_0-i)|\xi_0|^2\right)^{-1}\left(\overline{\zeta}_0((s_0+i)\zeta_0-2i\xi_0)-(k_0-i)|\xi_0|^2\right)^{-1}=1$$

- $z_{45}z_{42}w_{14}w_{34}=1$ *is equivalent to*

$$\frac{\zeta_0^2\zeta_1(\xi_0-\zeta_0)^2\left((s_0-i)\overline{\zeta}_0-t+i\right)\left(\overline{\xi}_0((k_0+i)\xi_0-2i\xi_1)-(k_1-i)|\xi_1|^2\right)\left((s_0-i)\overline{\zeta}_0-(s_1-i)\overline{\zeta}_1\right)}{(s_0+i)^2(\zeta_0-1)|\zeta_0|^4(\zeta_0-\xi_1)(\zeta_0-\zeta_1)\left(-(k_0-i)|\xi_0|^2(\xi_1-\zeta_0)+(k_1-i)|\xi_1|^2(\xi_0-\zeta_0)+(s_0-i)|\zeta_0|^2(\xi_1-\xi_0)\right)}=1$$

- $z_{21}z_{31}z_{61}w_{21}w_{31}w_{61}=1$ *is equivalent to*

$$-(\xi_0-1)\left(-\overline{\xi}_2((k_2+i)\xi_2-2i)+t-i\right)\left(i|\zeta_0|^2+\zeta_0\left(-s_0\overline{\zeta}_0+t-i\right)\right)\left(|\zeta_2|^2+\overline{\zeta}_2(-2-is_2\zeta_2)+it+1\right)$$

$$\left(i|\zeta_2|^2+\zeta_2\left(-s_2\overline{\zeta}_2+t-i\right)\right)\zeta_2^{-1}\left(i|\zeta_0|^2-s_0|\zeta_0|^2+t-2i\zeta_0+i\right)^{-1}\left(-i|\zeta_2|^2+s_2|\zeta_2|^2-t+2i\zeta_2-i\right)^{-1}$$
$$\left(|\zeta_2|^2-\overline{\zeta}_2(is_2+\zeta_2+1)+it+1\right)^{-1}\left((k_0-i)(\xi_2-1)|\xi_0|^2-(k_2-i)(\xi_0-1)|\xi_2|^2+(t-i)(\xi_0-\xi_2)\right)^{-1}=1$$

- $z_{31}z_{41}z_{21}w_{31}w_{41}w_{21}=1$ *is equivalent to*

$$(\xi_0-1)(\xi_0-\zeta_0)\left(-\overline{\zeta}_0((s_0+i)\zeta_0-2i)+t-i\right)\left(-i|\zeta_0|^2+\overline{\zeta}_0(-s_0\zeta_0+2i)+t-i\right)\left(i|\zeta_0|^2+\zeta_0\left(-s_0\overline{\zeta}_0+t-i\right)\right)$$
$$\left(i|\zeta_2|^2+\zeta_2\left(-s_2\overline{\zeta}_2+t-i\right)\right)\zeta_2^{-1}(\xi_0-\xi_2)^{-1}\left((i-s_0)|\zeta_0|^2+t-2i\zeta_0+i\right)^{-1}\left(-i|\zeta_0|^2+\overline{\zeta}_0(-s_0+i(\zeta_0+1))+t-i\right)^{-1}$$
$$\left((i-s_2)|\zeta_2|^2+t-2i\zeta_2+i\right)^{-1}\left((k_0-i)(\zeta_0-1)|\xi_0|^2-(s_0-i)(\xi_0-1)|\zeta_0|^2+(t-i)(\xi_0-\zeta_0)\right)^{-1}=1$$

- $z_{31}z_{51}z_{41}w_{31}w_{51}w_{41}=1$ *is equivalent to*

$$(\xi_0-1)\zeta_0^2(\xi_0-\zeta_0)\left(-\overline{\zeta}_0((s_0+i)\zeta_0-2i)+t-i\right)\left(-i|\zeta_0|^2+\overline{\zeta}_0(-s_0\zeta_0+2i)+t-i\right)\left((k_1-i)\overline{\xi}_1-(s_0-i)\overline{\zeta}_0\right)$$
$$\left((s_0-i)\overline{\zeta}_0-(s_1-i)\overline{\zeta}_1\right)\zeta_2^{-1}(\xi_0-\xi_2)^{-1}\left(-i|\zeta_0|^2+\overline{\zeta}_0(-s_0+i(\zeta_0+1))+t-i\right)^{-1}\left(\overline{\xi}_1((k_1+i)\xi_1-2i\zeta_0)-(s_0-i)|\zeta_0|^2\right)^{-1}$$
$$\left((s_0-i)|\zeta_0|^2+\overline{\zeta}_1(2i\zeta_0-(s_1+i)\zeta_1)\right)^{-1}\left((k_0-i)(\zeta_0-1)|\xi_0|^2-(s_0-i)(\xi_0-1)|\zeta_0|^2+(t-i)(\xi_0-\zeta_0)\right)^{-1}=1$$

- $z_{31}z_{51}z_{61}w_{31}w_{51}w_{61}=1$ *is equivalent to*

$$-(\xi_0-1)\zeta_0^2\left(-\overline{\xi}_2((k_2+i)\xi_2-2i)+t-i\right)\left(-i|\zeta_2|^2+\overline{\zeta}_2(-s_2\zeta_2+2i)+t-i\right)\left((k_1-i)\overline{\xi}_1-(s_0-i)\overline{\zeta}_0\right)$$
$$\left((s_0-i)\overline{\zeta}_0-(s_1-i)\overline{\zeta}_1\right)\zeta_2^{-1}\left(|\zeta_2|^2-\overline{\zeta}_2(is_2+\zeta_2+1)+it+1\right)^{-1}\left(\overline{\xi}_1((k_1+i)\xi_1-2i\zeta_0)-(s_0-i)|\zeta_0|^2\right)^{-1}$$
$$\left(\overline{\zeta}_0(\zeta_0+is_0\zeta_0)+\overline{\zeta}_1(-is_1\zeta_1-2\zeta_0+\zeta_1)\right)^{-1}\left((k_0-i)\xi_0(\xi_2-1)\overline{\xi}_0-(k_2-i)(\xi_0-1)|\xi_2|^2+(t-i)(\xi_0-\xi_2)\right)^{-1}=1$$

- $z_{13}z_{63}z_{23}w_{13}w_{63}w_{23}=1$ *is equivalent to*

$$(s_2^2+1)(\xi_0-1)^2(\xi_2-1)\overline{\zeta}_2^2(\zeta_0-\zeta_2)\left((s_0-i)|\zeta_0|^2-t+2i\zeta_0-i\right)\left(|\zeta_0-1|^2-i\left(-s_0|\zeta_0|^2+2\,\mathrm{Im}(\zeta_0)+t\right)\right)$$
$$\left((k_0-i)|\xi_0|^2+\overline{\xi}_2(2i\xi_0-(k_2+i)\xi_2)\right)\left((1-ik_0)|\xi_0|^2+(1+ik_2)|\xi_2|^2-2\xi_2\overline{\xi}_0\right)(t^2+1)^{-1}(\zeta_0-1)^{-2}(\xi_0-\xi_2)^{-2}$$
$$\left(\overline{\xi}_2((k_2+i)\xi_2-2i)-t+i\right)^{-1}\left(|\xi_0|^2+\overline{\xi}_0(-2-ik_0\xi_0)+it+1\right)^{-1}\left(|\xi_0-1|^2-i\left(-k_0|\xi_0|^2+2\,\mathrm{Im}(\xi_0)+t\right)\right)^{-1}$$

$$\left((s_0-i)|\zeta_0|^2+\overline{\zeta}_2(2i\zeta_0-(s_2+i)\zeta_2)\right)^{-1}=1$$

- $z_{13}z_{43}z_{23}w_{13}w_{43}w_{23}=1$ *is equivalent to*

$$(s_0+i)(s_2-i)(\xi_0-1)^2\overline{\zeta}_0\overline{\zeta}_2\left((s_0-i)|\zeta_0|^2-t+2i\zeta_0-i\right)\left(|\zeta_0-1|^2-i\left(-s_0|\zeta_0|^2+2\operatorname{Im}(\zeta_0)+t\right)\right)$$

$$\left((k_0-i)|\xi_0|^2+\overline{\zeta}_0(2i\xi_0-(s_0+i)\zeta_0)\right)\left((1-ik_0)|\xi_0|^2+(1+ik_2)|\xi_2|^2-2\xi_2\overline{\xi}_0\right)(t^2+1)^{-1}(\xi_0-\xi_2)^{-1}(\xi_0-\zeta_0)^{-1}$$

$$\left(\overline{\zeta}_0((s_0+i)\zeta_0-2i)-t+i\right)^{-2}\left(|\xi_0|^2+\overline{\xi}_0(-2-ik_0\xi_0)+it+1\right)^{-1}\left(|\xi_0-1|^2-i\left(-k_0|\xi_0|^2+2\operatorname{Im}(\xi_0)+t\right)\right)^{-1}=1$$

- $z_{13}z_{43}z_{53}w_{13}w_{43}w_{53}=1$ *is equivalent to*

$$(s_0+i)(s_2-i)(\xi_0-1)(\zeta_0-1)^2\overline{\zeta}_0\overline{\zeta}_2(\xi_0-\xi_1)\left((k_0-i)|\xi_0|^2+\overline{\zeta}_0(2i\xi_0-(s_0+i)\zeta_0)\right)\left((s_0-i)|\zeta_0|^2\right.$$

$$\left.+\overline{\zeta}_1(2i\zeta_0-(s_1+i)\zeta_1)\right)\left((1-ik_0)|\xi_0|^2+(1+ik_2)|\xi_2|^2-2\xi_2\overline{\xi}_0\right)\left(k_1|\xi_1|^2-s_0|\zeta_0|^2+i|\xi_1-\zeta_0|^2+2\operatorname{Im}\left(\zeta_0\overline{\xi}_1\right)\right)$$

$$(s_1+i)^{-1}(t-i)^{-1}\overline{\zeta}_1^{-1}(\xi_0-\xi_2)^{-1}(\xi_0-\zeta_0)^{-1}(\xi_1-\zeta_0)^{-1}(\zeta_0-\zeta_1)^{-1}\left(\overline{\zeta}_0((s_0+i)\zeta_0-2i)-t+i\right)^{-2}$$

$$\left(|\xi_0|^2+\overline{\xi}_0(-2-ik_0\xi_0)+it+1\right)^{-1}\left(k_0|\xi_0|^2-k_1|\xi_1|^2-i|\xi_0-\xi_1|^2-2\operatorname{Im}\left(\xi_0\overline{\xi}_1\right)\right)^{-1}=1$$

- $z_{13}z_{53}z_{63}w_{13}w_{53}w_{63}=1$ *is equivalent to*

$$(s_2^2+1)(\xi_0-1)(\xi_2-1)\left(\overline{\zeta}_2\right)^2(\xi_0-\xi_1)(\zeta_0-\zeta_2)\left((k_0-i)|\xi_0|^2+\overline{\xi}_2(2i\xi_0-(k_2+i)\xi_2)\right)$$

$$\left((s_0-i)|\zeta_0|^2+\overline{\zeta}_1(2i\zeta_0-(s_1+i)\zeta_1)\right)\left((1-ik_0)|\xi_0|^2+(1+ik_2)|\xi_2|^2-2\xi_2\overline{\xi}_0\right)$$

$$\left(k_1|\xi_1|^2-s_0|\zeta_0|^2+i|\xi_1-\zeta_0|^2+2\operatorname{Im}\left(\zeta_0\overline{\xi}_1\right)\right)(s_1+i)^{-1}(t-i)^{-1}\overline{\zeta}_1^{-1}(\xi_0-\xi_2)^{-2}(\xi_1-\zeta_0)^{-1}(\zeta_0-\zeta_1)^{-1}$$

$$\left(\overline{\xi}_2((k_2+i)\xi_2-2i)-t+i\right)^{-1}\left(|\xi_0|^2+\overline{\xi}_0(-2-ik_0\xi_0)+it+1\right)^{-1}\left((s_0-i)|\zeta_0|^2+\overline{\zeta}_2(2i\zeta_0-(s_2+i)\zeta_2)\right)^{-1}$$

$$\left(k_0|\xi_0|^2-k_1|\xi_1|^2-i|\xi_0-\xi_1|^2-2\operatorname{Im}\left(\xi_0\overline{\xi}_1\right)\right)^{-1}=1$$

The system's solutions will be the possible points of our complex hyperbolic polyhedron, if we want to look for the face-pairings we will have to check the Cartan angular compatibility and compute the face-pairing transformations. Explore the equations system will need a refined work that we expect to do after. With this work, we expect that the reader understands the difficulties in the Falbel algorithm to construct $\mathrm{PU}(2,1)-$representations of fundamental groups of links complements on the sphere. These difficulties are mainly: high-level computations, no guarantee to the existence of solutions in the equation systems, the exponential growth of the number of equations and variables, the representation obtained could not be the whole representation variety. On the other hand, with the previous equation system we can approximate the dimension of the representation variety.

Theorem 5.2.11. *The representation variety* $\mathrm{Rep}(\pi_1(\mathbb{S}^3 \setminus \mathscr{B}), \mathrm{PU}(2,1))$ *has at most real dimension* 14.

Proof. Let $F : \mathbb{R}^{19} \times \mathbb{R}^{14} \to \mathbb{R}^{14}$ be the real function given by

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{y} - g(\mathbf{x}))$$

where $\mathbf{x}$ denotes the real parts of the points $t, p_1, p_2, q_1, q_2, q_0', q_1'$ and q_2' in the complex polyhedron 5.9 and $g : \mathbb{R}^{19} \to \mathbb{R}^{14}$ is the function given by the left side of the edge and face equations. As one can notice, the function F is well defined, continuous and differentiable as a real function, even more, if $\mathbf{x}_0$ is a solution of the Edge-Face equation system, the vector $(\mathbf{x}_0, \mathbf{1})$ satisfies $F(\mathbf{x_0}, \mathbf{1}) = 0$. Due to the fact that the submatrix $\frac{\partial F}{\partial \mathbf{y}}$ is invertible, by the Implicit Function Theorem, there exists an open neighborhood U around $\mathbf{x}_0$ and $h : U \to \mathbb{R}^{14}$ such that $F(\mathbf{x}, h(\mathbf{x})) = 0$ for all $\mathbf{x} \in U$. From the previous, we have that the dimension of $\mathrm{Rep}(\pi_1(\mathbb{S}^3 \setminus \mathscr{B}), \mathrm{PU}(2,1))$ has to be at most 14.

□

The previous upper bound for the dimension of the character variety will help us to understand more about it, but mainly, it will be interesting to compare this dimension in the forward research.

A The Complex Hyperbolic Boundary

Introduction

The following section set the notions for the understanding of the Heisenberg structure on the boundary of the Complex hyperbolic space, we set these notions in the dimension two case but we can refer to [73] for the higher dimensional version. For a more detailed explanation of all the properties and geometry of the Siegel domain and its boundary, we refer to [127] and Chapter 2 of [40].

A.1 The Boundary

Consider the space $\mathbb{C}^{2,1}$ that is $\mathbb{C}^3$ with the Hermitian metric defined by

$$H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

we recall that the projectivization of the negative space for the Hermitian product is a model of the Complex Hyperbolic space and the projectivization of the null space is a model of its boundary.

We will say that a point $z \in \partial \mathbb{H}^2_{\mathbb{C}}$ is finite if there exists a lift of z of the form

$$z = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}$$

where $2\,\mathrm{Re}(z_1) + |z_2|^2 = 0$. If we take $\zeta = \dfrac{z_2}{\sqrt{2}} \in \mathbb{C}$, the previous equation implies that $2\,\mathrm{Re}(z_1) = -2|\zeta|^2$. So, we can write $z_1 = -|\zeta|^2 + iv$ for some $v \in \mathbb{R}$. Therefore there exists an identification of the boundary of the Siegel domain with the one point compactification of $\mathbb{C} \times \mathbb{R}$.

Consider the map T from $\mathbb{C} \times \mathbb{R}$ to $\mathrm{GL}(3, \mathbb{C})$ given by

$$T(\zeta, v) = \begin{pmatrix} 1 & -\sqrt{2}\zeta & -|\zeta|^2 + iv \\ 0 & 1 & \sqrt{2}\zeta \\ 0 & 0 & 1 \end{pmatrix}$$

It just take a few computations to check that T fixes infinity and sends the origin to the point (ζ, v), also it is easy to check that $T(\zeta, v)$ is an element of $\mathrm{PU}(2, 1)$.

We have to mention that the set of matrices of the form $T(\zeta, v)$ has a group structure with the matrix product and that this product satisfies:

$$T(\zeta, v) \cdot T(\xi, t) = T(\zeta + \xi, v + y + 2\,\mathrm{Im}(\overline{\xi}\zeta).$$

We can translate this group structure to $\mathbb{C} \times \mathbb{R}$ in order that T become a group homomorphism, the group product in $\mathbb{C} \times \mathbb{R}$ is given by

$$(\zeta, v) * (\xi, t) = (\zeta + \xi, v + t + 2\,\mathrm{Im}(\overline{\xi}\zeta).$$

We have to mention that the previous group law gives $\mathbb{C} \times \mathbb{R}$ the structure of the 3−dimensional Heisenberg group $\mathfrak{H}$.

A.1.1 Horospherical coordinates

Take a fix $u \in \mathbb{R}_+$ and consider the set of points $z \in \mathbb{H}^2_{\mathbb{C}}$ for which $\langle z, z \rangle = -2u$ for a standard lift. This set is called the *horosphere of height* u, it is not hard to check that, as the boundary, this set carries the Heisenberg group structure. Like as we did for the boundary, we can identify the set of horospheres with the space $\mathbb{C} \times \mathbb{R} \times \mathbb{R}_+$. This implies, that every point z in the Siegel domain can be viewed as a point $(\zeta, v, u) \in \mathfrak{H} \times \mathbb{R}_+$, these coordinates are know as *Horospherical coordinates* of z.

It the later will be useful to identify the finite boundary points with the horosphere of height zero, which means that $(\zeta, v) = (\zeta, v, 0) \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty\}$.

We define the *horoball* U_t of height t as the union as horospheres of height $u > t$, this is a topological ball of dimension 4. Thus $\mathbb{H}^2_{\mathbb{C}}$ is itself the horoball U_0.

A.1.2 The Cygan metric

Consider $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty\}$ with the Heisenberg group structure, denoted by $\mathfrak{H}$. The Heisenberg norm is given by

$$|(\zeta, v)| = \left||\zeta|^2 - iv\right|^{1/2}.$$

The previous norm gives rise to a metric on $\mathfrak{H}$ given by

$$\rho_0((\zeta_1, v_1), (\zeta_2, v_2)) = \left|(\zeta_1, v_1)^{-1} * (\zeta_2, v_2)\right|.$$

The previous metric is know as the *Cygan metric*. This metric can be extended to an incomplete metric on $\overline{\mathbb{H}^2_{\mathbb{C}}} \setminus \{\infty\}$ using the horospherical coordinates as follows:

$$\rho_0((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)) = \left||\zeta_1 - \zeta_2| + |u_1 - u_2| - iv_1 + iv_2 + i\,\mathrm{Im}(\zeta_1\overline{\zeta}_2)\right|^{1/2}$$

.

We have to recall, that the previous function restricted to a fixed horosphere coincide with the Cygan metric.

A *Cygan sphere* of radius $r \in \mathbb{R}_+$ and center $(\zeta_0, v_0) \in \mathfrak{H}$ is defined and denoted by

$$S_r(\zeta_0, v_0) = \{(\zeta, v, u) : \rho_0((\zeta, v, u), (\zeta_0, v_0)) = r\}.$$

We have to recall that not all the isometries of $\mathbb{H}^2_{\mathbb{C}}$ is an isometry for the Cygan metric, there exists elements of $\mathrm{PU}(2,1)$ that distort the Cygan metric.

Consider the subgroup of $\mathrm{PU}(2,1)$ that is the stabilizer of the point at infinity, this subgroup is known as *Heisenberg similarities*. The Heisenberg translations are elements of the Heisenberg similarities and the group of Heisenberg translations is a normal subgroup of the Heisenberg similarities group. Other elements of the Heisenberg similarities are:

- *Heisenberg rotations,* these are elements that fix ∞ and the origin $(0,0)$ and are given by

$$(\zeta, v) \mapsto (e^{i\theta}\zeta, v).$$

- *(Real) Heisenberg dilatations,* these elements also fix ∞ and $(0,0)$ and are given by
$$(\zeta, v) \mapsto (r\zeta, r^2 v).$$

- *(Complex) Heisenberg dilatations,* these are a product of a rotation and a real dilatations.

It is not hard to see that the set of complex Heisenberg dilatations is isomorphic to $\mathbb{R}_+ \times \mathrm{U}(1)$. The group of Heisenberg similarities is the semi-direct product of the complex dilatations and the Heisenberg translations.

The next lemma shows how a map in $\mathrm{PU}(2,1)$ not fixing ∞ distorts the Cygan metric on the boundary.

Lemma A.1.1 (Lemma 4.6 in [127])**.** *Let B be any element of $\mathrm{PU}(2,1)$ that does not fix ∞. Then there exists a positive real number r_B depending only on B so that for all $z, w \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \{\infty, B^{-1}(\infty)\}$ we have*

1.
$$\rho_0\left(B(z), B(w)\right) = \frac{r_B^2 \rho_0(z,w)}{\rho_0\left(z, B^{-1}(\infty)\right)\rho_0\left(w, B^{-1}(\infty)\right)},$$

2.
$$\rho_0\left(B(z), B(\infty)\right) = \frac{r_B^2}{\rho_0\left(z, B^{-1}(\infty)\right)}.$$

The previous Lemma implies that B sends the Cygan sphere of radius r_B with centre $B^{-1}(\infty)$ to the Cygan sphere of radius r_B with centre $B(\infty)$, that motivates a definition of *isometric sphere* of B, as an analogous of the isometric sphere for the Möbius transformations.

Definition A.1.2. For an element $\gamma \in \mathrm{PU}(2,1)$ that doesn't fix ∞. The *isometric sphere* of γ is the Cygan sphere of radius r_γ and centre $\gamma^{-1}(\infty)$.

Another important transformation of the Heisenberg space that doesn't preserve the Cygan metric.

Definition A.1.3. The *Koranyi inversion* ι is a map on $\mathfrak{H} \setminus \{(0,0)\}$ into itself given by

$$(\zeta, v) \mapsto \left(\frac{-\zeta}{|\zeta|^2 - iv}, \frac{-v}{|\zeta|^4 + v^2} \right).$$

This map can be extended to whole $\partial \mathbb{H}^2_{\mathbb{C}}$ sending $\iota(0,0) = \infty$ and viceversa.

The previous map can be extended to whole $\mathbb{H}^2_{\mathbb{C}}$ using horospherical coordinates and this extension is an element of $\mathrm{PU}(2,1)$.

A.2 Boundaries of Totally Geodesic Subspaces

We have to recall that for the two dimensional complex hyperbolic space there are two types of totally geodesic subspaces: the complex lines and the Lagrangian spaces.

In the following we will describe how it looklike the boundary of each one of this subspaces.

A.2.1 Complex Lines

Let L be a complex line, without lossing generality, we can think that this line is passing through the point at infinity and by the origin. So we can say that this line is spanned by the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We can assure that the points, on the previous complex line, have horospherical coordinates $(0, v, u)$, *i.e.*, the intersection of L with the finite part of the boundary corresponds only on the vertical axis $\{(0, v) : v \in \mathbb{R}\} \subset \mathfrak{H}$. Applying a Heisenberg translations we can obtain that the boundary of another complex line passing through infinity is a vertical line in $\mathfrak{H}$. This are called *infinite chain* or *infinite* $\mathbb{C}-$*circle.*

Now, assume that L doesn't pass through infinity anymore. By simplicity assume that this line is spanned by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

It is not hard to check that the horospherical coordinates of the points the previous line are of the form $(\zeta, 0, 1 - |\zeta|^2)$. We have that this line will intersect the finite part of the boundary when $|\zeta| = 1$, which implies that the points have to belong to the circle $\{(e^{i\theta}, 0) : \theta \in [0, 2\pi)\}$. Applying Heisenberg dilatations and translations we can obtain a general boundary that is a ellipse in $\mathfrak{H}$. This ellipses are called *finite chain* or *finite* $\mathbb{C}-$*circle.*

A.2.2 Lagrangian Subspaces

Let R be a real subspace, without lose of generality we can think that passes through $(0, 0)$ and ∞. We know that canonically this space is composed by points in the Siegel domain with real entries, whose horospherical coordinates are of the form $(x, 0, u)$ with $x \in \mathbb{R}$. So the finite part of the boundary that intersects R are the points $\{(x, 0) : x \in \mathbb{R}\}$.

We can apply Heisenberg rotations and Heisenberg translations to obtain a general boundary of R. These boundaries are known as *infinite* $\mathbb{R}-$*circles.*

For a totally real subspace not passing through ∞, it is more complicated. The have to look at the real subspace fixed by the complex involution. In [127] there are the computations for this case, they shows that these kind of boundaries are Lemniscates in $\mathfrak{H}$. These boundaries are known as *finite* $\mathbb{R}-$*chains.*

B Gromov Hyperbolic Groups

Introduction

The following Appendix present the basic notions on Gromov Hyperbolic Spaces, their Boundaries and the Topology on them, for that reason we will omit the proof of several affirmations but feel free to see [89] and [33] for further reference and lectures.

B.1 Definition and Basic Notions of Hyperbolicity

Definition B.1.1 (Geodesic Space)**.** A metric space (X, d) is said to be *geodesic*, if any two points $x, y \in X$ can be joined by a geodesic segment $[x, y]$ naturally parametrized by the distance.

Some natural geodesic space to had in mind are $\mathbb{R}^n$ and $\mathbb{C}^n$.

Definition B.1.2. A geodesic metric space (X, d) is called $\delta-$hyperbolic, where $\delta \geq 0$, if for any geodesic triangle in X each side of the triangle is contained in the $\delta-$neighborhood of the union of the other sides. We will say that a geodesic metric space X is *hyperbolic* if is $\delta-$hyperbolic for some $\delta \geq 0$.

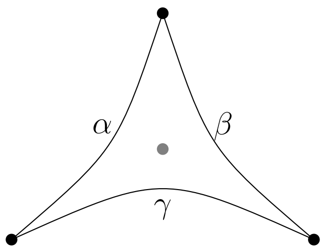


Figure B.1 – Geodesic traingle.

Example B.1.3. Some examples of $\delta-$hyperbolic spaces are:

- Any geodesic space of finite diameter, let say D, is $D-$hyperbolic.
- The real line $\mathbb{R}$ of an infinite simplicial tree is $0-$hyperbolic.
- The real hyperbolic space, $\mathbb{H}^2$, is $\log(\sqrt{2}+1)-$hyperbolic.

Definition B.1.4 (Cayley Graph)**.** Let Γ a group and $S \subset \Gamma$ a generating set, with the property that if $g \in S$, then $s^{-1} \in S$. Let $\Delta = \Delta(\Gamma, S)$ the graph with set of vertices Γ and we connect two vertices, $g, h \in \Gamma$ if $g^{-1}h \in S$. We call $\Delta(\Gamma, S)$ the *Cayley graph* of Γ with respect of S.

Definition B.1.5. A finitely generated group G is said to be *word-hyperbolic* if there is a generating set S such that the Cayley graph $\Gamma(G, S)$ is hyperbolic respect to word metric d_w, where

$$d_w(g_1, g_2) = \text{ shortest lenght of } g_1^{-1}g_2.$$

Remark B.1.6. If G is a *word-hyperbolic* group, then $\Gamma(G, S)$ is hyperbolic for any generating set, but the hyperbolicity constant depends on the choice of S.

Example B.1.7. Examples of word-hyperbolic groups are finite groups and $\mathbb{Z}$. But $\mathbb{Z} \times \mathbb{Z}$ is not word-hyperbolic. Another key example is the fundamental group of the sphere with two handles.

Remark B.1.8. The finitely generated word-hyperbolic group are also finitely presented.

B.2 Boundaries and Bordifications

Definition B.2.1. Let (X, d) be a metric space. For $x, y, z \in X$ we define

$$(y, z)_x := \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

We will call $(y, z)_x$ the *Gromov product* if y, z respect x.

Remark B.2.2. The previous definition is a way to measure how long two geodesics are close together.

Definition B.2.3. We will say that two geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to X$ are equivalent $\gamma_1 \sim \gamma_2$ if there is $K > 0$ such that for any $t \geq 0$

$$d(\gamma_1(t), \gamma_2(t)) \leq K.$$

Remark B.2.4. We can say that two equivalent geodesic rays are the ones where the rays get close enough for more time.

Definition B.2.5. Let $x \in X$ be a fixed point in (X, d) an hyperbolic metric space. The relative geodesic boundary of X respect to x as

$$\partial_x^g X = \{[\gamma] : \gamma : [0, \infty] \to X \text{ is a geodesic ray with } \gamma(0) = x\}.$$

The geodesic boundary of X is

$$\partial_x^g X = \{[\gamma] : \gamma : [0, \infty] \to X \text{ is a geodesic ray }\}$$

Definition B.2.6. Let $x \in X$ be a fixed point in (X, d) an hyperbolic metric space. We say that the sequence $(x_n)_{n \geq 1}$ of points in X converges to infinity if

$$\lim_{i,j \to \infty} (x_j, x_i)_x = \infty.$$

Remark B.2.7. Two sequences $(x_k), (y_k)$ converge to infinity are equivalent if

$$\lim_{i,j\to\infty} (x_j, y_j)_x = \infty.$$

Definition B.2.8. Let (X, d) be a $\delta-$hyperbolic metric space. The boundary of X is defined as follows:

$$\partial X := \{[(x_n)] : (x_n)_{n\geq 1} \text{ is a sequence converging to infinity}\}.$$

We can ask us what is the relation between these two "boundaries". Consider the following two maps:

$$\begin{aligned} \iota_x : \quad \partial^g_x X \to &\quad \partial^g X \\ [\gamma] \mapsto &\quad [\gamma] \end{aligned}$$

$$\begin{aligned} \iota : \quad \partial^g X \to &\quad \partial X \\ [\gamma] \mapsto &\quad [(\gamma(n))_{n\geq 1}] \end{aligned}$$

A metric space (X, d) is said to be proper if all the metric closed balls in X are compact.

Proposition B.2.9. *Let (X, d) be a proper $\delta-$hyperbolic metric space. Then*

1. *For any $x \in X$, $\iota_x : \partial^g_x X \to \partial^g X$ is a bijection.*

2. *The map $\iota : \partial^g X \to \partial X$ is a bijection.*

3. *For any non-equivalent geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to X$ there exists a bi-infinity geodesic $\gamma : \mathbb{R} \to X$ such that $\gamma|_{[0,\infty)}$ is equivalent to γ_1 and $\gamma|_{(-\infty,0]}$ is equivalent to γ_2 (after a re-parametrization of γ_2 from $\gamma_2(0)$).*

In the following we will say that a geodesic ray $\gamma : [0, \infty) \to X$ connects $x = \gamma(0)$ with $p \in \partial X$ if $p = [\gamma(n)]$ and similarly to the bi-infinite geodesics.

Until now, we know that there exists bijections between the two notions of "boundary" for a hyperbolic space, but we does not know if the two of them have the same topology or they differ. In what follows, we will introduce the topology for the three boundaries and show that the previous maps provide a homemorphism.

Definition B.2.10. Let (X, d) be a $\delta-$hyperbolic metric space and $x \in X$ a base-point. For any $p \in \partial_x^g X$ and $r \geq 0$, we define

$$V(p, r) := \left\{ q \in \partial_x^g X : \liminf_{t\to\infty} (\gamma_1(t), \gamma_2(t)_x) \geq r \right\}$$

where γ_1, γ_2 are geodesic rays for p, q starting from x.

We have that $V(p, r)$ is a basis for the topology of $\partial_x^g X$.

The previous basis can be extended to $X \cap \partial X$.

Definition B.2.11. Let (X, d) be a $\delta-$hyperbolic metric space and let $x \in X$ be a basepoint. For any $p \in \partial X$ and $r \geq 0$ we define

$$U(p, r) := \left\{ q \in \partial X : \liminf_{i,j} (x_i, y_j)_x \geq r \right\}$$

for some sequence that represents p and q.

We have that $U(p, r)$ is a basis for the topology of ∂X.

Proposition B.2.12. *Let (X, d) be a proper $\delta-$hyperbolic space. Then:*

1. *For any* $x \in X$ *and* $y \in X$, $(\iota_y)^{-1} \circ \iota_x : \partial_x{}^g X \to \partial_y{}^g X$ *is an homeomorphism.*

2. *For* $x \in X$, $\iota \circ \iota_x : \partial^g_x X \to \partial X$ *is an homeomorphism.*

3. ∂X *and* $X \cup \partial X$ *are compact.*

Definition B.2.13 (Visual Metrics)**.** Let (X, d) be a $\delta-$hyperbolic proper space. Let $a \geq 1$ and $x_0 \in X$ be a basepoint. We will say that a metric d_a on ∂X is a *visual metric* with respect to the base point x_0 and visual parameter a, if there exists $C > 0$ such that the following holds:

1. The metric d_a induces the canonical topology on ∂X.

2. For any two distinct points, $p, q \in \partial X$, for any bi-infinite geodesic γ converging to p and q, and any $y \in \gamma$ with $d(x_0, y) = d(x_0, \gamma)$ it holds:

$$\frac{1}{C} a^{-d(x_0,y)} \leq d_a(p, q) \leq C a^{-d(x_0,y)}.$$

Example B.2.14.

- If X is an $\mathbb{R}-$tree (see [33]), ∂X is the set of ends of the tree and every visual metric is of the form $d_a(p, q) = a^{-d(x_0,y)}$ where d is the metric on X.

-

Theorem B.2.15. *Let* (X, d) *be a proper* $\delta-$*hyperbolic metric space. Then:*

1. *Suppose* d' *and* d'' *are two visual metrics on* ∂X *with respect the same visual parameter* a *and base-point* x_0' *and* x_0''*. Then* d' *and* d'' *are Lipschitz-equivalent.*

2. *Suppose that* d' *and* d'' *are two visual metrics on* ∂X *with different visual parameters and base-points. Then* d' *and* d'' *are Holder-equivalent.*

Definition B.2.16. A function $f : X \to Y$ from metric spaces (X, d_0) and (Y, d_1) is called a quasi-isometry if there is $C > 0$ such that:

1. For any $y \in Y$, there exists $x \in X$ such that $d_1(y, f(x)) \leq C$.

2. For any $x, x' \in X$, it satisfies

$$\frac{1}{C} d_0(x, x') - C \leq d_1(f(x), f(x')) \leq C d_0(x, x') + C.$$

Proposition B.2.17. *Let X, Y be two proper geodesic spaces and suppose $f : X \to Y$ is a quasi-isometry, then:*

1. X is hyperbolic if and only if Y is hyperbolic.

2. The map f extends to a canonical homeomorphism $\tilde{f} : \partial X \to \partial Y$.

Definition B.2.18. Let G a word-hyperbolic group and S a generating set. We define the boundary of G as $\partial G := \partial \Gamma(G, S)$.

From the fact that the change on the generating set induces quasi-isometries of the Cayley graphs, the topology of ∂G does not depend on S.

Definition B.2.19. We will say that a group G acts on a geodesic space X *geometrically* if the action is isometric, cocompact and properly discontinuous.

Remark B.2.20.

- If X admits a geometric action of a group G, then G is proper.
- Finitely generated groups act geometrically on its Cayley graphs.

Theorem B.2.21. *Let G be a group. Then G is word-hyperbolic if and only if G admits a geometric action on proper hyperbolic metric space (X, d). Moreover, in this case ∂G is homeomorphic to ∂X.*

C Orbifolds and $(G,X)-$structures.

Introduction

The following Appendix set the notions of an Orbifold and their geometric structures, we based this notes on [46] and we give explicit and detailed definitions and examples because of the lack of literature for this theme.

C.1 Topology of Orbifolds

An *orbifold* is a Hausdorff second-countable space Y so that each point has a neighborhood homeomorphic to the quotient space of an open subset U of $\mathbb{R}^n$ by an action of a finite group.

Definition C.1.1. Let $y \in Y$ and $V \subset Y$ a neighborhood such that $\tilde{V} \subset \mathbb{R}^n$ open and G_1 acting on $\tilde{V}$, such that $V = \tilde{V}/G_1$. We will say that $(\tilde{V}, G_1)$ is a model pair if the following condition is satisfied:

If $U \subset Y$ is another neighborhood for y and $V \subset U$. Let U be modeled by $(\tilde{U}, G_2)$ and the inclusion map $\phi_{V,U} : V \to U$ lifts to an embedding $\tilde{\phi}_{V,U} : \tilde{V} \to \tilde{U}$ equivariant to a homomorphism $\psi_{V,U} : G_1 \to G_2$ so that the following diagram conmutes

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{\phi}_{V,U}} & \tilde{U} \\ \downarrow & & \downarrow \\ \tilde{V}/G_1 & \longrightarrow & \tilde{U}/\psi_{V,U}(G_1) \\ \downarrow & & \downarrow \\ & & \tilde{U}/G_2 \\ \downarrow & & \downarrow \\ V & \xrightarrow[\phi_{V,U}]{} & U \end{array}$$

We say that V is a *model* and $(\tilde{V}, G_1)$ is a *model pair.*

Remark C.1.2. Let $g_{\in}G_2$. If $(\tilde{\phi}_{V,U}, \psi_{V,U})$ satisfies the above condition, then $(g\tilde{\phi}_{V,U}, g\psi_{V,U}g^{-1})$ also satisfies the condition. So the pairs $(\tilde{\phi}_{V,U}, \psi_{V,U})$ is unique up to elements of G_2.

Remark C.1.3. If $\phi_{V,U} : V \to U$ and $\phi_{U,W} : U \to W$ are inclusion maps, then we are forced to have

$$\begin{aligned} \tilde{\phi}_{V,W} &= g\tilde{\phi}_{U,W}\tilde{\phi}_{V,U}, \\ \psi_{V,W} &= g\psi_{U,W}\psi_{V,U}g^{-1}, \end{aligned}$$

where $g \in G_3$ and G_3 is a finite group associated to W.

Definition C.1.4. A maximal family of covering $\mathscr{O}$ with models satisfying the equations on C.1.3 is said to be an orbifold structure on Y. We will say that Y is the underlying space of $(Y, \mathscr{O})$.

Remark C.1.5. We will denote by X_M the underlying space for an orbifold M.

Definition C.1.6. Given two orbifolds M and N, an *orbifold-map* is a map $f : X_M \to X_N$ so that for each point $x \in X_N$ and neighborhood (U, G) and an inverse image of x, let say y with model neighborhood (V, G'), there exists a smooth map $\tilde{f} : V \to U$ inducing f, equivariantly respect to an homomorphism $\psi : G' \to G$. An *orbifold-diffeomorphism* is an orbifold-map that has an inverse that is an orbifold-map.

Definition C.1.7. An orbifold with boundary Y is a Hausdorff second-countable space so that each point has a neighborhood modeled by an open subset intersected with the upper half space H^n and a finite group. The *interior* of Y is the set of points modeled by open balls and its complement is called the *boundary*.

Definition C.1.8. A *singular* point x of an orbifold is a point of the underlying space which model neighborhood has a nontrivial element of the group fixing a point corresponding to x. A point in a orbifold is called *regular* if it has a model neighborhood homeomorphic to a ball.

Remark C.1.9. The set of regular points is an open dense subset of the underlying space.

Definition C.1.10. A *suborbifold* of an orbifold N is an imbedded subset Y of X_N with an orbifold structure, so that for each point $x \in Y$ and a neighborhood V modeled on (V', G), the neighborhood $V \cap P$ is modeled on $(V' \cap P, G|_P)$ where P is a submanifold of $\mathbb{R}^n$ where G acts and $G|_P$ denotes the image subgroup of the restriction homomorphism to groups acting on P.

Example C.1.11. Let M be a manifold and Γ is a discrete group acting on M properly but not necessarily freely. Then M/Γ has an orbifold structure as follows: let $x \in M/\Gamma$ and $\tilde{x} \in M$ a point corresponding to x. Then there is $I_{\tilde{x}} \subset \Gamma$ a subgroup that fixes $\tilde{x}$. There is an open ball U containing $\tilde{x}$ on which $I_{\tilde{x}}$ acts and for any $g \in \Gamma \setminus I_{\tilde{x}}$, $g(U) \cap U$ is empty, then $U/I_{\tilde{x}}$ is a model for x.

Remark C.1.12. Given two orbifolds M and N, the product space $X_M \times X_N$ has an orbifold structure.

Definition C.1.13. A *homotopy* of two orbifold-maps $f_1, f_2 : M \to N$ is an orbifold-map $F : M \times [0, 1] \to N$ such that

$$\begin{aligned} F(x, 0) &= f_1(x), \\ F(x, 1) &= f_2(x) \end{aligned}$$

for every $x \in M$.

Definition C.1.14. An *isotopy* $f : M \to M$ is a self orbifold-diffeomorphism, *i. e.,* is an automorphism so that there is a homotopy $F : M \times [0, 1] \to M$ in which

1. $F_0(x) = F(x, 0)$ is the identity.
2. $F(x, 1) = f(x)$.
3. $F(x, t)$ is an orbifold-diffeomorphism for every $t \in [0, 1]$.

Remark C.1.15. Two orbifold-maps $f_1, f_2 : M \to N$ are *isotopic* if there exists a homotopy from f_1 to f_2 and every F_t is an orbifold-diffeomorphism.

Definition C.1.16. A covering $\{O_i\}$ of an orbifold is said to be a *nice covering* if it satisfies:

1. Each O_i is connected and open.
2. O_i has a model pair $(\tilde{O}_i, G_i)$ so that $\tilde{O}_i$ is simply-connected.
3. The intersection of any finite collection of O_i has the above two properties.

Proposition C.1.17. *An orbifold M has a nice locally finite covering.*

C.2 Coverings and Fibers on Orbifolds

Definition C.2.1. A covering orbifold M is an orbifold $\tilde{M}$ with a surjective orbifold-map $p : X_{\tilde{M}} \to X_M$ such that each point x_M has a neighborhood U, called *elementary neighborhood,* with a homeomorphism $\phi : \tilde{U}/G_U \to U$ and a subset $\tilde{U} \subset \mathbb{R}^n$ or H^n with group G_U acting on it, so that each component V_i of $p^{-1}(U)$ has a homomorphism $\tilde{\phi}_i : \tilde{U}/G_i \to V_i$ and G_i is a subgroup of G_U.

Remark C.2.2. We will call a *fiber* of a point $x \in M$ to the inverse image $p^{-1}(x) \subset \tilde{M}$.

Definition C.2.3. Given an orbifold-map $f : Y \to Z$ and (Z_1, p_1) a covering of Z. If $\tilde{f} : Y \to Z_1$ is an orbifold-map that satisfies

$$p_1\tilde{f} = f$$

and $\tilde{f}$ lifts for every model in a consistent way for Z, then $\tilde{f}$ is said to be a *lifting.*

Remark C.2.4. Two covering orbifolds $(Y_1, p_1), (Y_2, p_2)$ of Y are *isomorphic* if there is an orbifold-diffeomorphism $f : Y_1 \to Y_2$ so that $p_2 f = p_1$. A *covering transformation* or *deck transformation* of Y_1 is a covering isomorphism of Y_1.

Definition C.2.5. A covering (Y_1, p_1) is *regular* if the automorphism group acts transitively on fibers over regular points.

Proposition C.2.6. *Let (Y_1, p_1) and (Y_2, p_2) coverings over an orbifold Y. Let $f : Y_1 \to Y_2$ be a covering morphism so that $f : Y_1^r \to Y_2^r$ is a covering isomorphism where Y_i^r is the inverse image of the regular part Y^r of Y. Then f is a covering isomorphism.*

Definition C.2.7. A *path* is a smooth orbifold-map from an interval to an orbifold.

Proposition C.2.8. *Let Y an orbifold and $p : Y' \to Y$ an orbifold covering.*

- *Let $x \in Y$, $x' \in p^{-1}(x)$ and $f : I \to Y$ a path such that $f(0) = x$, the f lifts to a unique path $f' : I \to Y'$ in Y' so that $f'(0) = x'$.*
- *Let $f_1 : Z \to Y'$ and $f_2 : Z \to Y'$ orbifold-maps lifting $f : Z \to Y$. If $f_1(x) = f_2(x)$ for a regular point $x \in Y$, then $f_1 = f_2$.*

Definition C.2.9. A *mirror point* is a singular point of an orbifold with model pair (U, G) where G is an order two group acting on U fixing an hyperplane meeting U. The set of mirror points is said to be the *mirror set.*

Remark C.2.10. We can form an orbifold M^d covering M so that there are two points in the inverse image of each regular point of M. This covering is called the *2-fold cover* and has no mirror points. The construction is as follows:

Let $V_i, i = 1, 2, \cdots$ be model neighborhoods covering M and V_i has model pair (U_i, G_i). We know that U_i has an induced orientation from $\mathbb{R}^n$. For each i define $(U_i \times \{-1, 1\}, G_i)$ where the action of G_i is given by

$$g(x, l) = (g(x), sgn(g)l)$$

and $sng(g)$ is given depending on whether g preserve the orientation.

For each $V_i \cap V_j \to V_i$ we simply define

$$\tilde{\phi}'_{V_i \cap V_j, V_i} : V_i \cap V_j \times \{-1, 1\} \to V_i \times \{-1, 1\}$$

equal to $\tilde{\phi}_{V_i \cap V_j, V_i} \times Id_{\{-1,1\}}$.

Proposition C.2.11. *Let V be a $n-$orbifold whiich is a quotient space of a $n-$ball $\tilde{V}$ by a finite group G acting on it. Then the following statements hold:*

1. *A connected covering orbifold V_1 of V is isomorphic to $\tilde{V}/G'_V$ for a subgroup $G'_V \subset G_V$ with covering map*

$$p : \tilde{V}/G'_V \to V = \tilde{V}/G.$$

2. *Given two covering orbifolds $\tilde{V}/G_1$ and $\tilde{V}/G_2$, a covering morphism $\tilde{V}/G_1 \to \tilde{V}/G_2$ is induced by an element $g \in G$, so that $gG_1g^{-1} \subset G_2$.*

3. *The covering automorphism group of a covering orbifold V' is given by $N(G'_V)/G'_V$ where G'_V is a subgroup corresponding to V' and $N(G'_V)$ is the normalizer of G'_V in G_V.*

Let G_i, $i \in I$ a collection of subgroups of a finite group G_V acting on an open set $V \subset \mathbb{R}^n$. Then $p_i : V/G_i \to V/G_V$ form a collection of orbifold-covering maps.

Then

$$p : V \times \prod_{i \in I} G_i \setminus G_V \to V,$$

where $G_i \setminus G_V$ denotes the right action cosets of G_i in G_V, is a covering-map and G_V acts on it by

$$\gamma(v, G_i\gamma_i)_{i\in I} = (\gamma(v), G_i\gamma_i\gamma^{-1})_{i\in I}.$$

Definition C.2.12. The *orbifold-fiber-product* is defined

$$p^f : V^f \longrightarrow V/G_V,$$

where $V^f = \big(V \times \prod_{i\in I} G_i \setminus G_V\big)/G_V$

Definition C.2.13. A *universal covering orbifold* is a connected covering $(\tilde{Y}, p_Y)$ with a regular base point $\tilde{y}$ mapping to y so that for any covering $p_Z : Z \to Y$ where Z is connected orbifold with regular base point z over y, there is an orbifold-covering-morphism $q : \tilde{Y} \to Z$ so that $q(\tilde{y}) = z$ and $p_Z q = p_Y$.

The group of automorphisms of an universal cover for Y is said to be the *fundamental group* of Y and it is denoted by $\pi_1(Y)$.

Proposition C.2.14 (Thurston[1]). *Let Y be a connected orbifold. Then there exists an universal covering orbifold $\tilde{Y}$ unique up to covering isomorphisms. Moreover, the fundamental group of $\tilde{Y}$ acts transitively on the inverse images of the base point y.*

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Proposition C.2.15. *Let $p_1 : \tilde{Y} \to Y_1$ and $p_2 : \tilde{Y} \to Y_2$ universal covering orbifold maps. Then any orbifold-map $f : Y_1 \to Y_2$ overing an orbifold-diffeomorphism $g : Y \to Y$ lifts to an orbifold-diffeomorphism $\tilde{f} : \tilde{Y} \to \tilde{Y}$. The lift is unique if we decide to value $\tilde{f}(y_0)$ among $p_2^{-1}(f(p_1(y_0)))$ for a base point y_0 of $\tilde{Y}$. If g is the identity, then $\tilde{f}$ is an automorphism.*

Corollary C.2.16.

1. *The fundamental group acts transitively on each fiber of an universal cover $\tilde{Y}$,* i.e., *an the inverse image of a regular point $x \in Y$. Moreover, $\tilde{Y}/\pi_1(Y) = Y$.*

2. *Each covering space $p_1 : Y_1 \to Y$ is isomorphic to a covering map $p' : \tilde{Y}/\Gamma \to Y$ where p' is induced from the universal covering map $p : \tilde{Y} \to Y$ and Γ is a subgroup of $\pi_1(Y)$.*

3. *Teh isomorphism classes of covering spaces of Y are in one to one correspondence with the conjugacy classes of subgroups of $\pi_1(Y)$.*

4. *The group of automorphisms of a covering space $\tilde{Y}/\Gamma$ is isomorphic to $N(\Gamma)/\Gamma$ where $N(\Gamma)$ is the normalizer of Γ in $\pi_1(\Gamma)$.*

5. *A covering $Y' \to Y$ is regular if and only if Y' is isomorphic to $\tilde{Y}/\Gamma$ for a normal subgroup Γ of $\pi_1(Y)$.*

6. *Let Y_1 and Y_2 orbifolds with universal covering orbifolds $\tilde{Y}_1$ and $\tilde{Y}_2$, respectively. A lift $f : \tilde{Y}_1 \to \tilde{Y}_2$ of an orbifold-diffeomorphism $g : Y_1 \to Y_2$ is an orbifold diffeomorphism.*

Proposition C.2.17. *Let Y be an orbifold and $f : Y \to Y$ and $g : Y \to Y$ orbifold diffeomorphism with homotopy $H : Y \times I \to Y$. Then for any choice of lift $\tilde{f} : \tilde{Y} \to \tilde{Y}$ of f, there is a unique lift $\tilde{H} : \tilde{Y} \times I \to \tilde{Y}$ which becames a homotopy between $\tilde{f}$ and a lift $\tilde{g} : \tilde{Y} \to \tilde{Y}$ of g.*

Remark C.2.18. Given M, N two orbifolds and an orbifold-diffeomorphism $f : M \to N$ which lifts to a diffeomorphism $\tilde{f} : \tilde{M} \to \tilde{N}$, this induce a homomorphism $\tilde{f}_* : \pi_1(M) \to \pi_1(N)$, if $g \in \pi_1(M)$ then $\tilde{f}_*(g) = \tilde{f} g \tilde{f}^{-1}$.

Proposition C.2.19. *If $\tilde{f}_2 : \tilde{M} \to \tilde{M}$ is a diffeomorphism homotopic to $\tilde{f}_1$ by a homotopy $h : \tilde{M} \times [0,1] \to \tilde{N}$ equivariant with respcet $\tilde{f}_{1*} : \pi_1(M) \to \pi_1(N)$ then $\tilde{f}_{2*} = \tilde{f}_{1*}$.*

Definition C.2.20. A *good* orbifold is an orbifold with a covering orbifold that is a manifold. A *very good* orbifold is an orbifold with a finite regular that is a manifold.

Remark C.2.21. A good orbifold Y is always orbifold-diffeomorphic to M/Γ where M is a simply-connected manifold and Γ is a discrete group acting on M properly. A good orbifold M has a covering that is simply-connecte manifold $\tilde{M}$. Then it is a universal covering orbifold. If $Y = M/\Gamma$ and M is simply-connected, then $\pi_1(Y)$ equals Γ.

C.3 $(G, X)-$structures on Orbifolds

Definition C.3.1. A $(G, X)-$structure on an orbifold M is a collection of charts $\phi_U : U \to X$ for each model pair (U, H_U) so that ϕ_U conjugates the action of H_U by a finite subgroup of G_U of G on $\phi_U(U)$ by an isomorphism $i_U : H_U \to G_U$ and the inclusion map $U \to V$ is always realized by an element $g \in G$ and the homomorphism $G_U \to G_V$ is given by a conjugation by g.

Definition C.3.2. If maximal collection of pairs (ϕ_U, i_U) for M, we will said that M is a $(G, X)-$*manifold.*

Remark C.3.3. A $(G, X)-$structure on M induces a $(G, X)-$structure on its covering orbifolds.

Definition C.3.4. A $(G, X)-$map f between two $(G, X)-$manifolds M, N is a map so that for each point $x \in M$ and a point $y \in N$ so taht $x = f(y)$, and a neighborhood U of x modeled by $(\tilde{U}, H_U)$ with chart ϕ_U and an isomorphism $i_U : H_U \to G_U \subset G$, there is a neighborhood V of y modeled by a pair $(\tilde{V}, H_V)$ with chart ϕ_V and a isomorphism $i_V : H_V \to G_V$, so that f lifts to a map $\tilde{f} : \tilde{V} \to \tilde{U}$ equivariant with respect to a homomorphism $G_V \to G_U$ given by conjugation $g \to \vartheta g \vartheta^{-1}$ by some $\vartheta \in G$.

Theorem C.3.5. *A $(G, X)-$orbifold M is a good orbifold. There exists an inmersion D from the universal covering manifold $\tilde{M}$ to X, so that*

$$D\vartheta = h(\vartheta)D$$

for $\vartheta \in \pi_1(M)$ holds for a homeomorphism $h : \pi_1(M) \to G$, where D is a local $(G, X)-$map. Moreover, any such inmersion equals gD for $g \in G$, with the associated homomorphism ghg^{-1}.

Let us assume that M is a compact $(G, X)-$orbifold with an universal cover $\tilde{M}$.

Definition C.3.6. A pair (D, h) of inmersion $D : \tilde{M} \to X$ equivariant with respect to a homomorphism $h : \pi_1(M) \to G$ is said to be a *development pair,* where D is called *developing map* and h *holonomy representation.*

Remark C.3.7. Given such a development pair (D, h) for an orbifold, we can induce charts to $\tilde{M}$ and hence a $(G, X)-$structure on $\tilde{M}$. Since a deck-transformation is a $(G, X)-$map, we see that $M = \tilde{M}/\pi_1(M)$ has a induced $(G, X)-$structure from $\tilde{M}$.

Remark C.3.8. We say that two development pairs (D, h) and (D', h') are $G-$equivalent if $D' = \vartheta D$ and $h' = \vartheta h \vartheta^{-1}$, for some $\vartheta \in G$.

Let $\mathscr{M}(M)$ the set of all the $(G, X)-$structures on M. Let $\mu_1, \mu_2 \in \mathscr{M}(M)$, they are equivalent if there is an isotopy $\phi : M \to M$, so that the induced $(G, X)-$struture $\phi_*(\mu_1)$ equals μ_2.

Definition C.3.9. The *deformation space* of $(G, X)-$structes on M is defined to be the set $\mathscr{M}(M)/\sim$.

Remark C.3.10. We recall that the previous definition doesn't give a topology for $\mathscr{M}(M)/\sim$.

In order to give a topology on the deformation space, consider the isotopy-equivalence space $S(M_0)$ of $(G, X)-$strutures for a compact orbifold M_0, *i. e.,* is the space of equivalence classes of pairs $(D, \tilde{f} : \tilde{M}_0 \to \tilde{M})$ where $\tilde{f}$ is a diffeomorphism equivariant with respect to an isomorphism $\pi_1(M_0) \to \pi_1(M)$ and $D : \tilde{M} \to X$ is an inmersion equivariant with respect to a homomorphism $\pi_1(M) \to G$. Two pairs $(D, \tilde{f})$ and $(D', \tilde{f}')$ are isotopy-equivalent if and only if there are diffeomorphisms $\tilde{\phi} : \tilde{M} \to \tilde{M}'$ lifting $\phi : M \to M'$ with $D'\tilde{\phi} = D$ and an isotopy $H : \tilde{M} \times [0, 1] \to \tilde{M}'$ equivariant with respect to the isomorphism $\tilde{\phi}_* : \pi_1(M) \to \pi_1(M')$ so that $H_0 = \tilde{\phi}\tilde{f}$ and $H_1 = \tilde{f}'$.

We can endowed this space with the C^s-topology, *i.e.,* a sequence of functions converge if it does on every compact subset of $\tilde{M}_0$ uniformly in C^s-sense.

There is a natural $G-$action on $S(M_0)$ given by

$$\gamma(D, \tilde{f}) = (\gamma D, \tilde{f})$$

where $\gamma \in G$.

Let $\mathscr{D}(M_0)$ the quotient of $S(M_0)$ under the $G-$action. This space is in one to one correspondence with $\mathscr{M}(M_0)/\sim$.

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