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ON HEREDITARY PROPERTIES AND THEIR CHARACTERIZATION THROUGH FORBIDDEN SUBSTRUCTURES.

T E S I S

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Acknowledgments.

Introduction.

The concept of a mathematical graph was not yet defined by 1852. Nonetheless, it was in this year that, after colouring a map of the counties of England, a British mathematician (Francis Guthrie) came across what we can now consider the first partition problem in graph theory; the four colour problem. Francis asked whether or not any map drawn in the plane may have its regions coloured with four colours, in such a way that any two regions with a common boarder have different colours.

This particular problem remained unsolved for over a hundred years. When translated to graph theoretical language, the query asks if it is true that any planar graph admits a proper 4-coloring of its vertices. In general, partition problems of our interest ask if we can split the set of vertices of a graph in such a way that certain restrictions over the classes of vertices are satisfied. The aim of the first chapter is to familiarize the reader with this kind of problems, by giving several results in this area of graph theory.

The partition problems studied throughout this dissertation are stated in the converse way to the four colour problem; can we find a characterization for the graphs that admit a given partition? It is in Chapters two, three and four that we propose and solve three of this kind of problems.

In a similar way to how bipartite graphs are those graphs with no odd cycles, in Chapter five we introduce an original way of characterizing classes of graphs that are defined by admitting a given partition. This technique can be used in general to find a characterization of any hereditary property.

Chapter 1

Preliminaries.

Whenever a mathematician starts studying a new mathematical object, the most basic question will always arise: how can we tell whenever two of these are essentially the same?

Of course the concept of being "essentially the same" will depend on the kind of objects we are dealing with. And though as innocent and simple this query might seem, in most cases it turns out to be a really hard question to answer. From this simple question, many more will usually arise. For instance, if two objects are not essentially then we can ask ourselves the following questions: if we are given objects A and B, can we "fit" object A into object B? or can we "squeeze" object A onto object B? And so, we will stumble upon many more of these, and a vast amount of theory developed around them.

In order to give a simple example, let us go back to our basic algebra courses. The first object we encounter as mathematicians are sets. For our own luck as first semester students, these questions will be trivial to answer as long as we know the cardinality of the sets we are dealing with; given two sets A and B, we can "fit" set A into set B if there are at most as many elements in A as in B. Nevertheless a really tough question to answer does come up if we are curious enough: is there a set that can be fitted into the real numbers, and that the natural numbers fit in it, but it is not essentially the same to neither of them? In other words, is there a set A such that $|\mathbb{N}| < |A| < \mathbb{R}$? This is called the continuum hypothesis, and it was interesting enough to be one of the 23 Hilbert's problems (a list of unsolved problems by 1900 that strongly influenced the mathematics of the last century).

1.1 Definitions and examples.

The mathematical objects we will be dealing with throughout this work will be simple graphs and digraphs (we will only call them graph and digraph respectively). A *graph* G, is an orderer pair (V_G, E_G) where the elements of V_G are called *vertices*, and the element of E_G are unordered pairs of vertices called *edges*. Similar to a graph, a *digraph* D, is an orderer pair (V_D, A_D) where the elements of V_G are also called *vertices*, and the element of E_G are ordered pairs of vertices, $a = xy$ called *arcs*, where x is the *tail* of a and y the head of a.

Once we have the notion of graphs and digraphs, let us move on to a fundamental definition for this thesis.

Consider two graphs G and H, we will say that there is a *homomorphism* from G to H if there is a function $\varphi : V_G \to V_H$ such that for any two vertices $x, y \in V_G$, if $xy \in E_G$ then $\varphi(x)\varphi(y) \in E_H$.

In order to keep notation as simple as possible, we will write the homomorphism as $\varphi : G \to H$. If the homomorphism also satisfies that for any two vertices $x, y \in V_G$, if $\varphi(x)\varphi(y) \in E_H$ then $xy \in E_G$, we will call φ a *full-homomorphism*. If φ happens to be a bijective full-homomorphism, we will say that φ is an *isomorphism*.

In favor of tying up loose ends allow us to say that a graph G fits into graph H if there is an injective full-homomorphism $\varphi : G \to H$, if φ is now surjective then G squeezes onto H. And finally, if it is an isomorphism then G and H are "essentially the same". Having stated that, let us now leave our early vocabulary aside, and stick to proper mathematical language.

Example 1.1.1. Consider the 5-cycle $C_5 = (\{v_0, v_1, v_2, v_3, v_4\}, \{v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_0\})$, the path on five vertices $P_5 = (\{w_0, w_1, w_2, w_3, w_4\}, \{w_0w_1, w_1w_2, w_2w_3, w_3w_4\})$ and the functions,

- $\varphi_1 : V_{P_5} \to V_{C_5}$ such that $\varphi_1(w_i) = v_i$,
- $\varphi_2 : V_{P_5} \to V_{C_5}$ such that $\varphi_2(w_i) = v_{(i \mod 2)}$,
- $\varphi_3 : V_{C_5} \to V_{C_5}$ such that $\varphi_3(v_i) = v_{(i+1 \mod 5)}$ and
- $\varphi_4 : V_{C_5} \to V_{C_5}$ such that $\varphi_4(v_i) = v_i$ for $i \neq 4$ and $\varphi_4(v_4) = v_2$.

While $\varphi_1, \varphi_2, \varphi_3$ induce homomorphisms, φ_4 does not since $v_0v_4 \in E_{C_5}$ but $\varphi_4(v_0)\varphi_4(v_4) = v_0v_2 \notin E_{C_5}$. More over, φ_1 and φ_2 are not full-homomorphims but φ_3 is, and since it is bijective it is an isomorphism.

For the following section, will use a quite recent and thorough survey written by Pavol Hell [13] as guideline for introducing several results and some problems from which this dissertation thrives.

1.2 M-Partitions.

For a given graph H, the *H-homomorphism problem* addresses the question of whether or not a graph G is homomorphic to H.

With motive of illustrating a strong dichotomy amongst homomorphism problems consider the following problems:

- The *bipartition problem*: is a given graph G bipartite?
- The *3-colouring problem*: can a given graph G be coloured with 3 colours?

At first glimpse neither of them might seem as homomorphism problems. But in this scenario, since we are dealing with simple graphs, the first problem is equivalent to the K_2-homomorphism problem (2-colouring problem) whereas the second one is equivalent to the K_3-homomorphism problem. The dichotomy of the k-colouring problem, lies in the computational complexity; every k-colouring problem belongs to P or to NP-complete, but to no other class inbetween. For instance, the bipartition problem is polynomial time solvable, while for 3-colouring problem is NP-complete. In fact, Hell and Nesetril proved the following theorem in [17].

Theorem 1.2.1. *If H is bipartite, then the H-homomorphism problem is polynomial time solvable. Otherwise the H-homomorphism problem is NP-complete.*

Since graphs can be seen as symmetric digraphs, we may think of a generalized version of Theorem 1.2.1 for digraphs. It turns out that the dichotomy problem for digraph homomorphisms is equivalent to a conjecture published over two decades ago. It is called the Feder-Vardi conjecture which was finally proved independently by Bulatov [2] and Zhuk [8] last year.

By definition, the k-colouring problem is a partition problem. One may also see the H-homomorphism problem as a partition problem: a graph G is homomorphic to a graph H if and only if V_G admits a partition $V_G = \bigcup_{v \in V_H} S_v$ such that for any four vertices $x, y \in V_H$, $u \in S_x$ and $w \in S_y$, if $uw \in E_G$ then $xy \in E_H$. In a similar manner one can think of the full-homomorphism problem as a partition problem. It is clear then that in both cases, every class S_v of the partition will be an independent set of vertices.

What happens now if we would like to find partitions of vertices where some of the classes form a clique? For example let us consider the class of graphs satisfying that their set of vertices can be partitioned into an independent set and a clique. If a graph belongs to this class it is called a *split graph*. The problem of deciding whether or not a graph is a split graph, turns out to be linear time solvable [19]. Since we are dealing with loopless graphs, it is clear that the split graph recognition problem it is different to any H-homomorphism or full-homomorphism problem. Nonetheless there is a broader class of problems that include all of them.

Consider an $m \times m$ matrix M over $\{0, 1, *\}$. A partition $S_1, S_2, \ldots, S_m$ of V_G induces an *M-partition* of the graph G if for any two vertices $v \in S_i$ and $u \in S_j$ the following hold true:

- $uv \in E_G$ if $M(i,j) = 1$,
- $uv \notin E_G$ if $M(i,j) = 0$,
- if $M(i,j) = *$, u and v may or may not be adjacent.

The *M-partition problem* consists of deciding whether or not a graph G admits an M-partition. While working with symmetric matrices we can restrict the problem to graphs with no loss of generality, but if M is not symmetric the M-partition problem will be bounded to non symmetric digraphs.

Example 1.2.2. One can easily verify that the problems early stated are equivalent to the M-partition problem associated to the following matrices:

$$\text{bipartition problem } M_B = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad \text{3-colouring problem } M_3 = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$$

$$\text{and split graph recognition problem } M_S = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}.$$

In general the k-colouring problem will be equivalent to the M-partition problem where M is a $k \times k$ matrix with 0's over the diagonal and $*$'s everywhere else. On the other hand, if M_H is the adjacency matrix of the graph H

then, the M_H-partition problem is equivalent to the H-full-homomorphism problem. And finally, it is not hard to notice that if we replace every 1 in M_H by an $*$ then, the M_H partition problem corresponds to the H-homomorphism problem. As stated before, the M-partition problem wraps a wider class of problems than the H- homomorphism and full-homomorphism problems. All we have to do in order to convince ourselves that the previous statement is true, is to consider a matrix either with a non 0 diagonal, or a matrix with both 1's and $*$'s off the diagonal.

It is worth mentioning that in our definition of an M-partition, we do not require for all parts S_i to have at least one element. The version of the M-partition problem where we want for all parts S_i to be non empty is called the *surjective version*. Throughout this dissertation we will work with the general case, so we refer the interested reader to the survey [13] for results on the surjective version.

In contrast with the surjective version, whenever we have a matrix M with an asterisk on the diagonal, the M-partition problem becomes trivial; we can map every vertex to S_i where $M(i,i) = *$. Hence we will consider that our matrices will have diagonal entries over $\{0,1\}$.

In the same way that the H-homomorphism problem belongs to a larger class of problems (the M-partition problems). The property of admitting an M-partition belongs to a broader class of properties, hereditary properties. We say that a graph property $\mathcal{P}$ is *hereditary*, if for any graph $G \in \mathcal{P}$, $L \in \mathcal{P}$ for every induced subgraph L of G. We will denote that L is a induced subgraph of G by $L < G$. If $L \nless G$ we will say that G is *L-free*. Moreover, if $L \nless G$ for any L in a set of graphs A, we will say that G is *A-free*.

Although we are not going any further into hereditary properties, considering the property $\mathcal{P}_M$ (the class of graphs that admit an M-partition) as a hereditary property will give us a valuable tool to attack the problem.

Since a single vertex will always belong to $\mathcal{P}_M$ for any matrix M then, it makes sense to define a *minimal obstruction* to M-partition as a graph G that does not admit an M-partition but every proper induced subgraph of G does. Now, the set

$$F_M = \{\text{the set of minimal obstructions to } M\text{-partition}\}$$

characterizes the M-partitionable graphs in the sense that a graph G admits an M-partition if and only if it is F_M-free.

Example 1.2.3. Going back to two of our first examples, and remembering our first course in graph theory we know that $F_{M_B} = \{C_n : n \text{ is odd}\}$, while in [19] they prove that $F_{M_S} = \{2K_2, C_4, C_5\}$.

Again, we find a dichotomy that leads to an open problem called *The Characterization Problem*:

Problem 1.2.4. *Which matrices M have the property that cardinality of the set of minimal obstructions to M-partition is finite?*

By executing a brute force algorithm, one can verify if a graph with k vertices is an induced subgraph of a graph with n vertices in $k^2k!\binom{n}{k}$ time. Which is polynomial time solvable for fixed k. Hence for any other matrix M with a finite set of minimal obstructions, the M-partition problem will be polynomial time solvable. On the contrary, not necessarily all matrices M such that the M-partition problem is polynomial time solvable will have a finite set of minimal obstructions. We can draw upon one of our previous examples to notice this, the M_B-partition problem is polynomial time solvable while there are an infinite amount of minimal obstructions to it. Which leaves us with our second open problem called *The Complexity Problem*:

Problem 1.2.5. *Which matrices M have the property that the M-partition problem can be solved in polynomial time?*

Most of the research done around the M-partition problem stems from these problems.

In order to mention some results regarding The Characterization Problem, let us introduce some notation. Consider a square matrix M over $\{0, 1, *\}$, we will denote by k_M the amount of 0's in the main diagonal and by l_M the amount of 1's. The matrix obtained by replacing the 0's by 1's and viceversa will be indicated by $\overline{M}$.

Though the following proposition is a quite obvious observation, it allows us to assume without loss of generality that $l_M \geq k_M$ whenever comes in hand.

Proposition 1.2.6. *A partition of V(G) is an $\overline{M}$-partition of G if and only if it is an M-partition of $\overline{G}$*

Going back to The Characterization Problem, it is proved in [20] that for any $\{0, 1\}$-matrix M, there is a finite amount of minimal obstructions. Furthermore, in [21] we can find the following bound for the size of minimal obstructions.

Theorem 1.2.7. *Let M be a symmetric $\{0,1\}$-matrix, then all minimal obstructions to M-partition have at most $(k_M+1)(l_M+1)$ vertices, and there are at most two minimal obstructions with exactly $(k_M+1)(l_M+1)$ vertices. Moreover, if both $k_M > 0$ and $l_M > 0$ then there is at most one minimal obstruction with $(k_M+1)(l_M+1)$ vertices.*

Actually, this bound cannot be improved.

Example 1.2.8. ([13]) Consider the following matrices,

$$M_{bC} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The M_{bC}-partition problem is known as the *biclique recognition problem.* In this case $l_{M_{bC}} = 2$ and $k_{M_{bC}} = 0$, so $(l_{M_{bC}}+1)(k_{M_{bC}}+1) = 3$ and there are exactly two minimal obstructions of this size (K_3 and $K_1 \cup K_2$). In our second matrix we have $l_N = 1 > 0$, $k_N = 2 > 0$ and there is exactly one minimal obstruction of size $(l_N+1)(k_N+1) = 6$, namely $K_{3,3}$.

It was later proved in [15] that we do not need a symmetric matrix for the result to be true.

Theorem 1.2.9. *Let M be a $\{0,1\}$-matrix, then all digraph minimal obstructions to M-partition have at most $(k_M+1)(l_M+1)$ vertices, and this bound is best posible.*

Before getting to the last results with respect to The Characterization Problem allow us to introduce one more definition. We will say that a matrix M is *unfriendly* if $M_{ii} = M_{jj} \neq *$ and $M_{ij} = M_{ji} = *$, for some i and j; otherwise we say that M is *friendly.* Going back to our early examples we have two unfriendly matrices M_B and M_3, while M_{bC} and M_S are friendly.

In [15], they study The Characterization Problem for small matrices, and we can find the following result.

Theorem 1.2.10. *Let M be a symmetric matrix over $\{0,1,*\}$ of size at most five. Then there is finite set of minimal obstructions to M if and only if M is friendly.*

A similar result for non symmetric matrices is proved in [16] by finding the set of minimal obstruction for all 2×2 matrices.

Proposition 1.2.11. *The set of minimal obstructions to a 2×2 matrix M is infinite if and only if $M = M_B$ or $M = \overline{M_B}$*

For the moment these are all the results we are visiting regarding The Characterization Problem. Concerning The Complexity Problem, we already have a result for the class of matrices that represent a Homomorphism Problem in graphs (Theorem 1.2.1). The following, is a similar result for small matrices proved in [15].

Theorem 1.2.12. *Let M be a symmetric $\{0,1,*\}$-matrix of size at most four. If M contains, as a principal submatrix, the pattern of 3-colouring, or its complement, then the M-partition problem is NP-complete. Otherwise, the M-partition problem is polynomial time solvable.*

As we can see, in order to find a dichotomy in the complexity of the M-partition problem, and to relax the hypotheses of M representing a graph homomorphism problem, there is a high prize to pay; the result now needs forneighbour M to be of size at most four. Which strongly suggests that the general problem is a really hard problem to tackle. Thus, the tactic now is to find results in restricted graph classes, for which we refer the interested reader again to [13].

1.3 Forbidden Subpatterns.

A *pattern*, $G_\leq = (V_\leq, E)$, is a graph G whose set of vertices is linearly ordered by $\leq$. For any $xy \in E$ such that $x \leq y$, we say that x is a left neighbour of y and y is a right neighbour of x. In order to distinguish when a set is linearly ordered we will write $\langle$ and $\rangle$, instead of $\{$ and $\}$ respectively. For instance $A = \langle a, b, c\rangle$ and $B = \langle b, a, c\rangle$ are the same as sets, but are different as linearly ordered sets.

We say that a pattern $L_{\leq_1}$ *occurs* in $G_{\leq_2}$ if there is an injective order preserving function $\varphi : V_L \to V_G$ such that it induces a full-homomorphism $\varphi : L \to G$; otherwise $G_{\leq_2}$ is $L_{\leq_1}$-*free*. Consider a set of patterns FP and a graph G, we say that G admits an FP-*free ordering* if there is a pattern $G_\leq$ such that for any $L_< \in FP$, $G_\leq$ if $L_<$-free. Whenever the induced graph of the edges of a pattern $(V_\leq, E)$ contains all vertices V, we will denote the pattern by only the set of edges E. The decision problem that asks whether or not a graph G admits an FP-free ordering is denoted by $ORD(FP)$.

Example 1.3.1. Consider the following patterns,

- $G_{<_1} = (\langle v_0, v_1, v_2, v_3\rangle, \{v_0v_1, v_1v_2, v_2v_3, v_0v_3\})$,
- $H_{<_2} = (\langle v_0, v_1, v_2, v_3\rangle, \{v_0v_1, v_1v_2, v_2v_3\})$, and
- $L_{<_3} = (\langle v_0, v_1, v_2\rangle, \{v_0v_1, v_1v_2\})$.

Note that $G_{<_1}$ is the ordered cycle on four vertices, while $H_{<_2}$ and $L_{<_3}$ are the ordered paths on four and three vertices respectively. Whenever the set of vertices of a pattern are ordered according to the indices of the vertices (i.e. $v_i \leq v_j$ if and only if $i \leq j$), we can describe the pattern only by specifying the adjacencies. For instance, we may also write the patterns in this example as $G_{<_1} = \{v_0v_1, v_1v_2, v_2v_3\}$, $H_{<_2} = \{v_0v_1, v_1v_2, v_2v_3, v_0v_3\}$, and $L_{<_3} = \{v_0v_1, v_1v_2\}$. Finally we can see that $H_{<_2}$ is $G_{<_1}$-free since $v_0v_3 \notin E_{G_{<_1}}$,while $L_{<_3}$ does occur in $H_{<_2}$.

Recall that any hereditary property $\mathcal{P}$ is characterized through a set of minimal obstructions $F_{\mathcal{P}}$. By considering the set $\mathbb{O}(F_{\mathcal{P}}) = \{G_{\leq} : G \in F_{\mathcal{P}}$ and $\leq$ is a linear ordering of $V_G\}$, we notice that for any hereditary property $\mathcal{P}$, there is at least one set FP such that for any graph G, $G \in \mathcal{P}$ if and only if G admits an FP-free ordering. A set of *forbidden subpatterns* for a property $\mathcal{P}$ is any set of patterns FP such that $G \in \mathcal{P}$ if and only if G admits an FP-free ordering. It is clear that if $\mathcal{P}$ admits a finite set of minimal obstructions $F_{\mathcal{P}}$ then there is a finte set of forbidden subpatterns for $\mathcal{P}$, namely $\mathbb{O}(F_{\mathcal{P}})$. On the contrary, can we find a finite set of forbidden subpatterns for $\mathcal{P}$ when $F_{\mathcal{P}}$ is infinite? Though it is an easy question to answer, let us first convince ourselves that there are hereditary properties with a non trivial set of forbidden subpatterns. The trivial set being $\mathbb{O}(F_{\mathcal{P}})$.

An easy example would be split graphs. In this case $F_{\mathcal{P}} = \{2K_2, C_4, C_5\}$, so $\mathbb{O}(F_{\mathcal{P}})$ would have around 2(4!)+5! (minus symmetries) patterns, nonetheless we have the following proposition.

Proposition 1.3.2. *Let G be a graph then, G is a split graph if and only if G admits a FP_S-free ordering. Where $FP_S = \{(\langle v_1, v_2, v_3\rangle, \{v_1v_2\}), \{v_1v_2, v_1v_3\}\}$.*

Proof. Suppose $G = (I, K)$ is a split graph where I is an independent set, and K is a clique. Let $<$ be any linear ordering of V_G such that, if $v \in I$ and $u \in K$ then, $v < u$. It is not hard to see that $G_<$ is FP_S-free. On the other hand, if $G_{\leq}$ is an FP_S-free ordering of G, let $I = \{x \in V_G\colon$ x has no left neighbours} and $K = V_G \setminus I$. It is easy to verify that I is an independent set. Now, let $x, y \in K$ and suppose $x \leq y$, since $x \notin I$ then x has a left neighbour z. Since $G_{\leq}$ is FP_S-free, we conclude that $xy \in E_G$. ■

Now that we have convinced ourselves that in fact, there are properties for which there is a set of forbidden subpatterns different from $\mathbb{O}(\mathcal{F}_{\mathcal{P}})$, let us study bipartite graphs.

Proposition 1.3.3. *Let G be a graph then, G is bipartite if and only if G admits a FP_B-free ordering. Where $FP_B = \{\{v_1v_2, v_2v_3\}, \{v_1v_2, v_1v_3, v_2v_3\}\}$.*

Proof. Consider a bipartite graph $G = (X, Y)$ and any linear ordering, $\leq$, of V_G such that, if $x \in X$ and $y \in Y$ then $x \leq y$. It is not hard to notice that $G = (X, Y)_{\leq}$ is $\{v_1v_2, v_2v_3\}$-free, and since it is bipartite then is triangle free. Suppose now that $H_<$ is FP_B-free and with out lost of generality we can assume that H has no isolated vertices. The bipartition of H will be $X = \{v \in V_H : v$ is the right end of an edge in $<\}$ and $Y = \{v \in V_H : v$ is the left end of an edge in $<\}$. Since $H_<$ is $\{v_1v_2, v_2v_3\}$-free, proving that $X \cup Y = V_H$ and that both, X and Y, are independent sets it straight forward. ■

This is our first example of a property with an infinite set of minimal obstructions but with a finite set of forbidden subpatterns. Furthermore, in [14], within the proof of another proposition, the next result is proved. Before mentioning it, we introduce a useful notation used in set theory. Consider a set A, we will denote by $A^{[k]}$ the subsets of A with k elements.

Proposition 1.3.4. *Let G be a graph then, G is k-colourable if and only if G admits an FP_k-free ordering. Where $FP_k = \{H_< : H_< = \{v_1v_2, v_2v_3, \ldots, v_kv_{k+1}\} \cup E', E' \subseteq \{v_1, v_2, \ldots, v_{k+1}\}^{[2]}\}$.*

In fact, one of our problems generalizes Propositions 1.3.2 and 1.3.4 using a very similar proof to the one in [14] that proves the later proposition.

Allow us to give another example where there is an infinite set of minimal obstructions and a finite set of forbidden subpatterns.

Example 1.3.5. A *chordal* graph is one with no induced cycles of length 4 or more. Of course then, the property of being a chordal graph $\mathcal{CH}$ has a minimal obstruction set $F_{\mathcal{CH}} = \{C_n : n \geq 4\}$. It is well known that a graph is chordal if and only if it has a perfect elimination ordering. In other words, a graph G is chordal if and only if, it admits a $\{v_1v_2, v_2v_3\}$-free ordering.

The *complement* $G^c_<$ of a pattern $G_<$, is the pattern $\overline{G}_<$. The *dual* $H^*_<$ of the pattern $H_<$ will be the pattern $H_{<^*}$, where $<^*$ is the inverse order of $<$. In order to give a list of characterizations by forbidden subpatterns of different families, let us introduce some notation for eight patterns on three vertices,

- $ch = \{v_1v_2, v_1v_3\}$,
- $cp = \{v_1v_2, v_2v_3\}$,
- $cl = \{v_1v_2, v_2v_3, v_1v_3\}$.

With the previous definitions, the symbols $ch^c, ch^*, ch^{c*}, cp^c$ and cl^c are clear now.

Having introduced this notation, we have the following examples from a theorem found in [12].

Example 1.3.6. The following sets of forbidden supatterns characterize the graph families on the right.

$\{ch\}$	Chordal graphs
$\{cp\}$	Comparability graphs
$\{ch, cp\}$	Arborescence comparability graphs
$\{ch, cl\}$	Forests
$\{ch, ch^c\}$	Threshold graphs
$\{ch, cp^c\}$	Interval graphs
$\{ch, ch^{c*}\}$	Split graphs
$\{cp, cl\}$	Bipartite graphs

A graph G, is a *comparability* graph if there is an orientation of the edges A_G such that if $xy, yz \in A_G$ then $xz \in A_G$. An *arborescence comparability* graph is graph G that admits an orientation of the edges A_G such that if $xy, yz \in A_G$ then $xz \in A_G$, and if $xy, zy \in A_G$ then $xz \in E_G$. A graph G is a *threshold* graph is there is a real number S, and for every vertex $v \in V_G$ there is a real weight a_v such that $vu \in E_G$ if and only if $a_v + a_u \geq S$. And finally, an *interval* graph, is a graph whose set of vertices represent intervals of the real line, and there is an edge between two vertices if and only if, the corresponding intervals intersect.

Proving that these sets in fact characterize the corresponding graph classes is not a hard task, therefore the interested reader may use it as an exercise to familiarize with characterization through forbidden subpatterns.

Contrary to the relation between the cardinality of the set of minimal obstructions of a property $\mathcal{P}$, and the complexity of the decision problem associated. A finite set of forbidden subpatterns does not guaranty that the decision problem can be solved in polynomial time (recall that there are $n!$ possible ordering of a graph on n vertices!). Nonetheless a really neat result is proven [14], where they define the family $F_k = \{H_< : |V_H| = k\}$.

Proposition 1.3.7. *Let $\mathcal{P}$ be a property characterized by a set of forbidden subpatterns $FP \subseteq F_3$, then the decision problem associated to $\mathcal{P}$ is polynomial time solvable.*

They actually find a master algorithm that for any graph H, and a set of forbidden supatterns $FP \subseteq F_3$, returns an FP-free pattern $H_<$, or reports that such pattern does not exist. Furthermore, they prove that this result cannot be extended to larger sizes of forbidden patterns, which can be seen as a corollary to Proposition 1.3.4: for every $k \geq 4$ there is a set $FP \subseteq F_k$ such that ORD(FP) is NP-complete; namely $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$.

Chapter 2

A small problem on full-homomorphisms.

This chapter will address the C_n-full-homomorphism and P_n-full-homomorphism problems. We will do so by finding the corresponding sets of minimal obstructions, let us denote them by $\mathcal{F}_{C_n}$ and $\mathcal{F}_{P_n}$ respectively. In fact, for any graph H allow us to denote by $\mathcal{F}_H$ the set of minimal obstructions to the H-full-homomorphism problem.

It was not until we found the sets of minimal obstructions, that we also found a closely related result by Ball, Nesetril and Pultr in [18]. Fortunately for us, they found the connected minimal obstructions in $\mathcal{F}_{C_n}$, and they gave a recursive formula for the disconnected graphs in $\mathcal{F}_{P_n}$. Since we include the disconnected graphs in $\mathcal{F}_{C_n}$, and we find an exact formula for the disconnected obstructions in $\mathcal{F}_{P_n}$, we consider that our results complete theirs.

A standard way to denote when a graph H is homomorphic to a graph G, is $H \to G$. Throughout this chapter we are only studying full-homomorphims. Thus, for a pair of graphs G, H we will write $H \to G$ if there is a full-homomorphism that maps H to G, and $H \not\to G$ otherwise.

2.1 Full-homomorphisms and points determining graphs.

In [6] Sumner defined a *point determining* graph as a graph for which non adjacent vertices have distinct neighbourhoods. We will strongly use this concept throughout this section, for instance we have the following results.

Proposition 2.1.1. *Let G be a graph, and let H be a point determining graph. If $\varphi : H \to G$ is a full-homomorphism then, φ is injective.*

Proof. Suppose φ in not injective. Consider distinct vertices $u, v \in V_H$ such that $\varphi(u) = \varphi(v)$. Let $G_0 = \varphi[H]$, since φ is a full-homomorphism then, $\varphi^{-1}[N_{G_0}(\varphi(x))] = N_H(x)$ for every $x \in V_H$. Thus $N_H(u) = N_H(v)$, and since $\varphi(u) = \varphi(v)$, $uv \notin E_H$. Therefore H is not point determining. ∎

Given an equivalence relation r on the vertices of graph G. The *quotient graph* is defined as ${}^{G}\!/_{r} = (V', E')$, where $V' = {}^{V_G}\!/_{r}$ and $[x]_r[y]_r \in E'$ if and only if there are $w \in [x], z \in [y]$ such that $wz \in E$.

Consider an arbitrary graph G and let us define the following relation on V_G,

$$x \sim y \text{ if and only if } N(x) = N(y) \text{ and } xy \notin E$$

It is not hard to notice that $\sim$ is an equivalence relation. So we can consider the quotient graph ${}^{G}\!/_{\sim}$. It is also clear to see that the equivalence classes in $\sim$ are independent sets, and ${}^{G}\!/_{\sim}$ is a point determining graph.

Let us now consider a set $X_G = \{x_1, x_2, \ldots, x_k\}$ of representatives for the equivalence classes of $\sim$, and define the following functions,

$$\phi_{X_G} : {}^{G}\!/_{\sim} \to G$$

$$[x_i] \mapsto x_i$$

$$\Phi_G : G \to {}^{G}\!/_{\sim}$$

$$v \mapsto [v]_\sim.$$

Remark 2.1.2. By doing an exhaustive search over the vertices, one can find the equivalence classes of the relation $\sim$ in polynomial time, and therefore, for given any graph H one can find ${}^{H}\!/_{\sim}$ again in polynomial time.

Lemma 2.1.3. *Let G be a graph then, ϕ_{X_G} and Φ_G are full-homomorphisms.*

Once again, the statement is directly inferred from the definition of ${}^{G}\!/_{\sim}$ and the relation $\sim$. And since the composition of full-homomorphisms is a full homomorphism then, from the previous lemma we obtain the following result.

Proposition 2.1.4. *For any two graphs G, H the following are equivalent:*

- *there is a full-homomorphism* $\varphi_1 : H \to G$,
- *there is a full-homomorphism* $\varphi_2 : {}^{H}\!/\!_{\sim} \to G$,
- *there is a full-homomorphism* $\varphi_3 : H \to {}^{G}\!/\!_{\sim}$ *and*
- *there is a full-homomorphism* $\varphi_4 : {}^{H}\!/\!_{\sim} \to {}^{G}\!/\!_{\sim}$.

As a direct corollary we have the following.

Corollary 2.1.5. *For any graph* G *we have the equality,* $\mathcal{F}_G = \mathcal{F}_{G/\sim}$

Suppose that $H \not\to G$ and H is not point determining, from Proposition 2.1.4, ${}^{H}\!/\!_{\sim} \not\to G$. Consider now $L = \phi_{X_H}\left[{}^{H}\!/\!_{\sim}\right]$ for any set of representatives X_H. Since ${}^{H}\!/\!_{\sim}$ is point determining and ϕ_{X_H} is a full-homomorphism, then L is point determining and $L < H$. Moreover, from Proposition 2.1.1, $\phi_{X_H} : {}^{H}\!/\!_{\sim} \to L$ is an isomorphism. Hence $L \not\to G$ and, since H is not point determining $L \neq H$. So L is a proper induced subgraph of H that does not admit a full-homomorphism to G. Therefore H is not a minimal obstruction to the G-full-homomorphism problem. Hence, we have the following proposition.

Proposition 2.1.6. *If a graph* H *is such that* $H \in \mathcal{F}_G$ *for any graph* G *then,* H *is point determining.*

Propositions 2.1.4 and 2.1.6, and Corollary 2.1.5, give us enough arguments to focus on the full-homomorphisms problem restricted to point determining graphs. Thus, from now on we will work with point determining graphs, unless stated otherwise.

In [6] the following proposition is proved.

Proposition 2.1.7. *For every non trivial point determining graph* G *there is a vertex* $v \in V_G$ *such that* $G - v$ *is point determining. Moreover, if* G *is connected then, there are two distinct vertices with that property.*

Later in [7] the *nucleus* of a graph is defined as: $G^o := \{v \in V_G \colon G - v$ is point determining$\}$.

Proposition 2.1.8. *Let* H *be a point determining graph. If* H *is* k*-regular, then* $H^o = H$.

Proof. Proceeding by contrapositive, suppose that $H^o \neq H$ and set $x \in V_H$ such that $H - x$ is not point determining. Let $r, s \in V_{H-x}$ such that $N_{H-x}(s) = N_{H-x}(r)$. Since H is point determining, we can assume without loss of generality that $xr \in E_H$ and $xs \notin E_H$. Hence, $d_H(s) = d_{H-x}(s) = d_{H-x}(r) = d_H(r) - 1$. Thus, H is not k-regular. ■

Proposition 2.1.9. *If $H \in \mathcal{F}_G$ for some graph G. Then, $|V_H| \leq |V_G| + 1$.*

Proof. Recall that the set of minimal obstructions $\mathcal{F}_G$ consists of point determining graphs (Proposition 2.1.6). Consider $H \in \mathcal{F}_G$ and $x \in V_H$ such that $H - x$ is point determining. Since H is a minimal obstruction, then there is a full-homomorphism $\varphi : H - x \to G$. From Proposition 2.1.1 φ is injective and therefore $|V_{H-x}| \leq |V_G|$. ■

For a graph G we will use the following notation, $M(G) = \{H \in \mathcal{F}_G \colon |V_H| = |V_G| + 1\}$. In [21] the following proposition is proved.

Proposition 2.1.10. *For any graph G, $|M(G)| \leq 2$.*

Proposition 2.1.11. *Let $H \in M(G)$ for some graph G. Then there is a vertex $v \in V_H$ such that $H - x \cong G$.*

Proof. Since H is point determining, then H^o is not empty. Consider any $v \in H^o$ and the full-homomorphism $\varphi : (H - v) \to G$. Since φ is injective and $|V_{H-v}| = |V_G|$ then φ is an isomorphism. ■

In order to distinguish a vertex such that $H - x \cong G$ we will write $H = G \odot x$. In favor of simplifying the following proof, but with no loss of reliablity, allow us to write $H - x = G$ instead of $H - x \cong G$.

Lemma 2.1.12. *If G is a k-regular connected graph and $k \geq 2$ then,*

$$M(G) \subseteq \{H = G \odot x \colon N_H(x) - y = N_H(y) - x, xy \in E_H, \text{ for some } y \in V_G\}.$$

Proof. Consider $H = G \odot x \in M(G)$. Since G is connected, if $d(x) > 0$ then H is connected. By Proposition 2.1.7, $|H^o| \geq 2$ and therefore there is a vertex $y \in G \cap H^o$. Since $H - y$ is point determining, then $H - y \cong G$, so $d_{H-y}(v) = k$ for any vertex $v \in V_{H-y}$. Also, $d_{G-y}(v) = k - 1$ if and only if $v \in N_G(y) = N_{H-x}(y)$. On the other hand, $k - 1 = d_{H-y}(v) - 1 = d_{G-y}(v)$ if and only if $v \in N_{H-y}(x)$. Hence, $v \in N_{H-y}(x)$ if and only if $v \in N_{H-x}(y)$ for any $v \in V_{(H-\{x,y\})}$. Thus, $N_H(x) - y = N_{H-y}(x) = N_{H-x}(y) = N_H(y) - x$, and since H is point determining then $xy \in E_H$.

Let us now see that in fact $d(x) > 0$. Since G is k regular, and $k \geq 2$, G has no leafs. Let $v \in V_G$ and $\varphi : H - v \to G$ be a full-homomorphism. From Proposition 2.1.8, $G - v$ is point determining and therefore $\varphi_{\restriction G-v}$ is injective. So let $L = \varphi[G - v]$, then $|V_L| = |V_G| - 1$. Since G is connected then $\varphi(x)$ must have a neighbour in L or belong to L. But L has no isolated vertices because $L \cong G - v$ and G has no leafs. Therefore $\varphi(x)$ must have a neighbour in L. And since φ is a full-homomorphism, then x cannot be and isolated vertex. ■

Proposition 2.1.13. *Let G be a k-regular connected graph and $k \geq 2$ then:*

$$M(G) = \begin{cases} \{K_{n+1}\} \text{ if } G \cong K_n, \\ \varnothing \text{ otherwise.} \end{cases}$$

Proof. It is not hard to see that $K_{n+1} \in M(K_n)$, hence due to Lemma 2.1.12, $\{K_{n+1}\} = M(K_n)$. Suppose that $H \in M(G)$ and G is not complete. By Lemma 2.1.12, $H = G \odot x$ and there is a vertex $y \in V_G$ such that $N_H(x) - y = N_H(y) - x$ and $xy \in E_H$. Since G is not complete, there is a vertex $z \in V_G$ such that $zy \notin E_G$. Therefore $yz, xz \notin E_H$, and there is a full-homomorphism $\varphi : H - z \to G$. Given that G is k-regular, and $d_{H-z}(x) = d_{H-z}(y) = k + 1$, we can find two vertices $r, s \in N_{H-z}(y)$ such that $\varphi(r) = \varphi(s)$. Hence $N_{H-z}(r) = N_{H-z}(s)$, so $N_{G-z}(r) = N_{H-z-x}(r) = N_{H-z-x}(s) = N_{G-z}(s)$. Therefore $G - z$ is not point determining, contradicting the Proposition 2.1.8. Thus $M(G) = \varnothing$. ■

Theorem 2.1.14. *Let $H \in M(G)$ for some graph G. Then,*

$$\mathcal{F}_H = (\mathcal{F}_G \setminus \{H\}) \cup M(H).$$

Proof. Let us first prove that $\mathcal{F}_G \setminus \{H\} \cup M(H) \subseteq \mathcal{F}_H$. By definition of $M(H)$ we only have to prove that $\mathcal{F}_G \setminus \{H\} \subseteq \mathcal{F}_H$. If there is a graph $L \in \mathcal{F}_G$ such that $L \to H$, since $H \in \mathcal{F}_G$ then $L = H$, otherwise, H would not be a minimal obstruction. Consider now $L_1 < L$ and the full homomorphism $\varphi : L_1 \to G$. From Proposition 2.1.11, $G < H$ and therefore $L_1 \to H$. So each minimal obstruction from G that is not H, is a minimal obstruction of H. Let us now prove the opposite inclusion. Since $H = G \odot x$, then every graph A such that $A \to G$ also $A \to H$. Hence, every minimal obstruction of H is an obstruction of G. All we have to prove now is that if $L \in \mathcal{F}_H \setminus M(H)$, and $L_1 < L$ then $L_1 \to G$. Consider the full-homomorphism $\varphi : L_1 \to H$. Since $|V_L| \leq |V_H|$, then $|V_{\varphi[L_1]}| < |V_H|$ so $\varphi[L_1] \to G$, and therefore $L_1 \to G$. ■

Corollary 2.1.15. *Consider two graphs G, G' such that $M(G) \cap M(G') \neq \varnothing$ then $G \cong G'$. Equivalently if $G \ncong G'$, then $M(G) \cap M(G') = \varnothing$.*

Corollary 2.1.16. *If $H \in M(G)$ for some graph G, then*

$$|\{L \in \mathcal{F}_H \colon |V_L| = |V_H|\}| \leq 1.$$

Proof. Set $S = \{L \in \mathcal{F}_H \colon |V_L| = |V_H|\}$. By Theorem 2.1.14, $\mathcal{F}_H = (\mathcal{F}_G \setminus \{H\}) \cup M(H)$. Hence, $S = M(G) \setminus \{H\}$. So by Proposition 2.1.10 $|S| \leq 1$. ■

Recall that the orbit of a vertex $y \in V_H$ is the set of vertices $x \in V_H$ such that there is an automorphism $\varphi \in A(H)$ such that $\varphi(y) = x$, and it is denoted by $o(y)$. If $x \in o(y)$ for some pair of vertices $x, y \in V_H$ then, $H - x \cong H - y$.

Lemma 2.1.17. *Let H be a non complete vertex transitive graph, then $\mathcal{F}_H = \mathcal{F}_{H-x} \setminus \{H\}$ for any vertex $x \in V_H$.*

Proof. Since H is vertex transitive then $H - x \cong H - y$ for any pair of vertices $x, y \in V_H$. So $L \to H - x$ for any $L < H$. And since H is point determining then $H \not\to H - x$. So $H \in M(H - x)$. By Proposition 2.1.8, $M(H) = \varnothing$. Then, Theorem 2.1.14 allows us to conclude the proof. ■

Proposition 2.1.18. *Let C_n be the cycle graph on n vertices with $n \geq 5$ then:*

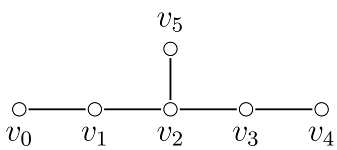
$$\mathcal{F}_{C_n} = \mathcal{F}_{P_{n-1}} \setminus \{C_n\}.$$

Proof. All we have to do is notice that, for any vertex $x \in V_{C_n}$, $C_n - x \cong P_{n-1}$ and then conclude using Lemma 2.1.17. ■

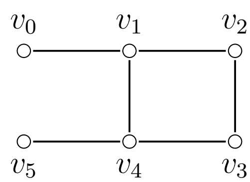
2.2 P_n-full-homomorphism.

We now proceed to find the minimal obstructions for the n path. For a positive integer $k \geq 3$, fix $\mathcal{C}_k := \{C_m \colon 3 \leq m \leq k, m \neq 4\}$. Following the notation used in [18], let us define the following graphs,

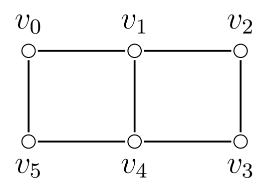
- $T_0 = (\{v_0, v_1, v_2, v_3, v_4, v_5\}, \{v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_2v_5\})$,
- $A = (\{v_0, v_1, v_2, v_3, v_4, v_5\}, \{v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_1v_4\})$,
- and $B = (\{v_0, v_1, v_2, v_3, v_4, v_5\}, \{v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_1v_4, v_0v_5\})$.



Graph T_0



Graph A



Graph B

For a path on n vertices we define $LF(n)$ as the set of linear forest that belongs to $\mathcal{F}_{P_n}$. We first find the elements in $\mathcal{F}_{P_n} \setminus LF(n)$.

Proposition 2.2.1. *The set of minimal obstructions to the P_n-full-homomorphism is,*

$$\mathcal{F}_{P_n} = LF(n) \cup \mathcal{C}_{n+1} \cup L(n).$$

Where $L(n)$ is defined as follows,

- *$T_0 \in L(n)$ if $n \geq 7$,*
- *$A \in L(n)$ if $n \geq 6$*
- *and $B \in L(n)$ if $n \geq 5$.*

Proof. It is not hard two notice either of the following,

- $\mathcal{F}_{P_1} = \{P_2\}$,
- $\mathcal{F}_{P_2} = \{K_3, P_1 + P_2\}$,

- $\mathcal{F}_{P_3} = \mathcal{F}_{P_2}$ (since P_3 is not point determining and $PD(P_3 = P_2)$),
- $\mathcal{F}_{P_4} = \{P_4 + P_1, 2P_2, K_3, C_5\}$,
- and if $n \geq 5, LF(n) \cup \mathcal{C}_{n+1} \cup L(n) \subseteq \mathcal{F}_{P_n}$.

Hence, all is left to prove is that $\mathcal{F}_{P_n} \subseteq LF(n) \cup \mathcal{C}_{n+1} \cup L(n)$ for all $n \geq 5$. Consider a graph $H \in \mathcal{F}_{P_n}$, if it has no induced cycles of order $k \leq n+1$, since $|V_H| \leq n+1$ (Proposition 2.1.9), H must be a forest. Let $T < H$ be a connected component of H and consider $P = (v_0, \ldots, v_{r-1}) < T$ a path of maximum length in T. If $r \in \{1, 2\}$ then $T = P$. Suppose that there is a vertex $x \in V_T \setminus V_P$ such that $v_1 x \in E_T$. Since P is a path of maximum length, then x and v_0 must both be leaves. Then $N_H(x) = N_T(x) = \{v_1\} = N_T(v_0) = N_H(v_0)$ which contradicts the fact that H is point determining (Corollary 2.1.6). Analogously we can prove that v_{r-2} cannnot have neighbours outside P. Since T is a connected component of H, if $P \neq T$, then there must be a vertex $x \in V_T \setminus V_P$ such that $xv_2 \in E_T$ and $r \geq 5$. It is not hard to see that if $n \in \{5, 6\}$ then $2P_2 + P_1 \in \mathcal{F}_{P_n}$ and $2P_2 + P_1 < H[\{v_0, v_1, v_2, v_3, v_4, x\}]$. Thus no such vertices can exist and then T must be a path. But if $n \geq 7$ then $H[\{v_0, v_1, v_2, v_3, v_4, x\}] \cong T_0$, and because H is a minimal obstruction, then $H \cong T_0$ or T is a path, thus H is a linear forest. Consider now the case where H has an induce cycle. Since H has at most $n+1$ vertices, if H has an induced cycle of order $k \neq 4$, then $H \in \mathcal{C}_{n+1}$. Suppose then that $C_4 = (v_0, v_1, v_2, v_3) < H$. Since H is point determining, there must be vertices $x, y \in V_H$ such that $x \in N(v_1) \setminus N(v_3)$ and $y \in N(v_2) \setminus N(v_0)$. Since H is triangle free then $H[\{v_0, v_1, v_2, v_3, x, y\}] \cong A$ or $H[\{v_0, v_1, v_2, v_3, x, y\}] \cong B$. Hence, if $n \geq 6$, $H \in L(n)$. Of course then, $n \geq 5$ since $|V_H| \geq 6$. All is left to prove is that if $n = 5$ then $H[\{v_0, v_1, v_2, v_3, x, y\}] \cong B$. For which one only has to notice that $P_4 + P_1 \in \mathcal{F}_{P_5}$ and $P_4 + P_1 < A$. Therefore $\mathcal{F}_{P_n} \subseteq LF(n) \cup \mathcal{C}_{n+1} \cup L(n)$ for all $n \geq 5$. ■

In order to complete the characterization of $\mathcal{F}_{P_n}$ let us study the set $LF(n)$. Consider a path $P = (v_0, v_1, \ldots, v_{n-1})$, if $m < n$ and $0 \leq r$ we will denote by $P_{[r,m]}$ the subpath induced by $\{v_r, v_{r+1}, \ldots, v_m\}$.

Lemma 2.2.2. *Let $L = \sum_{k=1}^{m} P_{n_k}$ be a linear forest then, there is an injective full-homomorphism*

$$\varphi : L \to P_n \text{ if and only if, } (m-1) + \sum_{k=1}^{m} n_k \leq n.$$

Proof. If $m = 1$ it is trivial that if $(m-1) + \sum_{k=1}^{m} n_k = n_1 \leq n$, then there is an injective full-homomorphism $\varphi : P_{n_1} \to P_n$. Proceeding by induction suppose the implication holds for $m-1$. If

$$(m-1) + \sum_{k=1}^{m} n_k \leq n \text{ then, } (m-1) - 1 + \sum_{k=1}^{m-1} n_k \leq n - n_k - 1.$$

Hence, there is an injective full-homomorphism $\varphi : L - P_{n_m} \to P_{[0,n-n_k-2]}$. And by mapping P_{n_m} to $P_{[n-n_k,n-1]}$ we get an injective full-homomorphism from L to P_n.

Consider now an injective full-homomorphism $\varphi : L \to P_n$. It is clear that $\varphi[P_{n_k}] = P_{[i_k,i_k+n_k-1]}$ for some $i_k \in \{0, \dots, n - n_k - 1\}$. Define the sets,

$$\overline{\varphi}(P_{n_k}) = \begin{cases} \{v_i, \dots, v_{i+n_k-1}\}, & \text{if } i + n_k - 1 = n, \\ \{v_i, \dots, v_{i+n_k}\}, & \text{else.} \end{cases}$$

Since φ is an injective full-homomorphism, then $\overline{\varphi}(P_{n_k}) \cap \overline{\varphi}(P_{n_l}) = \varnothing$ for any $k \neq j$. Then

$$n \geq \sum_{k=1}^{m} |\overline{\varphi}(P_{n_k})| \geq (m-1) + \sum_{k=1}^{m} n_k.$$

■

A trained eye is not needed to notice that for a linear forest $L = \sum_{k=1}^{m} P_{n_k}$, the variables in $(m-1) + \sum_{k=1}^{m} n_k$ are the connected components of L, $m = c(L)$, and the order of L, $|V_L| = \sum_{k=1}^{m} n_k$.

Notice that for a linear forest L and $x \in V_L$. If x is an isolated vertex then $c(L - x) = c(L) - 1$, else $c(L) \leq c(L - x) \leq c(L) + 1$.

Lemma 2.2.3. *Suppose $L = \sum_{k=1}^{m} P_{n_k} \in LF(n)$ for some path P_n. Then,*

$$n + 1 \leq (m-1) + \sum_{k=1}^{m} n_k = (m-1) + |V_L| \leq n + 2.$$

Moreover, if L has no isolated vertices then, $(m-1) + |V_L| = n + 1$.

Proof. By Lemma 2.2.2, if $(m-1) + \sum_{k=1}^{m} n_k \leq n$ then $L \to P_n$. To prove the second inequality, recall that by Proposition 2.1.7 there is a vertex

$x \in V_L$ such that $L - x$ is point determining. Then, there is an injective full-homomorphism $\varphi : L - x \to P_n$. Again, by Lemma 2.2.2,

$$c(L - x) - 1 + |V_L| - 1 = c(L - x) - 1 + |V_{L-x}| \leq n.$$

From the observation previous to this lemma, $c(L) - 1 \leq c(L - x)$ then, $c(L) - 1 + |V_L| \leq n + 2$. But if L has no isolated vertices then $c(L) \leq c(L - x)$ thus, $c(L) - 1 + |V_L| \leq n + 1$. ■

Consider a linear forest $L = \sum_{k=1}^{m} P_{n_k}$, in order to simplify writing, we define $m_i = |\{k \in \{1, \ldots, m\} \colon n_k = i\}|$. Note that $m_i = 0$ for all $i \geq m$.

Lemma 2.2.4. *If $L = \sum_{k=1}^{m} P_{n_k} \in LF(n)$ for some path P_n then the following hold,*

- *$n_k \in \{1, 2, 4\}$ for all $k \in \{1, \ldots, m\}$,*
- *$m_1 \leq 1$,*
- *and if $n_k = 4$ for some k then, $m_1 = 1$.*

Proof. Since P_3 is not point determining then $n_k \neq 3$ for al $k \in \{1, \ldots, m\}$. If $P_{n_k} = (v_0, \ldots, v_{n_k - 1})$ where $n_k = 5$ or $n_k \geq 7$ for some $k \in \{1, \ldots, m\}$ then, $L - v_2$ is point determining and $c(L - v_2) = c(L) + 1$ thus,

$$|V_{L-v_2}| + c(L - v_2) - 1 = |V_L| - 1 + (c(L) + 1) - 1 = |V_L| + c(L) - 1.$$

By Lemma 2.2.3, $|V_L| + c(L) - 1 \geq n + 1$, so $|V_{L-v_2}| + (c(L - v_2) - 1) \geq n + 1$. Hence by Lemma 2.2.2 we conclude that $L - v_2 \not\to P_n$, contradicting the fact that L is a minimal obstruction. Before proving that $n_k \neq 6$ for any $k \in \{1, \cdots\}$ we need to prove the third statement of the proposition. Now, if L is a linear forest with two copies of P_1 it would not be point determining so $L \notin LF(n)$ for any $n \in \mathbb{N}$.

Suppose that $P_{n_k} = (v_0, \ldots, v_{n_k - 1})$ with $n_k \in \{4, 6\}$ for some $k \in \{1 \ldots, m\}$ and $L \in LF(n)$ for some path P_n. Then, $P_{n_k} - v_1$ must be point determining. Thus, $P_{n_j} = P_1$ for some $n_j \in \{1 \ldots, m\}$; otherwise $L - v_1 \not\to P_n$.

And finally, suppose that $P_{n_k} = (v_0, \ldots, v_5)$. From the last paragraph we know that $P_{n_j} = P_1$ for some $j \in \{1, \ldots, m\}$. Then, from Lemma 2.2.3, $(m - 1) + |V_L| = n + 2$. With some simple calculations one can see that $c(L - v_5) - 1 + |V_{L-v_5}| = n + 1$, and $L - v_5$ is point determining. From Lemma 2.2.2 we conclude that $L - v_5 \not\to P_n$, which contradicts the fact that L is a minimal obstruction for the P_n full-homomorphism. Hence $n_k \in \{1, 2, 4\}$ for al $k \in \{1, \ldots, m\}$. ■

Almost as a Corollary of these technical Lemmas, we get the following Proposition.

Proposition 2.2.5. *Let $L = \sum_{k=1}^{m} P_{n_k}$ be a linear forest then, $L \in LF(n)$ for the path P_n, if and only if, either of the following hold,*

- *$|V_L| + m - 1 = n + 1$ and $n_k = 2$ for all $k \in \{1, \dots, m\}$,*
- *or, $|V_L| + m - 1 = n + 2$, $n_k \in \{1, 2, 4\}$ with $m_1 = 1$.*

Proof. Suppose $L = \sum_{k=1}^{m} P_{n_k} \in LF(n)$ and $|V_L| + m - 1 = n + 1$. Due to Lemma 2.2.3, L has no isolated vertices. Using Lemma 2.2.4 we conclude that $n_k = 2$ for all $k \in \{1, \dots, m\}$. By Lemma 2.2.3, if $|V_L| + m - 1 \neq n + 1$ then, $|V_L| + m - 1 = n + 2$. Then again, by Lemma 2.2.4 $n_k \in \{1, 2, 4\}$ with $m_1 \leq 1$ and if $m_4 \geq 1$ then $m_1 = 1$. In order to prove that $m_1 = 1$, all we have to observe is that there must exist $k \in \{1 \dots, m\}$ such that $n_k \neq 2$. We will proceed by contradiction, asume $n_k = 2$ for all $k \in \{1, \dots, m\}$. Then, for all vertices $x \in V_L$, $c(L - x) = c(L)$. Hence $c(L - x) - 1 + |V_{L-x}| = n + 1$. By using Lemma 2.2.2 we notice that $L - x \not\to P_n$, contradicting the fact that $L \in LF(n)$.

In order to prove the remaining implication, let us first suppose that $|V_L| + m - 1 = n + 1$ and $n_k = 2$ for all $k \in \{1, \dots, m\}$. It is clear that the linear forest $L = \sum_{k=1}^{m} P_{n_k}$ is then point determining and for any vertex $x \in V_L$, $c(L - x) = c(L)$. Hence $c(L - x) - 1 + |V_{L-x}| = 1$. By using Lemma 2.2.2 we conclude that $L \not\to P_n$ but $L - x \to P_n$ for any $x \in V_L$ therefore, $L \in LF(n)$. We now suppose that $|V_L| + m - 1 = n + 2$, $n_k \in \{1, 2, 4\}$ with $m_1 = 1$. Clearly, L is point determining, and because of Lemma 2.2.2, $L \not\to P_n$. Now we have to prove that $L - x \to P_n$ for any $x \in V_L$.

Claim 2.2.6. *For any vertex $x \in V_L$, $|V_{PD(L-x)}| + c(PD(L - x)) - 1 \leq n$.*

If the claim is true, from Lemma 2.2.2, $PD(L - x) \to P_n$ for any vertex x. Then, by Proposition 2.1.4, we would conclude that $L - x \to P_n$, and we would be done.

We have three cases when proving the claim:

- if x is an isolated vertex then, $c(L - x) = c(L) - 1$ and $L - x$ is point determining. Hence $PD(L - x) \cong L - x$ and $|V_{L-x}| + c(L - x) - 1 = n$.

- If x belongs to a copy of P_2 or is a cut-vertex of a copy of P_4 then, we would have two isolated vertices in $L-x$ so $|V_{PD(L-x)}| \leq |V_L| - 2$, and $c(PD(L-x)) \leq c(L)$. Therefore $|V_{PD(L-x)}| + c(PD(L-x)) - 1 \leq |V_L| - 2 + c(L) - 1 = n$.

- Finally, consider the case where x is the initial (or final) vertex of a copy of P_4 and suppose $n_l = 4$. Then, by removing x from P_{n_l} we are left with a copy of P_3, $P_{n_l'} = (x_1, x_2, x_3)$. Hence x_1 and x_3 have the same neighbourhood; namely $\{x_2\}$. Thus, x_1 and x_3 belong to the same equivalence class in $PD(L-x)$. Therefore $|V(PD(L-x))| \leq |V_L| - 2$, and since x was not a cut vertex in L, $c(L) \geq c(PD(L))$. Thus, $|V_{PD(L-x)}| + c(PD(L-x)) - 1 \leq |V_L| - 2 + c(L) - 1 = n$.

■

The following Proposition is basically a rephrasing of Proposition 2.2.5, but it gives us a better insight of the structure of the linear forests in $LF(n)$.

Proposition 2.2.7. *Consider a linear forest L then, $L \in LF(n)$ if and only if either of the following hold,*

- *$L = m_2P_2$ where $3m_2 = n + 2$ and $m_2 \in \mathbb{Z}^+$,*

- *or $L = P_1 + m_2P_2 + m_4P_4$ where m_2 and m_4 are positive integer solutions to the equation $3m_2 + 5m_3 = n + 2$.*

Proof. Consider a linear forest $L = \sum_{k=1}^{m} P_{n_k}$. All we have to do is use Proposition 2.2.5 and notice that $|V_L| + (m-1) = \sum_{i=1}^{m} im_i + \left(\sum_{i=1}^{m} m_i\right) - 1$. ■

We have now explicitly found the set of minimal obstruction to the P_n-full-homomorphism problem, and with Proposition 2.1.18 we also know the structure of $\mathcal{F}_{C_n}$.

Chapter 3

A small problem on homomorphisms.

Though there is only a subtle difference in the definitions of full-homomorphism and homomorphism, and every full-homomorphism is a homomorphism, the corresponding associated decision problems turn out to be completely different. For instance, from Proposition 2.1.9 we know that for any graph H, the H-full-homomrphism problem has a finite amount of minimal obstructions hence, is polynomial time solvable. While from Proposition 1.2.1 we know that the H-homomorphism problem is polynomial time solvable when H is bipartite and NP-complete otherwise. In fact let us start with a proposition that proves that the H-homomorphism problem is polynomial time solvable when H is bipartite.

Proposition 3.0.8. *Let H be a non-empty bipartite graph then, there is a homomorphism $\varphi : G \to H$ is and only if G is bipartite.*

Proof. Obviously, if G is bipartite, there is a homomorphism $\varphi_1 : G \to K_2$. Since H is not an empty graph, then there is an edge $xy \in E_H$. Hence we can map G to xy. On the other hand, consider a graph G and a homomorphism $\varphi : G \to H$. Since H is bipartite, consider the homomorphism $\varphi_b : H \to K_2$. Then, $\varphi_b \circ \varphi : G \to K_2$ is a graph homomorphism, so G must be bipartite. ■

Clearly then, if H is a bipartite graph, the H-homomorphism problem is polynomial time solvable.

As in chapter one, we could set the task to give a characterization of the graphs that admit a homomorphism to the n-path, and of those that admit a homomorphism to the n-cycle. Since the n-path and every even-cycle are bipartite graphs, the task becomes trivial because of Proposition 3.0.8. So we are focusing on the odd-cycle homomorphism.

If there was a way of describing the structure of the minimal obstructions to the C_{2n+1}-homomorphism-problem, we could likely be able to find a polynomial time algorithm that solves C_{2n+1}-homomorphism-problem. Since we believe this thesis will not prove that $P = NP$, the purpose of this chapter will be to find a finite set of forbidden subpatterns to the C_{2n+1}-homomorphism, $FP_{C_{2n+1}}$.

For a given pattern $G_< = (V_G, E_G)$, we define the following set $\mathcal{SP}(G_<) = \{H_< = (V_G, E')\colon V_G$ has the same linear order in $H_<$ as in $G_<$, and $E_G \subseteq E'\}$. In other words, $\mathcal{SP}(G_<)$ is the set of super-patterns of $G_<$ whose underlying graph contain G as a spanning subgraph, and their vertices have the same ordering.

Example 3.0.9. Consider the ordered path on k vertices $P_{k<}$. The characterization of k-partite graphs given in Proposition 1.3.4 can be rewritten as: let G be a graph then, G is k-colourable (partite) if and only if G admits an FP_k-free ordering. Where $FP_k = \mathcal{SP}(P_{(k+1)<})$.

Note that any two orderings of K_3 are the same, so we will denote an ordering of K_3 as K_3 whenever there ir no ambiguity. Also, since the 3-cycle is K_3, $FP_{C_3} = \mathcal{SP}(P_{4<})$. So we will be working with the odd cycles on more than four vertices (we will not repeat this assumption unless needed).

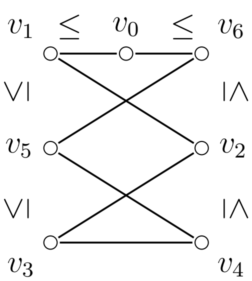
For an odd cycle $C_{2n+1} = (v_0, \dots, v_{2n})$ we define the *alternating ordering* of C_{2n+1}, as:

$$\langle v_n, v_{n+2}, v_{n-2}, v_{n+4}, v_{n-4}, \dots, v_{2n-1}, v_1, v_0, v_{2n}, v_2, v_{2n-2}, v_4, \dots, v_{n-1}, v_{n+1}\rangle$$
$$\text{if } n \equiv 1 \pmod 2, \text{ and}$$
$$\langle v_{n+1}, v_{n-1}, v_{n+3}, v_{n-3}, \dots, v_{2n-1}, v_1, v_0, v_{2n}, v_2, v_{2n-2}, v_4, \dots, v_{n-2}, v_{n+2}, v_n\rangle$$

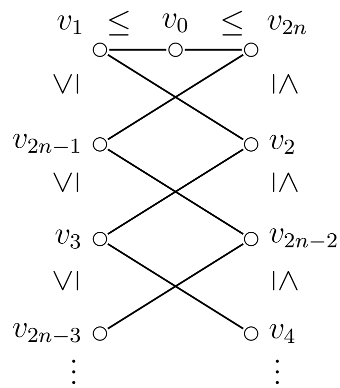
if $n \equiv 0 \pmod 2$. In both cases we will denote it by $C_{2m+1,al}$.

With the previous pattern definition, we define $S(2n+1) = \{C_{2m+1,al}\colon 1 \leq m < n\}$, the set of alternating orderings of odd cycles on at most $2n-1$ vertices. And allow us to define one more set, for $n \geq 2$ we define $PL(2n+1) = \{P_<\colon\ P = P_l = (v_1, v_2, \dots v_l),\ 6 \leq l < 2n+1,\ v_1 < v_2 < v_3$ and

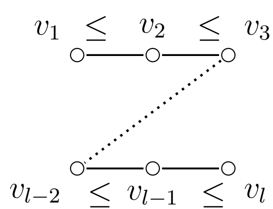
$v_{l-2} < v_{l-1} < v_l\}$. So $PL(2n+1)$ is a set of patterns whose underlying graph is a path on l vertices, where $6 \leq l \leq 2n+1$, and the three inicial vertices are ordered from smaller to larger index, so as the last three vertices of the path.



$S(7) = \langle v_3, v_5, v_1, v_0, v_6, v_2, v_4 \rangle$.



The “top end” of $S(2n+1)$.



A general element $P_< \in PL(2n+1)$.

Proposition 3.0.10. *For any $n \geq 2$, a graph G admits a C_{2n+1}-homomorphism if and only if it admits $FP_{C_{2n+1}}$-free ordering. Where*

$$FP_{C_{2n+1}} = S(2n+1) \cup PL(2n+1) \cup \mathcal{SP}(P_{4<}) \cup \{K_3\}.$$

Proof. We will assume without loss of generality that the graphs considered do not have isolated vertices.

Let $\varphi : G \to C_{2n+1}$ be a graph homomorphism. Consider the ordering of V_G obtained by choosing an arbitrary linear order for the vertices inside each $\varphi^{-1}(v_i)$, $i \in \{0, \ldots, 2n\}$, and for vertices with different images, x and y, $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$ in the alternating ordering of C_{2n+1}

$(C_{2n+1,al})$. Let us name this ordering $G_{\leq}$. Using the fact that the partition of V_G induced by φ is cyclical, it becomes trivial to notice that $G_{\leq}$ must then be $S(2n+1)$-free. By noticing that the longest ordered path in $C_{2n+1,al}$ is of length three, and every $\varphi^{-1}(v_i)$, $i \in \{0, \ldots, 2n\}$ is an independent set, we conclude that $G_{\leq}$ is $\mathcal{SP}(P_{4<})$-free. And finally, combining the fact that φ induces a cyclical partition, and that the longest ordered path in $C_{2n+1,al}$ is of length three, it is not hard to observe that $G_{\leq}$ is also a $PL(2n+1)$-free ordering.

Now, suppose there is an ordering $G_< = \langle v_1, \ldots, v_n \rangle$ of the vertices in G without seeing any forbidden pattern in $\mathcal{F}_{C_{2n+1}}$. Let us now define the partition that will induce a C_{2n+1}-homomorphism.

- Let X_0 be the set of vertices v_j, $1 \leq j \leq n$ that have left and right neighbours;
- let X_1 (Y_1) be the set of vertices v_j, $1 \leq j \leq n$ having a right (left) neighbour in X_0;
- let $X_{i+1} = \{v \in V_G \setminus \left(\bigcup_{j=1}^{i} X_j\right) \colon v$ has a right neighbour in $Y_i\}$;
- let $Y_{i+1} = \{v \in V_G \setminus \left(\bigcup_{j=1}^{i} Y_j\right) \colon v$ has a left neighbour in $X_i\}$;
- and let X_n (Y_n) be the set of vertices v_l, $1 \leq j \leq n$, in $V_G \setminus \left(\bigcup_{j=1}^{n-1} X_j\right)$ (in $V_G \setminus \left(\bigcup_{j=1}^{n-1} Y_j\right)$) having no right (left) neighbours.

If it happened that $X_0 = \varnothing$, G would be $\mathcal{SP}(P_{3<})$-free. Hence, by Proposition 1.3.3 we could map G to any edge of C_{2n+1}. So we will assume that X_0 is not an empty set.

We affirm that $(X_0, X_1, Y_2, X_3, \ldots, X_n, Y_n, X_{n-1}, Y_{n-2}, \ldots, Y_1)$ is a cyclical partition of G, and hence, it induces a homomorphism to the $2n+1$-cycle. Note that if $n \equiv 0 (\text{mod } 2)$, the partition would look as follows, $(X_0, X_1, Y_2, X_3, \ldots, Y_n, X_n, Y_{n-1}, X_{n-2}, \ldots, Y_1)$. Since both cases are analogous, we will only work with the first case, which happens if $n \equiv 1 (\text{mod } 2)$ (fact which is never used throughout the proof).

We start by proving that $(X_0, X_1, Y_2, X_3, \ldots, X_n, Y_n, X_{n-1}, Y_{n-2}, \ldots, Y_1)$ is indeed a partition of the vertex set of G. Since all the vertices with left and right neighbours belong to X_0, every vertex not in X_0 fails to have either left neighbours or right neighbours, and hence, $\bigcup_{i=1}^{n}(X_i \cup Y_i) = V(G)$. By definition of sets X_i's and Y_i's, it is not hard to verify that $X_i \cap X_j = \varnothing$, $X_i \cap Y_j = \varnothing$, $X_0 \cap X_j = \varnothing$ and $X_0 \cap Y_j = \varnothing$ for each i, j, $1 \leq i < j \leq n$.

Since every vertex in X_i (Y_i), $1 \leq i \leq n$ have no left (right) neighbours they are all independent sets. Assume now that X_0 is not an independent set, then there are two vertices $v_i, v_j \in X_0$ such that $v_i v_j \in E_G$. Suppose with no loss of generality that v_i is a left neighbour of v_j, and consider the vertices v_k, v_l that are a left neighbour of v_i and a right neighbour of v_j, respectively. This leads to a contradiction since the induced subpattern by (v_k, v_i, v_j, v_l) is an element of $SP(P_{4<})$, and $G_<$ is $SP(P_{4<})$-free. Thus, $(X_0, X_1, Y_2, X_3, \ldots, X_n, Y_n, X_{n-1}, Y_{n-2}, \ldots, Y_1)$ is a partition of V_G into independent sets.

We now want to verify the desired non-adjacencies. For the same reason that X_i (Y_i), $1 \leq i \leq n$ are independent sets, there cannot be an edge between any two sets X_i and X_j (Y_i and Y_j), $1 \leq i, j \leq n$. Actually, by definition of the X_i's and Y_i's there can only be edges between the following sets: X_1 or Y_1 and X_0; Y_i and X_i ,X_{i+1} or X_{i-1}, $1 \leq i \leq n$. In order to conclude our proof we only have to verify that there will be no edges between sets X_i and Y_i for any $1 \leq i < n$. Suppose for a contradiction that $xy \in E_G$ with $x \in X_i$ and $y \in Y_i$ with $1 \leq i < n$. By construction of each of the X_i's and Y_i's, we can find two internally disjoint paths T_x and T_y such that T_x is an xx_0-path and T_y is a yy_0-path, where $x_0, y_0 \in X_0$. Let z_y be a left neighbour of y_0 and z_x a right neighbour of x_0, and y_1 (x_1) be the right (left) neighbour of y_0 (x_0) in T_y (T_x). We now have two cases:

- if $\{z_y, y_0, y_1\} \cap \{x_1, x_0, z_x\} = \varnothing$, consider the subpattern induced by $z_y y_0 T_y y x T_x x_0 z_x$, Q_{2i+1}. Since Q_{2i+1} is an ordering of the path on $2i+1$ vertices with $6 \leq 2i+1 < 2n+1$, and $z_y < y_0 < y_1$ and $x_1 < x_0 < z_x$ then $Q_{2i+1} \in PL(2n+1)$. Which leads to a contradiction because $G_<$ is $PL(2n+1)$-free.

- On the other hand, if $\{z_y, y_0, y_1\} \cap \{x_1, x_0, z_x\} \neq \varnothing$ by joining T_x and T_y through a common vertex in $\{z_y, y_0, y_1\}$ and $\{x_1, x_0, z_x\}$, and the edge xy we assumed existed, we would have either an alternating ordering of C_{2i+1} or K_3. In both cases we end up with a contradiction.

Therefore, $(X_0, X_1, Y_2, X_3, \ldots, X_n, Y_n, X_{n-1}, Y_{n-2}, \ldots, Y_1)$ is a cyclic partition of G.

■

Chapter 4

A small problem on M-partitions.

A natural question that arises when studying the M-partition problem is, if we know some information about the N-partition problem, for some principal submatrix N of M, what can we tell about the M-partition problem? For example, we know that if N is the matrix associated to the 3-colouring problem, or its complement, then the N-partition problem is NP-complete. Theorem 1.2.12 tells us that if M is a symmetric $\{0, 1, *\}$-matrix of size at most four and contains N as a principal submatrix then, the M-partition problem is also NP-complete.

It would be ideal to figure out a way of finding the elements in F_M, if we knew those in F_N for all, or some, principal submatrices N of M (distinct to M). Or the same question but now asking for a set of forbidden subpatterns. The main result of this chapter is to answer the last question in a particular case.

Consider two square matrices M, N over $\{0, 1, *\}$. We define the $*$-*sum* of M and N as,

$$M * N := \begin{bmatrix} M & * \\ * & N \end{bmatrix}$$

Our result constructs a set of forbidden subpatterns for the $(M * N)$-partition problem, given the forbidden patterns of the M- and N-partition problem.

Example 4.0.11. If M_k is the matrix of the k-colouring problem then, $M_k * M_l = M_{l+k}$. This is why we say our result generalizes Hell's, Mohar's and

Rafies's observation made in [14].

In order to gain a different perspective on the $(M * N)$-partition problem allow us to introduce some definitions. Consider hereditary properties $\mathcal{P}$ and $\mathcal{Q}$, a $(\mathcal{P}, \mathcal{Q})$-colouring of a graph G, is partition of $V_G = (A, B)$ such that $G[A] \in \mathcal{P}$ and $G[B] \in \mathcal{Q}$. We say that a graph G belongs to the class $\mathcal{P} \circ \mathcal{Q}$ if G is $(\mathcal{P}, \mathcal{Q})$-colourable.

Remark 4.0.12. Recall that the class of graphs that admit an M-partition is denoted by $\mathcal{P}_M$. A graph admits an $(M * N)$-partition if and only if is $(\mathcal{P}_M, \mathcal{P}_N)$-colourable.

The complexity of the $(\mathcal{P}, \mathcal{Q})$-colouring problem has been well studied. For example, Kratochvíl and Schiermeyer conjectured in [9] that if $\mathcal{P}$ and $\mathcal{Q}$ are also additive graph properties, recognizing graphs in $\mathcal{P} \circ \mathcal{Q}$ is NP-hard, which was later proved by Farrugia in [3]. Other particular cases have been studied in [11], [4] and [1].

Let us now proceed to our work. In order to simplify notation, whenever we have a graph $G = (V, E)$ and a subset of vertices $U \subseteq V$ we will write indifferently (U, E) or $(U, E_{G[U]})$. Same notation will be used when talking about ordered graphs. Throughout this chapter the set of vertices of any pattern is ordered by the indices of its vertices; for any $v_i, v_j \in V$ $v_i \leq v_j$ if and only if $i \leq j$.

Proposition 4.0.13. *Let M and N be two square matrices over $\{0, 1, *\}$ with forbidden subpatterns FP_M and FP_N respectively. Then, the set*

$$FP = \{(\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_s\}, E) \colon\ (\{v_1, v_2, \ldots, v_r\}, E) \in FP_M \text{ and } (\{v_r, v_{r+1}, \ldots, v_s\}, E) \in FP_N\}$$

characterizes the $*$-sum of M and N by forbidden subpatterns.

Proof. Let us first consider a graph G that admits an $(M * N)$-partition. It is clear to see that $V_G = V_M \cup V_N$ where $V_M \cap V_N = \varnothing$ and $G[V_M], G[V_N]$ admit an M and N partition respectively. Consider now the ordering of G, $G_{\leq} = (\{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}, E)$ where $(\{v_1, v_2, \ldots, v_m)\}, E)$ is FP_M-free and $(\{v_m, v_{m+1}, \ldots, v_n\}, E)$ is FP_N-free. We can suppose that there is a subpattern, $H_{\leq} = (\{u_1, u_2, \ldots, u_r, u_{r+1}, \ldots, u_s\}, E)$ of $G_{\leq}$ such that $(\{u_1, u_2, \ldots, u_r\}, E) \in FP_M$, otherwise $G_{\leq}$ would already be FP-free and there would be nothing left to prove. Since $(\{v_1, v_2, \ldots, v_m\}, E)$ is FP_M-free then we have that $u_r \geq v_m$ so $(\{u_r, u_{r+1}, \ldots, u_s\}, E) \subseteq (\{v_m, v_{m+1}, \ldots, v_n\}, E)$ which is FP_N-free, allowing us to conclude that $H_{\leq} \notin FP$. Therefore $G_{\leq}$ is FP-free.

Let $G_{\leq} = (\{v_1, v_2, \ldots, v_n\}, E)$ be an ordered FP-free graph. We have to find a partition (V_M, V_N) of V_G such that $G[V_M]$ and $G[V_N]$ admit an M and N partition respectively. In order to do this, let us define the following assignment for an ordered graph $H_{\leq} = (\{w_1, w_2, \ldots, w_l\}, D)$,

$$m(H_{\leq}) = \max\{k \in \mathbb{N}\colon\ (\{w_1, \ldots, w_k\}, D)\ is\ FP_M - free\}.$$

Using this function, we propose the following recursive definition:

- $G_0 = G_{\leq}$, $U_0 = \{v_1, \ldots, v_{m(G_0)}\}$ and $W_0 = \{v_{m(G_0)+1}\}$;
- $G_{l+1} = G_{\leq} - \{W_l\}$, $U_{l+1} = \{v_1, \ldots, v_{m(G_{l+1})}\}$ and $W_{l+1} = W_l \cup \{v_{m(G_{l+1}+1)}\}$.

Suppose that the recursion stops at time t. By definition of the assignment m, it is not hard to notice the U_t is FP_M-free. Let us proceed by contradiction and suppose that W_t is not FP_N-free, and let $\{u_1, u_2, \ldots, u_l\}$ be a subset of W_t such that $(\{u_1, u_2, \ldots, u_l\}, E) \in FP_N$. Let a be the minimum integer such that $u_1 \in W_a$. By definition of W_a, we can find a subset $\{w_1, \ldots, w_k\} \subseteq U_a$ that satisfies that $(\{w_1, \ldots, w_k, u_1\}, E) \in FP_M$. But then, by definition of FP,

$$(\{w_1, \ldots, w_k, u_1, u_2, \ldots, u_l\}, E) \in FP.$$

But $(\{w_1, \ldots, w_k, u_1, u_2, \ldots, u_l\}, E)$ is a subpattern of $G_{\leq}$, which leads to a contradiction since $G_{\leq}$ is FP-free. ■

Exactly the same proof can be made for finding a set of forbidden subpatterns for the $(\mathcal{P}, \mathcal{Q})$-colouring problem if we know the ones for $\mathcal{P}$ and $\mathcal{Q}$, we have the following theorem.

Theorem 4.0.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two properties characterized through sets of forbidden subpatterns $FP_{\mathcal{P}}$ and $FP_{\mathcal{Q}}$ respectively. Then, the set*

$$FP = \{(\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_s\}, E)\colon\ (\{v_1, v_2, \ldots, v_r\}, E) \in FP_{\mathcal{P}}\ and\ (\{v_r, v_{r+1}, \ldots, v_s\}, E) \in FP_{\mathcal{Q}}\}$$

characterizes the $(\mathcal{P}, \mathcal{Q})$-colouring problem by forbidden subpatterns.

Chapter 5

A little extra.

The idea of characterizing hereditary properties through sets of minimal obstructions detaches from the definition of hereditary property itself. Furthermore, with a small observation one can notice that hereditary properties can be characterized through sets of forbidden subpatterns. In the same way that this happens, we introduce a new way of characterizing hereditary properties.

5.1 First definitions and observations.

We say that G_O is an *orientation* of the graph G, if G_O is an oriented graph (digraph) whose underlying graph is G. If an orientation H_O is an induced oriented subgraph of another oriented graph G_O, we say that H_O *occurs* in G_O; otherwise G_O is H_O-*free*. We can naturally extend the later definition to sets of oriented graphs, consider a set of digraphs $\mathbb{H}$, we say a graph G admits an $\mathbb{H}$-*free* orientation if there is an orientation G_O such that H_O does not occur in G_O for any $H_O \in \mathbb{H}$.

Let $\mathcal{P}$ be hereditary property. A set of oriented graphs $FO_{\mathcal{P}}$ such that for any graph G, $G \in \mathcal{P}$ if and only if G admits an $FO_{\mathcal{P}}$-free orientation, will be called a set of *forbidden oriented graphs* for $\mathcal{P}$.

Example 5.1.1. Let $\mathcal{P}$ be a hereditary, property and consider the set

$$\mathbb{TO}(\mathcal{P}) = \{H_O \colon H_O \text{ an orientation of } H, H \in F_{\mathcal{P}}\}.$$

It is clear then that for any graph G, $G \in \mathcal{P}$ is and only if, G admits an $\mathbb{TO}(\mathcal{P})$-free orientation. This set is called the *trivial* set of forbidden oriented graphs for $\mathcal{P}$.

As in the case of forbidden subpatterns, if $\mathcal{P}$ is characterized with a finite set of minimal obstructions then, there is at least one finite set of forbidden oriented graphs for $\mathcal{P}$, namely $\mathbb{TO}(\mathcal{P})$. The following Propositions give two examples of properties with an infinite set of minimal obstructions, but a finite set of forbidden oriented graphs.

Proposition 5.1.2. *The set $FO = \{\overrightarrow{P_3}\} \cup o(K_3)$, where $\overrightarrow{P_3}$ is the directed path on 3 vertices, and $o(K_3)$ is the set of the two possible oriented tournaments with 3 vertices, characterizes P_B through forbidden oriented graphs.*

Proof. If $G = (X, Y)$ is a bipartite graph, turn all the edges to an arc with tail in X and head in Y. It is easy to verify that the obtained orientation is FO-free. Now, suppose that there is an FO-free orientation H_O for a graph H. Suppose that H has no isolated vertices, and set $X = \{v \in V_H :$ there is a vertex $u \in V_H$ such that $vu \in A(H_O)\}$ and $Y = \{v \in V_H :$ there is a vertex $u \in V_H$ such that $uv \in A(H_O)\}$. It is clear to see that X and Y are independent sets. ■

If we now take a look at $FO = \{P_3^0\} \cup o(K_3)$, where P_3^0 is the orientation of P_3 where the middle vertex has indegree 2, we have the following result.

Proposition 5.1.3. *Let $\mathcal{P}_u$ be the class of triangle-free graphs where each connected component has at most one cycle then, $FO = \{P_3^0\} \cup o(K_3)$ is a set of forbidden oriented graphs for $\mathcal{P}_u$.*

Proof. Without loss of generality, we can assume that G is a connected graph. For any $G \in \mathcal{P}_{\mathrm{u}}$ and any orientation G_O of G, G_O is $o(K_3)$-free. If G is a cycle, it suffices to orient it in a cyclic way. If G is a tree, we can set an arbitrary vertex v to root the tree on v, and orient the edges from father to son. If G is neither a cycle nor a tree, we can orient the only cycle C in G in a cyclic way. Then we can contract C to a single vertex; since G is unicyclic, the resulting graph G/C is a tree. Root G/C in the vertex corresponding to C and orient the edges from father to son. Since $E_G = E_C \cup E_{G/C}$, the combination of the previous orientations induce an orientation of G, which we can easily verify is a P_3^0-free orientation of G. Consider now a graph G that does not belong to $\mathcal{P}_{\mathrm{u}}$.

Suppose that G is a connected graph with at least two cycles. Consider any orientation G_O of G, it suffices to prove that G_O has a vertex that is the head of two arcs and then G_O cannot be FO-free. Let C be a cycle in

G. Then C must be oriented in a cyclic way, otherwise G_O is not P_3^0-free. If there is a path T between two vertices $x, y \in V_C$, we can assume without loss of generality that T is internally disjoint to C. Each x and y are either the head or the tail of one and only arc of T. If they were both the tail, then some vertex z in T must be the head of two arc which would lead to a contradiction. Hence, we assume that y is the head of its corresponding arc in T. Since T has no internal vertex in C then y is the head of two arcs, one in C and one in T, hence G_O is not P_3^0-free. On the other hand, if C has no internal paths, there must be another cycle C'. If $|V_C \cap V_{C'}| > 1$ we can find a cycle with an internal path. If $|V_C \cap V_{C'}| = 1$ then, since C' must also be oriented in a cyclic way, the intersecting vertex x must then be the head of two arcs. Finally, suppose $V_C \cap V_{C'} = \varnothing$, in the same way as previous cases C' must be oriented in a cyclic way, and since G is connected, there must be a CC'-path T. If T is not oriented in a directed way, then there is a vertex $x \in V_T$ that is a head of two arcs. Suppose then that T is oriented from C to C', then the meeting vertex of T and C' is the head of two arcs. ■

We now know the relation between sets of forbidden subpatterns and minimal obstructions, and sets of forbidden oriented graphs and minimal obstructions. But what about the relation between sets of forbidden subpatterns of forbidden oriented graphs;

- is there a property $\mathcal{P}$ that admits a finite set of forbidden subpatterns but no finite set of finite forbidden oriented graphs?
- Is there a property $\mathcal{P}$ that admits a finite set of forbidden oriented graphs but no finite set of finite subpatterns?

Unfortunately we have not been able to answer the second question. In fact, we do not know if there is a property that does not admit a characterization by finite set of forbidden subpatterns. On the other hand, the remainder of the chapter is dedicated to answer the first query. Two very important tools we use are Symbolic Dynamics and Königs infinite tree Lemma.

5.2 Symbolic Dynamics and Königs infinite tree Lemma.

Symbolic Dynamics object of study are Shift Spaces. Though these spaces result to be really interesting structures to study, we will only mention basic definitions and results that are needed for this dissertation. We refer the interested reader to [10] for a deeper study of Shift Spaces.

In this context, an *alphabet* is a finite set $\mathcal{A}$ whose elements are called *symbols* (equivalently *letters*). The *full $\mathcal{A}$-shift* is the set

$$A^{\mathbb{Z}} = \{x\colon x : \mathbb{Z} \to \mathcal{A}\}.$$

A more intuitive way to think of the elements in $A^{\mathbb{Z}}$ is to consider them as bi-infinite sequences of symbols in $\mathcal{A}$; each of these elements is called a *point* of the full shift. A *block* (or *word*) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. The *length* of a word u is the number of symbols it contains. Thus, we denote by $\mathcal{A}^n$ the set of words over $\mathcal{A}$ of length n. The set of all blocks over $\mathcal{A}$ is $\mathcal{A}^* = \bigcup_{n\geq 0} \mathcal{A}^n$, where the only word with no symbols is the *empty word* denoted by ϵ. A *subword* of a word $u = u_1 \dots u_k$, is a block $v = u_l u_{l+1} \dots u_j$ where $1 \leq l \leq j \leq k$.

If x is a point in $\mathcal{A}^{\mathbb{Z}}$, we denote the coordinates of x from i to j by $x_{[i,j]} = x(i) \dots x(j)$, where $i \leq j$. This definition is extended to denote by $x_{[i,\infty]} = x(i)x(i+1)\dots$, the *right-infinite* sequence of x from position i.

We now have an idea of where the word "Symbolic" in Symbolic Dynamics comes from, let us now take a look at the Dynamics in this shift spaces.

The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ is the function,

$$\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$$

$$\sigma(y)_i = y_{i-1}.$$

The composition of σ with itself k times is denoted by σ^k. A *periodic* point $x \in \mathcal{A}^{\mathbb{Z}}$ is a point such that $\sigma^k(x) = x$ for some $k \in \mathbb{N}$. Is such k exists, we say that x has *period k*. If x fails to be periodic, we say that x is an *aperiodic* point.

Example 5.2.1. Consider the full shift over the alphabet $\{-1, 1\}$. The constant functions $x_i = 1$ and $y_i = -1$ for all $i \in \mathbb{Z}$ are periodic points of period one. Points with such period are called *fixed* points. The point $z_i = -1$ if i is odd and $z_i = 1$ otherwise, is a point of period two. The point $w_i = -1$ for all $i \neq 0$ and $w_0 = 1$ is an aperiodic point.

Anyone who has studied dynamical systems, knows that in most cases dynamic occurs in a space with some structure; a topological space, an algebraic structure, a vector space. This will not be the exception.

Consider two points x, y in the full shift. Define $n(x,y) = \min\{k \in \mathbb{N} \colon x_k \neq y_k \text{ or } x_{-k} \neq y_{-k}\}$, and

$$d(x,y) = \frac{1}{2^{n(x,y)}}.$$

In Chapter 6 of [10] following results are proven.

Theorem 5.2.2. *Let $\mathcal{A}^{\mathbb{Z}}$ be a full shift then, the following are true,*

- *$(\mathcal{A}^{\mathbb{Z}}, d)$ is a compact metric space,*
- *the shift σ is a continuous function,*
- *the neighbourhood of a point x in the topology induced by d, have the structure $V(x,k) = \{y \in \mathcal{A}^{\mathbb{Z}} \colon y_{[-k,k]} = x_{[-k,k]}\}$*

From a heuristically point of view, the metric d tells us that two points of $\mathcal{A}^{\mathbb{Z}}$ are close if they agree on a large window around 0.

A *shift space* X, is a closed metric subspace of $\mathcal{A}^{\mathbb{Z}}$ such that $\sigma[X] \subseteq X$. On the other hand, consider a set of blocks $\mathcal{F}$ and consider the set $X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} \colon \text{ is } b \text{ if a subword of } x, b \notin \mathcal{F}\}$. Again, in Chapter 6 of [10] the following Theorem is proven.

Theorem 5.2.3. *For every shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ there is a set of words $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.*

This Theorem is of special interest to us since it gives a combinatorial perspective to a topologically defined object. And given this Theorem we can define a *Shift of Finite Type* (SFT) as a shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some finite set $\mathcal{F}$.

Consider two shift spaces (X, σ_1) and (Y, σ_2). An *homomorphism* between X and Y, is a continuous function $\phi : X \to Y$ such that $\phi(\sigma_1(x)) = \sigma_2(\phi(x))$. If ϕ is a surjective function, we say that X is a *factor*.

A *sofic* shift, is a shift X such that there is a factor $\phi_Y \to X$, where Y is a SFT. Obviously every SFT is a sofic shift.

One can also find combinatorial characterizations of SFT, sofic shifts and shift homomorphisims in [10]. For now, let us give two more (and final) results for shift spaces (also proven in [10]).

Proposition 5.2.4. *Let $\{X_i\}_{i \in S}$ be a family of sofic shifts. If S is a finite set then the shifts $\bigcup_{i \in S} X_i$ and $\bigcap_{i \in S} X_i$ are both sofic shifts.*

Proposition 5.2.5. *The set of periodic points of a sofic shift X, is dense in X.*

Finally, we mention Königs infinite tree Lemma which is proven in [5].

Theorem 5.2.6 (Königs infinite tree Lemma)**.** *If a denumerable point set of an infinite graph splits into denumerably many sets $E_1, E_2, \ldots$ which are finite and non-empty and such that every point of E_{n+1}, $n \in \mathbb{N}$, is joined with a point of E_n by an edge, then there is, in the graph, an infinite path $a_1, a_2, \ldots$ that contains from each set E_n a point a_n. Note that it is not necessary that the sets E_n be disjoint.*

5.3 Main result.

Remark 5.3.1. Consider a hereditary property $\mathcal{P}$ with set of minimal obstructions $F_{\mathcal{P}}$. If FO characterizes $\mathcal{P}$ through forbidden oriented graphs then, for any $D \in FO$, there is a graph $H \in F_{\mathcal{P}}$ such that the underlying graph of D is an induced subgraph of H.

Let $Q_O = (\{v_1, v_2, \ldots, v_n\}, A)$ be an orientation of the path P_n. If there are two positive integers, k and l, such that $Q_O[\{v_{k-l}, \ldots, v_k\}] \cong Q_O[\{v_k, \ldots, v_{k+l}\}]$, we will say that the orientation Q_O contains a *square* of the oriented path $Q_O[\{v_{k-l}, \ldots, v_k\}]$. If no such integers exists, we say that Q_O is *square-free* for any subpath.

Lemma 5.3.2. *Let $\mathcal{P}$ be a hereditary property such that $F_{\mathcal{P}} = \{C_n : n \in N_{\mathcal{P}}\}$ where $N_{\mathcal{P}} \subseteq \mathbb{N}$. Consider a finite set of forbidden oriented graphs FO for $\mathcal{P}$. If* $\max\{m \in \mathbb{N} :$ *there is an orientation of C_m in FO*$\} = k_0 \in \mathbb{N}$ *then, any FO-free orientation of a path Q is square-free for any subpath of size l, for $l \in \{m \in \mathbb{N}\colon\ m > k_0 + 1, m - 1 \in N_{\mathcal{P}}\}$.*

Proof. Let us proceed by contradiction. Consider an FO-free orientation, Q_O, of the path on n vertices. Suppose now that $Q_O[\{v_{k-l+1}, \ldots, v_k\}] \cong Q_O[\{v_k, \ldots, v_{k+l-1}\}]$ for some $l > k_0 + 1$ and $l - 1 \in N_{\mathcal{P}}$. Let $C_O = (u_0, \ldots, u_{l-1} = u_0)$ be the cycle on $l-1$ vertices with the following orientation: $u_i u_{i+1} \in A(C_O)$ if $v_{k+i} v_{k+i+1} \in A(Q_O)$ otherwise, $u_{i+1} u_i \in A(C_O)$ for $0 \leq i \leq l-2$. Since $l-1 \in N_{\mathcal{P}}$ then $C_{l-1} \notin \mathcal{P}$ thus, C_O is not FO-free. Now, $l > k_0+1$ so $l-1 > k_0$ and then $C_O \notin FO$. Hence there is a proper suborientation $H_O < C_O$ such that $H_O \in FO$. Consider a vertex $u_{-i} \in V(C_O) \setminus V(H_O)$

where $0 \leq i \leq l-2$, and the subpath $R = C_O[\{u_{-i+1}, \ldots, u_{-i+(l-2)}\}]$ where indices are taken $\bmod(l-1)$. Clearly $H_O < R$. Finally, consider the function

$$\varphi : V(R) \to V(Q_O)$$

$$\varphi(u_{-i+j}) = v_{k-i+j} \text{ for all } 1 \leq j \leq l-2.$$

It is not hard to notice that $\varphi : V_R \to V_{Q_O}$ is a well defined injective function with $\varphi[V(R)] \subseteq V(Q_O)[\{v_{k-l+1}, \ldots, v_{k+l-1}\}]$. Moreover, since $Q_O[\{v_{k-l+1}, \ldots, v_k\}] \cong Q_O[\{v_k, \ldots, v_{k+l-1}\}]$ and because of the construction of C_O, φ induces an injective full-homomorphism of oriented graphs $\varphi : R \to Q_O$. Hence $H_O \cong \varphi[H_O] < Q_O$ therefore, Q_O is not FO-free. ■

Consider the set of minimal obstructions $F_{\mathcal{P}}$ for the hereditary property $\mathcal{P}$, we will say that a graph G contains an *overlap* of $F_{\mathcal{P}}$ if there is a graph $H \in F_{\mathcal{P}}$ such that for every connected component H_i of H, $H_i < G$. Otherwise, we will say that G is *$F_{\mathcal{P}}$-overlap-free*.

Remark 5.3.3. Note that if G is $F_{\mathcal{P}}$-overlap-free then G is $F_{\mathcal{P}}$-free, and if $F_{\mathcal{P}}$ consists of connected graphs, then G is $F_{\mathcal{P}}$-overlap-free if and only if G is $F_{\mathcal{P}}$-free.

Suppose now that FO characterizes $\mathcal{P}$ through forbidden oriented graphs. We say that a graph G admits an *FO-overlap-free orientation* if there is an orientation G_O such that G_O is FO-overlap-free as a digraph.

Proposition 5.3.4. *Let $\mathcal{P}$ be a hereditary property closed under disjoint unions of graphs. Then, the following hold:*

- *$F_{\mathcal{P}}$ consists of connected graphs,*
- *G is $F_{\mathcal{P}}$-free if and only if it is $F_{\mathcal{P}}$-overlap-free,*
- *G admits an FO free orientation if and only if, G admits an FO-overlap free orientation.*

Proof. We will prove the first statement by contrapositive. Assume that there is a disconnected graph $H = (H_1, H_2) \in F_{\mathcal{P}}$, where there are no edges between H_1 and H_2. Then H_1,$H_2 \in \mathcal{P}$ but $H_1 + H_2 \notin \mathcal{P}$. Hence, by Remark 5.3.3, G is $F_{\mathcal{P}}$-free if and only if, it is $F_{\mathcal{P}}$-overlap-free. Let us prove the contrapositive of the last statement. Since every oriented graph D, free of a connected oriented graph D', D is D' overlap-free, we may assume that FO consists of disconnected oriented graphs. Suppose there is a graph G that admits an FO-free orientation, but not an FO-overlap-free orientation. Clearly then, $G \in \mathcal{P}$. Let $FO_G = \{L_O \in FO : |V(L_O)| \leq |V(G)|\}$, $k = |FO_G|$, and $l = \max\{n \in \mathbb{N}$: there is an oriented graph $L_O \in FO_G$ with n connected components$\}$. Consider the graph $H = G^{lk} = \sum_{i=1}^{lk} G$, and any orientation of it, H_O. Then, every connected component of H_O, is an orientation of G. Thus, every connected component of H_O contains and FO_G-overlap. Think of the elements in FO_G as the pigeonholes, and of each connected component of H_O as a pigeon. Hence, there must be an element $L_O \in FO_G$ that occurs as an overlap in l orientated copies of G in H_O. Since L_O has at most l connected components, then L_O occurs in H_O. Thus, $H \notin \mathcal{P}$. ■

Before proceeding to further results allow us to introduce some notation and definitions. Let us denote by $\mathcal{Q}$ and $\mathcal{Q}_O$ the set of non trivial paths and orientations of non trivial paths respectively. Consider now the following function,

$$c : \mathcal{Q}_O \to \{-1, 1\}^*$$

$$Q_O = (\{v_1, \ldots, v_m\}, A) \mapsto a_1 a_2 \ldots a_{m-1}$$

$$\text{where } a_i = \begin{cases} 1 \text{ if } v_i v_{i+1} \in A \\ -1 \text{ if } v_{i+1} v_i \in A. \end{cases}$$

In order to simplify notation, whenever there is no ambiguity, we will drop the O subscript notation that differentiates oriented from non oriented graphs.

Consider $l = \sum_{k=1}^{m} Q^k$ an orientation of a linear forest and let us define the shift space

$$X_l = \bigcup_{k=1}^{m} X_{c(Q^k)}, \text{ where } X_{c(Q^k)} \subseteq \{-1, 1\}^{\mathbb{Z}} \text{ is the } c(Q^k)\text{-free subshift.}$$

Suppose that $\mathcal{P}$ is a hereditary property for which $F_{\mathcal{P}}$ is a family of cycles. Then, by Remark 5.3.1, if FO defines $\mathcal{P}$ through forbidden oriented graphs we can assume that $FO = LF \cup C$ where LF is a family of oriented linear forests, and the elements of C are oriented cycles. Thus, for such a property, let us define the subshift $X_{\mathcal{P}} = \bigcap_{l \in LF} X_l$.

Lemma 5.3.5. *Let $\mathcal{P}$ be a hereditary property whose minimal obstructions are cycles. If there is a finite set of forbidden oriented graphs $FO = LF \cup C$ that characterizes $\mathcal{P}$ then, $X_{\mathcal{P}}$ is a non empty sofic shift.*

Proof. For each $l \in LF$ the shift X_l is a finite union of shifts of finite type (SFT), since SFT are sofic shifts, then by Proposition 5.2.4, each X_l is a sofic shift. By hypothesis $|LF| \in \mathbb{N}$, again by Proposition 5.2.4 we conclude that $X_{\mathcal{P}}$ is a sofic shift.

Let $\mathcal{Q}'_{\mathcal{O}}$ be the set of FO-overlap-free orientations of odd order paths, and consider the function

$$\varphi : \mathcal{Q}'_{\mathcal{O}} \to \{-1,1\}^{\mathbb{Z}}$$
$$Q = (v_1, \ldots, v_{2n+1}) \mapsto x, \text{ where}$$
$$x_i = -1,\ i \in \mathbb{Z} \setminus \{-n, \ldots, n\} \text{ and } x_{[-n,n]} = c(Q)$$

For $i \in \mathbb{Z}^+$, consider the following set, $S_i = \{\varphi(Q_i) : Q_i$ is an FO-overlap free orientation of the path on $2i+1$ vertices$\}$. By Proposition 5.3.4, for each $i \geq 1$, S_i is not an empty set. Define now the infinite graph $\mathfrak{G} = (\bigcup_{i\geq 1} S_i, E)$ where $xy \in E$ if and only if $x \in S_i$, $y \in S_{i+1}$ and $x_{[-2i,2i]} = y_{[-2i,2i]}$. It is important to notice that the vertices of $\mathfrak{G}$ are points of the full shift $\{-1,1\}^{\mathbb{Z}}$.

Claim 5.3.6. *For any $y \in S_{i+1}$ there is a point $x \in S_i$ such that $xy \in E$.*

Proof. Since $Q_y = c^{-1}(y_{[-2(i+1),2(i+1)]})$ is an FO-overlap-free orientation of the path on $2(i+1)+1$ vertices, and $Q'_y = c^{-1}(y_{[-2i,2i]}) < Q_y$ then, Q'_y is an FO-overlap-free orientation of the path on $2i+1$ vertices. Thus $\varphi(Q'_y) \in S_i$ and $\varphi(Q'_y)_{[-2i,2i]} = y_{[-2i,2i]}$. Therefore $\varphi(Q'_y)y \in E$ □

Using König's Lema (Theorem 5.2.6), we can find an infinite path $R = \{x_n\}_{n\geq 1}$ in $\mathfrak{G}$. By definition of the graph $\mathfrak{G}$, R induces a sequence in $\{-1,1\}^{\mathbb{Z}}$ such that $x_{n[-n,n]} = x_{m[-n,n]}$ for any $m \geq n \geq 1$. Hence, R is a convergent sequence in the full shift, and since $(\{-1,1\}^{\mathbb{Z}}, d)$, is a compact metric space (Theorem 5.2.2),

$$lim_{n\to\infty} x_n = x_p \in \{-1,1\}^{\mathbb{Z}}.$$

We now want to prove that $x_p \in X_{\mathcal{P}}$. Suppose the opposite, then there is a linear forest $l = \sum_{k=1}^{m} Q^k \in LF$ such that $x_p \notin X_l$. Then there is an integer r such that for all Q^k, $c(Q^k)$ appears as a subword of $x_{p[-r,r]}$. Since $x_{n[-n,n]} = x_{m[-n,n]}$ for any $m \geq n \geq 1$, and $lim_{n\to\infty} x_n = x_p$ then, $x_{n[-r,r]} = x_{p[-r,r]}$ for any $n \geq r$. Which contradicts the fact that $c^{-1}(x_{n[-n,n]})$ is an FO-overlap-free orientation of the path on $2n+1$ vertices.

Therefore $X_{\mathcal{P}}$ is a non empty sofic shift. ■

Lemma 5.3.7. *Let $\mathcal{P}$ be a hereditary property such that $F_{\mathcal{P}} = \{C_n : n \in N_{\mathcal{P}}\}$ where $N_{\mathcal{P}} \subseteq \mathbb{N}$ is such that for any positive integer m there is $k_m \in \mathbb{N}$ such that $mk_m \in N_{\mathcal{P}}$. If FO is a set of forbidden oriented graphs for $\mathcal{P}$ with $|FO| \in \mathbb{N}$ then there is an aperiodic shift space Y, such that $X_{\mathcal{P}} \subseteq Y$.*

Proof. Consider the fullshift $X = \{-1,1\}^{\mathbb{Z}}$, and let $k_0 = \max\{m \in \mathbb{N} :$ there is an orientation of C_m in $FO\}$. Consider the set of words $F_2 = \{ww : |w| \in N_{\mathcal{P}}$ and $|w| > k_0 + 1\}$, and the shift $Y = X_{F_2}$. Recall that $X_{\mathcal{P}} = \bigcap_{l \in LF} X_l$, where $X_l = \bigcup_{k=1}^{m} X_{c(Q^k)}$ and $l = \sum_{k=1}^{m} Q^k$. Consider a point $x \in X_{\mathcal{P}}$. By definition of $X_{\mathcal{P}}$, $c^{-1}(x_{[-l,l]})$ is an FO-free orientation of the path on $2l+1$ vertices for any $l > 0$. Hence, by Lemma 5.3.2, $c^{-1}(x_{[-l,l]})$ is square-free for any subpath of size t, for $t \in \{m > k_0 + 1 : m \in N_{\mathcal{P}}\}$. Thus, $x_{[-l,l]}$ is F_2 free for any $l \geq 1$. And since $x_{[0]} \in \{-1,1\}$ we conclude that $x \in Y$.

Consider a periodic point $x \in \{-1,1\}^{\mathbb{Z}}$ with period m and $k_m \in \mathbb{N}$ such that $mk_m \in N_{\mathcal{P}}$. Then $x_{[1,mk_m]} = x_{[mk_m+1,2mk_m]}$, so $x_{[1,2mk_m]} \in F_2$ and therefore $x \notin Y$. Concluding that $X_{\mathcal{P}} \subseteq Y$, where Y is an aperiodic subshift. ■

Theorem 5.3.8. *Let $\mathcal{P}$ be a hereditary property with set of minimal obstructions, $F_{\mathcal{P}} = \{C_n : n \in N_{\mathcal{P}}\}$, where $N_{\mathcal{P}} \subseteq \mathbb{N}$ is such that for any positive integer m there exists $k_m \in \mathbb{N}$ such that $mk_m \in N_{\mathcal{P}}$. If FO characterizes $\mathcal{P}$ through forbidden oriented graphs then, $|FO| = \aleph_0$.*

Proof. Once again, we will prove the statement by contradiction. Suppose there is a finite set of forbidden oriented graphs, FO, that characterizes $\mathcal{P}$. By Lemma 5.3.5 $X_{\mathcal{P}}$ is a non empty sofic shift. By Lemma 5.3.7 $X_{\mathcal{P}} \subseteq Y$ where Y is an aperiodic shift. But periodic points in sofic subshifts are a dense set (Proposition 5.2.5), since $X_{\mathcal{P}} \neq \varnothing$ there is a periodic point $x \in X_{\mathcal{P}} \subseteq Y$ which is our final contradiction. ■

Example 5.3.9. A property with an infinite set of minimal obstructions $F_{\mathcal{P}} = \{C_n : n \in N_{\mathcal{P}}\}$ such that there is an integer m, whose set of multiples $M_m \cap N_{\mathcal{P}} = \varnothing$ is the class of bipartite graphs P_B. Which from Proposition 5.1.2 we already know has a finite set of forbidden oriented graphs.

Corollary 5.3.10. *For neither of the properties, forests, chordal graphs, evenhole-free graphs there is a finite set of forbidden oriented graphs*

Remark 5.3.11. If $\mathcal{P}$ is a property such that $F_{\mathcal{P}} = \mathfrak{G}_{\mathcal{P}} \cup C$, where $\mathfrak{G}_{\mathcal{P}}$ is a finite set of minimal obstructions, and C is a family of cycles that satisfies the hypothesis of Theorem 5.3.8, one can do a little modification to Lemmas 5.3.2, 5.3.5, 5.3.7 and Proposition 5.3.4, to conclude that $F_{\mathcal{P}}$ cannot be characterized with a finite set of forbidden oriented graphs. For example, there is no set, FO, of forbidden oriented graphs such that $|FO| \in \mathbb{N}$ and FO characterizes the property of being a linear forest.

Proposition 5.3.12. *There are properties that can be characterized through a finite set of forbidden subpatterns but with no finite set of forbidden oriented graphs.*

Proof. From Example 1.3.6 we know that forests and chordal graphs can be characterized with a finite set of forbidden subpatterns. But neither of these can be characterized with a finite set of forbidden oriented graphs (Corollary 5.3.10). ■

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