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CONNECTIONS BETWEEN TWO INITIAL DIRICHLET TYPE PROBLEMS FOR PARABOLIC EQUATIONS

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Contents

Chapter 0

Introducción

Los primeros resultados de este trabajo tienen que ver con algunas relaciones entre la solubilidad del problema inicial L^p Dirichlet y una condición de L^q regularidad, en cilindros de la forma $\Omega_T = D\times(0,T) \subset \mathbb{R}^n\times\mathbb{R}$, donde D es un dominio Lipschitz acotado en $\mathbb{R}^n$, $1 < p, q < \infty$ y $T > 0$.

Para estos resultados requerimos que los coeficientes de los operadores parabólicos a considerar, sean independientes de la variable temporal. De esta manera, garantizamos que las soluciones a operadores adjuntos puedan obtenerse a partir de las soluciones al operador original vía una reflexión en la variable temporal (y recíprocamente).

Tanto el problema L^p Dirichlet y la condición de regularidad surgen una vez que el problema clásico de Dirichlet es planteado. El problema clásico de Dirichlet para un operador parabólico $\mathcal{L} = \operatorname{div}(A(x,t)\nabla) - \partial_t$ en un cilindro Lipschitz Ω_T, es el siguiente:

Dada un función en la frontera $f \in C(S_T)$, buscamos una función $u \in C^{2,1}$ cumpliendo:

$$\begin{cases} \mathcal{L}u(X) = 0 & X \in \Omega_T, \\ \lim\limits_{\substack{X\to Q \\ X\in\Omega}} u(X) = f(Q) & Q \in S_T. \end{cases}$$

Para una descripción completa de este problema, véase la sección 2.2. Se sabe que el problema clásico de Dirichlet tiene solución (véase [14]). Es natural preguntarse si podemos plantear y resolver un problema de Dirichlet en el que el dato en la frontera pertenezca a un espacio distinto de funciones.

El problema L^p Dirichlet, escrito $(D)_p$ de manera corta, se define pidiendo una condición que regule el comportamiento en la frontera de la solución al problema clásico. En consecuencia, dada un dato en la frontera g que pertenezca a L^p, la condición antes mencionada permite a la solución u alcanzar el dato en la frontera g en una manera similar al problema clásico (véase la sección 2.3).

El problema de Regularidad, escrito $(R)_p$ abreviadamente, requiere de otra condición que controle el comportamiento en la frontera del gradiente en las variables espaciales de la solución al problema clásico. Debido a esto, dado un dato en la frontera h en un espacio de Sobolev apropiadamente definido, la anterior condición permite a la solución u alcanzar al dato en la frontera h (véase la sección 2.3).

En la segunda parte establecemos un resultado ligeramente más débil en un dominio que varía con el tiempo. Probamos que la condición anteriormente mencionada de L^q regularidad implica la condición A_∞ de la medida parabólica, lo cual genera la solubilidad del problema L^p Dirichlet para cierta $p > 1$, no necesariamente cumpliendo $1/q + 1/p = 1$. En esta segunda parte, la restricción de la variable temporal no es necesaria.

Para la primera parte, inicialmente probamos que la condición $(R)_p$ que definimos en este contexto, implica una estimación para la norma $L^{p'}$ de la función maximal no tangencial de la solución al problema inicial de Dirichlet, donde $1 < p < \infty$, y $p' = p/(p-1)$. Este resultado es el análogo parabólico de [28, Theorem 5.4], y esencialmente significa que la solubilidad del problema inicial de L^p regularidad implica la solubilidad del problema inicial $L^{p'}$ Dirichlet en cilindros Lipschitz, con $1/p + 1/p' = 1$.

Esencial para este resultado son una desigualdad de Poincaré con peso (Lemma 3.2), una desigualdad reversa de Harnack local para una familia muy específica de funciones (Lemma 2.12 y Remark 3.5) y una técnica de *reflexión local en la variable temporal* del argumento de cierta familia de soluciones.

También establecemos un resultado recíproco parcial, el cual es una adaptación a ecuaciones parabólicas del resultado en [38]. Este resultado afirma que si $1 < p < \infty$, las condiciones $(D)_{p'}$ y $(R)_q$ juntas implican la condición $(R)_p$, siempre que $1 < q < p$.

Las adaptaciones que realizamos no son consecuencia inmediata del caso de ecuaciones elípticas. Por ejemplo, al contrario del operador elíptico en [28, 38], el operador parabólico no es autoadjunto y por la naturaleza evolutiva de las ecuaciones parabólicas, ciertas estimaciones básicas para soluciones, medida parabólica y función de Green tienen un *desplazamiento en la variable del tiempo* lo que requiere una argumentación distinta a la de ecuaciones elípticas.

Una característica particular en la prueba del resultado principal de la segunda parte, es el uso de otra forma de una desigualdad de Poincaré para funciones anulándose en una porción de la frontera. De esta manera, podemos evadir el uso de la bien conocida comparabilidad de la función de Green y la medida parabólica asociada al operador parabólico en dominios no cilíndricos (ver por ejemplo [37, Lemmata 2.8 and 2.9]).

Chapter 1

Introduction

The first results of this dissertation deal with some connections between the solvability of the initial L^p Dirichlet problem and a condition of L^q regularity, on cylinders of the form $\Omega_T = D \times (0, T) \subset \mathbb{R}^n \times \mathbb{R}$, where D is a bounded Lipschitz domain in $\mathbb{R}^n$, $1 < p, q < \infty$ and $T > 0$.

For these results we require the main coefficients of the parabolic operators to be independent of the time variable. This way, we guarantee that the solutions to adjoint equations may be obtained from solutions to the original equation via a reflection in the time variable (and conversely).

Both the L^p Dirichlet problem and a condition of L^q regularity arise once the Classical Dirichlet problem is posed. The Classical Dirichlet problem for $\mathcal{L} = \operatorname{div}(A(x,t)\nabla) - \partial_t$ in a Lipschitz cylinder Ω_T is the following:

Given a boundary data $f \in C(S_T)$, we look for a function $u \in C^{2,1}$ satisfying:

$$\begin{cases} \mathcal{L}u(X) = 0 & X \in \Omega_T \\ \lim\limits_{\substack{X \to Q \\ X \in \Omega}} u(X) = f(Q) & Q \in S_T. \end{cases}$$

For the complete description of this problem, see section 2.2. It is known that the Classical Dirichlet problem is solvable (see e.g. [14]). It is natural to ask if we can pose and solve a Dirichlet problem in which the boundary data belongs to a different space of functions.

The L^p Dirichlet problem, in short $(D)_p$, is defined by requiring a condition that restricts the boundary behavior of the solution to the Classical problem. Accordingly, given a boundary data g belonging to L^p, the previously mentioned condition allows the solution u to attain the boundary data g in a similar way to the Classical problem (see section 2.3).

The Regularity problem, in short $(R)_p$, requires another condition that controls the boundary behavior of the spatial gradient of the solution to the Classical problem. Likewise, given a boundary data h in a properly defined Sobolev space, the aforementioned condition enables the solution u to the boundary data h (see section 2.3).

In the second part we establish a slightly weaker result on a time-varying domain. We prove that the previously mentioned condition of L^q regularity implies the A_∞ property of parabolic measure, which yields the solvability of the L^p Dirichlet problem for certain $p > 1$, not necessarily satisfying $1/q + 1/p = 1$. In this second part, the time variable independence constraint is not needed.

For the first part, we first prove that the condition $(R)_p$ that we define in this setting, implies an estimate for the $L^{p'}$ norm of the non-tangential maximal function of a solution of the initial Dirichlet problem, where $1 < p < \infty$, and $p' = p/(p-1)$. This result is the parabolic analogue of [28, Theorem 5.4], and essentially means that the solvability of the initial L^p regularity problem implies the solvability of the initial $L^{p'}$ Dirichlet problem on Lipschitz cylinders, with $1/p + 1/p' = 1$.

Essential to this first result are a weighted Poincaré type inequality (Lemma 3.2), a local backward Harnack inequality for a very specific family of solutions (Lemma 2.12 and Remark 3.5) and a *local reflection in time variable* technique of the argument of certain family of solutions.

We also establish a partial reciprocal result, which is an adaptation to parabolic equations of the result in [38]. This result asserts that if $1 < p < \infty$, conditions $(D)_{p'}$ and $(R)_q$ together imply condition $(R)_p$, provided $1 < q < p$.

The adaptations we provide are not straightforward consequences from the situation for elliptic equations. For instance, unlike the elliptic operator in [28, 38], the parabolic operator is not self adjoint, and by the evolutionary nature of the parabolic equations, certain basic estimates for solutions, parabolic measure and Green's function have a *shift in the time variable* which requires different argumentations than those for elliptic equations.

The second part's main result is that a condition of L^q regularity implies the A_∞ property of parabolic measure, which yields the solvability of the L^p Dirichlet problem for certain $p > 1$, not necessarily satisfying $1/q + 1/p = 1$.

A particular technical feature in the proof of the last main result is the use of yet another particular form of Poincaré's inequality for functions vanishing on a portion of the boundary. In this way, we are able to avoid the use of a well-known comparability of Green's function and parabolic measure associated to the parabolic operator on non-cylindrical domains (see e.g. [37, Lemmata 2.8 and 2.9]).

1.1 Brief historical background

The goal of this section is to provide a brief and non exhaustive historical view of the subject studied in this thesis, making a strong emphasis on the work most closely linked to this thesis. We leave for the next chapter the precise definitions of some of the descriptions that we provide in the remaining of this chapter.

To understand the origins of the Dirichlet and regularity problem we can look at R. A. Hunt and R. L. Wheeden's paper [23]. They established a Fatou-type theorem for harmonic functions on a Lipschitz domain. Namely, on a domain $D \subset \mathbb{R}^n$ with Lipschitz (non-smooth) boundary, a bounded harmonic function has non-tangential boundary limit almost everywhere. Along the way they developed some fundamental estimates about the harmonic measure ω of starlike Lipschitz domains, and what is known as kernel function nowadays. Along the lines of their work, while proving these estimates, they actually established the solvability of a sort of $L^p(d\omega)$ Dirichlet problem.

Years later, in the setting of a Lipschitz domain $D \subset \mathbb{R}^n$, in the fundamental seminal work [9] B. E. J. Dahlberg showed the mutual absolute continuity of the surface measure σ and the harmonic measure ω. In connection to that, he also proved in [10] that the Poisson kernel, written in its form of Radon-Nikodym derivative $\kappa = \dfrac{d\omega}{d\sigma}$, belongs to the class $RH_2(\Delta)$ of *reverse Hölder weights*, for $\Delta \subset \partial D$ a small surface ball. This last condition turns out to be equivalent to the solvability of the L^2 Dirichlet problem.

In fact, the previous characterization implies the existence of $\epsilon_0 > 0$ such that for $2 - \epsilon_0 \leq p < \infty$ and $f \in L^p(\partial D, d\sigma)$ one has

$$\|(u)^*\|_{L^p(\Delta_r, d\sigma)} \lesssim \|f\|_{L^p(\partial D, d\sigma)},$$

where u is the solution to the Dirichlet's problem with boundary data f and $(u)^*$ is the maximal non-tangential function of u. A similar estimate holds in the case that D is a C^1 domain and for $1 < p < \infty$.

The regularity problem was first studied by D.S. Jerison and C.E. Kenig in [24]. They proved via the Green's theorem and a Rellich-type identity that the regularity problem for the Laplace equation in Lipschitz domains is valid for $p = 2$ (see [24, Theorem 3]).

It is worthwhile to mention that in [19], E.B. Fabes and S. Salsa prove the heat equation analogue of [9], that is, the caloric and surface measure on the lateral boundary of a Lipschitz cylinder $\partial D \times (0, \infty)$ are mutually absolutely continuous and the caloric Poisson kernel $\kappa = \frac{d\omega}{d\sigma}$ satisfies a reverse Hölder inequality of index 2 for small surface balls. This leads to the solvability of the initial-Dirichlet problem for the heat equation with given lateral data in L^p, $p \geq 2$.

Back to the Laplace's equation setting and D a C^1 domain, E.B. Fabes, M. Jodeit, Jr. and N.M. Riviére in [17] solved the L^p Dirichlet problem for $1 < p < \infty$, even providing a representation formula in the form of a double layer potential. Using this explicit form of the solution they were able to obtain gradient estimates near the boundary, when the boundary data has derivatives in $L^p(\partial D)$ for the same range $1 < p < \infty$. The similar problem on Lipschitz domains remained open for some time afterwards.

In [8], R.R. Coifman, A. Macintosh and Y. Meyer showed that the Cauchy Integral was a bounded operator for L^p, $1 < p < \infty$, on Lipschitz curves. Combining their result with some arguments found in [17], G. Verchota in [41] proved some Maximal functions and pointwise convergence results concerning layer potentials. For D, a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, the classical layer potentials for Laplace's equation were shown to be invertible operators in $L^2(\partial D, d\sigma)$ and various subspaces of $L^2(\partial D, d\sigma)$. For $1 < p \leq 2$ and data in $L^p(\partial D, d\sigma)$ with first derivatives in $L^p(\partial D, d\sigma)$ it was shown that there exists a unique harmonic function u that solves the Dirichlet problem for the given data and such that the non-tangential maximal function of the gradient ∇u belongs to $L^p(\partial D, d\sigma)$.

As far as the consequences of the solvability of the L^p regularity problem (that for shortness we denote by $(R)_p$), work of C. E. Kenig and J. Pipher [28] established several results concerning $(R)_p$ in the case of an elliptic operator $\mathcal{L}$ with non smooth coefficients over the n-dimensional unit ball B. The first one says that the solvability of $(R)_p$ depends only on the behavior of the coefficient matrix for $\mathcal{L}$ near the boundary:

Theorem 1.1. *[28, Theorem 5.1] Suppose that* $\mathcal{L}_0 = \mathrm{div}(A_0 \nabla u)$ *and* $\mathcal{L}_1 = \mathrm{div}(A_1 \nabla u)$ *and* $0 < \delta < 1$ *and* $A_0 \equiv A_1$ *in* $B \setminus B_\delta$*. Then* $(R)_p$ *is solvable for* L_1 *if* $(R)_p$ *is solvable for* L_0*.*

The second result is the analog of the fact that for the L^p Dirichlet problem, the solvability of $(D)_p$ for some $1 < p < \infty$ implies the solvability of $(D)_q$ for all $q \geq p$ and states the following:

Theorem 1.2. *[28, Theorem 5.2] Suppose that $(R)_p$ is solvable for $\mathcal{L}$ in B for some $1 < p < \infty$. Then $(R)_q$ is solvable for $(R)_q$ for $\mathcal{L}$ for all $1 < q \leq p$.*

The third result is also an analogue for the Dirichlet problem:

Theorem 1.3. *[28, Theorem 5.3] Suppose that $(R)_p$ is solvable for $\mathcal{L}$ in B for some $1 < p < \infty$. Then there exists and $\epsilon > 0$ such that $(R)_q$ is solvable for $\mathcal{L}$ for $p < q \leq p + \epsilon$.*

And the fourth result establishes a connection of the $(R)_q$ and the L^p Dirichlet problem:

Theorem 1.4. *If $(R)_p$ is solvable for $\mathcal{L}$, then $(D)_{p'}$ is solvable for $\mathcal{L}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

This relation is established by making use of some fundamental properties of solutions to obtain integral estimates for the gradient of a solution close to the boundary and then applying these estimates to a particular family of solutions, along with a known property relating the elliptic measure and the Green's function.

In this thesis, we establish the parabolic analogue of this last result (see Theorem 2.18). The nature of parabolic equations requires to pose a proper estimate for a Regularity problem that involves the change in time, one in which the tangential gradient in space as well a half order derivative in time direction of the solution are included. This and other issues mentioned earlier lead to some auxiliary inequalities (for instance Lemma 2.12) and new argumentation.

Another important relation between the regularity and the L^p Dirichlet problem was pointed out by Z. Shen in [38]. He proved what may be viewed as a partial converse of Theorem 1.3, again in the setting of an elliptic operator $\mathcal{L}$ with bounded measurable coefficients over a Lipschitz domain D.

Theorem 1.5. *[38, Main Theorem] Suppose that $(D)_{p'}$ is solvable for $\mathcal{L}$. Then either $(R)_p$ is solvable for $\mathcal{L}$ or $(R)_q$ is not solvable for any $1 < q < \infty$.*

The key ingredients for this theorem are a reverse Holder inequality for solutions vanishing locally at a small surface ball on the boundary and what can be called a good λ type of inequality.

A parabolic analogue is also established in this thesis in Theorem 2.19. This also requires to introduce some adaptations and auxiliary estimates (for instance Theorem 3.1).

We now describe an important source of ideas for the regularity condition that we adopt in this work. Credit should be given to works on Lipschitz cylinders as well as on time-varying non-cylindrical domains.

In the heat equation setting, R. Brown in [3] constructed a solution for the initial L^2 Dirichlet problem as a corollary of their solution to the initial Neumann problem. He also considered an initial Dirichlet problem in a Lipschitz cylinder with data in the space $H^{1,\frac{1}{2}}$ (see [3, Definition 2.12]), a space of functions with one spatial derivative and half of a time derivative in L^2, and he proved that the solution to this problem satisfies

$$\|\mathcal{N}(\nabla u)\|_{L^2(S_T,d\sigma)} \lesssim \|f\|_{H^{1,\frac{1}{2}}(\partial D,d\sigma)}.$$

In simple words, he proved $(R)_2$ for the heat equation in a Lipschitz cylinder. Brown's definition of $(R)_2$ is the first hint to our definition of a Regularity problem for a parabolic operator.

In [22], S. Hofmann and J. L. Lewis, Lewis considered the Dirichlet problem for the heat equation in domains $\Omega \subset \mathbb{R}^n$ given by regions above a time varying graph. They defined a regularity problem in which the boundary data belongs to a Sobolev space having a tangential spatial gradient and half of a time derivative in L^2. Solutions to this problem could be represented as caloric layer potentials and satisfied optimal regularity estimates. In [21], such regularity problem was proven uniquely solvable in a range $1 < p < p_0$ where p_0 depends only on the geometric features of the domain Ω and the dimension n.

Another weight property related to the RH_q classes is the so called A_∞ property. One may think of $A_\infty(\sigma)$ as a class of measures which enjoy a mutual absolute continuity with the surface measure σ. In fact it is well known that $\mu \in A_\infty(\sigma)$ if and only if $\mu \ll \sigma$ and the Radon-Nikodym derivative $\dfrac{d\omega}{d\sigma}$ belongs to RH_q for certain $1 < q < \infty$. In a sense, one may write $A_\infty(\sigma) = \bigcup_{q>1} RH_q$.

It is well known via some counterexamples, that the parabolic measure may be singular with respect to the surface measure, even in a smooth cylinder. And so a strategy to prove solvability of certain initial L^p Dirichlet problems is to establish that the parabolic measure belongs to $A_\infty(\sigma)$. Also, some recent criteria have been proved in order to establish this property, and these results involve quadratic expressions related to area integral estimates, or Littlewood-Paley theory (both adapted to the parabolic setting). The origins of this circle of ideas can be traced back to the work [27, 29] on equations of elliptic type (see also [12]). Moreover, in the reference [11] it has been established a connection with A_∞ and the solvability of the so called BMO-Dirichlet problem. The parabolic versions of these results are in [32, 33, 13].

The last result in this dissertation may be viewed as a continuation of these ideas. We establish that under the assumption that $(R)_q$ holds for some $1 < q < \infty$, one may obtain the A_∞ property of parabolic measure. Although this result is far from optimal on Lipschitz cylinders (because of the results we obtained in the first chapters of this thesis), as far as we know is the only result that relates the solvability of initial $(R)_q$ regularity problem with initial L^p Dirichlet problem on non-cylindrical domains. It also makes the point that the initial $(R)_q$ regularity problems may be closely related to quadratic expressions of Littlewood-Paley type.

Chapter 2

Preliminary definitions

2.1 Lipschitz cylinders and $Lip\left(1, \frac{1}{2}\right)$ cylinders.

2.1.1 Lipschitz cylinders.

Points in $\mathbb{R}^{n+1}$ will be denoted by $X = (x, t) = (x', x_n, t)$, where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \equiv \mathbb{R}^n$ is referred to as the *space variables* and $t \in \mathbb{R}$ is conceived as the *time variable*.

An open and bounded domain $D \subset \mathbb{R}^n$ is a *Lipschitz domain* if its boundary is given locally by Lipschitz functions. This means that for every $P = (p_1, \ldots, p_{n-1}, p_n) \in \partial D$, there is a new local coordinate system (x', x_n), $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, and with respect to these coordinates, one can find

- A function $\phi = \phi_P : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $|\phi(x') - \phi(y')| \leq m_P|x' - y'|$ for certain $m_P > 0$,
- A *rectangle of radius* $r > 0$ of the form $R = R(P, r) = \{(x', x_n) : |x_i - p_i| < r, \quad |x_n - p_n| < 2nm_p r\}$,

with the following significance. In this new local coordinate system (x', x_n), one has

(i) $2R \cap \partial D = 2R \cap \{(x', x_n) : x_n = \phi(x')\}$,

(ii) $2R \cap D = 2R \cap \{(x', x_n) : x_n > \phi(x')\}$,

where $2R$ is the rectangle concentric to R with twice its radius.

By compactness of ∂D we can choose a finite number of rectangles $R_1, \cdots, R_N$ with the same radius r_0 covering ∂D and a finite number of Lipschitz functions $\phi_1, \cdots, \phi_N$ satisfying the conditions above with an absolute and unique Lipschitz constant m. In fact one can always take $0 < r_0 < 1$. Once this constant is fixed, one can define local geometric objects within Lipschitz cylinders, whose definition we recall shortly.

An *infinite Lipschitz cylinder with constants m and r_0* is an open set of the form $\Omega = D \times \mathbb{R}$ where D is a Lipschitz domain with constants m and r_0 as described above.

2.1.2 $Lip\left(1, \frac{1}{2}\right)$ cylinders.

Fix $r_0 > 0$ and $m > 0$. A function $\psi : \mathbb{R}^n \to (-m\, r_0, m\, r_0)$ is said to satisfy the $Lip(1, 1/2)$ condition with constant m if

$$|\psi(x', t) - \psi(y', s)| \leq m\left(|x - y| + |t - s|^{\frac{1}{2}}\right)$$

for all $(x', t), (y', s) \in \mathbb{R}^n$. We define two very similar kind of non - cylindrical domains on which we will work. For the first one we follow [21, 22]. We define regions Ω above the graph of a compactly supported *function $\psi : \mathbb{R}^n \to \mathbb{R}$ of Lip(1,1/2) type* with Lipschitz constant $m > 0$. More precisely

$$\Omega = \left\{(x', x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : x_n > \psi(x', t)\right\}. \tag{2.1}$$

For the second type of non - cylindrical domain,

$$Z = \left\{X \in \mathbb{R}^{n+1} : |x_i| < r_0,\ i = 1, \cdots, n-1, \quad |x_n| < 2nmr_0, \quad t \in \mathbb{R}\right\}.$$

A domain $\Omega \subset \mathbb{R}^{n+1}$ is said to be $Lip(1, 1/2)$ cylinder with constants m and r_0 if there exists cylinders $Z_1, \cdots, Z_N$ and functions $\psi_1, \cdots, \psi_N$ satisfying the $Lip(1, 1/2)$ condition such that

(i) $2Z_i \cap \partial\Omega = 2Z_i \cap \{(x', x_n, t) : x_n = \psi_i(x', t)\}$,

(ii) $2Z_i \cap \Omega = 2Z_i \cap \{(x', x_n, t) : x_n > \psi_i(x', t)\}$,

where $2Z$ is the cylinder concentric to Z with twice its radius. We assume that $\operatorname{diam}(\Omega) = \sup_{\tau\in\mathbb{R}} \operatorname{diam}(\Omega \cap \{t = \tau\})$ is finite, where diam is the euclidean diameter. We also assume that there exists $x_0 \in \mathbb{R}^n$ such that $(x_0, \tau) \in \Omega$ for all $\tau \in \mathbb{R}$.

It is important to notice that a Lipschitz cylinder is a $Lip(1, 1/2)$ cylinder, since it suffices to take $\psi(x', t) = \psi(x)$ for every t.

2.1.3 Further definitions.

The following definitions apply to Lipschitz cylinders, *Lip* $(1, 1/2)$ cylinders and the region above the graph of a *Lip*$(1, 1/2)$ graph. We denote by S the *lateral boundary of* Ω, defined as $S = \partial D \times \mathbb{R}$.

If $T > 0$, we let Ω_T denote the bounded domain $\Omega \cap \{0 < t < T\}$ and by S_T we denote its lateral boundary $S \cap \{0 \leq t < T\}$. The *parabolic boundary of* Ω_T is defined by $\partial_p \Omega_T = S_T \cup (\Omega \cap \{t = 0\})$, and the *parabolic center of* Ω_T is defined as $\Xi = (0, T + 1)$, which will play a role in some future estimates. The fact that we will consider mainly solutions to $\mathcal{L}u = 0$ that vanish at $\Omega \times \{t = 0\}$, the *bottom of* Ω_T, is because of our interest to deal with *initial Dirichlet problems*.

On S or S_T we can consider the surface measure σ given by the product measure $d\sigma = d\widetilde{\sigma} \times dt$, where $\widetilde{\sigma}$ denotes the surface measure on the Lipschitz domain D and dt is the Lebesgue measure on $\mathbb{R}$.

For $r < r_0/10$ and $Q = (q, s) \in S$, where r_0 depends on the geometric features of Ω, we define the *Carleson boxes*, *surface balls* and *right and left corkscrew points* (in that order) as

$$\Psi_r(Q) = \left\{ X = (x', x_n, t) \in \Omega : \begin{array}{c} |x_i - q_i| < r, \, i = 1, \cdots, n-1 \\ \psi(x', t) < x_n < \psi(x', t) + 4nmr \\ |s - t| < r^2 \end{array} \right\},$$

$$\Delta_r(Q) = S \cap \overline{\Psi}_r(Q),$$

$$\overline{\mathcal{A}}_r(Q) = (q', \psi(q', s) + 6nmr, s + 2r^2), \quad \underline{\mathcal{A}}_r(Q) = (q', \psi(q', s) + 6nmr, s - 2r^2).$$

Sometimes we use Δ instead of $\Delta_r(Q)$ and by $k\Delta$ we mean the surface ball $\Delta_{rk}(Q)$. The *parabolic cubes in* $\mathbb{R}^{n+1}$ are defined by $\mathcal{Q}_r(X) = \left\{ Y = (y, s) \in \mathbb{R}^{n+1} : |x - y| < r, \; |t - s| < r^2 \right\}$, $0 < r < r_0$.

In order to define the conditions $(R)_p$ and $(D)_p$, we introduce the *non tangential approach regions* $\Gamma_\alpha(Q) = \{X \in \Omega : \delta(X, Q) \leq (1 + \alpha)\delta(X)\} \cap \Psi_{r_0}(Q)$. Here, $\delta(X) = \delta(X; S)$ is the parabolic distance from X to the lateral boundary S and is given by $\delta(X; S) := \inf_{Q \in S} \delta(X; Q)$, where the *parabolic distance* between $X = (x, t) \in \mathbb{R}^{n+1}$ and $Y = (y, s) \in \mathbb{R}^{n+1}$ is $\delta(X; Y) = |x - y| + |t - s|^{1/2}$.

The previous notations and definitions are of importance throughout this work. The following ones derive from these, but are only used in chapter 5.

For $X = (x, t) \in \Omega$ the *Carleson region adapted to* X, denoted by $\Psi(X)$, is defined as the Carleson region $\Psi_{\theta\delta(X)}(\widehat{X})$, where the center of this Carleson region $\widehat{X}$ is the vertical projection of X on $\partial\Omega$ and $\theta > 0$ is the constant from the definition of $Q(X)$ in 2.16.

For $Z = (z, \tau) \in \partial\Omega$ and $0 < \kappa < 1$ define *generalized* κ*-scaled Carleson regions* as

$$\kappa\Psi_r(Z) \equiv \left\{(x, x_n, t) \in \Omega : |x' - z'| < \kappa r,\ |t - \tau| < \kappa r^2,\ z_n < x_n < z_n + 4nmr\right\}. \tag{2.2}$$

For $0 < \epsilon < 1$, the ϵ*-top portion* of a Carleson region Ψ_r (or a κ-scaled Carleson region $\kappa\Psi_r$) is defined as $\Psi_r^\epsilon = \Psi_r \cap \{X \in \Omega : \delta(X) \geq \epsilon r\}$ (or $\kappa\Psi_r^\epsilon = \kappa\Psi_r \cap \{X \in \Omega : \delta(X) \geq \epsilon r\}$ respectively).

Observe that if $\Delta = \Delta_r(Q)$ or $X \in \Omega$, then the notations $\Psi = \Psi(\Delta)$, $\overline{\mathcal{A}}(\Delta)$, $\overline{\mathcal{A}}(\Psi)$, $\kappa\Psi(\Delta)$, $\kappa\Psi(X)^\epsilon$, etc. still make sense with an obvious meaning. For X we make the convention of writing $\overline{\mathcal{A}}(X) = \overline{\mathcal{A}}(\Psi(X))$.

Also, for $0 < r < r_0$ define the *local non tangential approach region* as $\Gamma_\alpha^r(Q) = \Gamma_\alpha(Q) \cap \{X \in \Omega : \delta(X, Q) < r\}$. The aperture $\alpha > 0$ is chosen and fixed in such a way that $\Gamma_\alpha(Q) \subset \Omega$ for every $Q \in \partial\Omega$ (this can be done because of the Lip(1,1/2) property of $\partial\Omega$), and so we simply use the notation $\Gamma(Q)$ or $\Gamma^r(Q)$ for these non-tangential regions.

2.2 Solutions.

The functions we will consider are solutions to operators $\mathcal{L} = \operatorname{div}(A(x,t)\nabla) - \partial_t$, where ∇ denotes the gradient with respect to space variables only, $\partial_t = \dfrac{\partial}{\partial t}$, and where the coefficients of $A(x, t)$ form a symmetric matrix of functions $(a_{i,j})$ which are assumed to be smooth or infinitely differentiable, and satisfy the ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x,t)\xi_j\xi_i \leq \frac{1}{\lambda}|\xi|^2 \quad \text{for all } (x,t) \in \mathbb{R}^{n+1} \text{ and } \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \tag{2.3}$$

and where $|\xi|$ denotes the Euclidean norm of ξ.

The smoothness assumption for the coefficients is adopted in order to have a well defined concept of solution to the equation $\mathcal{L}u = 0$, and the only quantitative information that will arise in the constants of the results and estimates invoked, is the ellipticity constant λ, the dimension, and geometric constants of the domain D. This will potentially make it possible to use limiting arguments to extend the results to weaker notions of solutions (see e.g. [5]). One further assumption is that solutions to $\mathcal{L}u = 0$ are always supposed to vanish at the bottom of Ω, hence we will only be considering initial Dirichlet problems.

We emphasize that for the results concerning some connections between the initial L^p Dirichlet problem and a condition of L^q regularity, namely Theorems 2.18 and 2.19, we have assumed that the main coefficients of the operator $\mathcal{L}$ are independent of t. To justify this assumption, recall the definition of adjoint solutions associated to $\mathcal{L}$. These are functions $v \in C^{2,1}$ which are solutions to the equation $\mathcal{L}^* v = 0$, where $\mathcal{L}^* = \operatorname{div}(A(x)\nabla) + \partial_t$.

With the change of variable $t \to -t$ one can see that if $\mathcal{L}u = 0$ on Ω then $v(x,t) = u(x,-t)$ is solution to $\mathcal{L}^* v = 0$ on Ω. And even tough $\mathcal{L}^*$ has the same ellipticity than $\mathcal{L}$, they may be very different operators. A couple of times in our argumentations we employ auxiliary adjoint solutions arising from the application of this *reflection mapping* and compare them in the adjoint variable with the Green's functions for $\mathcal{L}$ in Ω. This is also why the proofs of theorems 2.18 and 2.19 work on cylindrical domains.

2.2.1 Parabolic and surface measure.

It is well known that a Lipschitz cylinder is a regular domain for any parabolic operator that satisfies condition (2.3) (this may be obtained using parabolic capacity, see e.g. [14]). That is, for every $f \in C(S_T)$ there exists a unique solution $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ such that

$$\begin{cases} \mathcal{L}u(X) = 0 & X \in \Omega \\ \lim\limits_{\substack{X \to Q \\ X \in \Omega}} u(X) = f(Q) & Q \in S_T. \end{cases} \tag{2.4}$$

From this, through the Riesz Representation Theorem, we can derive the notion of the parabolic measure. The *$\mathcal{L}$-parabolic measure at $X \in \Omega$* is the unique Borel measure $\omega^X = \omega^X_{\mathcal{L}}$ defined on S_T such that the solution to (2.4) is represented at $X \in \Omega$ by

$$u(X) = \int_{S_T} f(Q)\, d\omega^X(Q). \tag{2.5}$$

Define the surface measure for any Borel set $F \subseteq S$ by setting

$$\sigma(F) = \int_F d\sigma_t dt, \tag{2.6}$$

where σ_t is the $(n-1)$-dimensional Hausdorff measure of $F_t \equiv F \cap (\mathbb{R}^n \times \{t\})$, and dt denotes integration with respect to 1-dimensional Hausdorff measure. Observe that in particular σ is finite on surface cubes $\Delta \subset \partial\Omega$, and in fact if Δ has radius $0 < r \le r_0$ then $\sigma(\Delta) \approx r^{n+1}$. In this work we will only consider the **initial Dirichlet problem**, that means that the boundary data vanishes at $t = 0$.

2.2.2 Green Function.

The Green function of $\mathcal{L}$ on Ω with pole at $X = (x,t) \in \Omega$ is denoted by $G(X;Y)$ and defined as

$$G(X;Y) = \Gamma(X;Y) - \int_{S_T} \Gamma(Z;Y)\, d\,\omega_X(Z), \tag{2.7}$$

where $\Gamma(X;Y)$ is the fundamental solution of $\mathcal{L}$ (see e.g. [1] or [37]).

2.2.3 Properties of solutions.

Now we state some fundamental theorems of the theory, which are quite useful for our purposes. The constants playing a role in each of the following results depend only on the ellipticity constant, the dimension and the geometric features of the Ω, such as m, the Lipschitz constant of D.

Theorem 2.1 (Harnack's Inequality)**.** *[31, Theorem 2][18, Theorem 1] Let u be a non-negative solution of $\mathcal{L}u = 0$ in Ω_T. Let D' be a convex subdomain of D such that $\delta = dist(D', \partial D) > 0$. Then for all $x, y \in D'$ and $0 < s < t \leq T$ we have*

$$u(y,s) \leq u(x,t) \exp\left[c\left(\frac{|x-y|^2}{t-s} + \frac{t-s}{R} + 1\right)\right],$$

where $c = c(n,\lambda)$ and $R = min\{1, s, \delta^2\}$.

Theorem 2.2 (Carleson estimate)**.** *[15, Theorem 0.3][18, Theorem 2] Let $Q = (q,s) \in S$ and $0 < r < \min\{r_0, \sqrt{s}\}$. Then for any non-negative solution of $\mathcal{L}u = 0$ in Ω vanishing continuously on $\Delta(Q, 2r)$, we have*

$$\sup_{\Psi_r(Q)} u \leq c\, u(\overline{\mathcal{A}}_r(Q)),$$

where the constant $c = c(n, \lambda, m, r_0) > 0$.

Theorem 2.3. *[26, Lemma 1.1] Let $Q = (q,s) \in S_T$ and $0 < r < \frac{1}{2}\min\{r_0, \sqrt{s}, \sqrt{T-s}\}$. Then, for each $X \in \Psi_{\frac{r}{2}}(Q)$ we have*

$$\omega^X(\Delta_r(Q)) \geq c,$$

where $c = c(n, \lambda, m) > 0$.

In the next two Theorems, G denotes the Green function in the domain $D \times \{-1 < t < T+2\}$.

Theorem 2.4. *[15, Theorem 1.4] Let $Q = (q, s) \in S_T$ and $0 < r < \frac{1}{2}\min\{r_0, \sqrt{s}, \sqrt{T-s}\}$. Then, for each $X = (x, t) \in \Omega_T$ with $s + 4r^2 \le t \le T$ we have*

$$c^{-1} r^n G(X; \overline{\mathcal{A}}_r(Q)) \le \omega^X(\Delta_r(Q)) \le c\, r^n G(X; \underline{\mathcal{A}}_r(Q)),$$

where $c = c(n, \lambda, m, r_0, T) > 0$.

It's worth noting that Theorem 2.4 differs from its elliptic analog in the order of the argument variables (see [6, Lemma 2.2]).

Theorem 2.5. *[15, Corollary 2.3] Let $Q = (q, s) \in S_T$ and $0 < r < \frac{1}{2}\min\{r_0, \sqrt{s}, \sqrt{T-s}\}$. Then we have*

$$c^{-1} \le \frac{G(\Xi; \underline{\mathcal{A}}_r(Q))}{G(\Xi; \overline{\mathcal{A}}_r(Q))} \le c,$$

where $c = c(n, \lambda, m, r_0, T) > 0$.

Theorem 2.6 (Local comparison principle)**.** *[15, Theorem 1.6] Let $Q \in S_T$ and u, v be two positive solutions of $\mathcal{L}u = 0$ in $\Psi_{2r}(Q)$ with $u = v = 0$ continuously on $\Delta_{2r}(Q)$. Then, for each $X \in \Psi_{\frac{r}{8}}(Q)$ and $0 < r < \frac{1}{2}\min\{r_0, \sqrt{s}, \sqrt{T-s}\}$, we have*

$$\frac{u(X)}{v(X)} \le c\frac{u(\overline{\mathcal{A}}_r(Q))}{v(\underline{\mathcal{A}}_r(Q))},$$

where the constant $c = c(n, \lambda, m, r_0) > 0$.

Theorem 2.7 (Hölder continuity)**.** *[37, Theorem 1.3] Let $\mathcal{L}u = 0$ in Ω. Then there exists $\alpha = \alpha(n, \lambda) > 0$ such that $u \in C^\alpha(\Omega)$. Furthermore, if $Q_{2r}(x, t) \subset \Omega$ and $(z, w), (y, s) \in Q_r(x, t)$, then*

$$|u(z, w) - u(y, s)| \le \left(\frac{|z - y|}{r} + \frac{|w - s|}{r^2}\right)^\alpha r \left(\fint_{Q_r(x,t)} |\nabla_y u|^2\, dy\, ds\right)^{\frac{1}{2}}.$$

From now on, we will use the notation $\lesssim$ or $\gtrsim$ to write an inequality where the constants involved depend only on known features such as dimension, ellipticity and the Lipschitz character of Ω.

From Theorem 2.7 we can derive Hölder continuity at the boundary for solutions to $\mathcal{L}u = 0$ that vanish continuously on $\Delta_{3r}(Q)$ for some $Q \in S$.

Proposition 2.8. *Let $\mathcal{L}u = 0$ in Ω vanishing on $\Delta_{3r}(Q)$ for some $Q \in S$. Then there exists $\alpha = \alpha(n, \lambda) > 0$ such that*

$$|u(X)| \lesssim \left(\frac{\delta(X)}{r}\right)^\alpha r \left(\fint_{\Psi_{2r}(Q)} |\nabla_y u|^2\, dy'\, dy_n\, d\rho\right)^{\frac{1}{2}}, \tag{2.8}$$

for every $X \in \Psi_r(Q)$.

Proof. First consider the case when $\Omega = \mathbb{R}^{n+1}_+$ (we will use the + super-index to indicate that we work with the Carleson box and the surface ball in the setting of $\mathbb{R}^{n+1}_+$) and let $\widetilde{u}$ be the function obtained by reflecting u across $x_n = 0$ as an odd function of x_n. Then $\widetilde{\mathcal{L}}\widetilde{u} = 0$ where $\widetilde{\mathcal{L}}$ has the same ellipticity than $\mathcal{L}$. If $(x', x_n, t) \in \Psi_r(q', 0, s)$, applying Theorem 2.7 to $\widetilde{u}$, we get

$$|\widetilde{u}(x', x_n, t) - \widetilde{u}(x', 0, t)| \leq \left(\frac{x_n}{r}\right)^{\alpha} r \left(\fint_{Q_{2r}(q',0,s)} |\nabla_y \widetilde{u}|^2 \, dy' \, dy_n \, d\rho \right)^{\frac{1}{2}}.$$

With a change of variables and the definition of $\widetilde{u}$

$$|u(x', x_n, t)| \leq \left(\frac{x_n}{r}\right)^{\alpha} r \left(\fint_{\Psi^+_{2r}(q',0,s)} |\nabla_y u|^2 \, dy' \, dy_n \, d\rho \right)^{\frac{1}{2}}. \tag{2.9}$$

Now, if Ω is a Lipschitz cylinder and u is a solution of $\mathcal{L}u = 0$ vanishing on $\Delta_{3r}(Q)$ for some $Q = (q', \psi(q', s), s)$ we let $v(x', x_n, t) = u(x', x_n + \psi(x', t), t)$ for each $(x', x_n, t) \in \Psi_{3r}(Q)$. Then $\overline{\mathcal{L}}v = 0$ in $\Psi^+_{3r}(q', 0, s)$ where $\overline{\lambda} = \overline{\lambda}(\lambda, m)$ and v vanishes on $\Delta^+_{3r}(q', 0, s)$. From (2.9), for each $(x', x_n, t) \in \Psi_r(q', 0, s)$, we have

$$|v(x', x_n, t)| \lesssim \left(\frac{x_n}{r}\right)^{\alpha} r \left(\fint_{\Psi^+_{2r}(q',0,s)} |\nabla_y v|^2 \, dy' \, dy_n \, d\rho \right)^{\frac{1}{2}}.$$

Putting $X = (x', x_n + \psi(x', t), t) \in \Psi_r(Q)$, by a change of variables we obtain Proposition 2.8. □

There is also a version of the Theorem 2.9 adapted for solutions vanishing on a portion of the boundary. The form of this result that we now present may be found in [30, Lemma 5], and the ideas for its proof are the same as sketched therein.

Theorem 2.9. *Let $\mathcal{L}u = 0$ in Ω vanishing on $\Delta_{2r}(Q)$ for some $Q \in S$. Then there exist $\alpha = \alpha(n, \lambda) > 0$ and $C = C(n, \lambda) > 0$ such that*

$$|u(X)| \lesssim \left(\frac{\delta(X)}{r}\right)^{\alpha} \sup_{\Psi_r(Q)} u, \tag{2.10}$$

for every $X \in \Psi_r(Q)$.

Some of the results stated above have a counterpart that holds for adjoint solutions, that is, solutions to $\mathcal{L}^* v = 0$, where $\mathcal{L}^* v = \operatorname{div} A \nabla v + \partial_t v$. If needed, we will explicitly mention each of the results that we may use, and at this point we also mention it as an instance of the *reflecting technique* that we adopt in a couple of results later in the thesis.

Proposition 2.10. *Let v be a non-negative solution of $\mathcal{L}^*v = 0$ in Ω_T. Let D' be a convex subdomain of D such that $\delta = dist(D', \partial D) > 0$. Then for all $x, y \in D'$ and $0 < s < t \leq T$ we have*

$$v(x,t) \leq v(y,s)\exp\left[c\left(\frac{|x-y|^2}{t-s} + \frac{t-s}{R} + 1\right)\right],$$

where $c = c(n, \lambda)$ and $R = min\{1, s, \delta^2\}$.

Proof. Define $u(x,t) = v(x,-t)$ and apply Proposition 2.1. □

Theorem 2.11. *Let $Q \in S_T$ and v_1, v_2 be two positive solutions of $\mathcal{L}^*v = 0$ in $\Psi_{2r}(Q)$ with $v_1 = v_2 = 0$ continuously on $\Delta_{2r}(Q)$. Then, for each $X \in \Psi_{\frac{r}{8}}(Q)$ and $0 < r < \frac{1}{2}\min\{r_0, \sqrt{s}, \sqrt{T-s}\}$, we have*

$$\frac{v_1(X)}{v_2(X)} \leq c\frac{v_1(\underline{\mathcal{A}}_r(Q))}{v_2(\overline{\mathcal{A}}_r(Q))},$$

where the constant $c = c(n, \lambda, m, r_0) > 0$.

Proof. Define $u_i(x,t) = v_i(x,-t)$ for $i = 1, 2$ and $-T < t < 0$. Then u_i are positive solutions to $\mathcal{L}u = 0$ in $D \times (-T, 0)$. Inserting u_i in Theorem 2.6, we deduce the desired estimate. □

Finally, we include an adapted version of what might as well be called *local backward Harnack inequality*.

Lemma 2.12. *[2, Theorem 1][7, Theorem 13.7] Let Ω_T be a Lipschitz cylinder. Pick $Q = (q, s) \in S_T$ with $0 < r < \frac{1}{2}\min\left\{r_0, \sqrt{s}, \sqrt{T-s}\right\}$. Assume u is a non-negative solution to $\mathcal{L}u = 0$ in $\Psi_{2r}(Q)$ which continuously vanishes on $\Delta_{2r}(Q)$. Then, for $0 < \rho \leq \frac{1}{2}r$ we have*

$$u(\overline{\mathcal{A}}_\rho(Q)) \lesssim \left(1 + \frac{M_r}{m_r}\right)u(\underline{\mathcal{A}}_\rho(Q)),$$

where $M_r = \sup_{\Psi_{2r}(Q)} u$ and $m_r = u(\underline{\mathcal{A}}_r)$.

Lemma 2.12 may not seem very strong in general, but for very specific solutions this inequality does not depend on u or r, which will prove to be useful later on, as shown in the next section.

2.3 Conditions $(D)_p$ and $(R)_p$

We say that *the L^p Dirichlet problem has solution for $\mathcal{L}$ in Ω_T*, denoted by $(D)_p$, if the solution to (2.4) satisfies also

$$\|(u)^*\|_{L^p(S_T)} \leq c\|f\|_{L^p(S_T)}, \quad \text{where the constant } c > 0 \text{ is independent of } f, \tag{2.11}$$

and where $(u)^*$ is the *non tangential maximal function* of u given by

$$(u)^*(Q) = \sup_{X\in\Gamma(Q)} |u(X)|. \tag{2.12}$$

Estimate (2.11) guarantees that the solution u attains the boundary data f as established in the following Theorem.

Theorem 2.13. *[37, Theorem 6.3] Suppose $(D)_p$ holds for $\mathcal{L}$ for some $1 < p < \infty$. Then given $f \in L^p(S_T)$, there exists a unique u, $\mathcal{L}u = 0$ in Ω_T such that,*

i) $\lim_{t\to 0^+} u(x,t) = 0$ *uniformly on compact sets of* Ω_T.

ii) *For almost every* $Q \in S_T$,

$$\lim_{\substack{X\to Q \\ X\in\Gamma(Q)}} u(X) = f(Q).$$

iii) $u^* \in L^p(S_T, d\sigma)$ *and estimate* (2.11) *holds for* u *and* f.

The regularity condition associated to the Dirichlet problem that we describe, is meant to generalize the problems for the heat equation considered in [3, 4, 21, 22].

Based on the definitions for similar problems for elliptic equations from [28] and the heat equation in [3], we now define a regularity condition for the initial Dirichlet problem associated to $\mathcal{L}u = 0$ on the Lipschitz cylinder Ω_T. Given the nature of a parabolic equation we will use a Sobolev-type space over S with the usual derivatives in the (space) tangent directions, and a half order derivative in time direction.

The mixed norm space $W_p^{1,\frac{1}{2}}(S_T)$ is defined as the closure of the set

$$\left\{ g = f|_{S_T} : f \in C_0^\infty(\mathbb{R}^n \times (0,\infty)) \right\},$$

with respect to the norm

$$\|f\|_{W_p^{1,\frac{1}{2}}(S_T)} = \|f\|_{L^p(S_T)} + \|f\|_{1,\frac{1}{2},p}\,,$$

where

$$\|f\|_{1,\frac{1}{2},p} = \left(\int_{S_T} (|\nabla_{tan} f|^p)d\,\sigma\,dt + \int_{S_T} |\partial_t^{\frac{1}{2}} f|^p \,d\sigma \right)^{\frac{1}{p}}, \tag{2.13}$$

and where $\nabla_{tan} f = \nabla f - \nu(\nabla f \cdot \nu)$ is the *tangential gradient* of f, ∇f is the spatial gradient of f, ν is the exterior normal unit vector to ∂D and

$$\partial_t^{\frac{1}{2}} f(x,s) = \left(\int_{-\infty}^{T} \frac{|f(x,s) - f(x,t)|^2}{|s-t|^2}\, dt \right)^{\frac{1}{2}}.$$

This definition is taken from [39, p. 1034].

We say that the $(R)_p$ *condition holds for* $\mathcal{L}$ *in* Ω_T whenever the following estimate holds

$$\|\mathcal{N}(\nabla u)\|_{L^p(S_T)} \leq c\|f\|_{1,\frac{1}{2},p}, \tag{2.14}$$

for each $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and u the corresponding solution to (2.4).

We say that the $(R)_p$ *problem is solvable for* $\mathcal{L}$ *in* Ω_T whenever

$$\|(u)^*\|_{L^p(S_T)} + \|\mathcal{N}(\nabla u)\|_{L^p(S_T)} \leq c\|f\|_{1,\frac{1}{2},p}\,, \tag{2.15}$$

for each $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ regarded as an element of $W_p^{1,\frac{1}{2}}(S_T)$ and u the corresponding solution to (2.4).

Here, the *modified non-tangential maximal function* of a continuous function v defined on Ω is defined as

$$\mathcal{N}v(Q) = \mathcal{N}_\alpha v(Q) = \sup_{X \in \Gamma_\alpha(Q)} \left(\fint_{Q(X)} |v|^2 \, dY \right)^{\frac{1}{2}}, \tag{2.16}$$

where $Q(X) \equiv Q_{c^{-1}\delta(X)}(X)$ is such that $Q(X) \subset \Omega$ and $\alpha = \alpha(m) > 0$ is fixed. We are adopting the notation $\fint_{Q(X)} v\, dY$ for the *integral average* $\frac{1}{\delta(X)^{n+1}} \int_{Q(X)} v\, dY$. Integral averages with measure different to the Lebesgue measure on $\mathbb{R}^{n+1}$ will be used later on in this work, and its meaning should be clear from the context. Also, in later arguments $\mathcal{N}(\nabla u)$ will be used instead of $\mathcal{N}(|\nabla u|)$.

Remark 2.14. *Given $\alpha, \beta > 0$, using standard arguments (see e.g. [40, §6.2-6.4] or the original argumentation in [20, p. 166]) one may prove that for any function h defined on Ω_{T+1}*

$$\|\mathcal{N}_\alpha(h)\|_{L^p(S_T)} \approx \|\mathcal{N}_\beta(h)\|_{L^p(S_T)}.$$

This will become useful when proving Theorem 2.18 that we state below.

Remark 2.15. *In the process of solving an initial L^2 regularity problem for the heat equation, the definition of the mixed norm space $W_p^{1,\frac{1}{2}}(S_T)$ is meant to generalize the space adopted in [3, p. 352-353]. Another extension is adopted in [4], using parabolic Riesz potentials, following a definition from [16]. This extension is also adopted by [21, 22] in the setting of non-cylindrical domains. In any case, for $p = 2$ the definitions coincide, by Plancherel's theorem, as observed for instance in [21, p. 353].*

Based on the norm (2.13) we can establish a regularity estimate very similar to condition $(R)_p$ and inspired on [3, Theorem 6.1]. With this condition we are able to obtain a solution for every boundary data in $W_p^{1,\frac{1}{2}}(S_T)$ with non tangentially boundary convergence. For that purpose first we need the following uniform estimate for solutions.

Lemma 2.16. *Let v be a solution to $\mathcal{L}v = 0$ and $\Omega' \subset\subset \Omega_T$. Then for $p > 1$,*

$$\sup_{\Omega'} |v|^p \lesssim \fint_{\Omega'} |v|^p \, dX.$$

Proof. Consider the covering $\{Q^X\}_{X\in\Omega'}$ where each cube has the form $Q^X = Q_{\kappa\delta(X)}(X)$ and $0 < \kappa < 1$ depends on $\mathrm{dist}(\Omega', \partial\Omega_T)$. By the Besicovitch covering theorem (see [25, p. 483]) we can find a sequence of cubes $\{Q_i\}_{i=1}^{\infty}$ with finite overlap covering Ω'. By local boundedness (see [31, Theorem 3]) we have,

$$\sup_{\Omega'} |v|^p \leq \sum_{i=1}^{\infty} \sup_{Q_i} |v|^p \lesssim \sum_{i=1}^{\infty} \fint_{\widetilde{Q}_i} |v|^p \, dY,$$

where every $\widetilde{Q}_i$ is a cube contained in Q_i and with the appropriate time shift such that $|\widetilde{Q}_i| \approx |Q_i|$. Since $\Omega' \subset\subset \Omega_T$, the radii of the cubes Q_i and $\widetilde{Q}_i$ are uniformly bounded below and therefore $|\widetilde{Q}_i| \approx |Q_i| \approx |\Omega'|$. With this in mind,

$$\sup_{\Omega'} |v|^p \lesssim \frac{1}{|\Omega'|} \sum_{i=1}^{\infty} \int_{\widetilde{Q}_i} |v|^p \, dY \lesssim \frac{1}{|\Omega'|} \int_{\Omega'} |v|^p \left(\sum_{i=1}^{\infty} \chi_{\widetilde{Q}_i} \right) dY \lesssim \frac{1}{|\Omega'|} \int_{\Omega'} |v|^p \, dY,$$

where the last inequality is due to the finite overlap property inherited from the family of cubes $\{Q_i\}_{i=1}^{\infty}$. □

Theorem 2.17. *Suppose that for some $1 < p < \infty$, $(R)_p$ is solvable for $\mathcal{L}$ in Ω_T. Then given $f \in W_p^{1,\frac{1}{2}}(S_T)$, there exists a unique u, $\mathcal{L}u = 0$ in Ω_T such that,*

i) For almost every $Q \in S_T$,

$$\lim_{\substack{X\to Q \\ X\in\Gamma(Q)}} u(X) = f(Q).$$

ii) $\mathcal{N}(\nabla u) \in L^p(S_T, d\sigma)$ and estimate (2.15) holds for u and f.

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of smooth functions converging to f in $L^p(S_T)$ and in $W_p^{1,\frac{1}{2}}(S_T)$. After passing to a subsequence we can assume that f_k converges to f almost everywhere on S_T. Let $\{u_k\}_{k=1}^{\infty}$ be the corresponding sequence of solutions to (2.4) with boundary data f_k.

For $0 < \epsilon < \operatorname{diam}(\Omega_T)/2$ define the *non tangential truncated region*

$$\Gamma^\epsilon(P) = \Gamma(P) \setminus \{x \in D : |x - P| < \epsilon\}$$

and the corresponding non tangential truncated maximal function

$$(u)^*_\epsilon(P) = \sup\{|u(x)| : x \in \Gamma^\epsilon(P)\}.$$

We will demonstrate that $\{u_k\}_{k=1}^\infty$ is uniformly Cauchy in every open set compactly contained in Ω_T with an estimate that allows us to prove that the limit function will be a solution to $\mathcal{L}u = 0$. Let $\Omega \subset\subset \Omega_T$ be an open set and $\delta' = \operatorname{dist}(\Omega', \partial\Omega_T)$. For $P \in S_T$ we have,

$$\int_{S_T} |(u_k - u_l)^*(P)|^p d\sigma \geq \int_{S_T} \frac{1}{|\Gamma(P)|} \int_{\Gamma(P)} |(u_k - u_l)(X)|^p \, dX \, d\sigma \geq \frac{1}{|\Omega_T|} \int_{S_T} \int_{\Gamma^\epsilon(P)} |(u_k - u_l)(X)|^p \, dX \, d\sigma.$$

Now we make two observations:

a) For $X \in \Gamma^\epsilon(P)$ the set $\mathcal{B}_\epsilon(X) = \{P \in S_T : X \in \Gamma^\epsilon(P)\}$ has measure $\sigma(\mathcal{B}_\epsilon(X))$ comparable to a surface ball of radius $\delta(X)$.

b) $\chi_{\Gamma^\epsilon(P)}(X) = \chi_{\mathcal{B}_\epsilon(X)}(P)$.

Hence, by Fubini's Theorem we obtain,

$$\begin{aligned}
\int_{S_T} |(u_k - u_l)^*(P)|^p d\sigma &\geq \frac{1}{|\Omega_T|} \int_{S_T} \int_{\Omega_T} |(u_k - u_l)(X)|^p \chi_{\Gamma^\epsilon(P)}(X) \, dX \, d\sigma \\
&\geq \frac{1}{|\Omega_T|} \int_{\Omega_T} |(u_k - u_l)(X)|^p \sigma(\mathcal{B}_\epsilon(X)) \, dX \\
&\gtrsim \fint_{\Omega'} |(u_k - u_l)(X)|^p \delta(X)^{n+1} \, dX \gtrsim \fint_{\Omega'} |(u_k - u_l)(X)|^p \, dX,
\end{aligned}$$

where in the last inequality we have used that $\delta(X) \geq \delta'$ because $X \in \Omega'$. By lemma 2.16,

$$\|(u_k - u_l)^*\|^p_{L^p(S_T)} \gtrsim \fint_{\Omega'} |(u_k - u_l)(X)|^p \, dX \gtrsim \sup_{\Omega'} |u_k - u_l|^p.$$

As a consequence of the regularity estimate (2.15), there exists u, $\mathcal{L}u = 0$, such that $\{u_k\}_{k=1}^\infty$ converges uniformly to u. The uniform convergence assures that this limit is unique and does not depend on the choice of the sequence converging to f.

Now, to establish *ii*) we notice that for $Q \in S_T$,

$$\mathcal{N}^\epsilon(\nabla u - \nabla u_k)(Q) \leq \liminf_{l \to \infty} \mathcal{N}^\epsilon(\nabla u_l - \nabla u_k)(Q)$$

and that

$$\|(u-u_k)^*_\epsilon\|_{L^p(S_T)} \leq \liminf_{l\to\infty} \|(u_l-u_k)^*_\epsilon\|_{L^p(S_T)}.$$

For fixed $l, k \in \mathbb{N}$,

$$\|(u_l-u_k)^*_\epsilon\|_{L^p(S_T)} + \|\mathcal{N}^\epsilon(\nabla u_l - \nabla u_k)\|_{L^p(S_T)} \lesssim \|f_l - f_k\|_{L^p(S_T)} + \|f_l - f_k\|_{W_p^{1,\frac{1}{2}}(S_T)}$$

and as $l \to \infty$ and $\epsilon \to 0$ we have,

$$\|(u-u_k)^*\|_{L^p(S_T)} + \|\mathcal{N}(\nabla u - \nabla u_k)\|_{L^p(S_T)} \lesssim \|f - f_k\|_{L^p(S_T)} + \|f - f_k\|_{W_p^{1,\frac{1}{2}}(S_T)}.$$

On the other hand,

$$\|(u)^*\|_{L^p(\Omega_T)} + \|\mathcal{N}(\nabla u)\|_{L^p(S_T)} \leq \|(u-u_k)^*\|_{L^p(\Omega_T)} + \|(u_k)^*\|_{L^p(\Omega_T)} + \|\mathcal{N}(\nabla u - \nabla u_k)\|_{L^p(S_T)} + \|\mathcal{N}(\nabla u_k)\|_{L^p(S_T)}.$$

By letting $k \to \infty$ we get *ii*). Finally, it remains to check *i*) with a rather standard technique. For $P \in S_T$ define

$$\Theta_f(P) = \limsup_{\substack{X\to P\\ X\in\Gamma(P)}} u(X) - \liminf_{\substack{X\to P\\ X\in\Gamma(P)}} u(X).$$

Note that $\Theta_f(P) \leq 2(u)^*(P)$ and $\Theta_{f_k}(P) = 0$ for $P \in S_T$. Therefore,

$$\Theta_f(P) \leq \Theta_{f_k}(P) + \Theta_{(f-f_k)}(P) = \Theta_{(f-f_k)}(P).$$

Hence, for $\alpha > 0$

$$\sigma(\{P \in S_T : \Theta_f(P) > \alpha\}) \leq \sigma(\{P \in S_T : \Theta_{(f-f_k)}(P) > \alpha\}) \lesssim \frac{1}{\alpha^p}\int_{S_T} |(u-u_k)^*|^p d\sigma$$

$$\lesssim \frac{1}{\alpha^p}\|f - f_k\|_{L^p(S_T)} + \frac{1}{\alpha^p}\|f - f_k\|_{W_p^{1,\frac{1}{2}}(S_T)}.$$

Letting $k \to \infty$ and $\alpha \to 0$ we have proved *i*).

□

2.3.1 Connections between the $(D)_p$ and $(R)_p$ conditions.

With all the previous definitions and remarks, we are now in position to make precise our goal in this work. We aim to prove the following two results relating the conditions $(D)_p$ and $(R)_q$.

Theorem 2.18. *[36, Theorem 1.3] Let Ω_T be a Lipschitz cylinder and $\mathcal{L} = \operatorname{div}(A(x)\nabla) - \partial_t$ an operator satisfying (2.3). If condition $(R)_p$ holds for $\mathcal{L}$ in Ω_{T+1}, then condition $(D)_{p'}$ holds for $\mathcal{L}$ in Ω_T, $1/p + 1/p' = 1$.*

Theorem 2.19. *[36, Theorem 1.4] Let Ω_T be a Lipschitz cylinder and $\mathcal{L} = \operatorname{div}(A(x)\nabla) - \partial_t$ an operator satisfying (2.3). Suppose $1 < p < \infty$, and that $1/p + 1/p' = 1$. If conditions $(D)_{p'}$ and $(R)_q$ hold for $\mathcal{L}$ in Ω_T for some $1 < q < p$, then condition $(R)_p$ holds for $\mathcal{L}$ in Ω_T.*

The next result is an immediate consequence of these theorems, a well-known property of the condition $(D)_p$ and the classical theory of Muckenhoupt weights and reverse Hölder inequalities (see e.g. [37, Theorem 6.1]).

Corollary 2.20. *[36, Corollary 1.5] Let $1 < q < \infty$ and assume that the $(R)_q$ condition is satisfied. Then there exists $\epsilon > 0$ such that the $(R)_s$ condition is satisfied for every $q < s \leq q + \epsilon$.*

Proof. If $(R)_q$ holds, by Theorem 2.18 we know $(D)_{q'}$ holds, with $1/q + 1/q' = 1$. But then there exists $\epsilon > 0$ such that $(D)_s$ holds for $s \in (q' - \epsilon, q')$. Noticing that $s' \in (q, q + \epsilon)$, where $s' = s/(s-1)$, by Theorem 2.19 we now know $(R)_{s'}$ for $s' \in (q, q + \epsilon)$. □

2.4 A_∞ condition and its connection with the $(R)_p$ condition

The A_∞ condition can be defined for any pair of measures ω and σ, but since in the entirety of this work we assume that ω is the parabolic measure with pole at $\Xi = (0, T+1)$ and σ is the surface measure on $\partial\Omega$, we specialize our definitions and descriptions to this pair of measures.

As in the classical case, we say that the measure ω is in the class $A_\infty(\sigma)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\sigma(F)}{\sigma(\Delta)} < \delta \quad \text{implies} \quad \frac{\omega(F)}{\omega(\Delta)} < \epsilon, \tag{2.17}$$

for every Borel set $F \subset \Delta_r(Q, s)$, with $0 < r < r_0$ and $(Q, s) \in \partial\Omega$. The general theory of Muckenhoupt weights can be adapted to the parabolic setting (see e.g. [37, Theorems 6.1 and 6.2]), and in particular the A_∞ property entails a scale invariant mutual absolute continuity between ω and σ, in the sense that $\omega \in A_\infty(\sigma)$ if and only if $\sigma \in A_\infty(\omega)$. Moreover $\omega \in A_\infty(\sigma)$ if and only if there exist constants $C > 0$ and $0 < \tau < 1$ such that for every surface ball $\Delta \subset \partial\Omega$ and every Borel set $E \subset \Delta$ one has

$$\frac{\omega(E)}{\omega(\Delta)} \leq C \left(\frac{\sigma(E)}{\sigma(\Delta)} \right)^{\tau}. \tag{2.18}$$

In exchange of the roughness of the ambient domain, we obtain a slightly weaker version of Theorem 2.18.

Theorem 2.21. *[35, Theorem 3.2] If Ω is the region above a Lip$(1, \frac{1}{2})$ graph and condition $(R)_q$ holds for some $1 < q < 2$, then $\omega \in A_\infty(d\sigma)$.*

Chapter 3

Poincaré type inequalities and some consequences

The first Poincaré type inequality that we state takes place in the boundary of a Lipschitz cylinder and has nothing to do with the properties of solutions of any parabolic operator. Rather, it is a first instance where the definition of the norm in (2.13) becomes convenient. The inequality contained in the following theorem is an auxiliary result to prove Lemma 4.5 which in turn helps to prove Theorem 2.19, and is inspired on a result in [4, p. 17-18]. It may be of independent interest.

Theorem 3.1. *Let $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$. Then there exists β_r such that for $p \geq 1$ and $r < r_0$, we have*

$$\int_{\Delta_r} |f(q,s) - \beta_r|^p \, d\,\sigma(q,s) \lesssim r^p \int_{\Delta_r} |\nabla_{tan} f(q,s)|^p \, d\,\sigma(q,s) + r^p \int_{\Delta_r} |\partial_t^{\frac{1}{2}} f(q,s)|^p \, d\sigma(q,s),$$

where $\Delta_r = \Delta_r(Q_0)$ with $Q_0 \in S_T$.

Proof. Let $Q_0 = (q_0, s_0) \in S_T$. Notice that $\Delta_r = \widetilde{\Delta}_r \times I_r$, where $\widetilde{\Delta}_r = \widetilde{\Delta}_r(q_0)$ is a surface ball with radius r on ∂D (the boundary of the Lipschitz domain D), and $I_r = (s_0 - r^2 + s_0 + r^2)$ is a time-interval. Define

$$\beta_r = \fint_{\Delta_r} f(q,s) \, d\,\sigma(q,s) \qquad \text{and} \qquad \beta_r(s) \equiv \beta(s) = \fint_{\widetilde{\Delta}_r} f(q,s) \, d\,\widetilde{\sigma}(q).$$

Observe that

$$\int_{\Delta_r} |f(q,s) - \beta_r|^p \, d\sigma(q,s) \lesssim \int_{\Delta_r} |f(q,s) - \beta_r(s)|^p \, d\sigma(q,s) + \int_{\Delta_r} |\beta_r(s) - \beta_r|^p \, d\sigma(q,s)$$
$$\equiv \mathcal{I} + \mathcal{II}.$$

In order to handle $\mathcal{I}$ we employ a Poincaré inequality valid for $g \in C^\infty(\mathbb{R}^n)$ (see [28, proof of Theorem 5.19]):

$$\int_{\widetilde{\Delta}_r} |g(q) - \gamma_r|^p \, d\sigma(q) \lesssim r^p \int_{\widetilde{\Delta}_r} |\nabla_{tan} g(q)|^p \, d\sigma(q),$$

where $\gamma_r = \fint_{\widetilde{\Delta}_r} g(q) \, d\sigma(q)$. By Fubini's theorem and the aforementioned Poincaré type inequality, we see that

$$\mathcal{I} \lesssim r^p \int_{\Delta_r} |\nabla_{tan} f(q,s)|^p \, d\sigma(q,s).$$

To handle $\mathcal{II}$, we notice that

$$\begin{aligned}
|\beta_r - \beta_r(s)| &\lesssim \fint_{\widetilde{\Delta}_r} \fint_{I_r} |f(q,\tau) - f(q,s)| \, d\tau \, d\widetilde{\sigma}(q) \\
&\lesssim \fint_{\widetilde{\Delta}_r} \left\{ \left(\fint_{I_r} \frac{|f(q,\tau) - f(q,s)|^2}{|\tau - s|^2} \, d\tau \right)^{\frac{1}{2}} \left(\fint_{I_r} |\tau - s|^2 \, d\tau \right)^{\frac{1}{2}} \right\} d\widetilde{\sigma}(q) \\
&\lesssim r \fint_{\widetilde{\Delta}_r} \partial_t^{\frac{1}{2}} f(q,s) \, d\widetilde{\sigma}(q).
\end{aligned}$$

Finally, integrating over Δ_r and applying Hölder's inequality and Fubini's theorem we obtain,

$$\begin{aligned}
\int_{\Delta_r} |\beta_r - \beta_r(s)|^p \, d\sigma(q,s) &\lesssim r^p \int_{\Delta_r} \left| \fint_{\widetilde{\Delta}_r} \partial_t^{\frac{1}{2}} f(q,s) \, d\widetilde{\sigma}(q) \right|^p d\sigma \\
&\lesssim r^p \int_{\Delta_r} \fint_{\widetilde{\Delta}_r} \left| \partial_t^{\frac{1}{2}} f(q,s) \right|^p d\widetilde{\sigma}(q) \, d\sigma(q,s) \\
&\lesssim r^p \int_{\Delta_r} \left| \partial_t^{\frac{1}{2}} f(q,s) \right|^p d\sigma(q,s).
\end{aligned}$$

□

In order to prove Theorem 2.18 we will go through a series of lemmas and observations concerning a very particular class of solutions to (2.4). The next result is the second Poincaré type inequality alluded in the title of the chapter.

Lemma 3.2. *Let Ω be a $Lip(1,1/2)$ cylinder, $Q = (q', q_n, s) \in S$ and $0 < r < \min\left\{r_0, \sqrt{s}\right\}$. If $u \in C^\infty(\Psi_r(Q)) \cap C(\overline{\Psi_r(Q)})$ and $u \equiv 0$ on $\Delta(Q,r)$, then*

$$\int_{\Psi_r(Q)} \delta(X)^\alpha u^2(X) \, dX \lesssim \frac{r^2}{1-\alpha} \int_{\Psi_r(Q)} \delta(X)^\alpha |\nabla u(X)|^2 \, dX,$$

for each $0 \le \alpha < 1$.

Proof. Fix $X' = (x', x_n', t) \in \Psi_r(Q)$. We first note that

$$u(X') = u(x', x_n', t) - u(x', \psi(x', t), t) = \int_{\psi(x',t)}^{x_n'} \frac{\partial}{\partial x_n} u(x', x_n, t)\, d x_n.$$

Setting $X = (x', x_n, t)$, by Cauchy–Schwarz inequality,

$$|u(X')| \le \int_{\psi(x',t)}^{x_n'} \frac{\delta^{\frac{\alpha}{2}}(X)}{\delta^{\frac{\alpha}{2}}(X)} |\nabla u(X)|\, d x_n \le \left(\int_{\psi(x',t)}^{x_n'} \delta^{\alpha}(X) |\nabla u(X)|^2\, d x_n \right)^{\frac{1}{2}} \left(\int_{\psi(x',t)}^{x_n'} \delta^{-\alpha}(X)\, d x_n \right)^{\frac{1}{2}}. \quad (3.1)$$

If $\alpha > 0$,

$$\int_{\psi(x',t)}^{x_n'} \delta^{-\alpha}(X)\, d x_n \lesssim \int_{\psi(x',t)}^{x_n'} \frac{d x_n}{(x_n - \psi(x', t))^{\alpha}} = \int_0^{x_n' - \psi(x',t)} y^{-\alpha}\, d y \le \int_0^{r} y^{-\alpha}\, d y = \frac{r^{1-\alpha}}{1-\alpha}. \quad (3.2)$$

By (3.1) and (3.2)

$$\begin{aligned}
\int_{\Psi_r(Q)} \delta^{\alpha}(X') u^2(X')\, d X' &\le \int_{\Psi_r(Q)} r^{\alpha} u^2(X')\, d X' \\
&\lesssim \frac{r^{1-\alpha} r^{\alpha}}{1-\alpha} \int_{\Psi_r(Q)} \int_{\psi(x',t)}^{x_n'} \delta^{\alpha}(X) |\nabla u(X)|^2\, d x_n\, d X' \\
&\le \frac{r}{1-\alpha} \int_{E_r(Q)} \int_{\psi(x',t)}^{\psi(x',t)+r} \int_{\psi(x',t)}^{\psi(x',t)+r} \delta^{\alpha}(X) |\nabla u(X)|^2\, d x_n\, d x_n'\, d(x', t) \\
&\le \frac{r^2}{1-\alpha} \int_{E_r(Q)} \int_{\psi(x',t)}^{\psi(x',t)+r} \delta^{\alpha}(X) |\nabla u(X)|^2\, d x_n\, d(x', t) \\
&= \frac{r^2}{1-\alpha} \int_{\Psi_r(Q)} \delta^{\alpha}(X) |\nabla u(X)|^2\, d X.
\end{aligned}$$

Here, $E_r(Q) = \{(x', t) : (x', x_n, t) \in \Psi_r(Q)\}$. If $\alpha = 0$, we argue the same way using (3.1). □

Remark 3.3. *Note that we have used the fact that* $x_n - \psi(x', t) \approx \delta(X)$ *when* Ω *is a* Lip$(1, 1/2)$ *cylinder. This may be proven with the aid of Lagrange multipliers.*

Proof of remark 3.3: Without loss of generality assume that $X_0 = (0, x_0, 0)$ and $\psi(0, 0) = 0$. It is clear that $\delta(X_0) \le x_0$. Notice that $\widetilde{\Gamma} = \widetilde{\Gamma}(0) = \{(x', x_n, t) \in \Omega : |x'| + |t|^{\frac{1}{2}} < m^{-1} x_n\}$ is contained in Ω and $X_0 \in \widetilde{\Gamma}$. As a consequence, $\delta(X, \partial\widetilde{\Gamma}) \le \delta(X_0)$, so it is enough to compute $\delta(X, \partial\widetilde{\Gamma})$. This problem reduces to minimize the function

$$f(x', x_n, t) = |x'|^2 + |x_0 - x_n| + t^{\frac{1}{2}}$$

in the surface of the cone

$$g(x', x_n, t) = |x'|^2 + t^{\frac{1}{2}} - m^{-1} x_n = 0.$$

Taking derivatives, we have

$$\nabla f(x', x_n, t) = (2x', 2(x_n - x_0), 1),$$
$$\nabla g(x', x_n, t) = \left(\frac{x'}{|x'|}, -m^{-1}, \frac{1}{2\sqrt{t}} \right).$$

Setting $c = m^{-1}$, the pair of equations $\nabla f = \lambda \nabla g$ and $g = 0$ becomes

$$\begin{cases} |x'| = \frac{\lambda}{2} = t^{\frac{1}{2}}, \\ \\ x_n = \frac{1}{2}(2x_0 - \lambda\, c), \\ \\ |x'|^2 + t^{\frac{1}{2}} = c\, x_n. \end{cases}$$

From this, we may conclude that $\lambda = \frac{2cx_0}{2+c^2}$ and

$$\delta(X, \partial\widetilde{\Gamma})^2 \approx \min_{\partial\widetilde{\Gamma}} f = \left(\frac{c^4 + 2c^2}{(2 + c^2)^2} \right) x_0^2 \approx x_0^2.$$

□

Lemma 3.2 was first thought only as a step towards proving that condition $(R)_p$ implies $(D)_{p'}$. Inspired by [11] we could find another application of the technique behind this lemma, namely Lemma 3.4.

This lemma is an important step for the argumentation of Theorem 2.21 and may be of independent interest when dealing with other initial Dirichlet-type problems for parabolic equations.

Lemma 3.4 (Poincaré-type inequality). *Suppose that $\Delta \subset \partial\Omega$ is the surface cube $\Delta \equiv \Delta_r(Q)$ with $Q_0 = (q_0, s_0) \in \partial\Omega$, $0 < 5r < r_0$. Let u be a positive solution to $Lu = 0$ on Ω vanishing continuously on 4Δ, and such that $u \leq 1$ in Ω, and $u(\underline{\mathcal{A}}_{4r}(Q_0)) \gtrsim 1$. For any $Q = (q, s) \in \Delta$ let $X = (x, t) \in \Gamma^r(Q)$, with $\rho = \delta(X)$. Then for any $0 < \varepsilon_0 < 1$ and any $0 < \kappa < 1$*

$$\frac{1}{\rho^2} \int_{Q(X)} |u(Y)|^2 \, dY \lesssim \int_{\kappa\Psi^{\varepsilon_0}(X)} |\nabla u(Y)|^2 \, dY + \varepsilon_0^{\gamma} \rho^n u^2(\overline{\mathcal{A}}_{2\rho}(Q)),$$

where $\gamma \in (0, 1)$ depends only on n, λ_1 and λ_2.

Proof. Let $0 < \varepsilon_0 < 1$, $0 < \kappa < 1$, and recall that $\rho = \delta(X)$ with $X \in \Omega$, and that $\operatorname{diam}(Q(X)) \approx \rho$, $|Q(X)| \approx \rho^{n+2}$. Applying the hypothesis to 2.12 we have

$$u(\overline{\mathcal{A}}_s(Q)) \lesssim u(\underline{\mathcal{A}}_s(Q)),$$

for $Q \in \Delta$ and $0 < s \leq r$.

By Hölder continuity at the boundary (see e.g. [30, Lemma 5, p. 521]) and Carleson estimate (Theorem 2.2), we have $u(Y) \lesssim \left(\frac{\delta(Y)}{\rho}\right)^{\gamma} u(\overline{\mathcal{A}}(X))$ for every $Y \in Q(X)$. Altogether, by the above observation and Harnack's inequality $\sup_{Q(X)} u \lesssim \inf_{\kappa\Psi} u$, where $\kappa\Psi$ is a generalized κ-scaled Carleson region associated to $\Psi \equiv \Psi(X)$. This yields by Lemma 3.2

$$\int_{Q(X)} u^2(Y)\, dY \lesssim \frac{1}{\rho^{\gamma}} \int_{\kappa\Psi} \delta(Y)^{\gamma} u^2(Y)\, dY \lesssim \frac{\rho^2}{\rho^{\gamma}} \int_{\kappa\Psi} \delta(Y)^{\gamma} |\nabla_Y u(Y)|^2\, dY.$$

Defining $\Psi_1 = \kappa\Psi^{\varepsilon_0}$ and $\Psi_2 = \Psi \setminus \Psi_1$ we continue our estimates as follows:

$$\begin{aligned}\int_{Q(X)} u^2(Y)\, dY &\lesssim \rho^{2-\gamma} \left[\int_{\Psi_1} + \int_{\Psi_2}\right] \delta(Y)^{\gamma} |\nabla u(Y)|^2\, dY \\ &\lesssim \rho^2 \int_{\Psi_1} |\nabla u(Y)|^2\, dY + \varepsilon_0^{\gamma} \rho^2 \int_{\Psi_2} |\nabla u(Y)|^2\, dY \equiv I + II.\end{aligned}$$

Term I is the first term in the right-hand-side of Theorem (3.4). To estimate term II we apply the parabolic version of boundary Caccioppoli's inequality (obtained as a variation of [31, p. 113]) to obtain

$$II \lesssim \varepsilon_0^{\gamma} \int_{\Psi_2^*} |u(Y)|^2\, dY,$$

where Ψ_2^* denotes a small dilation of Ψ_2. By the Carleson-type estimate in [37, Lemmata 2.3] we obtain $II \lesssim \varepsilon_0^{\gamma} \rho^{n+2} u^2(\overline{\mathcal{A}}_{2\rho}(Q))$. □

From this point on in this chapter as well as the whole next chapter the ambient domain will be that of a Lipschitz cylinder . If $Q = (q, s) \in S_T$ and $0 < r < \min\{r_0, \sqrt{s}, \sqrt{T - s}\}$, we define $\overrightarrow{Q}(r) = (q, s + r^2)$ and $\overleftarrow{Q}(r) = (q, s - r^2)$. Now, take $g \in C_0^{\infty}(\mathbb{R}^n \times (0, \infty))$ such that $g \equiv 1$ in $\Delta_r(\overleftarrow{Q}_0(5r))$ and $g \equiv 0$ in $\Delta_{2r}(\overleftarrow{Q}_0(5r))^c$ for some $Q_0 = (q_0, s_0) \in S_T$ with $0 < 6r < \min\{r_0, \sqrt{s_0}, \sqrt{T - s_0}\}$.

For the remaining of this chapter and section 4.1 of the following chapter, u will denote the solution to the Dirichlet problem over the domain $\Omega \cap \{-1 < t < 2T\}$, with boundary datum given by a function g as described above.

Remark 3.5. *Observe that with the notation just introduced, Lemma 2.12 yields*

$$u(\overline{\mathcal{A}}_\rho(Q)) \lesssim u(\underline{\mathcal{A}}_\rho(Q)), \qquad \rho \leq 2r,$$

for $Q = (q, s) \in S_T$ *with* $s_0 + 43r^2 \leq s \leq s_0 + 200r^2$ *and* $|q - q_0| \lesssim r$.

Proof. Note that $s_0 + 43r^2 \leq s$ implies $\Delta_{8r}(Q) \subset \Delta_{2r}(\overleftarrow{Q}_0(5r))^c$. Clearly $M_r \leq 1$. By Theorem 2.3, for $X \in \Psi_{\frac{r}{2}}(\overleftarrow{Q}_0(5r))$ we have

$$\omega^X(\Delta_r(\overleftarrow{Q}_0(5r))) \gtrsim 1.$$

As a consequence

$$u(X) = \int_{S_T} f\, d\omega^X \geq \omega^X(\Delta_r(\overleftarrow{Q}_0(5r))) \gtrsim 1.$$

With this, Carleson and Harnack inequality (Theorem 2.1) gives $m_r = u(\underline{\mathcal{A}}_{4r}(Q)) \gtrsim 1$ and the proof of this Remark is finished. □

Lemma 3.6. *Let* $Q = (q, s) \in S_T$ *with* $s_0 + 43r^2 \leq s \leq s_0 + 200r^2$. *Then, there exists* $0 < \alpha < 1$ *depending on* n *and* λ *only, such that*

$$\fint_{\Psi_r(Q)} u^2(X)\, dX \leq c\, \frac{1}{r^\alpha} \fint_{\Psi_r(Q)} \delta(X)^\alpha u^2(X)\, dX.$$

Proof. Consider $K = \Psi_r(Q) \setminus S_{\frac{1}{2}r}(Q)$ where $S_{ar}(Q) = \{X \in \Psi_r(Q) : \delta(X) < ar\}$, $0 < a < 1$. Note that $|K| \approx r^{n+2} \approx |\Psi_r(P)|$. By the Theorem 2.2, Remark 3.5 and Harnack's inequality give

$$\sup_{\Psi_r(Q)} u^2 \lesssim u^2(\overline{\mathcal{A}}_r(Q)) \lesssim u^2(\underline{\mathcal{A}}_r(Q)) \lesssim \inf_K u^2 \leq \fint_K u^2(X)\, dX. \tag{3.3}$$

This and Theorem 2.9 imply that for $X \in \Psi_r(Q)$ one has

$$u(X) \lesssim \left(\frac{\delta(X)}{r}\right)^\alpha \left(\fint_K u^2(Y)\, dY\right)^{1/2}. \tag{3.4}$$

Since $\delta(X) \approx \delta(Y)$, for $X, Y \in K$, we can use (3.3) and (3.4), to obtain

$$\begin{aligned}\fint_{\Psi_r(Q)} u^2(X)\, dX &\lesssim \fint_K \fint_K \left(\frac{\delta(X)}{r}\right)^{2\alpha} u^2(Y)\, dY\, dX \\ &\lesssim \frac{1}{r^\alpha} \fint_K \delta(Y)^\alpha u^2(Y)\, dY \lesssim \frac{1}{r^\alpha} \fint_{\Psi_r(Q)} \delta(Y)^\alpha u^2(Y)\, dY.\end{aligned}$$

The lemma follows. □

Lemma 3.7. *Let* $Q = (q, s) \in S_T$ *with* $s_0 + 43r^2 \leq s \leq s_0 + 200r^2$. *Then we have*

$$\fint_{\Psi_{2r}(Q)} u^2(X)\, dX \lesssim \fint_{\Psi_r(Q)} u^2(X)\, dX, \qquad \fint_{\Psi_r(Q)} |\nabla u(X)|^2\, dX \lesssim \fint_{\Psi_{\frac{3}{4}r}(Q)} |\nabla u(X)|^2\, dX. \tag{3.5}$$

Proof. The first assertion follows using (3.3). For the second assertion we use Caccioppoli's at the boundary inequality to have

$$\fint_{\Psi_r(Q)} |\nabla u(X)|^2 \, dX \lesssim r^{-2} \fint_{\Psi_{\frac{3}{2}r}(Q)} u^2(X) \, dX.$$

With this, the second assertion of this lemma may be derived from the first assertion and Lemma 3.2 for $\alpha = 0$. ◻

Lemma 3.8. *Let $Q = (q, s) \in S_T$ with $s_0 + 43r^2 \leq s \leq s_0 + 200r^2$. Then, there exists $\epsilon = \epsilon(n, \lambda, m) > 0$ such that*

$$\fint_{\Psi_r(Q)} |\nabla u(X)|^2 \, dX \lesssim \fint_{\Psi_r(Q) \setminus S_{\epsilon r}(Q)} |\nabla u(X)|^2 \, dX.$$

Proof. According to Lemma 3.7 and Caccioppoli's inequality at the boundary inequality, we have

$$\fint_{\Psi_r(Q)} |\nabla u(X)|^2 \, dX \lesssim \fint_{\Psi_{\frac{r}{2}}(Q)} |\nabla u(X)|^2 \, dX \lesssim \frac{1}{r^2} \fint_{\Psi_r(Q)} u^2(X) \, dX.$$

From Lemmas 3.6 and 3.2

$$\frac{1}{r^2} \fint_{\Psi_r(Q)} u^2(X) \, dX \lesssim \frac{1}{r^{2+\alpha}} \fint_{\Psi_r(Q)} \delta^\alpha(X) u^2(X) \, dX \lesssim \frac{1}{r^\alpha} \fint_{\Psi_r(Q)} \delta^\alpha(X) |\nabla u(X)|^2 \, dX.$$

Thus, we have

$$\begin{aligned}
\fint_{\Psi_r(Q)} |\nabla u(X)|^2 \, dX &\lesssim \frac{1}{r^\alpha} \fint_{\Psi_r(Q) \setminus S_{\epsilon r}(Q)} \delta^\alpha(X) |\nabla u(X)|^2 \, dX + \frac{1}{r^\alpha} \fint_{S_{\epsilon r}(Q)} \delta^\alpha(X) |\nabla u(X)|^2 \, dX \\
&\lesssim \fint_{\Psi_r(Q) \setminus S_{\epsilon r}(Q)} |\nabla u(X)|^2 \, dX + \epsilon^\alpha \fint_{S_{\epsilon r}(Q)} |\nabla u(X)|^2 \, dX \\
&\leq \fint_{\Psi_r(Q) \setminus S_{\epsilon r}(Q)} |\nabla u(X)|^2 \, dX + \epsilon^\alpha \fint_{\Psi_r(Q)} |\nabla u(X)|^2 \, dX.
\end{aligned}$$

Finally, choosing $\epsilon > 0$ very small we can hide the second term in the right hand side into the left hand side, and hence we conclude the desired estimate. ◻

Chapter 4

Connections between $(D)_p$ and $(R)_p$ conditions

As stated previously, our goal in this work is to prove some connections between the $(D)_p$ and $(R)_p$ conditions. At this moment, we are ready to prove Theorem 2.18.

Recall the definitions of *Carleson boxes, surface balls, corkscrew point and non tangential approach region* from subsection 2.1.3.

4.1 Proof of Theorem 2.18

We retain notations from the previous sections. In particular, recall that we have stated right before the Remark 3.5 that u denotes a solution over $\Omega \cap \{-1 < t < 2T\}$ with a very particular prescribed data function f. It is convenient now to impose some extra conditions on this data, namely $|\nabla f| \lesssim \frac{1}{r}$ and $|f_t| \lesssim \frac{1}{r^2}$.

We are interested in the norm of f as an element of $W_p^{1,\frac{1}{2}}(S_T)$ because once this norm is computed, the fact that $(R)_p$ is solvable will provide us a precise estimate of the L^p norm of $\mathcal{N}(\nabla u)$. Indeed, observe that with these new conditions on f, we have

$$\int_{S_T} |\nabla f(Q)|^p \, d\sigma(Q) \lesssim r^{n+1-p}, \qquad \left|\partial_t^{\frac{1}{2}} f(x,s)\right| \lesssim \left(\int_{s_0-29r^2}^{s_0-21r^2} \left(\frac{1}{r^2}\right)^2 dt\right)^{\frac{1}{2}} \lesssim \frac{1}{r}.$$

This implies that

$$\|f\|^p_{W^{1,1/2}_p(S_{T+1})} \lesssim r^{n+1-p}.$$

When attempting to prove that $(R)_p$ implies $(D)_{p'}$, using the techniques from [28], one finds some difficulties when trying to use the Theorem 2.4. Indeed, in the elliptic case, the selfadjoint property of second linear operators in divergence form, similar to $\mathcal{L}$, plays a role several times. For the parabolic operators this is not the case, and actually the Green's function is not symmetric in its arguments; that is, the order of the argument variables is essential.

One way to tackle this obstacle is to use an auxiliary solution v to the adjoint equation $\mathcal{L}^* v = 0$ which is defined in terms of the particular solution u, by a *reflection in time* change of variables, as mentioned earlier. The use of Theorem 2.11 is crucial in the estimates below and this explains the time independence of the matrix A.

To be more precise, we think of Ω as the extended domain $D \times (-\infty, \infty)$. For $X = (x, t) \in \overline{\Omega} \cap \{-2T < t < 2T\}$, define $\widetilde{X} = (x, \widetilde{t})$ to be the reflection of X with respect to the hyperplane for which $t = s_0 + 25r^2$, that is, $\widetilde{X} = (x, 2(s_0 + 25r^2) - t)$. Here it is important to recall that $Q_0 = (q_0, s_0) \in S_T$ has been fixed when defining the support of f.

Now define $v(X) = u(\widetilde{X})$. Observe that $\mathcal{L}^* v = 0$ in $\overline{\Omega} \cap \{-2T < t < 2T\}$ and that by the support definition of f we have $v(x, t) = 0$ if $2(s_0 + 25r^2) - t \geq s_0 - 21r^2$ or equivalently $t \leq s_0 + 71r^2$.

Take $Q \in \Delta_{\frac{r}{16}}(Q_0)$ and $\rho < \frac{r}{16}$. From Theorems 2.4, 2.5 and 2.11 , we have

$$\begin{aligned}
\frac{\omega^{\Xi}(\Delta_\rho(Q))}{\rho^{n+1}} &\lesssim \frac{G(\Xi, \underline{\mathcal{A}}_\rho(Q))}{\rho} = \frac{v(\underline{\mathcal{A}}_\rho(Q))}{\rho} \frac{G(\Xi; \underline{\mathcal{A}}_\rho(Q))}{v(\underline{\mathcal{A}}_\rho(Q))} \\
&\lesssim \frac{v(\underline{\mathcal{A}}_\rho(Q))}{\rho} \frac{G(\Xi; \underline{\mathcal{A}}_r(Q_0))}{v(\overline{\mathcal{A}}_r(Q_0))} \lesssim \frac{v(\underline{\mathcal{A}}_\rho(Q))}{\rho} G(\Xi; \overline{\mathcal{A}}_r(Q_0)) \\
&\lesssim \frac{v(\underline{\mathcal{A}}_\rho(Q))}{\rho} \frac{\omega^{\Xi}(\Delta_r(Q_0))}{r^n},
\end{aligned} \tag{4.1}$$

where we have used the fact that $v(\overline{\mathcal{A}}_r(Q_0)) = u(\underline{\mathcal{A}}_r(\widetilde{Q}_0)) \gtrsim 1$. On the other hand, by Carleson inequality 2.2, Remark 3.5 and Harnack inequality 2.1(similarly to (3.3)), we may conclude

$$v(\underline{\mathcal{A}}_\rho(Q)) = u(\overline{\mathcal{A}}_\rho(\widetilde{Q})) \lesssim \left(\fint_{\Psi_\rho(\widetilde{Q})} u^2(Y)\, dY \right)^{\frac{1}{2}}.$$

From this and lemmas 3.2 and 3.8 we obtain

$$v(\underline{\mathcal{A}}_\rho(Q)) \lesssim \rho \left(\fint_{\Psi_\rho(\widetilde{Q})\setminus S_{\epsilon\rho}(\widetilde{Q})} |\nabla u(Y)|^2 \, dY \right)^{\frac{1}{2}}. \tag{4.2}$$

Plugging (4.1) and (4.2) together, we get

$$\frac{\omega^{\Xi}(\Delta_\rho(Q))}{\rho^{n+1}} \lesssim \frac{\omega^{\Xi}(\Delta_r(Q_0))}{r^n} \left(\fint_{\Psi_\rho(\widetilde{Q})\setminus S_{\epsilon\rho}(\widetilde{Q})} |\nabla u(Y)|^2 \, dY \right)^{\frac{1}{2}}. \tag{4.3}$$

This suggests that we introduce the next two maximal functions

$$\mathcal{M}_\sigma \omega(Q) \equiv \mathcal{M}_{\sigma,\frac{r}{16}} \omega(Q) = \sup_{0<\rho<\frac{r}{16}} \frac{\omega^{\Xi}(\Delta_\rho(Q))}{\rho^{n+1}},$$

$$\mathcal{N}^\epsilon \phi(Q) \equiv \mathcal{N}^\epsilon_\alpha \phi(Q) = \sup_{X\in\Gamma_\alpha(Q)} \left(\fint_{\Psi_{\delta(X)}(P_X)\setminus S_{\epsilon\delta(X)}(P_X)} \phi^2(Y) \, dY \right)^{\frac{1}{2}},$$

where $P_X = (x', \psi(x',t), t)$ if $X = (x', x_n, t)$. Notice that we have included in the notation the aperture α. The reason will be clear shortly. With this definitions, (4.3) yields

$$\mathcal{M}_\sigma \omega(Q) \lesssim \frac{\omega^{\Xi}(\Delta_r(Q_0))}{r^n} \mathcal{N}^\epsilon(\nabla u)(\widetilde{Q}). \tag{4.4}$$

For the final part of the proof of Theorem 2.18, we make use of the following

Lemma 4.1.

$$\mathcal{N}^\epsilon_\alpha v(Q) \lesssim \mathcal{N}_\beta v(Q),$$

for some $\beta = \beta(n, \lambda, m, r_0) > \alpha$, where ϵ is as in Theorem 3.8.

Proof. First note that the region

$$\widetilde{\Psi}_r(Q) = \left\{ (x,t) \in \mathbb{R}^{n+1} : \begin{array}{c} |x_i - q_i| < r, \; i = 1, \cdots, n-1 \\ \psi(x',t) - r < x_n < \psi(x',t) + 4nmr \\ |s-t| < r^2 \end{array} \right\},$$

can be covered with $N = N(\epsilon, n, m)$ parabolic cubes $\mathcal{Q}$ of radius $r' \geq \epsilon c^{-2} r$, independently of r. For $X \in \Gamma_\alpha(Q)$, as we just observed, we can cover $\Psi_{\delta(X)}(P_X) \setminus S_{\epsilon\delta(X)}(P_X)$ with N parabolic cubes $\mathcal{Q}_i$ with radius $c^{-2}\delta(X_i)$ centered at points of $\Psi_{\delta(X)}(P_X) \setminus S_{\epsilon\delta(X)}(P_X)$. Now, from the fact that $\epsilon\delta(X) \leq \delta(X_i) \leq \delta(X)$, it follows that

$$\left|\Psi_{\delta(X)}(P_X) \setminus S_{\epsilon\delta(X)}(P_X)\right| \approx \left|\Psi_{\delta(X)}(P_X)\right| \approx |\mathcal{Q}_i|,$$

where the comparability constants depend again only on ϵ and n. As a consequence, for $\beta > 0$ big enough

$$\fint_{\Psi_{\delta(X)}(P_X)\setminus S_{\epsilon\delta(X)}(P_X)} |\phi|^2\, dY \lesssim c\sum_1^N \fint_{Q_i} |\phi|^2 dY \lesssim \mathcal{N}_\beta \phi(Q).$$

Taking the supremum over $X \in \Gamma_\alpha(Q)$, the inequality is proven. □

To finish the proof of Theorem 2.18, we can apply the previous lemma and (4.4), along with the Remark 2.14 to obtain by the very definition of $\mathcal{M}_\sigma\omega$ that $\omega^\Xi << \sigma$ and

$$\left(\fint_{\Delta_{\frac{r}{16}}(Q_0)} \left(\frac{d\omega^\Xi}{d\sigma}\right)^p d\,\sigma\right)^{\frac{1}{p}} \lesssim \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{p}} \frac{\omega(\Delta)}{r^n} \|\mathcal{N}(\nabla u)\|_{L^p(S_{T+1})} \lesssim \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{p}} \frac{\omega(\Delta)}{r^n} \|f\|_{W_p^{1,\frac{1}{2}}(S_{T+1})}$$
$$\lesssim \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{p}} \frac{\omega(\Delta)}{r^n} \left(r^{n+1-p}\right)^{\frac{1}{p}} \approx \frac{\omega(\Delta)}{r^{n+1}} \approx \fint_{\Delta_{\frac{r}{16}}(Q_0)} \left(\frac{d\omega^\Xi}{d\sigma}\right) d\sigma,$$

where $\Delta = \Delta_r(Q_0)$, thus finishing the proof.

4.2 Some observations about solutions

In order to prove Theorem 2.19, first we make a couple of observations about the behavior of solutions near the boundary. While the second observation depends on properties of solutions, the first one does not, and it depends purely on the geometric features of Ω_T. Here is our first observation:

Lemma 4.2. *For any function u such that ∇u exists almost everywhere and $r < r_0$, we have,*

$$\fint_{\Psi_{28r}(Q_0)} |\nabla u(Y)|\, d\,Y \lesssim \fint_{\Delta_{32r}(Q_0)} \mathcal{N}(\nabla u)(P)\, d\sigma(P),$$

where $Q_0 \in S_T$.

Proof. Inequality of lemma 4.2 can be broken into two assertions:

$$\fint_{\Psi_{28r}(Q_0)} |\nabla u(Y)|\, d\,Y \lesssim \fint_{\Psi_{28r}(Q_0)} \fint_{Q(X)} |\nabla u(Y)|\, dY\, dX, \tag{4.5}$$

$$\fint_{\Psi_{28r}(Q_0)} \fint_{Q(X)} |\nabla u(Y)|\, dY\, dX \lesssim \fint_{\Delta_{32r}(Q_0)} N(\nabla u)(P)\, d\sigma(P). \tag{4.6}$$

Taking into account that $\delta(X) \approx \delta(Y)$ if $Y \in \mathcal{Q}(X)$, (4.5) is proved as follows:

$$\fint_{\Psi_{28r}(Q_0)} \fint_{\mathcal{Q}(X)} |\nabla u(Y)|\, dY\, dX \approx \fint_{\Psi_{28r}(Q_0)} \frac{1}{\delta(X)^{n+2}} \int_{\Psi_{30r}(Q_0)} |\nabla u(Y)| \chi_{\mathcal{Q}(X)}(Y)\, dY\, dX$$
$$\approx \fint_{\Psi_{30r}(Q_0)} \int_{\Psi_{28r}(Q_0)} \frac{1}{\delta(X)^{n+2}} |\nabla u(Y)| \chi_{\mathcal{Q}(Y)}(X)\, dX\, dY$$
$$\gtrsim \fint_{\Psi_{28r}(Q_0)} |\nabla u(Y)|\, dY.$$

To prove (4.6) we observe that for $P = (p', p_n, s) \in \Delta_{32r}(Q_0)$,

$$\mathcal{N}(\nabla u)(P) = \fint_{p_n}^{p_n+112nmr} \mathcal{N}(\nabla u)(P) d\rho \geq \fint_{p_n}^{p_n+112nmr} \fint_{\mathcal{Q}(p',\rho,s)} |\nabla u(Y)| dY\, d\rho.$$

Integrating this, we have

$$\fint_{\Delta_{32r}(Q_0)} \mathcal{N}(\nabla u)(P) d\sigma(P) \geq \fint_{\Delta_{32r}(Q_0)} \fint_{p_n}^{p_n+112nmr} \fint_{\mathcal{Q}(p',\rho,s)} |\nabla u(Y)| dY\, d\rho\, d\sigma(P)$$
$$\gtrsim \fint_{\Psi_{28r}(Q_0)} \fint_{\mathcal{Q}(X)} |\nabla u(Y)|\, dY\, dX.$$

□

Here is the second observation of the behavior of solutions near the boundary:

Lemma 4.3. *Assume that u is a solution of $\mathcal{L}u = 0$ in Ω and that $u = 0$ continuously on $\Delta_{32r}(Q_0)$ for some $Q_0 = (q_0, s_0) \in S_T$. Then we have,*

$$|u(X)| \lesssim \frac{G(\Xi; \widetilde{X})}{G(\Xi; \overline{\mathcal{A}}_{12r}(\widetilde{Q_0}))} \left(\fint_{\Psi_{18r}(Q_0)} |u(Y)|^2\, dY \right)^{\frac{1}{2}},$$

for $X \in \Psi_{\frac{3}{2}r}(Q_0)$.

Proof. Let u_1 and u_2 be the solutions of $\mathcal{L}u = 0$ in $\Psi_{32r} = \Psi_{32r}(Q_0)$ with data $f_1 = \max\{u, 0\}$ and $f_2 = \max\{-u, 0\}$ on $\partial\Psi_{16r}(Q_0)$ respectively. Note that $u = u_1 - u_2$ in $\partial\Psi_{32r}$, by uniqueness $u = u_1 - u_2$ in Ψ_{32r}.

Due to the evolutive nature of Theorem 2.4 we perform a reflection to X with respect to the time variable of Q_0, as we did previously. We must recall that for this step the independence of the time variable is essential. More precisely, in this instance we define $v_i(X) = u_i(\widetilde{X})$ for $X \in \Psi_{32r}(\widetilde{Q_0})$ where $\widetilde{X} = (x, 2(s_0 + (32)^2 r^2) - t)$ if $X = (x, t)$. By the comparison principle

$$v_i(\widetilde{X}) \lesssim \frac{G(\Xi; \widetilde{X})}{G(\Xi; \overline{\mathcal{A}}_{12r}(\widetilde{Q_0}))} v_i(\underline{\mathcal{A}}_{12r}(\widetilde{Q_0})), \qquad i = 1, 2,$$

for $X \in \Psi_{\frac{3}{2}r}(Q_0)$. This is the same as

$$u_i(X) \lesssim \frac{G(\Xi; \widetilde{X})}{G(\Xi; \overline{\mathcal{A}}_{12r}(\widetilde{Q_0}))} u_i(\overline{\mathcal{A}}_{12r}(Q_0)), \qquad i = 1, 2. \tag{4.7}$$

Now, note that by Harnack's inequality we have

$$u_i(\overline{\mathcal{A}}_{12r}(Q_0)) \lesssim \inf_K u_i \le \left(\fint_K u_i^2(Y)\, dY \right)^{\frac{1}{2}} \lesssim \left(\fint_{\Psi_{18r}(Q_0)} u_i^2(Y)\, dY \right)^{\frac{1}{2}}, \qquad i = 1, 2, \tag{4.8}$$

where $K \subset \Psi_{18r}(Q_0)$ is an appropriate compact set to the right of $\overline{\mathcal{A}}_{16r}(Q_0)$. Putting (4.7) and (4.8) together we obtain the lemma. □

Back to the main goal, which is to prove Theorem 2.19, we now state two lemmas, whose proof is provided in the last section of this chapter.

Lemma 4.4. *Let $1 < p < \infty$. Assume that $(D)_{p'}$ is solvable in Ω_T for $\mathcal{L}$. Let u be a solution of $\mathcal{L}u = 0$ in Ω_T that vanishes continuously in $\Delta_{32r} = \Delta_{32r}(Q_0)$ with $0 < 16r < \frac{1}{2}\min\{r_0, \sqrt{s_0}, \sqrt{T - s_0}\}$, then we have*

$$\left(\fint_{\Delta_r} |\mathcal{N}(\nabla u)|^p\, d\sigma \right)^{\frac{1}{p}} \lesssim \fint_{\Delta_{32r}} \mathcal{N}(\nabla u)\, d\sigma.$$

Lemma 4.5. *Retaining the notation from the previous lemma, define*

$$E(\lambda) = \{Q \in \Delta_{2r} : \mathcal{M}_{\Delta_{2r}}(|\mathcal{N}(\nabla u)|^q)(Q) > \lambda\}. \tag{4.9}$$

Let $1 < q < p < \infty$ and suppose that $(D)_{p'}$ and $(R)_q$ are solvable in Ω_T. Then there exists constants $\epsilon, \gamma, \alpha > 0$ such that

$$\begin{aligned} |E(A\lambda)| \le \epsilon |E(\lambda)| &+ |\{Q \in \Delta_r : \mathcal{M}_{\Delta_{2r}}(|\nabla_{tan} f|^q)(Q) > \gamma\lambda\}| \\ &+ |\{Q \in \Delta_r : \mathcal{M}_{\Delta_{2r}}(|\partial_t^{\frac{1}{2}} f|^q)(Q) > \gamma\lambda\}|, \end{aligned} \tag{4.10}$$

for $\lambda \ge \lambda_0$, where $A = (2\epsilon)^{-\frac{q}{p}}$ and

$$\lambda_0 = \alpha \fint_{\Delta_{2r}} |\mathcal{N}(\nabla u)|^q\, d\sigma. \tag{4.11}$$

Assuming temporarily these results, we can now provide the proof of Theorem 2.19.

4.3 Proof of Theorem 2.19

Multiplying both sides of (4.10) by $\lambda^{\frac{p}{q}-1}$, integrating and using the p-boundedness of Hardy-Littlewood operator and (4.10), we get

$$\int_{\lambda_0}^{\Lambda} |E(A\lambda)|\lambda^{\frac{p}{q}-1}\,d\lambda \le \epsilon \int_{\lambda_0}^{\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\,d\lambda + c\int_{\Delta_{2r}} |\nabla_{tan} f|^p\,d\sigma + c\int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p\,d\sigma. \tag{4.12}$$

Working with the left hand side of (4.12), by a change of variables we find that

$$2\epsilon \int_{A\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\,d\lambda \le \epsilon \int_{\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\,d\lambda + c\int_{\Delta_{2r}} |\nabla_t f|^p\,d\sigma + c\int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p\,d\sigma. \tag{4.13}$$

Splitting the first integral of the right hand side, and noting that $A^{\frac{p}{q}} = \frac{1}{2\epsilon}$, (4.13) becomes

$$\int_{A\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\,d\lambda \lesssim \int_{\lambda_0}^{A\lambda_0} |E(\lambda)|\lambda^{\frac{p}{q}-1}\,d\lambda + \frac{1}{\epsilon}\int_{\Delta_{2r}} |\nabla_t f|^p\,d\sigma + \frac{1}{\epsilon}\int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p\,d\sigma. \tag{4.14}$$

From (4.14) and (4.9) we see that

$$\int_{A\lambda_0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\,d\lambda \lesssim |\Delta_{2r}|\lambda_0^{\frac{p}{q}} + \int_{\Delta_{2r}} |\nabla_t f|^p\,d\sigma + \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p\,d\sigma,$$

where ϵ has been incorporated to the constants of the inequality. From the last inequality we obtain

$$\int_{0}^{A\Lambda} |E(\lambda)|\lambda^{\frac{p}{q}-1}\,d\lambda \lesssim |\Delta_{2r}|\lambda_0^{\frac{p}{q}} + \int_{\Delta_{2r}} |\nabla_t f|^p\,d\sigma + \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p\,d\sigma.$$

Letting $\Lambda \to \infty$, from (4.10) we get

$$\int_{\Delta_{2r}} |\mathcal{N}(\nabla u)|^p\,d\sigma \lesssim |\Delta_{2r}|\lambda_0^{\frac{p}{q}} + \int_{\Delta_{2r}} |\nabla_t f|^p\,d\sigma + \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p\,d\sigma. \tag{4.15}$$

Substituting λ_0 as in (4.11), using the hypothesis $(R)_q$ via lema 4.4 and Theorem 2.18 and then Hölder's inequality we see that

$$\int_{\Delta_{2r}} |\mathcal{N}(\nabla u)|^p\,d\sigma \lesssim \int_{\Delta_{2r}} |\nabla_t f|^p\,d\sigma + \int_{\Delta_{2r}} |\partial_t^{\frac{1}{2}} f|^p\,d\sigma. \tag{4.16}$$

Finally, by a standard covering argument we obtain $(R)_p$ and the proof is finished.

4.4 Proofs of Technical Lemmas 4.4 and 4.5

Proof of Lemma 4.4. It is enough to prove the following two estimates:

$$\mathcal{N}(\nabla u)(Q) \lesssim \left(\frac{u}{\delta}\right)^*(Q) + \fint_{\Delta_{32r}} \mathcal{N}(\nabla u)\, d\sigma(Q), \qquad Q \in \Delta_r, \tag{4.17}$$

$$\left(\fint_{\Delta_r} \left|\left(\frac{u}{\delta}\right)^*(Q)\right|^p d\sigma(Q)\right)^{\frac{1}{p}} \lesssim \fint_{\Delta_{32r}} \mathcal{N}(\nabla u)(Q)\, d\sigma(Q), \tag{4.18}$$

where

$$\left(\frac{u}{\delta}\right)^*(Q) = \sup_{Y \in \Gamma_{20\alpha}(Q)} \left\{\frac{|u(Y)|}{\delta(Y)} : \delta(Y) \le r\right\}.$$

Let us begin by establishing (4.17). For $Q \in \Delta_r$ pick $X \in \Gamma(Q)$ with $\delta(X) \ge r$. This way, if $A = \{P \in \Delta_{32r} : X \in \Gamma(P)\}$ then we will have $|A| \gtrsim r^{n+1}$. Hence we have

$$\fint_{\Delta_{32r}} |\mathcal{N}(\nabla u)(P)|\, d\sigma(P) \ge \frac{1}{|\Delta_{32r}|} \int_A \left(\fint_{Q(X)} |\nabla u(Y)|^2 d\,Y\right)^{\frac{1}{2}} d\sigma(P) \gtrsim \left(\fint_{Q(X)} |\nabla u(Y)|^2 d\,Y\right)^{\frac{1}{2}}. \tag{4.19}$$

On the other hand, if $\delta(X) \le r$, by Caccioppoli's inequality we see that

$$\begin{aligned}\left(\fint_{Q(X)} |\nabla u(Y)|^2 d\,Y\right)^{\frac{1}{2}} &\lesssim \left(\frac{1}{|Q_{\frac{3}{4}\delta(X)}(X)|} \int_{Q_{\frac{3}{4}\delta(X)}(X)} \frac{|u(Y)|^2}{\delta(X)^2}\, d\,Y\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{1}{|Q_{\frac{3}{4}\delta(X)}(X)|} \int_{Q_{\frac{3}{4}\delta(X)}(X)} \frac{|u(Y)|^2}{\delta(Y)^2}\, d\,Y\right)^{\frac{1}{2}} \lesssim \left(\frac{u}{\delta}\right)^*(Q).\end{aligned} \tag{4.20}$$

Now (4.17) follows from (4.19) and (4.20).

We now focus on proving (4.18). Applying lemma (4.3) one deduces that

$$\left(\frac{u}{\delta}\right)^*(Q) \lesssim \frac{1}{G(\Xi; \overline{\mathcal{A}}_{12r}(\widetilde{Q}_0))} \left(\fint_{\Psi_{18r}(Q_0)} u^2(Y)\, d\,Y\right)^{\frac{1}{2}} \left(\frac{G(\Xi,\cdot)}{\delta}\right)^*(\widetilde{Q}), \tag{4.21}$$

for $Q \in \Delta_r(Q_0)$. By Theorem 2.4 we know that

$$\frac{G(\Xi, \overline{\mathcal{A}}_{\delta(X)}(\widetilde{Q}))}{\delta(X)} \lesssim \frac{\omega(\Delta_{\delta(X)}(\widetilde{Q}))}{\delta(X)^{n+1}}.$$

Using this and the adjoint version of Carleson estimate, for $X \in \Gamma(Q)$ with $\delta(X) \le r$ we have

$$\frac{G(\Xi; \widetilde{X})}{\delta(\widetilde{X})} \lesssim \frac{G(\Xi; \underline{\mathcal{A}}_{\delta(X)}(\widetilde{Q}))}{\delta(\widetilde{X})} \lesssim \frac{G(\Xi; \overline{\mathcal{A}}_{\delta(X)}(\widetilde{Q}))}{\delta(\widetilde{X})} \lesssim \frac{\omega(\Delta_{\delta(X)}(\widetilde{Q}))}{\delta(\widetilde{X})^{n+1}}.$$

Therefore we obtain,

$$\left(\frac{G(\Xi,\cdot)}{\delta(\cdot)}\right)^{*}(\widetilde{Q}) \lesssim \sup_{0<\rho<r}\left\{\frac{\omega(\Delta_\rho(\widetilde{Q}))}{\rho^{n+1}}\right\} \equiv \mathcal{M}_{\sigma,r}(\omega)(\widetilde{Q}), \tag{4.22}$$

for $Q \in \Delta_r(Q_0)$. By hypothesis $(D)_{p'}$ is solvable in Ω_T, so we know that

$$\left(\fint_{\Delta_r(\widetilde{Q}_0)}\left|\frac{d\omega}{d\sigma}\right|^p d\sigma\right)^{\frac{1}{p}} \lesssim \frac{\omega(\Delta_r(\widetilde{Q}_0))}{|\Delta_r(\widetilde{Q}_0)|}. \tag{4.23}$$

By (4.21), (4.22), (4.23) and the L^p boundedness of $M_{\sigma,r}(\omega)$)where this last maximal operator is the same we defined at the end of section 4.1) we get

$$\begin{aligned}
\left(\fint_{\Delta_r}\left|\left(\frac{u}{\delta}\right)^{*}(Q)\right|^p d\sigma(Q)\right)^{\frac{1}{p}} &\lesssim \left(\fint_{\Delta_r}\left(\frac{1}{G(\Xi;\overline{\mathcal{A}}_{12r}(\widetilde{Q}_0))}\left(\fint_{\Psi_{18r}(Q_0)}u^2(Y)\,dY\right)^{\frac{1}{2}}\left(\frac{G(\Xi,\cdot)}{\delta(\cdot)}\right)^{*}(\widetilde{Q})\right)^p d\sigma(Q)\right)^{\frac{1}{p}} \\
&\lesssim \frac{1}{G(\Xi;\overline{\mathcal{A}}_{12r}(\widetilde{Q}_0))}\left(\fint_{\Psi_{18r}(Q_0)}u^2(Y)\,dY\right)^{\frac{1}{2}}\left(\fint_{\Delta_r}(M_{\sigma,r}(\omega)(\widetilde{Q}))^p\,d\sigma(Q)\right)^{\frac{1}{p}} \\
&\lesssim \frac{1}{G(\Xi;\overline{\mathcal{A}}_{12r}(\widetilde{Q}_0))}\left(\fint_{\Psi_{18r}(Q_0)}u^2(Y)\,dY\right)^{\frac{1}{2}}\left(\fint_{\Delta_r(\widetilde{Q}_0)}\left(\frac{d\omega}{d\sigma}\right)^p(Q)\,d\sigma(Q)\right)^{\frac{1}{p}} \\
&\lesssim \frac{1}{G(\Xi;\underline{\mathcal{A}}_{12r}(\widetilde{Q}_0))}\left(\fint_{\Psi_{18r}(Q_0)}u^2(Y)\,dY\right)^{\frac{1}{2}}\frac{\omega(\Delta_r(\widetilde{Q}_0))}{|\Delta_r(\widetilde{Q}_0)|} \\
&\lesssim \frac{1}{r}\left(\fint_{\Psi_{18r}(Q_0)}u^2(Y)\,dY\right)^{\frac{1}{2}},
\end{aligned}$$

where the last two inequalities are consequence of Theorems 2.4 and 2.5 respectively. We can continue this sequence of inequalities making use of Poincaré's inequality in Lemma 3.2 with $\alpha = 0$, and obtain

$$\left(\fint_{\Delta_r}\left|\left(\frac{u}{\delta}\right)^{*}(Q)\right|^p d\sigma(Q)\right)^{\frac{1}{p}} \lesssim \left(\fint_{\Psi_{18r}(Q_0)}|\nabla u(Y)|^2\,dY\right)^{\frac{1}{2}}. \tag{4.24}$$

By Caccioppoli's inequality at the boundary, and arguing as in (4.8) we get

$$\begin{aligned}
\left(\fint_{\Psi_{18r}(Q_0)}|\nabla u(Y)|^2\,dY\right)^{\frac{1}{2}} &\lesssim r^{-1}\left(\fint_{\Psi_{19r}(Q_0)}u^2(Y)\,dY\right)^{\frac{1}{2}} \lesssim r^{-1}\sup_{\Psi_{19r}(Q_0)}|u| \\
&\lesssim r^{-1}\fint_{\Psi_{28r}(Q_0)}|u(Y)|\,dY \lesssim \fint_{\Psi_{28r}(Q_0)}|\nabla u(Y)|\,dY.
\end{aligned} \tag{4.25}$$

The proof is complete once we put together (4.24),(4.25) and apply lema 4.2. □

Proof of Lemma 4.5. Let $\epsilon > 0$ be a small constant to be chosen later. By the weak $(1,1)$ estimate for the maximal operator $\mathcal{M}_{\Delta_{2r}}$ we have $|E(\lambda)| \leq \epsilon|\Delta_r|$ for $\lambda \geq \lambda_0$ if we choose α big enough.

Now we apply the Calderón-Zygmund type decomposition (described in [38, p. 210]) and obtain a collection of disjoint cubes $\{Q_k\}_k$ contained in Δ_r such that $E(\lambda) = \bigcup_k Q_k$ and each Q_k is maximal. We may also choose ϵ small enough so that $64Q_k \subset \Delta_{2r}$.

The key statement of this proof is that there exists constants $\epsilon > 0, \gamma, \alpha > 0$ such that if Q_k is a cube that satisfies

$$F_k = \left\{ Q \in Q_k : \mathcal{M}_{\Delta_{2r}}(|\nabla_{tan} f|^q)(Q) \le \gamma\lambda, \quad \mathcal{M}_{\Delta_{2r}}(|\partial_t^{\frac{1}{2}} f|^q)(Q) \le \gamma\lambda \right\} \ne \emptyset, \tag{4.26}$$

then

$$|E(A\lambda) \cap Q_k| \lesssim \epsilon |Q_k|. \tag{4.27}$$

From this, setting $C_\lambda = \bigcup_k F_k$, we have

$$|E(A\lambda) \cap C_\lambda| \le \sum_k |E(A\lambda) \cap Q_k| \le \epsilon \sum_k |Q_k| = \epsilon |E(\lambda)|$$

and (4.10) follows. To prove (4.27), under the assumption (4.26), we notice that for $Q \in Q_k$

$$\mathcal{M}_{\Delta_{2r}}(|\mathcal{N}(\nabla u)|^q)(Q) \le \max\{\mathcal{M}_{2Q_k}(|\mathcal{N}(\nabla u)|^q)(Q), \beta\lambda\}. \tag{4.28}$$

For ϵ small enough $A \ge \beta$, so in view of (4.28), we get

$$|E(A\lambda) \cap Q_k| \le |\left\{ Q \in Q_k : \mathcal{M}_{2Q_k}(|\mathcal{N}(\nabla u)|^q)(Q) > A\lambda \right\}|. \tag{4.29}$$

Now, for each k consider the smooth function $\phi_k : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\phi_k = 1$ in $64Q_k$, $\phi_k = 0$ in $(66Q_k)^c$, $|\nabla\phi_k| \lesssim |Q_k|^{-\frac{1}{n+1}}$ and $|(\phi_k)_t| \lesssim |Q_k|^{-\frac{2}{n+1}}$. Let v_k be the solution to $\mathcal{L}v = 0$ in Ω_T with boundary data $\phi_k(f - \alpha_k)$ where $\alpha_k = \fint_{64Q_k} f \, d\sigma$. Let $\overline{p} > p$. By (4.29), we obtain

$$\begin{aligned} |E(A\lambda) \cap Q_k| &\le \left| \left\{ Q \in Q_k : \mathcal{M}_{2Q_k}(|\mathcal{N}(\nabla u - \nabla v_k)|^q)(Q) > \frac{A\lambda}{2^{q+1}} \right\} \right| \\ &\quad + \left| \left\{ Q \in Q_k : \mathcal{M}_{2Q_k}(|\mathcal{N}(\nabla v_k)|^q)(Q) > \frac{A\lambda}{2^{q+1}} \right\} \right| \\ &\lesssim \frac{1}{(A\lambda)^{\frac{\overline{p}}{q}}} \int_{2Q_k} |\mathcal{N}(\nabla u - \nabla v_k)|^{\overline{p}} d\sigma + \frac{1}{A\lambda} \int_{2Q_k} |\mathcal{N}(\nabla v_k)|^q d\sigma \equiv \mathcal{I} + \mathcal{II}. \end{aligned} \tag{4.30}$$

First, let's handle $\mathcal{II}$. By hypothesis $(R)_q$ is solvable, which yields

$$\mathcal{II} \lesssim \frac{1}{A\lambda} \int_{66Q_k} \left| \nabla_{tan} \phi(f - \alpha_k) \right|^q d\sigma + \frac{1}{A\lambda} \int_{66Q_k} \left| \partial_t^{\frac{1}{2}} \phi(f - \alpha_k) \right|^q d\sigma \equiv \mathcal{III} + \mathcal{IV}.$$

From the Poincaré inequality in Theorem 3.1, we can see that

$$\begin{aligned}\mathcal{III} &\lesssim \frac{1}{A\lambda}\int_{66Q_k}\left|\phi\nabla_{tan}f\right|^q d\sigma + \frac{1}{A\lambda}\int_{66Q_k}\left|(f-\alpha_k)\nabla_{tan}\phi\right|^q d\sigma\\ &\lesssim \frac{1}{A\lambda}\int_{66Q_k}|\nabla_{tan}f|^q d\sigma + \frac{1}{A\lambda}|Q_k|^{\frac{-q}{n+1}}\int_{66Q_k}|f-\alpha_k|^q d\sigma\\ &\lesssim \frac{1}{A\lambda}\int_{66Q_k}|\nabla_{tan}f|^q d\sigma + \frac{1}{A\lambda}\int_{66Q_k}\left|\partial_t^{\frac{1}{2}}f\right|^q d\sigma.\end{aligned}$$

In order to bound $\mathcal{IV}$, we first notice that

$$\partial_t^{\frac{1}{2}}(\phi_k(f-\alpha_k))(q,s) = \left(\int_{I_k}\frac{|\phi_k(f-\alpha_k)(q,\tau)-\phi_k(f-\alpha_k)(q,s)|^2}{|\tau-s|^2}\,d\tau\right)^{\frac{1}{2}}$$

$$\begin{aligned}&\lesssim \left\{\int_{I_k}\left(|\phi_k(q,\tau)|^2\frac{|f(q,\tau)-f(q,s)|^2}{|\tau-s|^2} + |f(q,s)-\alpha_k|^2\frac{|\phi_k(q,\tau)-\phi_k(q,s)|^2}{|\tau-s|^2}\right)d\tau\right\}^{\frac{1}{2}}\\ &\lesssim \left\{\int_{I_k}\frac{|f(q,\tau)-f(q,s)|^2}{|\tau-s|^2}\,d\tau\right\}^{\frac{1}{2}} + |f(q,s)-\alpha_k|\left\{\int_{I_k}\frac{|\phi_k(q,\tau)-\phi_k(q,s)|^2}{|\tau-s|^2}\,d\tau\right\}^{\frac{1}{2}}\\ &\lesssim \partial_t^{\frac{1}{2}}f(q,s) + |Q_k|^{-\frac{1}{n+1}}|f(q,s)-\alpha_k|,\end{aligned}$$

where I_k is the projection over the t axis of $66Q_k$. Consequently,

$$\mathcal{IV} \lesssim \frac{1}{A\lambda}\int_{66Q_k}|\nabla_{tan}f|^q d\sigma + \frac{1}{A\lambda}\int_{66Q_k}\left|\partial_t^{\frac{1}{2}}f\right|^q d\sigma.$$

The estimates for $\mathcal{III}$ and $\mathcal{IV}$ together with (4.26) give,

$$\mathcal{II} \lesssim \frac{\gamma|Q_k|}{A}.$$

Now, let's handle $\mathcal{I}$. Note that the hypothesis $(D)_{p'}$ and well-known properties of the L^p-Dirichlet problem implies $(D)_{\overline{p}'}$ for some $\overline{p} > p$. Also observe that $u - v_k - \alpha_k$ is a solution with boundary data $(f-\alpha_k)(1-\phi)$ and it vanishes on $64Q_k$. By lemma 4.4 we find that

$$\begin{aligned}\mathcal{I} &\lesssim \frac{|2Q_k|}{(A\lambda)^{\frac{\overline{p}}{q}}}\left(\fint_{64Q_k}|\mathcal{N}(\nabla u-\nabla v_k)|d\sigma\right)^{\overline{p}}\\ &\lesssim \frac{|2Q_k|}{(A\lambda)^{\frac{\overline{p}}{q}}}\left\{\left(\fint_{64Q_k}|\mathcal{N}(\nabla u)|^q d\sigma\right)^{\frac{\overline{p}}{q}} + \left(\fint_{64Q_k}|\mathcal{N}(\nabla v_k)|^q d\sigma\right)^{\frac{\overline{p}}{q}}\right\}\\ &\lesssim \frac{|2Q_k|}{(A\lambda)^{\frac{\overline{p}}{q}}}\left(\fint_{66Q_k}|\nabla_{tan}f|^q d\sigma + \fint_{66Q_k}\left|\partial_t^{\frac{1}{2}}f\right|^q d\sigma\right)^{\frac{\overline{p}}{q}},\end{aligned}$$

where the last inequality is due to $(R)_q$. Using (4.26) again,

$$\mathcal{I} \lesssim \frac{|Q_k|}{A^{\frac{\overline{p}}{q}}}.$$

Finally, since $A = (2\epsilon)^{-\frac{q}{p}}$,

$$|E(A\lambda) \cap Q_k| \lesssim \{\gamma \epsilon^{\frac{q}{p}-1} + \epsilon^{\frac{\overline{p}}{p}-1}\}\epsilon |Q_k|.$$

We fix $\epsilon > 0$ so small such that $\epsilon^{\frac{\overline{p}}{p}-1} < 1$, and then we choose $\gamma > 0$ such that $\gamma\epsilon^{\frac{q}{p}-1} < 1$ and (4.27) follows. □

Chapter 5

Connection between $(R)_p$ and A_∞ conditions

In this chapter, Ω is always regarded as the region above a $Lip(1, 1/2)$ graph as described in 2.1.

In section 5.1, we generalize condition $(R)_p$ for the region above a $Lip(1, 1/2)$ graph and prove a technical estimate about maximal functions.

Finally in section 5.2, we provide a simple argument that adapts a couple of ideas from [28, 11] in order to prove that the condition $(R)_q$ implies the A_∞ property of parabolic measure. Incidentally, a similar argumentation has been recently performed in [34] for non-divergence elliptic equations.

5.1 $(R)_p$ condition revisited

The reflection technique in the proof of Theorem 2.18 as well as Theorem 2.19 carries a strong dependence of the time flatness in a Lipschitz Cylinder. The norm 2.13 carries that same dependence.

Our purpose in this section is to generalize the norm defined in 2.13 via a map that flattens the boundary of a region above a $Lip(1, 1/2)$ graph.

Let $\pi : \partial\Omega \to \mathbb{R}^n$ denote the canonical projection $\pi(x, \psi(x, t), t) = (x, t)$ and set $\widetilde{f} = f \circ \pi^{-1}$.

For $1 < p < \infty$ define

$$\|\widetilde{f}\|_{W_p^{1,\frac{1}{2}}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla_x \widetilde{f}(x,t)|^p dx\,dt + \int_{\mathbb{R}^n} |\partial_t^{\frac{1}{2}} \widetilde{f}(x,t)|^p dx\,dt\right)^{1/p}, \tag{5.1}$$

where

$$\partial_t^{\frac{1}{2}} \widetilde{f}(x,s) = \left(\int_{-\infty}^{T} \frac{|\widetilde{f}(x,s) - \widetilde{f}(x,t)|^2}{|s-t|^2}\,dt\right)^{1/2} \qquad \text{for every } x \in \mathbb{R}^n.$$

Then we define $W_p^{1,\frac{1}{2}}(\partial\Omega)$, $1 < p < \infty$, as the classes of equivalence of functions (equal almost everywhere) such that

$$\|f\|_{W_p^{1,\frac{1}{2}}(\partial\Omega)} \equiv \|\widetilde{f}\|_{W_p^{1,\frac{1}{2}}(\mathbb{R}^n)} < \infty.$$

As expected, we say that *the condition* $(R)_q$ *holds* ($1 < q < \infty$) *for* L *in* Ω_T whenever the following estimate holds

$$\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq c\|f\|_{W_p^{1,\frac{1}{2}}(\partial\Omega)},$$

for each $f \in C_0^\infty$, with a constant $c > 0$ not depending on f, where u is the solution to (2.4) corresponding to the datum function f and the *modified non-tangential maximal function* $\mathcal{N}v$ for continuous v is defined as in (2.16).

For $0 < \epsilon < 1$ and $0 < \kappa < 1$, there is also the (ϵ,κ)-*modified non-tangential maximal function*, defined as

$$\widetilde{\mathcal{N}}_{\epsilon,\kappa} v(Q) = \sup_{X \in \Gamma(Q)} \left(\frac{1}{|\kappa\Psi^\epsilon(X)|} \int_{\kappa\Psi^\epsilon(X)} |v|^2\,dY\right)^{1/2}.$$

With a slight adaptation of the argument in [20, p. 166] we can prove the following:

Lemma 5.1. *For every* $0 < \epsilon < 1$ *there exists* $0 < \kappa < 1$ *such that for* $0 < p < \infty$ *one has* $\|\widetilde{\mathcal{N}}_{\epsilon,\kappa} v\|_p \lesssim \|\mathcal{N} v\|_p$.

Proof. Given $\epsilon > 0$, choose $\kappa = \alpha\epsilon/4$. We start making some constructions and computations for some objects within Carleson regions and cones.

First notice that for any $P \in \partial\Omega$ and any $0 < R < r_0$, this choice of κ implies that $\kappa\Psi_R^\epsilon(P) \subset \Gamma(P)$. Next, construct slices of $\kappa\Psi_R^\epsilon(P) \subset \Gamma(P)$ given by

$$S_j \equiv S_j(P) = \kappa\Psi_R^\epsilon(P) \cap \left\{(y, y_n, s) : 2^j\epsilon R < y_n \leq 2^{j+1}\epsilon R\right\}, \quad j = 0, 1, \ldots, m-1,$$

where m is chosen so that $2^m\epsilon R > A_1 R > 2^{m-1}\epsilon R$. Hence $m < \log_2 A_1 - \log_2 \epsilon + 1$, which is an upper bound for m independent of R, α and P. Note that $\kappa\Psi_R^\epsilon(P) \subset \bigcup_{j=0}^{m} S_j$. Finally observe that the ratio $\sigma(\kappa\Psi^\epsilon)/\sigma(\Psi)$ remains constant, regardless of the radius of Ψ, which denotes any Carleson region located anywhere on $\partial\Omega$. Denote this constant by ϱ, and note that it clearly depends on α and ϵ.

Define

$$E_\lambda = \left\{P \in \partial\Omega : \mathcal{N}v(P) > \frac{\sqrt{\varrho}}{2\sqrt{m}}\lambda\right\} \quad \text{and} \quad E^*_\lambda = \left\{P \in \partial\Omega : M\left(\chi_{E_\lambda}(P)\right) > \left(\frac{\epsilon}{16}\right)^{n+1}\right\},$$

where M denotes the Hardy-Littlewood maximal function on $\partial\Omega$ with respect to σ. We claim that with these definitions, if $P \in \partial\Omega$ is such that $\widetilde{\mathcal{N}}_{\epsilon,\kappa}v(P) > \lambda$ then $P \in E^*_\lambda$. Observe that this claim may be used in the first inequality of the following estimates:

$$\int \left|\widetilde{\mathcal{N}}_{\epsilon,\kappa}v\right|^p d\sigma = p\int_0^\infty \lambda^{p-1}\sigma(\{P \in \Omega : \widetilde{\mathcal{N}}_{\epsilon,\kappa}v(P) > \lambda\})\, d\lambda \lesssim p\int_0^\infty \lambda^{p-1}\sigma(E^*_\lambda)\, d\lambda$$
$$\lesssim \frac{C_n}{\epsilon^{n+1}}p\int_0^\infty \lambda^{p-1}\sigma(E_\lambda)\, d\lambda \approx \frac{C_n\sqrt{m}}{\epsilon^{n+1}\sqrt{\varrho}}\int |\mathcal{N}v|^p\, d\sigma,$$

where of course we have used Hardy-Littlewood's theorem in the second inequality. This would finish the proof. To establish the claim, pick $P \in \partial\Omega$ such that $\widetilde{\mathcal{N}}_{\epsilon,\kappa}v(P) > \lambda$. Then, for certain $Y \in \Gamma(P)$ one has

$$\frac{1}{|\kappa\Psi^\epsilon(Y)|}\int_{\kappa\Psi^\epsilon(Y)} |v|^2\, dX > \left(\frac{\lambda}{2}\right)^2.$$

Call $\tilde{P} \equiv P_Y$ the vertical projection of Y on $\partial\Omega$, and r the vertical distance from Y to P. Then

$$\sum_{j=0}^{m-1}\int_{S_j(\tilde{P})} |v|^2\, dX \geq \int_{\kappa\Psi^\epsilon(Y)} |v|^2\, dX > \frac{\lambda^2}{4}|\kappa\Psi^\epsilon(Y)| = \frac{\lambda^2}{4}\varrho|\Psi(Y)|.$$

Now choose cylinders of the form $C_j(\tilde{P}) \equiv C(X_j)$, $j = 0, 1, \ldots, m$, with X_j, $j = 1, \ldots, m$, in the main axis of the truncated cone $\Gamma^{A_1 r}(\tilde{P})$, satisfying $C_j(\tilde{P}) \supset S_j$. Since and $|C_j(\tilde{P})| < |\Psi(Y)|$, this implies

$$\sum_{j=0}^{m-1}\frac{1}{C_j(\tilde{P})}\int_{C_j(\tilde{P})} |v|^2\, dX > \frac{\lambda^2}{4}\varrho, \qquad \text{which yields} \quad m\left[\mathcal{N}v(\tilde{P})\right]^2 < \frac{\varrho}{4}\lambda^2.$$

In fact, the same argumentation establishes that for certain surface ball $\Delta(\tilde{P})$ of radius $r_1 = \alpha\epsilon r/2$, and $Q \in \Delta(\tilde{P})$ one has $\mathcal{N}v(Q) > \sqrt{\varrho}\lambda/(2\sqrt{m})$. This means that $\Delta(\tilde{P}) \subset E_\lambda$. Hence

$$M(\chi_{E_\lambda}(P)) \geq \frac{\Delta(\tilde{P})}{\Delta_{2\alpha r}(P)} = \left(\frac{\epsilon}{4}\right)^{n+1} > \left(\frac{\epsilon}{16}\right)^{n+1}.$$

Therefore $P \in E^*_\lambda$, and the claim is proved. □

5.2 Proof of Theorem 2.21

Proof. We will establish that ω is absolutely continuous with respect to σ, and that (2.18) holds. For this purpose we let Δ denote any surface ball of radius $0 < r < r_0/5$, and let $\Delta_s = \Delta_s(Q) \subset \Delta$ be any surface ball of radius s satisfying $r/20 < s < r/8$. Denote by $\Delta' \subset 5\Delta$ another surface cube of radius r such that it is shifted away in time variable t by a distance of r^2 from Δ.

Once we think of s as a fixed parameter, we consider the boundary datum given by a smooth function f satisfying $0 \le f \le 1$, $f \equiv 1$ on $\Delta_{s/2}$, $f \equiv 0$ on $\partial\Omega \setminus \Delta_s$, with $|\nabla_T f| \lesssim 1/s$ and $|\partial_t f| \lesssim 1/s^2$. Let u be the solution of the Dirichlet problem $Lu = 0$, $u|_{\partial\Omega} = f$.

By our assumption, we know that $\|\mathcal{N}(\nabla u)\|_{L^q(\partial\Omega)} \le \|f\|_{W_q^{1,\frac{1}{2}}(\partial\Omega)}$, which by the choice of f yields

$$\|\mathcal{N}(\nabla u)\|_{L^q(\partial\Omega)} \lesssim \frac{1}{s}\sigma(\Delta_s)^{1/q}. \tag{5.2}$$

On the other hand, the doubling property of ω (see e.g. [37, §3]) and the above construction yields

$$\frac{\omega(\Delta_s)}{\omega(\Delta)} \lesssim \frac{1}{\omega(\Delta)}\int f\, d\omega = \frac{u(\Xi)}{\omega(\Delta)}.$$

By [37, Lemma 2.6] and Harnack's inequality

$$\frac{u(\Xi)}{\omega(\Delta)} \lesssim u(\overline{\mathcal{A}}_r) \lesssim u(\underline{\mathcal{A}}'_r) \lesssim \left(\frac{1}{|Q(\underline{\mathcal{A}}'_r)|}\int_{Q(\underline{\mathcal{A}}'_r)} u^2\, dX\right)^{1/2}. \tag{5.3}$$

Here we have used the notation $\overline{\mathcal{A}}_r = \overline{\mathcal{A}}(\Delta)$, $\underline{\mathcal{A}}'_r = \underline{\mathcal{A}}(\Delta')$, and once and for all we set $\overline{\mathcal{A}}'_r = \overline{\mathcal{A}}(\Delta')$ and $\underline{\mathcal{A}}_r = \underline{\mathcal{A}}(\Delta)$. Now we use Theorem 3.4 to obtain for any $\varepsilon_0 \in (0,1)$ and any $0 < \kappa < 1$

$$\left(\frac{1}{|Q(\underline{\mathcal{A}}'_r)|}\int_{Q(\underline{\mathcal{A}}'_r)} u^2\, dX\right)^{1/2} \lesssim r\left(\frac{1}{|\kappa\Psi^{\varepsilon_0}(\underline{\mathcal{A}}'_r)|}\int_{\kappa\Psi^{\varepsilon_0}(\underline{\mathcal{A}}'_r)} |\nabla u|^2\, dX\right)^{1/2} + \varepsilon_0^{\gamma} u(\overline{\mathcal{A}}'_r). \tag{5.4}$$

To handle the second term in the right hand side, we invoke Lemma 2.12, which in our situation implies

$$u(\overline{\mathcal{A}}'_r) \le c u(\underline{\mathcal{A}}'_r),$$

because Harnack's inequality and [37, Lemma 2.6] imply that $m = u(\underline{\mathcal{A}}(\Delta')) \gtrsim 1$. Hence, as in the last estimate in (5.3), back in (5.4), we obtain

$$\left(\frac{1}{|Q(\underline{\mathcal{A}}'_r)|}\int_{Q(\underline{\mathcal{A}}'_r)} u^2\, dX\right)^{1/2} \lesssim r\left(\frac{1}{|\kappa\Psi^{\varepsilon_0}(\underline{\mathcal{A}}'_r)|}\int_{\kappa\Psi^{\varepsilon_0}(\underline{\mathcal{A}}'_r)} |\nabla u|^2\, dX\right)^{1/2} + \varepsilon_0^{\gamma}\left(\frac{1}{|Q(\underline{\mathcal{A}}'_r)|}\int_{Q(\underline{\mathcal{A}}'_r)} u^2\, dX\right)^{1/2}.$$

Picking the value of ε_0, depending on n, the ellipticity constants of L and geometric features of Ω, we can hide a small term in the left hand side, and back in (5.3) we get

$$\frac{\omega(\Delta_s)}{\omega(\Delta)} \lesssim r\widetilde{\mathcal{N}}_{\varepsilon_0,\kappa}(\nabla u)(Q), \qquad \text{for } Q \in \Delta'',$$

where Δ'' is a surface cube of radius approximately r, with the property that the center of $Q(\underline{\mathcal{A}}'_r)$ is contained in $\Gamma(Q)$, for every $Q \in \Delta''$. Integrating over Δ'' with respect to σ, by the Lemma 5.1 with the adequate values of ε_0 and κ, and using (5.2), we obtain

$$\sigma(\Delta)^{1/q}\frac{\omega(\Delta_s)}{\omega(\Delta)} \lesssim \sigma(\Delta_s)^{1/q},$$

because $r/s \leq 20$ and $\sigma(\Delta) \approx \sigma(\Delta'')$. Although the constants involved in the last estimate depends on ε_0, this does not affect the essence of the estimate, since ε_0 has already been fixed. Note that this is (2.18) with $E = \Delta_s$, an arbitrary surface ball of radius $s \in (r/20, r/8)$ within Δ.

With a covering lemma of Vitali-type one can get the result for $E \subset \Delta$ any open set. Finally, by regularity of both ω and σ, and invoking the continuity property of measures, we get (2.18) for any Borel measurable set $E \subset \Delta$, which is what we wanted to prove. □

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