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GEOMETRIC STUDY OF THE SPACE OF A SPIN VIA THE MAJORANA REPRESENTATION

TESIS
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Resumen

La presente tesis es el resultado de la investigación realizada junto con mi tutor y mis compañeros durante mis estudios de doctorado. Producto de este periodo de trabajo, recientemente publicamos un artículo de investigación [1], además de que estamos planeando redactar un par más en un futuro cercano.

El tema central de esta tesis, es el uso de la representación de Majorana para estudiar diversos problemas desde un punto de vista geométrico. La representación de Majorana nos permite asignarle a los estados de un espín s una constelación, es decir, un conjunto de $2s$ puntos sobre la esfera unitaria. Esta representación tiene la propiedad de que, si un estado es rotado, la constelación asignada a éste rota de la misma manera. Por lo tanto, la representación de Majorana es útil cuando se estudian problemas relacionados con rotaciones.

Esta tesis ha sido dividida en cuatro partes.

En el capitulo 1, hacemos una breve introducción y presentamos varias herramientas matemáticas que usaremos a lo largo de la tesis.

En el capítulo 2, explicamos cómo describir al espacio projectivo de Hilbert de un espín s como un haz fibrado, donde el grupo es $SO(3)$ y la base es el "espacio de formas", el espacio cociente obtenido al identificar los estados que difieren por la acción de un operador de rotación. Por medio de la métrica de Fubini-Study, definimos una métrica para el espacio de formas y realizamos un estudio de las geodésicas y del escalar de Ricci para ésta. Algunas aplicaciones físicas de esta construcción son discutidas brevemente.

En el capítulo 3, definimos la representación estelar para los Grassmanianos del espacio de un espín s, la cual es una generalización natural a la representación de Majorana. Siendo más precisos, explicamos una manera de asignarle a un k-plano por el origen (subespacio vectorial de dimension k contendido en el espacio de estados del espín) un conjunto de $k(2s+1-k)$ puntos en la esfera unitaria al que llamaremos constelación del k-plano. Al igual que con la representación de Majorana, si un k-plano es rotado, su constelación rota de la misma manera. Debido a esto, podemos usarla para encontrar las simetrias rotacionales de un k-plano, un problema que es de interés en el contexto de computo cuántico. Sin embargo, hay un detalle; diferentes k-planos puedes tener la misma constelación. Debido a esto, no basta con mirar la constelación de un k-plano para encontrar sus simetrías. Para resolver este problema, presentamos dos maneras diferentes de asignarle a un k-plano más de una constelación tales que, es posible identificar sus simetrías sólo conociendo estas constelaciones. El primer procedimiento es más simple, pero tiene el problema que solamente es válido para 2-planos.

El segundo es computacionalmente más complicado, pero es aplicable para k-planos en general. También encontramos cuántos k-planos tienen la misma constelación genéricamente.

Finalmente, en el capítulo 4, estudiamos la robustez del efecto Wilczek-Zee bajo ruido externo. Para esto, consideramos el ejemplo de resonancia cuadrupolar nuclear. En este tipo de sistemas, manipulando un campo magnetico adecuadamente, podemos inducir una curva cerrada en el espacio de 2-planos de un espín $s = 3/2$. Por medio del efecto Wilczek-Zee, la fase no abeliana (la holonomia de esta curva) puede ser usada para implementar compuertas cuánticas. En este capítulo, encontramos los efectos del ruido del campo magnético (modelandolo como un proceso estocástico) en las compuertas. Más concretamente, hayamos la distribución de probabilidad de las compuetas obtenidas y calculamos la distancia promedio entre la compuerta obtenida y la del caso ideal (si no hubiera ruido en el campo magnético). También encontramos que hay una especie de frecuencia resonante que amplifica los efectos del ruido, efecto que no se encuentra reportado en la literatura.

Abstract

This thesis is the product of the research done with my advisor and my colleagues as a PhD student. Some of our results have been recently published in an article [1], and we are planning to write a couple more in the near future.

The common theme of this thesis, is the usage of the Majorana representation to study certain problems from a geometrical point of view. The Majorana representation allows us to assign to the states of a spin s a constellation; a set of $2s$ points over the unitary sphere. A nice property of this representation is that if a stated is rotated by a certain rotation, its corresponding constellation rotates accordingly, and therefore it is useful when studying problems dealing with rotations.

We have divided this thesis in four main chapters.

In chapter 1, we make a brief introduction and present various mathematical tools that we use throughout the thesis.

In chapter 2, we explain how to describe the projective Hilbert space of a spin s as a fiber bundle, where the acting group is $SO(3)$ and the base space is “shape space”; the quotient state obtained by identifying two states that differ by the action of a rotation operator. In terms of the Fubini-Study metric, we define one for the shape space, and make a study of the geodesic and the Ricci scalar of this newly defined metric. Some applications of this construction are briefly discussed.

In chapter 3, we define the stellar representation for the Grassmannians of the space of a spin s, a natural generalization of the usual Majorana representation. To be more precise, we explain a way to assign to a k-plane through the origin (contained in the state space of the spin) a set of $k(2s+1-k)$ points over the unitary sphere, referred as the constellation of the k-plane. Just like with the usual Majorana representation, if a k-plane is rotated, its corresponding constellation rotates accordingly. Because of this fact, this representation is useful, for instance, to find the rotational symmetries of a k-plane, a problem that might be of interest in quantum computing. However, there is a caveat; two different k-planes might be assigned to the same constellation, and therefore, we are not able to deduce the symmetries of a k-plane just by looking at its constellation. To solve this issue, we present two different procedures to assign a k-plane more than constellation, in a way such that we can deduce the rotational symmetries of a k-plane just by knowing these constellations. The first procedure is simpler, but it only works for 2-planes. The second procedure is more complicated, but it works

for general k-planes. We also answer the question of how many k-planes have the same constellation generically.

In chapter 4, by considering the example of nuclear quadrupole resonance, we study the robustness of the Wilczek-Zee effect under external noise. In this type of systems, by manipulating a magnetic field, one can induce a closed curve in the space of 2-planes of a spin $s = 3/2$. By the Wilczek-Zee effect, the non-abelian geometric phase (holonomy of this curve) can be used to implement quantum gates. In this chapter, we find the effects of the noise of the magnetic field (modeled as a stochastic process) on this gates . To be more concrete, we find the probability distribution for the gates obtained, and we also compute the average distance between them and the one for the ideal case (when there is no noise present). We also find that there is some kind of resonance frequency that amplifies the effect of the noise, effect that is not reported in the literature and that has to be considered when implementing this types of gates.

Contents

Chapter 1

Introduction and preliminaries

Introduction

Traditionally, quantum mechanics is treated as an algebraic theory. However, this does not mean that one can not study it from a geometrical point of view. As a matter of a fact, there are certain problems that can be more conveniently studied via geometrical methods. In this thesis we present some examples of such problems.

Of particular importance in quantum mechanics, is the study of a spin s. One approach to work with it, is via the Majorana representation, a geometrical way to visualize the states of a spin s as a set of $2s$ points in the unitary sphere. Although it was introduced almost 90 years ago in 1932 [2], it had not gotten the attention it deserves, until recently, when applications to quantum computation are being found. More of this latter on this chapter.

In this thesis, we study three different problems relevant to quantum computation using the Majorana representation and other geometrical machinery such as fiber bundles. The problems we tackle are: determining the geometrical properties of the shape space of a spin s, defining the Majorana representation for the Grassmannian and studying the robustness of the Wilczek-Zee effect under external noise.

Before explaining in detail these problems, we present the necessary preliminaries for the rest of the thesis. References are also provided in case the reader wants to delve deeper into a certain topic.

1.1 Projective Hilbert space

In this section, we briefly introduce the notion of projective Hilbert space, that is, according to [3], "the true state space of a quantum system". Another standard reference for this topic is [4].

When studying quantum mechanics, it is often said that the space state of a physical system is a complex Hilbert space, $\mathcal{H}$. This statement is imprecise in a certain sense. Physicists tend to work only with normalized states. Furthermore, states that only differ by a phase are physically indistinguishable. Because of this, quantum mechanics can be completely formulated in the space known as projective Hilbert space $\mathbb{P}(\mathcal{H})$ (although we work mostly with kinematical problems, quantum dynamics can also be formulated in geometrical terms in $\mathbb{P}(\mathcal{H})$, c.f. [5–7]).

$\mathbb{P}(\mathcal{H})$ is the set of all *rays* $[\psi]$ in $\mathcal{H}$, that is, all the linear subspaces of the following form,

$$[\psi] = \{\lambda|\psi\rangle,\, \lambda \in \mathbb{C}\} = \mathrm{span}\{|\psi\rangle\}\,, \tag{1.1.1}$$

where $|\psi\rangle$ is an arbitrary state. The set $[\psi]$ is commonly known as the ray through $|\psi\rangle$ or the projective state associated to $|\psi\rangle$. Clearly, two vectors of $\mathcal{H}$ are in the same ray if and only if they differ by a factor. Since such vectors define the same physical state (as discussed previously), there is a one-to-one correspondence between the set of physical states of a system and $\mathbb{P}(\mathcal{H})$, making it, in a sense, the state space of a quantum system.

Much of the structure of $\mathcal{H}$ can be inherited to $\mathbb{P}(\mathcal{H})$. Given a linear operator A of $\mathcal{H}$, we define an action of A on $\mathbb{P}(\mathcal{H})$ as follows,

$$A[\psi] \equiv [A\psi] \equiv \mathrm{span}\{A|\psi\rangle\}\,. \tag{1.1.2}$$

Using the inner product of $\mathcal{H}$, we can canonically define an inner product[1] between two rays $[\phi]$ and $[\psi]$ in $\mathbb{P}(\mathcal{H})$

$$\langle[\phi],[\psi]\rangle = \frac{|\langle\phi|\psi\rangle|}{\sqrt{\langle\phi|\phi\rangle\langle\psi|\psi\rangle}}\,. \tag{1.1.3}$$

It is easy to prove that the previous expression does not depend on the particular choice of the elements $|\psi\rangle$ and $|\phi\rangle$ in each ray.

[1] Note that, because $\mathbb{P}(\mathcal{H})$ is not a vector space, the product of equation (1.1.3) is not bilinear. The same can be said about the product given in equations (1.2.4) and (1.2.3) further down.

In terms of this inner product, we can define the distance between $[\phi]$ and $[\psi]$ as follows (the Fubini-Study metric),

$$d_{\mathbb{P}(\mathcal{H})}([\psi],[\phi]) = \arccos\langle[\phi],[\psi]\rangle\,, \tag{1.1.4}$$

One can also verify that $d_{\mathbb{P}(\mathcal{H})}$ satisfies the axioms of a distance.

Physically, the Fubini-Study distance is related with the transition probability between $|\psi\rangle$ and $|\phi\rangle$ [8]; mathematically, it is the angle between these two vectors. Note that the maximal distance between two rays is $\pi/2$, and it is attained in the case that $|\psi\rangle$ and $|\phi\rangle$ are orthogonal to each other. Also, note that $d_{\mathbb{P}(\mathcal{H})}$ is invariant under the action of a unitary operator U, $d_{\mathbb{P}(\mathcal{H})}([\psi],[\phi]) = d_{\mathbb{P}(\mathcal{H})}([U\psi],[U\phi])$.

In what follows, we introduce two different formalisms to work with $\mathbb{P}(\mathcal{H})$ that we use throughout the thesis; by using affine coordinates and by representing it with density operators for pure states.

1.1.1 Representations of $\mathbb{P}(\mathcal{H})$

Representation in terms of affine coordinates

Consider an orthonormal basis for $\mathcal{H}$, $\{|e_1\rangle,\ldots,|e_N\rangle\}$ (from now on, for simplicity, we assume that $\mathcal{H}$ is finite dimensional). Given $N-1$ complex numbers $b_2,\ldots,b_N$, consider the mapping

$$(b_2,\ldots,b_N) \leftrightarrow [\psi]\,, \quad \text{with } |\psi\rangle = |e_1\rangle + \sum_{i=2}^{N} b_i|e_i\rangle\,. \tag{1.1.5}$$

The numbers $b_2,\ldots,b_N$ are known as affine coordinates (note that the assignation rule depends of the choice of the basis $\{|e_i\rangle,\ i=1,\ldots,N\}$), and they can be used to cover almost all $\mathbb{P}(\mathcal{H})$, except for the states $[\psi]$ such that $\langle\psi|e_1\rangle = 0$. In a similar fashion, we can define a series of coordinate patches for all the rays $[\psi]$ such that $\langle\psi|e_i\rangle \neq 0$, $i=2,\ldots,N$. The union of these patches cover the whole $\mathbb{P}(\mathcal{H})$.

We could write the Fubini-Study distance (1.1.4) and the corresponding metric in terms of these coordinates but the resulting expression (c.f. [9, eqs. (4.45) and (4.51)]) is not very illuminating and we do not use it in this thesis.

Representation in terms of density operators

The second representation of $\mathbb{P}(\mathcal{H})$ we consider, is the one in terms of density operators for pure states. We use this formalism mostly in chapter 2. A density operator ρ is, by definition, a positive semidefinite self-adjoint operator such that $\mathrm{Tr}(\rho) = 1$. Additionally, if ρ is idempotent, $\rho^2 = \rho$, we say that ρ describes a pure state. One can prove that this is the case if and only if there is a normalized $|\psi\rangle$ in $\mathcal{H}$ such that $\rho = |\psi\rangle\langle\psi|$. This result allows us to give a bijection between the set of density operators for pure states and $\mathbb{P}(\mathcal{H})$ as follows,

$$[\psi] \leftrightarrow \rho_\psi = |\psi\rangle\langle\psi| \,. \tag{1.1.6}$$

It is easy to prove that the previous assignment does not depend on the state defining $[\psi]$ (as long as it is normalized). In these terms, the left action of an operator A on the state associated to ρ defined in (1.1.2) can be represented as follows,

$$A \triangleright \rho = A\rho A^\dagger \,. \tag{1.1.7}$$

In a similar way, we can naturally define a right action;

$$\rho \triangleleft A = A^\dagger \triangleright \rho = A^\dagger \rho A \,. \tag{1.1.8}$$

In this representation, we can also describe tangent vectors in $\mathbb{P}(\mathcal{H})$ (via the push-forward of the bijection (1.1.6)) as self-adjoint operators. The following theorem allows us to characterize all self-adjoint operators that represent a tangent vector at a point ρ_ψ,

Theorem 1. *Consider a pure density operator ρ_ψ. A self-adjoint operator v represents a tangent vector at the point $[\psi]$ of $\mathbb{P}(\mathcal{H})$ if and only if all of the following conditions hold,*

(i) $\mathrm{Tr}(v) = 0$,

(ii) $\rho v + v\rho = v$.

Proof. Consider a curve $\rho_\psi(t)$ in the space of density operators. Denote by v the tangent vector $\dot{\rho}(0)$ (a dot denotes derivative w.r.t. t). By differentiating the condition $\mathrm{Tr}(\rho_\psi(t)) = 1$ and evaluating at $t = 0$, we obtain (i). By doing the same with $\rho^2 = \rho$ we obtain (ii). □

As we argue in the following paragraph, tangent vectors in $\mathbb{P}(\mathcal{H}_s)$ can also be represented by elements in $\mathcal{H}$. The resulting expressions are used in section 2.1.4.

Consider a curve $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$. If we define $|\phi(t)\rangle = e^{i\gamma(t)}|\psi(t)\rangle$, it is clear that ρ can also be written as $\rho(t) = |\phi(t)\rangle\langle\phi(t)|$. Since $|\phi(t)\rangle$ is assumed normalized, the equality $\Re\langle\dot{\phi}(t)|\phi(t)\rangle = 0$ holds, where $\Re$ denotes the real part. By suitably choosing the phase $\gamma(t)$, we can make the whole inner product $\langle\dot{\phi}(t)|\phi(t)\rangle$ zero. We say that the vector $|\dot{\phi}(t)\rangle$ such that this inner product is zero, represents the tangent vector $\dot{\rho}(t)$. This is the main idea of the following theorem.

Theorem 2. *Consider a state $|\psi\rangle$ and the density operator ρ_ψ. Given an operator v as in theorem 1 that represents a vector tangent at ρ_ψ, there exists a unique $|\psi_v\rangle$ in $\mathcal{H}$ that meets the following requirements,*

(a) $v = |\psi_v\rangle\langle\psi| + |\psi\rangle\langle\psi_v|$,

(b) $\langle\psi_v|\psi\rangle = 0$.

Proof. Let v denote an operator that satisfies the conditions *(i)* and *(ii)* of theorem 1. Note that this implies the equality $\mathrm{Tr}\,(v\rho) = 0$. Therefore, if we define $|\psi_v\rangle$ according to the following equality,

$$|\psi_v\rangle = v|\psi\rangle\,. \tag{1.1.9}$$

we can easily prove that $|\psi_v\rangle$ satisfies requirement *(b)*,

$$\langle\psi_v|\psi\rangle = \langle\psi|v|\psi\rangle = \mathrm{Tr}\,(v\rho) = 0\,.$$

By substituting $\rho = |\psi\rangle\langle\psi|$ and (1.1.9) in the condition *(ii)* of theorem 1, the requirement *(a)* follows immediately. The uniqueness of $|\psi_v\rangle$ can be proved by projecting both sides of *(a)* onto the state $\langle\psi|$. □

In the density matrix representation, the Fubini-Study distance (1.1.4) between ρ_ψ and ρ_ϕ is,

$$d_{\mathbb{P}(\mathcal{H})}(\rho_\psi, \rho_\phi) = \arccos\sqrt{\mathrm{Tr}\,(\rho_\phi\rho_\psi)}\,. \tag{1.1.10}$$

By considering the infinitesimal version of the previous equation, we can obtain the expression for the Fubini-Study metric. Given v_1 and v_2 tangent vectors at ρ_ψ, after some algebra, one can find that the induced inner product between them is,

$$h(v_1, v_2) = \frac{1}{2}\,\mathrm{Tr}\,(v_1 v_2)\,. \tag{1.1.11}$$

Given a metric, we can find its geodesics. In this case, the geodesic between the points ρ_{ψ_i} and ρ_{ψ_f} (that turns out to be unique if $\langle\psi_i|\psi_f\rangle \neq 0$) can be arc length parametrized as $\rho_{\psi(t)}$, where (a nice deduction of the following formula can be found in [10]),

$$|\psi(t)\rangle = \cos\left(\tfrac{t}{2}\right)|\psi_i\rangle + \sin\left(\tfrac{t}{2}\right)|\psi_i^{\perp}\rangle\,, \tag{1.1.12}$$

with

$$|\psi_i^{\perp}\rangle = \frac{|\psi_f\rangle - \langle\psi_i|\psi_f\rangle|\psi_i\rangle}{\sqrt{1-\langle\psi_i|\psi_f\rangle^2}}, \tag{1.1.13}$$

where we assumed w.l.o.g. that $|\psi_i\rangle$ and $|\psi_f\rangle$ are normalized and that $\langle\psi_i|\psi_f\rangle$ is real.

1.1.2 The geometric measure of entanglement

Consider the Hilbert space $\mathcal{H}$ of a composite system. Denote by $\mathcal{H}_i$ the Hilbert space associated to the i-th subsystem and by M the number of subsystems. Then, $\mathcal{H}$ can be written as the tensor product of the spaces of the subsystems, $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_M$. A state $|\psi\rangle$ of $\mathcal{H}$ is said to be separable if it can be written as the tensor product of states of each subsystem, $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_M\rangle$, with $|\psi_i\rangle \in \mathcal{H}_i$. The previous characterization can be used to define separable states in $\mathbb{P}(\mathcal{H})$ ($[\psi]$ is separable if and only if $|\psi\rangle$ is separable). Denote by $\mathbb{P}_{\text{sep}}$ the manifold of separable states in $\mathbb{P}(\mathcal{H})$,

$$\mathbb{P}_{\text{sep}} = \{[\psi], \text{ such that } [\psi] \text{ is separable}\}\,.$$

A projective state is said to be entangled if it is not separable. Entanglement can be considered as one of the main features of quantum mechanics, and has many applications, for instance, in the fields of quantum teleportation, quantum computation and quantum information [11].

Several ways to quantify the degree of entanglement of a state have been proposed in the literature [12–14]. In this thesis, we work with the one known as geometric measure of entanglement [15]. According to this definition, the degree of entanglement of a projective state $[\psi]$ is the distance (1.1.4) between $[\psi]$ and the closest separable state, that is,

$$E\left([\psi]\right) = \min_{[\phi]\in\mathbb{P}_{\text{sep}}} d_{\mathbb{P}(\mathcal{H})}([\phi],[\psi])\,. \tag{1.1.14}$$

States where the minimum is attained are called the separable states closest to $[\psi]$. In general, there might be more than one. From the previous definition, it is clear that $E\left([\psi]\right)$ is zero if and only if $[\psi]$ is separable. This measure of entanglement also turns out to be invariant under the action of *local unitary operators* (operators U of the form $U = U_1 \otimes \cdots \otimes U_M$, with U_i an unitary operator of $\mathcal{H}_i$) — this follows immediately from the fact that local unitary operators map separable states to separable states and that the Fubini-Study distance is invariant under the action of unitary operators.

In this thesis, we are interested in the subspace of symmetric states of a system of M indistinguishable particles (symmetric under the exchange of particles). We can study entanglement in terms of this subspace because the separable state closest to a symmetric state is also symmetric. Although this statement might be what the reader would expect, proving it is very complicated. See [16, 17] for some partial proofs and [18] for the general demonstration (in fact, in [18] the authors proved that the separable states closest to a symmetric one are necessarily symmetric, if the number of subsystems is greater than two). Clearly, the states that are both symmetric and separable, are those that can be written as $[\boldsymbol{\psi}]$ with $|\boldsymbol{\psi}\rangle = |\psi\rangle \otimes \cdots \otimes |\psi\rangle$, that is, the M-th tensor power of the one-particle state $|\psi\rangle$. Because of these facts, the entanglement of a symmetric state $|\psi\rangle$ can be written as,

$$E\left([\psi]\right) = \min_{[\phi] \in \mathbb{P}(\mathcal{H})} d_{\mathbb{P}(\mathcal{H})}([\boldsymbol{\phi}], [\psi])\,. \tag{1.1.15}$$

To conclude this section, we make a final definition. The state $[\psi]$ for which the entanglement is maximal is called maximally entangled state. These states are useful in the fields of quantum information and quantum computation; for instance, if some states are not entangled enough, they are not suitable to be used for quantum computation [19].

1.2 The Grassmannian

In this section, we present a generalization of the projective Hilbert space $\mathbb{P}(\mathcal{H})$. Recall that $\mathbb{P}(\mathcal{H})$ is defined as the set of rays (one dimensional subspaces) in $\mathcal{H}$. A natural generalization of this is the following: denote by $Gr_k(\mathcal{H})$ the manifold of all the k-planes[2] contained in $\mathcal{H}$. This manifold is called the Grassmannian. Clearly, the concept of the Grassmannian generalizes the one of projective Hilbert space, $Gr_1(\mathcal{H}) = \mathbb{P}(\mathcal{H})$.

In what follows, we present some geometrical structures of $Gr_k(\mathcal{H})$ that we will refer to later on. Some relevant references include [9, 20–22].

[2]Throughout this thesis, when we mention a k-plane we actually refer to a k-plane through the origin, that is, a k-dimensional subspace of $\mathcal{H}$

1.2.1 The space of k-frames

A k-frame $|\Psi\rangle$ is defined as a k-tuple of linearly independent vectors in $\mathcal{H}$,

$$|\Psi\rangle = (|\psi_1\rangle, \ldots, |\psi_k\rangle) \,.$$

To avoid confusion between a k-frame and an element of $\mathcal{H}$, we use uppercase Greek letters to denote a k-frame. Its elements, are denoted by the corresponding lowercase Greek letter, as in the previous equation. If the states defining $|\Psi\rangle$ are orthonormal among each other, we say that $|\Psi\rangle$ is an orthonormal k-frame. Denote by $L(\mathcal{H})$ the space of all k-frames (including the non-orthogonal ones).

As we are about to show, $L(\mathcal{H})$ can be naturally described as a principal fiber bundle. As base space, we take the Grassmannian $Gr_k(\mathcal{H})$, and define the projection operator π as,

$$\pi(\,|\Psi\rangle) = \mathrm{span}\{|\psi_1\rangle, \ldots, |\psi_k\rangle\} \in Gr_k(\mathcal{H}) \,. \tag{1.2.1}$$

With these definitions, k-frames that span the same k-plane are in the same fiber. With this in mind, consider the right action[3] of an invertible matrix $A \in GL_k(\mathbb{C})$ on the k-frame $|\Psi\rangle$, defined as follows,

$$|\Psi A\rangle = (|\phi_1\rangle, \ldots, |\phi_k\rangle) \,, \text{ where } |\phi_i\rangle = \sum_{j=1}^{k} |\psi_j\rangle A_{ji} \,. \tag{1.2.2}$$

Note that this action corresponds to a change of basis of the vectors spanning the k-plane. It is easy to show that any two k-frames that span the same k-plane are connected by the right action of a invertible matrix A. Because of this, the fibers are isomorphic to $GL_k(\mathbb{C})$.

The last mathematical structure we define, is that of an inner product between two k-frames. Define the product between $|\Phi\rangle$ and $|\Psi\rangle$ as follows,

$$\langle \Phi \,|\, \Psi \rangle = \mathrm{Det}\; F(\Phi, \Psi) \,, \tag{1.2.3}$$

where F is the $k \times k$ matrix whose entry ij is,

$$F_{ij}(\Phi, \Psi) = \langle \phi_i | \psi_j \rangle \,.$$

Note that flipping the argument of the inner product gives the complex conjugate, $\langle \Phi \,|\, \Psi \rangle = \langle \Psi \,|\, \Phi \rangle^*$.

[3] We could use the definition of principal fiber bundle in terms of left actions (as it is the usual physicist's convention), but we preferred to use the convention of [23] and consider right actions.

We can find a relationship between the inner product and the right action (1.2.2) by computing $\langle \Phi \,|\, \Psi A\rangle$. By definition, this is the determinant of the matrix $F(\Phi, \Psi A)$, whose entry ij is,

$$F_{ij}(\Phi, \Psi U) = \sum_{l=1}^{k} \langle \phi_j | \psi_l \rangle A_{li} = \sum_{l=1}^{k} F_{il}(\Phi, \Psi) A_{li}\,,$$

that is, $F(\Phi, \Psi A)$ is equal to the matrix product $F(\Phi, \Psi)A$. Because of this, by computing its determinant, we conclude,

$$\begin{aligned}\langle \Phi \,|\, \Psi A\rangle &= \text{Det } F(\Phi, \Psi A) = \text{Det } (F(\Phi, \Psi)A) = \text{Det } F(\Phi, \Psi)\,\text{Det } A \\ &= \text{Det } A \langle \Phi \,|\, \Psi \rangle\,,\end{aligned}$$

In the same way we can prove the equality $\langle \Phi A \,|\, \Psi\rangle = (\text{Det } A)^* \langle \Phi \,|\, \Psi\rangle$.

Of particular interest to us, is the subbundle of $L(\mathcal{H})$ of orthonormal k-frames, $M(\mathcal{H})$. For this subbundle the acting group is $U(k)$, the subgroup of $GL_k(\mathbb{C})$ of unitary matrices. In this case, if the k- orthonormal frames $|\Psi\rangle$ and $|\Psi'\rangle$ are in the same fiber ($|\Psi'\rangle = |\Psi U\rangle$ for a certain $U \in U(k)$) then, for any other orthonormal k-frame $|\Phi\rangle$, the equality $|\langle \Phi \,|\, \Psi\rangle| = |\langle \Phi \,|\, \Psi'\rangle|$ holds (recall that the modulus of the determinant of any unitary transformation is one). This observation allows us to define an inner product between two k-planes Π_Φ and Π_Ψ as,

$$\langle \Pi_\Phi, \Pi_\Psi \rangle = |\langle \Phi \,|\, \Psi\rangle|\,, \tag{1.2.4}$$

where $|\Phi\rangle$ and $|\Psi\rangle$ are any orthonormal k-frames that get projected (under π (1.2.1)) to Π_Φ and Π_Ψ respectively (note that the footnote 1 on page 2 also applies for this product). This formula is completely analogous to the one for $\mathbb{P}(\mathcal{H})$ (1.1.3), and it also allows us to define a distance between two k-planes ,

$$d(\Pi_\Phi, \Pi_\Psi) = \arccos(\langle \Pi_\Phi, \Pi_\Psi \rangle)\,. \tag{1.2.5}$$

We can give a simple geometrical interpretation of (1.2.4). Consider the projection operator $\mathcal{P}_\Phi$ associated to Π_Φ. Since, by definition, $|\Phi\rangle$ is an orthonormal k-frame that spans Π_Φ, we can write $\mathcal{P}_\Phi$ as $\mathcal{P}_\Phi = \sum_{l=1}^{k} |\phi_l\rangle\langle\phi_l|$. The restriction of $\mathcal{P}_\Phi$ to Π_Ψ, $\mathcal{P}_\Phi|_{\Pi_\Psi}$, is a linear mapping from Π_Ψ to Π_Φ that can be represented as a $k \times k$ matrix P (w.r.t. the orthonormal bases $|\Phi\rangle$ and $|\Psi\rangle$) with the following entries,

$$P_{ij} = \langle \phi_i | \mathcal{P}_\Phi | \psi_j \rangle = \langle \phi_i | \left(\sum_{l=1}^{k} |\phi_l\rangle\langle\phi_l| \right) |\psi_j\rangle = \langle \phi_i | \psi_j \rangle = F_{ij}(\Phi, \Psi)\,.$$

This means that $F(\Phi,\Psi)$ is the matrix representation of the operator $\mathcal{P}_\Phi|_{\Pi_\Psi}$. As such, using a well-known result from linear algebra, Det $F_{ij}(\Phi,\Psi)$ is the "complex volume" of the k-dimensional parallelepiped defined by the vectors $\mathcal{P}_\Phi|\psi_1\rangle,\ldots,\mathcal{P}_\Phi|\psi_k\rangle$. These observations allow us to characterize two k-planes that are orthogonal to each other.

Theorem 3. *Two k-planes Π_Φ and Π_Ψ are orthogonal to each other (their inner product $\langle\Pi_\Phi,\Pi_\Psi\rangle$ is zero) if and only if there exists a state $|\psi_0\rangle\in\Pi_\Psi$ such that $|\psi_0\rangle$ is orthogonal to all the elements of Π_Φ, that is, $|\psi_0\rangle$ is an element of the space orthogonal to Π_Φ.*

Proof. Suppose that the planes are orthogonal, $\langle\Pi_\Phi,\Pi_\Psi\rangle=0$. This happens if and only if the *kernel* of the mapping $\mathcal{P}_\Phi|_{\Pi_\Psi}$ is not trivial, that is, if there is a $|\psi_0\rangle\in\Pi_\Psi$ (different from zero) such that $\mathcal{P}_\Phi|_{\Pi_\Psi}|\psi_0\rangle=\mathcal{P}_\Phi|\psi_0\rangle=0$. This in turn happens if and only if $|\psi_0\rangle$ is orthogonal to the whole space Π_Φ. □

To conclude this section, in what follows, we mention three different ways to represent $Gr_k(\mathcal{H})$ that are relevant to the rest of the thesis.

1.2.2 Representations of $Gr_k(\mathcal{H})$

Representation in terms of affine coordinates

The affine coordinates for $Gr_k(\mathcal{H})$ are analogous to the ones for $\mathbb{P}(\mathcal{H})$ introduced in section 1.1. The main idea is to find a suitable section of $L(\mathcal{H})$ that is simple to parametrize. The procedure is explained in the following paragraph.

Take any basis (not necessarily orthonormal) for $\mathcal{H}$, $|e_1\rangle,\ldots,|e_N\rangle$ (where N denotes the dimension of $\mathcal{H}$). Just as any state in $\mathcal{H}$ can be represented as a column vector in $\mathbb{C}^N$ (w.r.t. this basis), any k-frame $|\Psi\rangle$ in $L(\mathcal{H})$ can be represented as a $N\times k$ matrix C, a matrix whose i-th column ($i=1,\ldots,k$) is the representation (as a column vector) of $|\psi_i\rangle$. To be more precise, the matrix C contains the coefficients of $|\psi_i\rangle$ in terms of this basis,

$$|\psi_i\rangle=\sum_{j=1}^{N}|e_j\rangle C_{ji}\,,\quad (i=1,\ldots,k)\,. \tag{1.2.6}$$

As notation, C is called the matrix representation of $|\Psi\rangle$ (w.r.t. the basis $|e_1\rangle,\ldots,|e_N\rangle$).

Since the vectors in the k-frame $|\Psi\rangle$ are linearly independent, appealing to a classical result from linear algebra (c.f. [24, VI, §9]), there exist a square submatrix of C of dimensions $k\times k$ (made by taking k rows of C) such that its determinant is different from zero. Suppose that the first k rows of C are those such that the determinant of the submatrix build by considering them is not zero. Denote by $\tilde{C}$ this submatrix ($\tilde{C}_{ij} = C_{ij}$, $i, j = 1, \ldots, k$). Then, C can be written in terms of $\tilde{C}$ as follows,

$$C_{N\times k} = \begin{pmatrix} \tilde{C}_{k\times k} \\ \hdashline D_{(N-k)\times k} \end{pmatrix} ,$$

where the subscripts indicate the size of a matrix. If we multiply C by the right by $\tilde{C}^{-1}$ (that exists by hypothesis) we obtain,

$$T_{N\times k} \equiv (C\tilde{C}^{-1})_{N\times k} = \begin{pmatrix} \mathbb{1}_{k\times k} \\ \hdashline B_{(N-k)\times k} \end{pmatrix} , \tag{1.2.7}$$

where $B = D\tilde{C}^{-1}$. Just like C, $C\tilde{C}^{-1}$ can be regarded as the matrix representation of a certain k-frame, call it $|\Phi\rangle$. A quick computation (considering equation (1.2.6)) shows that the elements $|\phi_i\rangle$ defining $|\Phi\rangle$ satisfy the equalities $|\phi_i\rangle = \sum_{l=1}^{N} |\psi_l\rangle(\tilde{C}^{-1})_{li}$, that is, $|\Phi\rangle$ and $|\Psi\rangle$ only differ by the right action of $\tilde{C}^{-1}$ ($|\Phi\rangle = |\Psi\tilde{C}^{-1}\rangle$) and hence, are in the same fiber. From this analysis, we can conclude the following: given a generic k-frame,[4] we can find another k-frame in the same fiber such that its matrix representation is of the form of the r.h.s. of (1.2.7). It is easy to verify that this k-frame is the only one in the fiber with such property. On the other hand, any matrix T of this form (1.2.7) (with arbitrary B) is the matrix representation of a k-frame (the k-frame $|\Phi\rangle$ given by the states $|\phi_i\rangle = \sum_{i=1}^{N} |e_j\rangle T_{ji}$ — one can easily show that these vectors are linearly independent for $i = 1, \ldots, k$). When a k-plane Π is given in terms of such matrix T, we say that Π is written in the *standard form* w.r.t. the basis $\{|e_1\rangle, \ldots, |e_N\rangle\}$.

Thanks to the previous observations, we can build a section of $L(\mathcal{H})$ (associated to the ordered basis $\{|e_1\rangle, \ldots, |e_N\rangle\}$). Given any matrix B of dimensions $(N-k)\times k$, consider the k-frame represented by the matrix T of (1.2.7). By taking the projection π (1.2.1) of the elements of this section, we can give coordinates to a sector (everything except from a set of measure zero, this is analogous to the case of affine coordinates defined in (1.1.5) for $\mathbb{P}(\mathcal{H})$) of $Gr_k(\mathcal{H})$; the entries of the matrix B serve as coordinates. From this analysis we can also conclude that the (complex) dimension of $Gr_k(\mathcal{H})$ is $k(N-k)$. By changing the order of the elements of the basis $|e_1\rangle, \ldots, |e_N\rangle$, and following the same construction, we can define additional coordinate

[4] Generic in the sense that the determinant of the submatrix $\tilde{C}$ build by considering the first k rows of the matrix associated to said k-frame is not zero.

patches such that their union covers the totality of the Grassmannian. This situation is completely analogous to the one of the affine coordinates for $\mathbb{P}(\mathcal{H})$.

Representation in terms of k-vectors

Given a Hilbert space $\mathcal{H}$, we define the space of k-vectors $\wedge^k(\mathcal{H})$ as the k-th exterior power of $\mathcal{H}$, that is, the linear span of vectors of the form $|\psi_1\rangle \wedge |\psi_2\rangle \wedge \cdots \wedge |\psi_k\rangle$, where all the states $|\psi_i\rangle$ are elements of $\mathcal{H}$ and the wedge product is defined in the usual way,

$$|\psi_1\rangle \wedge \cdots \wedge |\psi_k\rangle = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) |\psi_{\sigma_1}\rangle \otimes \cdots \otimes |\psi_{\sigma_k}\rangle \,, \tag{1.2.8}$$

where S_k denotes the permutation group of k objects. Besides elements of the form (1.2.8), $\wedge^k(\mathcal{H})$ also contains linear combinations of such elements. Notice that the dimension of $\wedge^k(\mathcal{H})$ is $\binom{N}{k}$.

We can define an inner product in $\wedge^k(\mathcal{H})$. To this end, it is enough to specify it for elements of the form (1.2.8). We do this in the following way,

$$\big\langle |\psi_1\rangle \wedge \cdots \wedge |\psi_k\rangle, |\phi_1\rangle \wedge \cdots \wedge |\phi_k\rangle \big\rangle = \left(\langle \psi_1| \wedge \cdots \wedge \langle \psi_k| \right)\left(|\phi_1\rangle, \ldots, |\phi_k\rangle \right) , \tag{1.2.9}$$

or in other words, the inner product of the two k-vectors is the wedge product of the dual vectors $\langle\psi_1| \ldots \langle\psi_k|$ evaluated in terms of the hermitean inner product at $(|\phi_1\rangle, \ldots, |\phi_k\rangle)$.

The Grassmannian $Gr_k\,(\mathcal{H})$ can be naturally embedded *projectively* in this space as follows. Consider any k-plane Π spanned by the vectors of a k-frame $|\Psi\rangle$. We assign to Π the following k-vector,

$$\Pi \to |\psi_1\rangle \wedge \cdots \wedge |\psi_k\rangle \,. \tag{1.2.10}$$

Some algebra reveals that, if we choose a different k-frame for Π, say $|\Psi A\rangle$, then, the resulting k-vector is $(\mathrm{Det}\ A)|\psi_1\rangle \wedge \ldots |\psi_k\rangle$, and therefore projectively equivalent to $|\psi_1\rangle \wedge \ldots |\psi_k\rangle$. In this sense, the representation of $Gr_k\,(\mathcal{H})$ in terms of elements in $\wedge^k(\mathcal{H})$ is a projective representation.

Also of importance is the fact that if $|\Psi\rangle$ is an orthonormal k-frame for Π_Ψ, and $|\Phi\rangle$ is one for Π_Φ, then, the inner product (1.2.4) can be written in the following way,

$$\langle \Pi_\Phi, \Pi_\Psi \rangle = \left| \big\langle |\psi_1\rangle \wedge \cdots \wedge |\psi_k\rangle, |\phi_1\rangle \wedge \cdots \wedge |\phi_k\rangle \big\rangle \right| \,. \tag{1.2.11}$$

We use this representation mostly in section 3.5.

1.2.3 The Wilczek-Zee connection for $M(\mathcal{H})$

The Wilczek-Zee (WZ) connection for $M(\mathcal{H})$ (the space of orthonormal k-frames) [25, 26] has gain much interest in the recent years as it provides a theoretical basis for the so-called *holonomic quantum computation* [27, 28]. In this section, we briefly introduce the WZ connection and its relation with quantum computation.

As it is well-known, one possible way to define a connection (c.f. [23, section 1.2]) is to specify the space of horizontal vectors tangent at a point. Because of this, first we characterize the tangent space at a point $|\Psi\rangle \in M(\mathcal{H})$. To this end, consider a curve of orthonormal k-frames $|\Phi(t)\rangle$ such that $|\Phi(0)\rangle = |\Psi\rangle$. Then, the tangent vector at $t = 0$ can be written as a k-tuple of vectors (in $\mathcal{H}$) $|\Xi\rangle \equiv (\xi_1 \equiv |\dot{\phi}_1(0)\rangle, \dots, \xi_k \equiv |\dot{\phi}_k(0)\rangle)$. By differentiating the orthonormality condition $\langle\phi_i(t)|\phi_j(t)\rangle = \delta_{ij}$ at $t = 0$, we obtain that the tangent space at $|\Psi\rangle$ consists of all the k-tuples $|\Xi\rangle$ such that,

$$\Re\langle\xi_i|\psi_j\rangle = 0\,, \text{ for } i, j = 1, \dots, k\,,$$

where denotes $\Re$ real part. We say that a tangent vector $|\Xi\rangle$ is horizontal if also the imaginary part of these products is zero, that is, if

$$\langle\xi_i|\psi_j\rangle = 0\,, \text{ for } i, j = 1, \dots, k\,. \tag{1.2.12}$$

It is immediate to verify that this definition of horizontal vectors satisfy the three necessary conditions to induce a connection. First of all, none of the vectors satisfying (1.2.12) are vertical (tangent to the fibers) as can be easily shown. Secondly, a quick computation reveals that the (complex) dimension of the space of horizontal vectors tangent to a point in $M(\mathcal{H})$ is $k(N-k)$, the same one as the one of $Gr_k(\mathcal{H})$. Finally, it is also trivial to prove that the pushforward of the right action of any unitary matrix $U(k)$ maps horizontal vectors to horizontal vectors.

This connection, as we are about to see, is the one induced by the Schrödinger evolution in the adiabatic limit. The framework is the following. Consider a time-dependent Hamiltonian $H(t)$ $(0 \le t \le T)$ such that a certain energy level is k-degenerate for all t. Associated to this Hamiltonian, we can define a curve Π_t in the Grassmannian $Gr_k(\mathcal{H})$ by considering the k-plane spanned by the states in the degenerate level at a time t.

Consider any orthonormal k-frame $|\Psi_0\rangle$ that spans Π_0. Via Schrödinger evolution, we can obtain a curve $|\Psi_t\rangle$ in the space of orthonormal k-frames by evolving each state of $|\Psi_0\rangle$ from the initial time to t. If the time dependence of the Hamiltonian is adiabatic [29, Chapter XVII-II], any state in the degenerate level remains in it throughout all the evolution. This implies that all the elements in the frame $|\Psi_t\rangle$ belong to the plane Π_t and, moreover, they span it, $\pi(|\Psi_t\rangle) = \Pi_t$. If the curve Π_t is closed ($\Pi_0 = \Pi_T$) then, $|\Phi_0\rangle$ and $|\Phi_T\rangle$ are in the same fiber. Because of this, they differ by the right action of a unitary matrix U_{tot}, $|\Phi_T\rangle = |\Phi_0 U_{\text{tot}}\rangle$. Now, we compute U_{tot}. To this end, denote by E_t the value of the energy of the degenerate level at time t, and consider the curve of k-frames $|\Phi_t\rangle$ given by the following equation,

$$|\Psi_t\rangle = |\Phi_t e^{-i\mathcal{E}_t}\rangle = (e^{-i\mathcal{E}_t}|\phi_{1t}\rangle, \ldots, e^{-i\mathcal{E}_t}|\phi_{kt}\rangle)\,,$$

where,

$$\mathcal{E}_t = \int_0^t E_\tau \, d\tau\,.$$

Clearly, $|\Psi_t\rangle$ and $|\Phi_t\rangle$ are always in the same fiber as they only differ by a phase factor. The claim is that the curve $|\Phi_t\rangle$ is an horizontal lift of Π_t for the Wilczek-Zee connection (1.2.12). We already argued that $\pi(|\Psi_t\rangle) = \pi(|\Phi_t\rangle) = \Pi_t$, what remains to be proved is that $|\Phi_t\rangle$ is horizontal. To verify this, consider the Schrödinger equation of the i-th element in the frame $|\Psi_0\rangle$ (as usual, a dot denotes time derivative),

$$i|\dot{\psi}_{ti}\rangle = H|\psi_{ti}\rangle \Rightarrow E_t|\phi_{ti}\rangle + i|\dot{\phi}_{ti}\rangle = H|\phi_{ti}\rangle\,,$$

where we used the equality $|\phi_{ti}\rangle = e^{-i\mathcal{E}_t}|\psi_{ti}\rangle$ to obtain the implication. By computing the inner product of this equation with $\langle\phi_{jt}|$ and noting that $\langle\phi_{it}|H(t)|\phi_{jt}\rangle = E_t\delta_{ij}$ (since both states are eigenstates of $H(t)$) we obtain,

$$\langle\phi_{tj}|\xi_{ti}\rangle = 0\,,$$

where we defined $|\xi_{ti}\rangle \equiv |\dot{\phi}_{ti}\rangle$. This equality implies that the tangent vectors to the curve $|\Phi_t\rangle$ satisfy the horizontality condition (1.2.12) and, therefore, $|\Phi_t\rangle$ is horizontal as claimed. Denote by $U_{\text{geo}} \in U(k)$ the holonomy of the curve $|\Phi_t\rangle$. Then, by the previous argument, $|\Phi_T\rangle = |\Phi_0 U_{\text{geo}}\rangle$. As an holonomy, U_{geo} only depends on the trace of the curve Π_t — not in its parametrization — and therefore is termed geometrical. This matrix is known as a *non-abelian geometric phase* and, in the particular case $k = 1$, it reduces to the well-known Berry's phase [30, 31]. As a side note, we mention that this construction for the non-abelian phase can be extended to the case where the evolution of the Hamiltonian is not adiabatic [32].

By remembering that $|\Psi_t\rangle$ and $|\Phi_t\rangle$ only differ by the phase factor $e^{-i\mathcal{E}_t}$, we can write $U_{\text{tot}} = e^{-i\mathcal{E}_T} U_{\text{geo}}$.

The result of the previous paragraph has an important application; if a state $|\eta_0\rangle$, initially in the degenerate level (so it can be written as $|\eta_0\rangle = \sum_{i=1}^{k} |\psi_{0i}\rangle B_i$ for certain complex numbers B_i, $i = 1, \ldots, k$) is evolved, after a time T, the resulting state $|\eta_T\rangle$ is given by the following expression,

$$|\eta_T\rangle = e^{-i\mathcal{E}_T} \sum_{i,j=1}^{k} |\psi_{0i}\rangle (U_{\text{geo}})_{ij} B_j \,, \tag{1.2.13}$$

that is, the unitary matrix that encodes the information of how the degenerate states mix among each other is — except for an overall factor $e^{-i\mathcal{E}_T}$ known as *dynamical phase* — U_{geo}. As stressed above, U_{geo} only depends[5] on the trace of the curve Π_t. Basically, this is the main idea behind the Wilczek-Zee effect: closed curves in $Gr_k(\mathcal{H})$ induce unitary transformations for the initial (and therefore final) k-plane of the curve.

The proposal for quantum computation known as *holonomic quantum computation* consists in using the Wilczek-Zee effect to implement *quantum gates* (unitary transformations) to mix states in the degenerated space. The main motivation is that, since the holonomy U_{geo} is of a geometrical nature, it is more robust against external noise. In section 4, we study a simple case to see if this is the case.

Finally, to conclude this section, we mention very briefly how to find the holonomy U_{geo} in terms of a section of $M(\mathcal{H})$, as this is the approach that tends to be easier when making actual computations. Denote the elements of the section generically by $|\Psi\rangle$, and suppose that $|\Psi(t)\rangle$ denotes the closed curve in the section that gets projected onto the curve in $Gr_k(\mathcal{H})$ we are considering. Then, the problem of finding U_{geo}, reduces to the one of finding a curve of unitary matrices $U(t)$ such that the curve of k-frames $|\tilde{\Psi}(t)\rangle = |\Psi(t)U(t)\rangle$ is horizontal and, that $U(t)$ starts at the identity matrix, $U(0) = \mathbb{1}$. In these terms, the non-abelian geometric phase, U_{geo} is equal to $U(T)$. The consideration of condition (1.2.12) leads to the following equation for U (all the time dependence has been dropped out to improve readability),

$$\langle\tilde{\psi}_i|\tilde{v}_j\rangle = 0 \Rightarrow \sum_{l,m,j=1}^{k} \langle\psi_l|U_{li}^{*}\left(|v_m\rangle U_{mj} + |\psi_m\rangle \dot{U}_{mj}\right) = 0\,. \tag{1.2.14}$$

Therefore, if we define the matrix A according to the equality

$$A_{lm} \equiv i\langle\psi_l|v_m\rangle = i\langle\psi_l|\dot{\psi}_m\rangle\,. \tag{1.2.15}$$

[5] As a matrix, U_{geo} also depends on the initial k-frame considered. However, it is easy to prove that the induced unitary transformation in the degenerate space (1.2.13) is unique, the only difference is w.r.t. what basis this transformation is written.

then, (1.2.14) can be written matricially as

$$\dot{U} = iAU \, . \tag{1.2.16}$$

By solving the previous matrix differential equation, we can find $U(T) = U_{\text{geo}}$. Note that A is an hermitian matrix (as can be easily verified) so that $U(T)$ is unitary.

1.3 Majorana Representation

In what follows, we introduce the Majorana representation of the projective Hilbert space of a spin-s, $\mathbb{P}(\mathcal{H}_s)$ [2]. Under this representation, a projective state $[\psi]$ is mapped into a set of (possibly coincident) $2s$ points (that we refer to as "stars") in the unitary sphere, the latter set known as the constellation of the state (which we denote by $\mathcal{C}_\psi$). This correspondence turns out to be one-to-one. Although the Majorana representation has not gained much attention among physicists, the particular case of $s = 1/2$ is very well known. Indeed, when $s = 1/2$, the Majorana representation reduces to the so-called Bloch sphere representation, where any spin-1/2 state is characterized by one point in the unitary sphere.

In the following paragraphs, we explain how this association between elements of $\mathbb{P}(\mathcal{H})$ and points in the sphere is made, but first we want to stress that the Majorana representation *commutes with rotations*. By this we mean that the constellation of a rotated state $D(R)[\psi]$ (here $D(R)$ denotes the representation of a three dimensional rotation R as a linear operator acting on $\mathcal{H}$) can be computed by rotating the stars in the constellation of $[\psi]$ by R. Note that the intrinsic properties of $[\psi]$ are the same as the ones of $D(R)[\psi]$ as, in certain sense, the physical properties of a state must not be modified if we decided to change the orientation of the axes used in the laboratory to describe said state. This makes the Majorana representation particularly useful to study some properties of a state. Some examples of this type of properties that had been studied with the Majorana representation are the degree of anticoherence of a state (c.f. section 1.3.5), the entanglement of a system of $2s$ spin-1/2 particles (c.f. section 1.3.2) and the sensibility of a state to detect rotations [33]. Besides this, the Majorana representation has also been used in other contexts, for instance, the one of geometric phases [34, 35].

In the literature, it is possible to find many different ways to introduce the Majorana representation (c.f. [9]). Here we follow two different approaches. The first one is the one that the author finds the most illuminating. The second one, is useful for a generalization of the Majorana representation we present on chapter 3.

For the first approach, the outline is the following. First, in section (1.3.1), we define the Majorana representation for the Hilbert space of completely symmetric states (under exchange of particles) of a system of $2s$ spin-1/2 particles. Since this space is, mathematically speaking, the space of a spin s (more on this on section 1.3.2), we use the previous construction to define the Majorana representation for the state space of an arbitrary spin-s, whether it can be seen as a system of $2s$ spin-1/2 particles or not.

The second approach is presented at section 1.3.4 and it is based on the so-called spin coherent states.

1.3.1 Majorana representation for a system of $2s$ spin-1/2 particles

Consider the Hilbert space of a system of $2s$ spin-1/2 particles (the tensor product of the spaces of $2s$ spin-1/2 systems). Denote by $\mathcal{H}^{2s}_{\text{sym}}$ the subspace of completely symmetric states (under exchange of particles). In what follows, we show how to assign a constellation (set of $2s$ points in the unitary sphere) to each state of $\mathcal{H}^{2s}_{\text{sym}}$ via the Majorana representation. This representation has been useful to study some physical properties of a state in $\mathcal{H}^{2s}_{\text{sym}}$, for instance, it has been successfully used in [36] to study the entanglement of symmetric states and to find the maximally entangled state (c.f. section 1.1.2).

Before going on, we fix some notation. As it is well known, every state of a single spin-1/2 system can be characterized (up to a global phase) by a single direction in the unitary sphere (via the Bloch sphere representation alluded to above). We denote by $|\pm\hat{z}^{½}\rangle$ the state characterized by the direction $\pm\hat{z}$ (that turns out to be an eigenstate of the spin angular momentum operator S in the direction $\hat{z}$ with eigenvalue $\pm 1/2$). States characterized by the direction $\hat{n}$, $|\hat{n}^{½}\rangle$, can be written (upto a phase) in terms of $|\pm\hat{z}^{½}\rangle$ as follows [37, Chapter 1],

$$
\begin{aligned}
|\hat{n}^{½}\rangle &= \cos\left(\tfrac{\theta}{2}\right)|\hat{z}^{½}\rangle + e^{i\phi}\sin\left(\tfrac{\theta}{2}\right)|-\hat{z}^{½}\rangle \\
&= \cos\left(\tfrac{\theta}{2}\right)\left(|\hat{z}^{½}\rangle + \zeta|-\hat{z}^{½}\rangle\right) ,
\end{aligned}
\tag{1.3.1}
$$

where (θ, ϕ) denotes the spherical coordinates associated to $\hat{n}$, and the complex number $\zeta = \tan(\theta/2)e^{i\phi}$ is the image, via the stereographic projection from the south pole, of $\hat{n}$ (assuming $\hat{n}$ is not the south pole). It is customary to say that the state $|\hat{n}^{½}\rangle$ "points at" $\hat{n}$.

Given $2s$ directions $\hat{n}_1, \ldots, \hat{n}_{2s}$, we define the state $|\psi\rangle$ in $\mathcal{H}^{2s}_{\text{sym}}$ obtained by symmetrizing the state $|\hat{n}_1^{½}\rangle \otimes \cdots \otimes |\hat{n}_{2s}^{½}\rangle$ as follows,

$$|\Psi\rangle = |\hat{n}_1, \ldots, \hat{n}_{2s}\rangle = \frac{1}{A_\Psi} \sum_{\sigma \in S_{2s}} |\hat{n}_{\sigma_1}^{½}\rangle \otimes \cdots \otimes |\hat{n}_{\sigma_{2s}}^{½}\rangle \,, \tag{1.3.2}$$

where S_{2s} denotes the permutation of group of a collection of $2s$ objects and A_Ψ is the following normalization factor,

$$A_\Psi^2 = (2s)! \sum_{\sigma \in S_{2s}} \langle \hat{n}_1^{½} | \hat{n}_{\sigma_1}^{½}\rangle \ldots \langle \hat{n}_{2s}^{½} | \hat{n}_{\sigma_{2s}}^{½}\rangle \,. \tag{1.3.3}$$

An orthonormal basis for $\mathcal{H}^{2s}_{\text{sym}}$ can be constructed by symmetrizing the different combinations of $2s$ tensor products of the states $|\pm \hat{z}^{½}\rangle$. The resulting basis consists of $2s+1$ states $|D_k\rangle$ ($k = 0, \ldots, 2s$) known as Dicke states, and are defined according to the following equation,

$$|D_k\rangle = |\underbrace{\hat{z}, \ldots, \hat{z}}_{2s-k}, \underbrace{-\hat{z}, \ldots, -\hat{z}}_{k}\rangle \,. \tag{1.3.4}$$

By using some combinatorics, the value of the constant A_Ψ (1.3.3) corresponding to $|D_k\rangle$ can be proved to be the following,

$$A_k = \sqrt{(2s)! k! (2s-k)!} \,. \tag{1.3.5}$$

Since the Dicke states constitute an orthonormal basis for $\mathcal{H}^{2s}_{\text{sym}}$, any state $|\psi\rangle$ can be written as a linear combination of them,

$$|\psi\rangle = \sum_{k=1}^{2s} B_k |D_k\rangle = \sum_{k=1}^{2s} \langle D_k|\psi\rangle |D_k\rangle \,. \tag{1.3.6}$$

After this digression on notation, we mention how to assign a constellation to each state in $\mathcal{H}^{2s}_{\text{sym}}$. As we prove in the following theorem, any state $|\psi\rangle$ in $\mathcal{H}^{2s}_{\text{sym}}$ can be uniquely written as factor times the completely symmetric state associated to some directions $\hat{n}_1, \ldots, \hat{n}_{2s}$ (1.3.2). The constellation of $|\psi\rangle$ consists on the "stars" in these directions.

Theorem 4. *For each state $|\psi\rangle$ in $\mathcal{H}^{2s}_{sym}$, there exists a unique set of $2s$ directions $\hat{n}_1, \ldots, \hat{n}_{2s}$ (with possible multiplicity) such that,*

$$|\psi\rangle = C |\hat{n}_1, \ldots, \hat{n}_{2s}\rangle \,,$$

where $|\hat{n}_1, \ldots, \hat{n}_{2s}\rangle$ is defined as in (1.3.2) and C is a complex factor.

The strategy we use here to prove the theorem, is the following. First, we take an arbitrary $|\hat{n}_1, \dots, \hat{n}_{2s}\rangle$, and write it in terms of the Dicke states as in (1.3.6). In this way we can find an expression for the coefficients B_k in terms of $\hat{n}_1, \dots, \hat{n}_{2s}$. Finally, we prove that said expression is invertible for arbitrary coefficients B_k and, therefore, the conclusion is that any state can be written in the way the theorem claims.

Proof of theorem 4. As we have already mentioned in the previous outline, first we find explicitly the coefficients B_k (1.3.6) for the expansion of the state $|\Psi\rangle = |\hat{n}_1, \dots, \hat{n}_{2s}\rangle$. For the moment, assume that no direction $\hat{n}_i$ points towards the south pole. By considering the equality $B_k = \langle D_k|\Psi\rangle$ and more combinatorics, we obtain the following expression for B_k,

$$B_k = \frac{(2s)!}{A_k A_\Psi} \sum_{\sigma \in S_{2s}} \prod_{i=1}^{k} \langle -\hat{z}^{½} | \hat{n}_{\sigma_i}^{½} \rangle \prod_{i=k+1}^{2s} \langle \hat{z}^{½} | \hat{n}_{\sigma_i}^{½} \rangle \, . \tag{1.3.7}$$

By substituting the equations (1.3.5) and (1.3.1) in the previous expression for B_k we obtain, after some straight-forward algebra

$$B_k = \frac{\lambda}{A_\Psi} \frac{\sqrt{(2s)!}}{\sqrt{k!(2s-k)!}} \sum_{\sigma \in S_{2s}} \prod_{i=1}^{k} \zeta_{\sigma_i} \, , \tag{1.3.8}$$

where $\zeta_i = \tan(\theta_i/2) e^{i\phi_i}$ is the image under stereographic projection of $\hat{n}_i$, and λ is defined as follows; $\lambda = \prod_{i=1}^{2s} \cos(\theta_i/2)$.

The next step is to invert (1.3.8) to write the complex numbers ζ_i in terms of the coefficients B_k and thus conclude the proof. The trick is to consider the following equalities between polynomials in the complex variable ζ,

$$\prod_{i=1}^{2s} (\zeta - \zeta_i) = \sum_{k=0}^{2s} \zeta^{2s-k} \frac{(-1)^k}{k!(2s-k)!} \sum_{\sigma \in S_{2s}} \prod_{i=1}^{k} \zeta_{\sigma_i} \, . \tag{1.3.9}$$

By multiplying (1.3.8) by $(-1)^k \zeta^{2s-k} \binom{2s}{k}^{1/2}$, summing the resulting equations from $k = 0$ to $k = 2s$ and considering (1.3.9) we obtain,

$$\sum_{k=0}^{2s} \zeta^{2s-k} (-1)^k \binom{2s}{k}^{1/2} B_k = \frac{\lambda (2s)!}{A_\psi} \prod_{i=1}^{2s} (\zeta - \zeta_i) \, . \tag{1.3.10}$$

Note that the polynomial in the l.h.s. of the previous equation can be defined solely in terms of the coefficients B_k. On the other hand, the roots of this polynomial are the complex numbers ζ_i ($i = 1, \dots, 2s$) associated to the directions $\hat{n}_i$. This is the relationship we were after. To find the "constellation" of a state $|\psi\rangle$ written as in (1.3.6), the recipe is the following. First, we construct the polynomial in the variable ζ in terms of the coefficients B_k as defined in the l.h.s. of equation (1.3.10). Then, we find the complex roots of said polynomial, $\zeta_1, \ldots, \zeta_{2s}$. Finally, we map these complex numbers in the unitary sphere using the stereographic projection.

In the case that l of the $2s$ directions $\hat{n}_1, \ldots, \hat{n}_{2s}$ point towards the south pole, by using the same reasoning, it is easy to prove that the degree of the polynomial at the l.h.s. of (1.3.10) is $2s - l$. Therefore, if, for a certain state, the degree of (1.3.10) turns out to be $2s - l$, then, $2s - l$ of the directions in the constellation of $|\psi\rangle$ are given by the roots of this (1.3.10), and the remaining l stars are in the south pole. □

Note that the constellation of the state $|\psi\rangle$ is the same as the one for $\alpha|\psi\rangle$ (here α denotes an arbitrary factor). Because of this property, we can unambiguously define the constellation of the projective state $[\psi]$ — just take the constellation of $|\psi\rangle$. As an example, one can readily compute the constellation of the Dicke states (1.3.4), the constellation of $|D_k\rangle$ consists of k stars in the south pole and $2s - k$ in the north pole.

Since the polynomial defined in the previous proof turns out to be very useful, we define the "Majorana polynomial" of a state $|\psi\rangle = \sum_{k=0}^{2s} B_k |D_k\rangle$ as,

$$P_\psi(\zeta) = \sum_{k=0}^{2s} \zeta^{2s-k} (-1)^k \binom{2s}{k}^{1/2} B_k \,. \tag{1.3.11}$$

Note that the previous mapping induces a linear isomorphism between $\mathcal{H}_{\text{sym}}^{2s}$ and the space of polynomials of degree at most $2s$. We use this mapping frequently throughout the thesis.

1.3.2 A system of $2s$ spin-1/2 particles as a spin-s

In this section, we prove that $\mathcal{H}_{\text{sym}}^{2s}$ is, in a formal sense, the space of a spin s. Mathematically speaking, the space of a spin s is defined as the vector space (unique up to isomorphisms) of dimension $2s + 1$ where an irreducible representation of $SO(3)$ acts. One can show that $\mathcal{H}_{\text{sym}}^{2s}$ satisfy these requirements. Indeed, in $\mathcal{H}_{\text{sym}}^{2s}$ we can *naturally* define the angular momentum operators S_i $(i = x, y, z)$ as follows,

$$S_i = \sum_{k=1}^{2s} S_i^{(1/2,k)} \,, \tag{1.3.12}$$

where $S_i^{(1/2,k)}$ denotes the angular momentum operator (in the direction i) that acts only in the k-th spin,

$$S_i^{(1/2,k)} = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{k-1 \text{ times}} \otimes S_i^{(1/2)} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{2s-k \text{ times}},$$

being $\mathbb{1}$ the identity operator and $S_i^{(1/2)}$ the angular momentum for the space of a single spin-1/2. From this definition, it is trivial to show that this operators are self-adjoint and that they satisfy the well-known commutation relationships between angular momentum operators (taking $\hbar = 1$),

$$[S_i, S_j] = i\epsilon_{ijk} S_k \,.$$

The representation of a 3-dimensional rotation R as a linear operator on $\mathcal{H}^{2s}_{\text{sym}}$, $D(R)$, can be obtained by considering the exponential map of a linear combination of the angular momentum operators (1.3.12),

$$D(R) = D(R)^{(1/2)} \otimes \cdots \otimes D(R)^{(1/2)} \,. \tag{1.3.13}$$

where $D(R)^{(1/2)}$ denotes the corresponding rotation operator for the space of a single spin s. One can also prove without much effort that the $2s+1$ Dicke states (1.3.4) are eigenstates of the operator $S^2 = S_x^2 + S_y^2 + S_z^2$ with eigenvalue $s(s+1)$, and of S_z

$$S_z |D_k\rangle = (s-k)|D_k\rangle \,,$$

Clearly, the possible eigenvalues of S_z are $-s, -s+1, \ldots, s-1, s$, and $|D_{s+m}\rangle$ is an eigenstate of S_z with eigenvalue m. A quick computation reveals the following equality,

$$S_\pm |D_{s+m}\rangle = \sqrt{s(s+1) - m(m\pm 1)}|D_{s+m\pm 1}, m \pm 1\rangle \,,$$

where $S_\pm = S_x \pm S_y$. From these facts we can conclude that the action defined in (1.3.13) is irreducible, and that $|D_{s+m}\rangle$ is the state usually denoted by $|s, m\rangle$.

This proves our claim; that $\mathcal{H}^{2s}_{\text{sym}}$ is the space of a spin s. Because of this, we can reformulate the results in the previous section to the case for a general spin-s simply by making the substitutions $|D_{s+m}\rangle \to |s, m\rangle$ and $k \to s + m$. In the following section, we show the results obtained in this fashion.

Finally, we prove that the proposed way to assign a constellation to a state defined in the previous section *commutes with rotations*, as claimed at the beginning of section 1.3. To this end, consider a rotation R in $SO(3)$, and take take the corresponding rotation operator $D(R)$ as in (1.3.12). Take an arbitrary state $|\psi\rangle$ written as $|\psi\rangle = |\hat{n}_1, \ldots, \hat{n}_{2s}\rangle$ (in virtue of theorem 4). Then, by definition, the constellation of $|\psi\rangle$ is made up by stars in the directions $\hat{n}_1, \ldots, \hat{n}_{2s}$. Since, for a certain phase γ, the equality $D(R)^{(1/2)}|\hat{n}^{1/2}\rangle = e^{i\gamma}|R\hat{n}^{1/2}\rangle$ holds, we can conclude the following expression,

$$D(R)|\hat{n}_1, \ldots, \hat{n}_{2s}\rangle = e^{2si\gamma}|R\hat{n}_1, \ldots, R\hat{n}_{2s}\rangle \, .$$

This means that the constellation of $D(R)|\psi\rangle$ consists on the stars in the directions $R\hat{n}_1, \ldots, R\hat{n}_{2s}$; the directions obtained by rotating the ones of the constellation of $|\psi\rangle$, just as claimed.

1.3.3 Majorana representation of a spin s

In this section, we formulate the results of the previous sections for a general spin s, whether it is the composite system of $2s$ spin-1/2 particles or not.

Given a spin-s state $|\psi\rangle$ written in the eigenbasis of the angular momentum operator S_z as follows,

$$|\psi\rangle = \sum_{m=-s}^{s} B_m |s, m\rangle \, ,$$

define its Majorana polynomial (written as a function of the complex variable ζ) as,

$$p_\psi(\zeta) = \sum_{m=-s}^{s} (-1)^{s-m} B_m \binom{2s}{s+m}^{1/2} \zeta^{s+m} \, . \tag{1.3.14}$$

In terms of this polynomial, we define the constellation of a state, as outlined in the following paragraph.

Given a state $|\psi\rangle$, call $\zeta_1, \ldots, \zeta_{2s}$ to the $2s$ roots of p_ψ (in case that the degree of p_ψ, n, is lower that $2s$, assume the missing roots $2s - n$ are at infinity). The constellation of $|\psi\rangle$, $\mathcal{C}_\psi$ is defined as the collection of the $2s$ directions in the sphere (with possible multiplicity) obtained by applying the stereographic projection to the complex numbers $\zeta_1, \ldots, \zeta_{2s}$ (the image of infinity is assumed to be the south pole). This procedure is schematically show in figure 1.1.

Since multiplication by scalars leaves the constellation of a state invariant, we also define the constellation of the projective state $[\psi]$ as $\mathcal{C}_\psi$. Note that this mapping between $\mathbb{P}(\mathcal{H}_s)$ and the space of $2s$ points in the sphere is one-to-one. Given a constellation $\mathcal{C} = \{\hat{n}_1, \ldots, \hat{n}_{2s}\}$, we denote by $[\hat{n}_1, \ldots, \hat{n}_{2s}]$

$$\sum_{m=-s}^{s} B_m \, |s, m\rangle \longrightarrow \sum_{m=-s}^{s} (-1)^{s-m} B_m \binom{2s}{s+m}^{1/2} \zeta^{s+m}$$

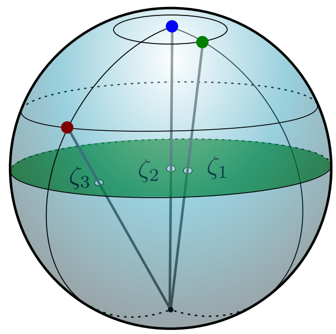


Figure 1.1: Schematic procedure to find the constellation of a state. The first step is to build the Majorana polynomial (1.3.14). The second step is to find the roots of this polynomial. Finally, via the stereographic projection, these roots are mapped onto the sphere. The obtained directions constitute the constellation.

the ray such that its constellation is $\mathcal{C}$, and by $|\hat{n}_1, \ldots, \hat{n}_{2s}\rangle$ to any normalized state in this ray.

As we have already proved, this procedure to define $\mathcal{C}_\psi$ *commutes with rotations*. Because of this, the Majorana representation is particularly useful to find all the rotational symmetries of a state — they are the same as the ones of the constellation. In figure 1.2 we show the constellation of certain states with rotational symmetries.

1.3.4 An approach to the Majorana representation by spin coherent states

Before presenting a second approach to introduce the Majorana representation, we define the spin coherent states. Some relevant references on this topic are [1, 38, 39].

We say that a spin-s state $|\psi\rangle$ is coherent in the direction $\hat{n}$ if and only if it is an eigenstate of the angular momentum operator $S \cdot \hat{n}$ with maximal projection, that is, if the equality $S \cdot \hat{n}|\psi\rangle = s|\psi\rangle$ holds. Since the spectrum of this type of operators is not degenerate, the coherent state in the direction $\hat{n}$ is, up to an overall factor, unique.

Perhaps, the simplest example of coherent state is the one pointing towards $\hat{z}$, $|s, s\rangle$. It is easy to see that the coherent state in any direction can be obtained by applying a suitable rotation to $|s, s\rangle$. Indeed, if R is any rotation of 3-dimensional space that maps $\hat{z}$ into $\hat{n}$ then, $D(R)|s, s\rangle$

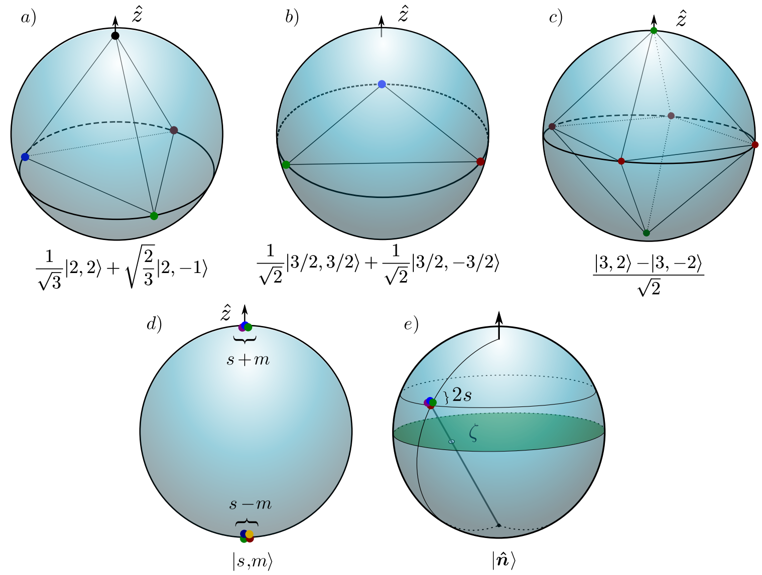


Figure 1.2: Example of the constellation of various states with rotational symmetries; $a)$ a regular tetrahedron, $b)$ an equilateral triangle at the equator, $c)$ an octahedron, $d)$ the eigenstates of S_z and $e)$ a coherent state (1.3.16).

is a coherent state in the direction $\hat{n}$ (to quickly prove this, recall that $S \cdot \hat{n} = D(R) S_z D(R)^\dagger$). From this, one can notice that, when working in $\mathcal{H}_{\text{sym}}^{2s}$, the coherent states are those where all the constituent spins "point in the same direction" — if we rotate the state $|s,s\rangle = |D_0\rangle = |\hat{z},\ldots,\hat{z}\rangle$ (see equations (1.3.2) and (1.3.4)) by $D(R)$, the resulting state is proportional to $|\hat{n},\ldots,\hat{n}\rangle$). This means that the constellation of the coherent states consist of only one direction (degenerated $2s$ times) and, therefore, the coherent states are the only separable ones in $\mathcal{H}_{\text{sym}}^{2s}$ (c.f. section 1.1.2).

The previous observations can be used to find an expression for a coherent state $|\hat{\boldsymbol{n}}\rangle$ associated to $\hat{n}$ in terms of the basis $|s,m\rangle$, $m = s,\ldots,-s$. Indeed, since the constellation of $|\hat{\boldsymbol{n}}\rangle$ has $2s$ stars in $\hat{n}$, its Majorana polynomial $p_{\hat{\boldsymbol{n}}}$ only has one root (with multiplicity $2s$). Therefore, $p_{\hat{\boldsymbol{n}}}$ can be written as,

$$p_{\hat{\boldsymbol{n}}}(\zeta) \propto (\zeta - \zeta_0)^{2s} = \sum_{m=-s}^{s} (-1)^{s-m} \zeta^{s+m} \zeta_0^{s-m} \binom{2s}{s-m}, \tag{1.3.15}$$

where ζ_0 denotes the image of $\hat{n}$ under the stereographic projection. By inverting the equation defining the Majorana polynomial (1.3.14), we see that,

$$|\hat{\boldsymbol{n}}\rangle = \frac{1}{A_{\hat{n}}} \sum_{m=-s}^{s} \binom{2s}{s-m}^{1/2} \zeta_0^{s-m} \,, \tag{1.3.16}$$

where $A_{\hat{n}}$ is the following normalization factor,

$$A_{\hat{n}} = \left(\sum_{m=-s}^{s} \binom{2s}{s-m} |\zeta_0|^{2(s-m)} \right)^{1/2} = (1+|\zeta_0|^2)^s \,.$$

Before going on, we want to mention three additional properties of the coherent states that we refer to later on. The first one, is that the space of coherent states $[\hat{\boldsymbol{n}}]$ is topologically a sphere, and the Fubini-Study distance (1.1.4) between the $[\hat{\boldsymbol{n}}]$ and $[\hat{\boldsymbol{m}}]$ is,

$$d_{\mathbb{P}(\mathcal{H})}([\hat{\boldsymbol{m}}],[\hat{\boldsymbol{n}}]) = \arccos\left(\cos^{2s}\tfrac{\theta}{2}\right) \,, \tag{1.3.17}$$

where θ denotes the angle between $\hat{m}$ and $\hat{n}$. This formula can be easily verified by computing the product between the states $|\hat{\boldsymbol{z}}\rangle = |\hat{z},\ldots,\hat{z}\rangle$ and $|\hat{\boldsymbol{n}}\rangle = |\hat{n},\ldots,\hat{n}\rangle$, and noting that the distance between $[\hat{\boldsymbol{n}}]$ and $[\hat{\boldsymbol{m}}]$ only depends of the angle between $\hat{m}$ and $\hat{n}$.

The second one, is that they maximize the magnitude of the vector v_ψ, defined for any $|\psi\rangle$ as follows,

$$v_\psi = \langle\psi|(S_x,S_y,S_z)|\psi\rangle \,. \tag{1.3.18}$$

In fact, it is easy to prove that for $|\hat{\boldsymbol{n}}\rangle$ the equality $v_{\hat{\boldsymbol{n}}} = s\hat{n}$ holds. The previous property implies immediately that coherent states also minimize the uncertainty relationship

$$\Delta S_x^2 + \Delta S_y^2 + \Delta S_z^2 = s(s+1) - v_\psi \cdot v_\psi \,. \tag{1.3.19}$$

The previous equality can be obtained with a little bit of algebra. In this sense, coherent states are the "most classical ones".

The third property we want to stress is that they provide a resolution of the identity operator $\mathbb{1}$,

$$\mathbb{1} = \frac{2s+1}{4\pi} \int_{S^2} |\hat{\boldsymbol{n}}\rangle\langle\hat{\boldsymbol{n}}| d\Omega \,, \tag{1.3.20}$$

where S^2 denotes the unitary sphere and $d\Omega$, the volume element of S^2. In section 3.1, we give a proof of this equality for a more general case.

Now we mention how to find define the Majorana representation via spin coherent states. Given a state $|\psi\rangle$, define the Husimi function H_ψ associated to it according the following equation,

$$H_\psi(\hat{n}) = \frac{2s+1}{4\pi} |\langle \hat{\boldsymbol{n}} | \psi \rangle|^2 \,. \tag{1.3.21}$$

Then, as we prove in the following theorem, we can characterize $\mathcal{C}_\psi$ in terms of the zeros of H_ψ.

Theorem 5. *Consider a state $|\psi\rangle$ and its constellation $\mathcal{C}_\psi$. Then, $\mathcal{C}_\psi$ is the set of all the directions $\hat{n}$ such that $H_\psi(-\hat{n}) = 0$.*

Proof. The proof is straightforward. First, write $|\psi\rangle = \sum_{-m}^{m} B_m |s, m\rangle$. Using the expression for a coherent state (1.3.16), we can compute the product $\langle -\hat{\boldsymbol{n}} | \psi \rangle$. Denote by ζ_0 the complex number associated to $\hat{n}$ (via stereographic projection) and, by ζ_{0A}, the one associated to $-\hat{n}$, the complex number *antipodal* to ζ_0. As it is well-known, $\zeta_{0A} = -1/\zeta_0^*$. Using these equations, we obtain the following,

$$\begin{aligned}
\langle -\hat{\boldsymbol{n}} | \psi \rangle &= \frac{1}{(1+|\zeta_0|^2)^s} \sum_{m=-s}^{s} B_m \binom{2s}{s-m}^{1/2} (\zeta_{0A}^*)^{s-m} \\
&= \frac{1}{(1+|\zeta_0|^2)^s} \sum_{m=-s}^{s} (-1)^{s-m} B_m \binom{2s}{s-m}^{1/2} \zeta_0^{m-s} \\
&= \frac{1}{\zeta_0^{2s}(1+|\zeta_0|^2)^s} \sum_{m=-s}^{s} (-1)^{s-m} B_m \binom{2s}{s-m}^{1/2} \zeta_0^{m+s} \\
&= \frac{p_\psi(\zeta_0)}{\zeta_0^{2s}(1+|\zeta_0|^2)^s} \,,
\end{aligned}$$

where we used (1.3.14) to deduce the last equality. Because of this, the Husimi function (1.3.21) H_ψ evaluated at $-\hat{n}$ is proportional to $|p_\psi(\zeta_0)|^2$. Since the proportionality factor is nowhere zero, $H_\psi(-\hat{n})$ is zero if and only if the Majorana polynomial evaluated at ζ_0 is zero, $p_\psi(\zeta_0) = 0$. This concludes the proof. □

The previous theorem allows us to give a physical interpretation of the directions of the stars in the constellation of a state $|\psi\rangle$. If there is a star in the direction $\hat{n}$ then, by the previous theorem, the equality $H_\psi(-\hat{n}) \propto |\langle -\hat{\boldsymbol{n}} | \psi \rangle|^2 = 0$ holds. This implies that $|\psi\rangle$ is orthogonal to $|-\hat{\boldsymbol{n}}\rangle$. Since $|-\hat{\boldsymbol{n}}\rangle$ is the only eigenstate (up to a phase) of the angular momentum operator

$-S \cdot \hat{n}$ with projection s, by Born rule, the probability of getting the value s when measuring the spin projection in $-\hat{n}$ of a system in the state $|\psi\rangle$ is zero. This result can be generalized for stars with multiplicity greater than one. Before stating the theorem, denote by $|\hat{\boldsymbol{n}}, m\rangle$ the usual eigenstate of $S \cdot \hat{n}$ with eigenvalue m. Now we state the theorem.

Theorem 6. *Consider a state $|\psi\rangle$ and its constellation $\mathcal{C}_\psi$. Then, there is a star in $\hat{n}$ in $\mathcal{C}_\psi$ with multiplicity l $(1 \leq l \leq 2s)$ if and only if $|\psi\rangle$ is orthogonal to $|-\hat{\boldsymbol{n}}, m\rangle$ for all m such that $s-l+1 \leq m \leq s$. In physical terms, the probability of producing the values $s, s-1, \dots, s-l+1$ when measuring the spin projection in the direction $-\hat{n}$ of a system in the state $|\psi\rangle$ is zero.*

Proof. First, we argue that it is enough to prove the theorem in the particular case that $\hat{n}$ points toward the south pole. Indeed, if the star in consideration is not in the south pole, we can always consider a rotation R that maps the point $\hat{n}$ to the south pole. Then, since the Majorana representation *commutes with rotations*, there are l stars in the constellation $D(R)|\psi\rangle$ in the south pole. Now, we can apply the particular case of the theorem we are about to prove. Finally, we rotate everything back to conclude the general case (it is easy to show that, up to a phase, $|\hat{\boldsymbol{n}}, m\rangle$ is equal to $D(R^{-1})|\hat{\boldsymbol{z}}, m\rangle$ for arbitrary m; also recall that the rotation operator $D(R)$ is unitary and therefore preserves the inner product of Hilbert space). We use this kind of argument frequently throughout the thesis.

Now we prove this particular case. Note that there are l stars in the south pole in $\mathcal{C}_\psi$ if and only if "infinity is a root of the Majorana polynomial" p_ψ (1.3.14) with multiplicity l. By convention, this means that the degree of p_ψ is $2s-l$, that is, the coefficients of the orders $\zeta^{2s}, \dots \zeta^{2s-l+1}$ of $p_\psi(\zeta)$ are zero. By considering (1.3.14), we note that this happens if and only if the coefficients of $|\psi\rangle$ (when expanded in the basis $|s, m\rangle$ with $m = -s, \dots, s$) corresponding to $|s, s\rangle, \dots |s, s-l+1\rangle$ are all zero. But, since this basis is orthonormal, this is equivalent to the statement that $|\psi\rangle$ is orthogonal to $|s, -s\rangle, \dots |s, -s+l-1\rangle$. This concludes the proof. □

1.3.5 Spin anticoherent states

In a certain sense, coherent states are *the most classical* ones, as they define one single direction — their Majorana constellation consist on a single $2s$-degenerate star. From here, it is natural to pose the opposite question, which states are the *most quantum* ones? Intuitively speaking, the constellation of such kind of states must consist of stars spread as *evenly as possible* over the unitary sphere.

There are many related problems that deal with the question of distributing a certain number of points uniformly over the surface, some examples include [40–42]. Here, we use the definition of anticoherent states given in [43]; we say that a state $[\psi]$ in $\mathbb{P}(\mathcal{H}_s)$ is anticoherent of order k if and only if, for any $m \leq k$, the quantity

$$\langle\psi|(S\cdot\hat{n})^m|\psi\rangle \quad \text{is independent on } \hat{n}.$$

The motivation of this definition is the following. Being the opposite to coherent states, we would like an anticoherent state $[\psi]$ to be the most *adirectional* possible. Naturally, this leads to the requirement that the vector (1.3.18) $v_\psi = \langle\psi|S|\psi\rangle$ is zero — otherwise, we could use it to define a direction. One can easily verify that the states with this property are the anticoherent states of order one. However, states that satisfy this property are not completely *adirectional*, for instance, $[\psi_0] = [s, 0]$ (for an integer spin s) is anticoherent of order zero, but it clearly defines two directions, $\hat{z}$ and $-\hat{z}$. These directions can be singled out by considering the uncertainty function over the unitary sphere

$$\Delta S^2_{\hat{n}} = \langle\psi_0|(S\cdot\hat{n})^2|\psi_0\rangle - (\langle\psi_0|S\cdot\hat{n}|\psi_0\rangle)^2 = \langle\psi_0|(S\cdot\hat{n})^2|\psi_0\rangle \,.$$

This function only has two zeros, $\hat{z}$ and $-\hat{z}$ (as $[\psi_0]$ is an eigenstate of S_z). If we want a state $[\psi]$ to be more *adirectional* than an anticoherent state of order one, we could impose the condition that, besides being anticoherent of order one, the uncertainty function $\Delta S^2_{\hat{n}} = \langle\psi|(S\cdot\hat{n})^2|\psi\rangle$ associated to it is independent of $\hat{n}$. In this way, no direction is associated to $[\psi]$ *up to second order* in S. Going on with this line of reasoning, we see that anticoherent states of order k do not define any direction *up to k-th order* in S.

A quick computation reveals that there are no anticoherent states for spin $s = 1/2$. For $s = 1, 3/2$, the highest order of anticoherence possible is one. In both of these cases, there is essentially one anticoherent state, for $s = 1$, the only anticoherent states are those whose constellation consists of antipodal stars; for $s = 3/2$, the stars of the constellations of the only anticoherent states define a maximal equilateral triangle (an equilateral triangle contained in a great circle of the sphere). For $s = 2$, the highest order is two [43]. In general, for a given s, the highest order of anticoherence possible is *at most* $\lfloor s \rfloor$ (where $\lfloor . \rfloor$ denotes the floor function) [44]. In [45] the authors prove that there are states of any order of anticoherence, provided s is big enough.

From a mathematical point of view, anticoherent states are interesting since their constellation tend to spread uniformly over the sphere [43, 46]. From a physical point of view, anticoherent states have applications to quantum metrology [47, 48] (there, the anticoherent states are refereed as “kings of quantumness”) and to quantum information [45].

Chapter 2

Geometry of the shape space of a spin s

As we have mentioned briefly in the previous chapter, when a spin state in $\mathbb{P}(\mathcal{H}_s)$ is rotated, its physical properties remain the same. Because of this, it is natural to identify states that only differ by a rotation. We call *shape space* the quotient space obtained via this identification . In this chapter, we present how to use this construction to describe $\mathbb{P}(\mathcal{H}_s)$ as a principal fiber bundle, where the base space is the shape space and the acting group is $SO(3)$. We also mention many geometrical properties derived from this particular description and the Fubini-Study metric.

The concept of shape space also appear in mathematics [49], and in other branches of physics, for instance, in classical mechanics [50] and in general relativity [51–53].

2.1 $\mathbb{P}(\mathcal{H}_s)$ as a fiber bundle

For the rest of this chapter, we work mainly with the representation of $\mathbb{P}(\mathcal{H}_s)$ in terms of pure density operators mentioned in section 1.1. As a matter of nomenclature, when we mention the constellation of the density matrix ρ, we are referring to the constellation of the state represented by ρ. Using (1.1.8), we can naturally define an action by the right of a rotation $R \in SO(3)$ on the ρ as follows,

$$\rho \triangleleft R = \rho \triangleleft D(R) = D(R)^\dagger \rho D(R) \,. \tag{2.1.1}$$

By of the results of section 1.3, the constellation of $\rho \triangleleft R$ can be obtained by rotating the one of ρ by R^{-1}. This situation occurs because we are working with right actions instead of left actions.

In terms of this action, we define the shape space $\mathscr{S}$ as the quotient space obtained by identifying states that differ by the action of a rotation. Note that, in terms of the Majorana representation, two states are in the same equivalence class if and only if their corresponding constellations can be connected by a rotation.

By considering the infinitesimal version of (2.1.1), we can define the action of $so(3)$ on $\mathbb{P}(\mathcal{H}_s)$. Indeed, by considering a rotation $D(R) = e^{-itS\cdot\hat{n}}$, we can compute the action of $\mathcal{A} = S \cdot \hat{n}$ on ρ as follows,[1]

$$\rho \triangleleft \mathcal{A} = d/dt(e^{it\mathcal{A}}\rho e^{-it\mathcal{A}})_{t=0} = i[\mathcal{A}, \rho] \equiv \mathcal{A}^\sharp(\rho). \tag{2.1.2}$$

The tangent vectors $\mathcal{A}^\sharp$ obtained in this way are called *fundamental vertical vectors* and it is easy to prove that they satisfy the conditions mentioned in theorem 1. If we fix $\mathcal{A}$ and repeat this procedure for all pure density operators ρ, we obtain a *fundamental vector field*, denoted by $\mathcal{A}^\sharp(\rho)$. On the other hand, if we fix ρ and vary $\mathcal{A}$ over $so(3)$, we obtain the *vertical tangent space* at the point ρ.

With this definition, we can naturally decompose $\mathbb{P}(\mathcal{H}_s)$ as a fiber bundle,[2] where the base space is $\mathscr{S}$, the group is $SO(3)$ and the projection operator π maps the state ρ to its equivalence class in $\mathscr{S}$. If two different states ρ and ρ' are in the same fiber, we say that ρ and ρ' have the same *shape*, but different *orientation*. As already argued, all the physical scalars associate to a state (for example, its entanglement or its degree of anticoherence) must induce a well-defined function over shape space, that is, its value must be the same for all the states with the same shape.

Generically speaking, the fibers are isomorphic to $SO(3)$. In the generic case, the constellation of a certain ρ has no rotational symmetries, so no rotation of $SO(3)$ fixes ρ. By applying to ρ all possible rotations, we obtain all the points in its fiber. In this way we can construct the following isomorphism φ_ρ between these fibers and $SO(3)$,

$$R \in SO(3) \overset{\varphi_\rho}{\longleftrightarrow} \rho \triangleleft R\,. \tag{2.1.3}$$

Note that, the fundamental vectors of equation (2.1.2) are the pushforward under this isomorphism of the elements of $so(3)$.

[1] Formally, the operators S_i are representations of the elements of $so(3)$, not the elements themselves. However, to simplify the notation, we denote them by the same symbol.

[2] We refer to $\mathbb{P}(\mathcal{H}_s)$ as a fiber bundle despite the fact that some fibers are not isomorphic to $SO(3)$. See the paragraph after (2.1.3) for more details.

However, if a state has some rotational symmetries, its fiber is no longer isomorphic to $SO(3)$. For instance, it is easy to verify that the fiber of the states $|s,m\rangle$, with $m \neq 0$ (for example, the coherent state $|s,s\rangle$) is isomorphic to the sphere S^2, while the one of $|s,0\rangle$ is isomorphic to the real projective space $\mathbb{RP}^2$. Further discussion about the topology of the fibers can be found in [54].

Since the *real* dimension of $\mathbb{P}(\mathcal{H}_s)$ is $4s$, the one of $\mathscr{S}$ is $4s-3$. This is relevant from a numerical point of view, for instance, to find the maximally entangled state (c.f. section 1.1.2) — the space where the numerical search is performed is reduced by three.

In general, it is not an easy task to assign coordinates to $\mathscr{S}$. A first approach to solve this problem, is to use the tensor representation of spin states to find such coordinates [55]. A second approach, is to use the Majorana representation to find particular sections of the total space $\mathbb{P}(\mathcal{H}_s)$ that can be easily parametrized. In the following paragraphs, we exemplify this second approach for two different values of s; $s=1$ and $s=3/2$. We skip the case $s=1/2$ since it is trivial; indeed, for $s=1/2$ shape space is only one point — the constellation of any state consists only in one star, so any two constellations of two different states can be trivially connected by a rotation. From this case we can also conclude that, in general, $\mathscr{S}$ is not a manifold but an *orbifold* [56].

First, we begin with the case of $s=1$, where the constellation of any state in $\mathbb{P}(\mathcal{H}_s)$ has two stars. By a suitable rotation, we can obtain a constellation where the two stars are in the xz plane and are bisected by the $\hat{z}$ axis. If we denote by q the angle between the original stars ($0 \leq q \leq \pi$), then, the directions of the ones obtained after applying the rotation are,

$$\hat{n}_1 = (\sin\tfrac{q}{2}, 0, \cos\tfrac{q}{2})\,, \quad \hat{n}_2 = (-\sin\tfrac{q}{2}, 0, \cos\tfrac{q}{2}). \tag{2.1.4}$$

Since the complex numbers associated to $\hat{n}_1$ and $\hat{n}_2$ (via the stereographic projection) are $\tan(q/2)$ and $-\tan(q/2)$ respectively, then, the Majorana polynomial for a state $|q\rangle$ with constellation made by the directions (2.1.4), is given by,

$$p_q(\zeta) \propto (\zeta - \tan\tfrac{q}{2})(\zeta + \tan\tfrac{q}{2})\,.$$

By inverting (1.3.14), and normalizing, we find, after a direct computation, that the state represented by $\rho(q)$ associated to this constellation is,

$$\rho(q) = |q\rangle\langle q|\,, \tag{2.1.5}$$

where ($|q\rangle$ is written as column vector w.r.t. the standard basis $\{|s=1,m\rangle$, $m=1,0,-1\}$)

$$|q\rangle = \frac{1}{\sqrt{3+\cos q}}(\cos\tfrac{q}{2}+1, 0, \cos\tfrac{q}{2}-1)\,.$$

The function $\rho(q)$ defines a section over $\mathbb{P}(\mathcal{H}_s)$. By construction, any state has the same shape as $\rho(q)$ (for a suitable q), and, if $q \neq q'$, then $\rho(q)$ and $\rho(q')$ have different shapes — this is obvious from the fact that rotations preserve the angle between the stars. Because of this, we conclude that, for $s=1$, $\mathscr{S}$ is topologically equivalent to the closed interval $[0,\pi]$. In the case $q=0$, the stars of equation (2.1.4) coincide, so $q=0$ corresponds to the "shape of coherent states". On the other hand, if $q=\pi$, the stars are antipodal and this case corresponds to the shape of the only anticoherent state of order one for $s=1$ (c.f. section 1.3.5).

Now, we make a similar analysis for $s=3/2$. In this case, the constellation of states has three stars. Since any three points in the sphere are coplanar, we can rotate them in a way such that they lie in a plane parallel to xy. Then, by applying a rotation around the $\hat{z}$ axis, we can make one of them intersect the meridian that goes from $\hat{z}$ to $\hat{x}$. The resulting constellation after these rotations is shown in figure 2.1. The directions of its stars are,

$$\begin{aligned}\hat{n}_1 &= (\sin\theta, 0, \cos\theta)\,,\\ \hat{n}_2 &= (\sin\theta\cos\phi_1, \sin\theta\sin\phi_1, \cos\theta)\,,\\ \hat{n}_3 &= (\sin\theta\cos\phi_2, \sin\theta\sin\phi_2, \cos\theta)\,.\end{aligned} \tag{2.1.6}$$

A careful analysis reveals that the rotations mentioned can be taken so that the equalities $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq 2\phi_1 < \phi_2 \leq 2\pi - \phi_1$ hold. This method to assign coordinates is illustrated in figure 2.1. Just like in the previous case, by considering the states which constellation is given by the directions (2.1.6), we can define a section of $\mathbb{P}(\mathcal{H}_s)$ and assign coordinates to $\mathscr{S}$. Some algebra reveals that this section can be parametrized as $\rho(\theta,\phi_1,\phi_2) = |\theta,\phi_1,\phi_2\rangle\langle\theta,\phi_1,\phi_2|$, where,

$$\begin{aligned}|\theta,\phi_1,\phi_2\rangle = \mathcal{N}^{-1}\Big(&2\sqrt{3}\cos^3\tfrac{\theta}{2}, \tfrac{1}{2}(e^{i\phi_1}+e^{i\phi_2}+1)\sin^2\theta\csc\tfrac{\theta}{2},\\ &\sin\tfrac{\theta}{2}\sin\theta(e^{i(\phi_1+\phi_2)}+e^{i\phi_1}+e^{i\phi_2}), 2\sqrt{3}\sin^3\tfrac{\theta}{2}e^{i(\phi_1+\phi_2)}\Big)\,,\end{aligned} \tag{2.1.7}$$

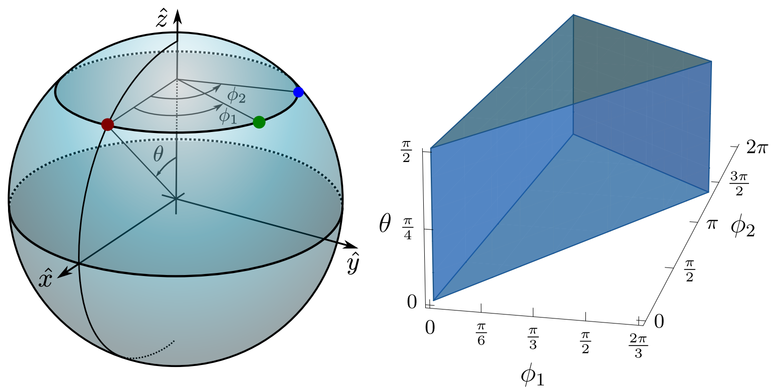


Figure 2.1: Left:Visual representation of the coordinates θ, ϕ_1 and ϕ_2 used to build the section (2.1.7). Right:Portion of the (ϕ_1, ϕ_2, θ) space needed to cover $\mathscr{S}$. Different points in the prism might get mapped to the same shape.

and $\mathcal{N}$ is the following normalization factor,

$$\begin{aligned}\mathcal{N}^2 = 9 &+ \cos\phi_1 + \cos\phi_2 + \cos(\phi_1 - \phi_2) \\ &- (\cos(\phi_1 - \phi_2) + \cos\phi_1 + \cos\phi_2 - 3)\cos 2\theta.\end{aligned}$$

Let us consider some examples. When $\theta = 0$, all the stars in the constellation of ρ coincide and, therefore, it is a coherent state. When $\theta = \pi/2$, $\phi_1 = 2\pi/3$ and $\phi_2 = 4\pi/3$, the stars describe an equilateral triangle, that is an anticoherent state of order one.

We also want to stress that, unlike in the previous case, the coordinates (θ, ϕ_1, ϕ_2) are not in a one-to-one correspondence with the elements of $\mathscr{S}$, for instance, the points of the section with $(\theta = \theta_0, \phi_1 = 0, \phi_2 = \pi)$ and $(\theta = \pi/2, \phi_1 = 0, \phi_2 = 2\theta_0)$ have the same shape — they both describe a shape where two stars coincide and the direction of the third one makes angle of $2\theta_0$ with the one of the first two, but they are different points in the prism of figure 2.1.

Similar procedures can be used to find sections to parametrize $\mathscr{S}$ for higher spins. For instance, one can consider a section where one star is in the north pole, a second one is in the xz plane and use the direction of the remaining stars as coordinates. Of course, just like the case of $s = 3/2$, this set of coordinates is not in a one-to-one correspondence with the points of $\mathscr{S}$.

2.1.1 Structures induced by the Fubini-Study metric

As we show in the paragraphs below, the decomposition of $\mathbb{P}(\mathcal{H}_s)$ as a fiber bundle — together with the Fubini-Study metric — allows us to define a series of geometrical structures in a straightforward way. Note that, the right action defined in (2.1.1), leaves the Fubini-Study metric (1.1.10) invariant. Indeed, using these equations, one can prove that the equality $d(\rho_1, \rho_2) = d(\rho_1 \triangleleft R, \rho_2 \triangleleft R)$ holds. By this invariance, any scalar defined in terms of this metric is constant for all the points in the same fiber. As a matter of nomenclature, recall that we say that a function of $\mathbb{P}(\mathcal{H}_s)$ defines a function in shape space if it attains the same value for different points in the same fiber.

Although most of the remaining of this section is mostly mathematical, we want to stress that we have reasons to believe that our results are of physical interest. We address the concrete applications of these constructions in a future work.

Some relevant background for the following calculations includes [23, Chapter 9], [57, Chapter 5] and [58].

First, we endow $\mathbb{P}(\mathcal{H}_s)$ with a connection. This can be done with the Fubini-Study metric h (1.1.11) in a very simple fashion; we say that a vector v tangent to ρ is horizontal if and only if it is perpendicular to all the fundamental vectors (2.1.2). We show in equation (C.2.3) of appendix C that this horizontality condition does meet all the necessary requirements to define a connection.

Denote by ω the $so(3)$ valued 1-form associated to this connection. By definition, this means that ω satisfies the following equalities,

$$\omega(\mathcal{A}^\sharp) = \mathcal{A} \text{ for all } \mathcal{A} \in so(3), \qquad \omega(v) = 0 \text{ for all horizontal vectors } v\,. \tag{2.1.8}$$

Also, denote by Ω the $so(3)$-valued curvature form for ω. Note that, because of the invariance of the Fubini-Study metric emphasized at the beginning of the section, the scalar $\mathrm{Tr}\,(\Omega^2)$ frequently considered in Yang-Mills theories,[3] defines a function in shape space.

With the help of the connection ω, we can decompose the tangent space at a point as the direct sum of the vertical and horizontal spaces. Note that

[3] If we write Ω in terms of coordinates q^A $(A = 1, \ldots, 4s)$ for $\mathbb{P}(\mathcal{H}_s)$ as,

$$\Omega = \frac{1}{2}\sum_{\alpha=1}^{3}\sum_{A,B=1}^{4s} S_\alpha\, \Omega^\alpha{}_{AB}\, dq^A \wedge dq^B\,, \text{ then, } \mathrm{Tr}\,(\Omega^2) = \sum_{\alpha,\beta=1}^{3}\sum_{A,B=1}^{4s} \Omega^\alpha{}_{AB}\Omega^{\beta AB} h(S^\sharp_\alpha, S^\sharp_\beta)\,.$$

these are orthogonal by definition. This decomposition allows us to define two new structures, a *vertical metric* for $SO(3)$ and an *horizontal metric* for $\mathscr{S}$, as we explain in what follows.

By considering the restriction of the Fubini-Study metric to the fiber of a state ρ, we can compute the induced metric over it. Some examples of this construction can be found in [59, 60]. In these terms, the vertical metric $k^{(\rho)}$ for $SO(3)$ is defined as the pullback of the induced one under the isomorphism φ_ρ (2.1.3). It is an easy exercise to prove that $k^{(\rho)}$ is a right invariant metric for $SO(3)$ and therefore, the Ricci scalar $R(k^{(\rho)})$ is the same for all the elements of $SO(3)$. One can also prove that $SO(3)$ with the metric $k^{(\rho\triangleleft r)}$ is isometric to $SO(3)$ with $k^{(\rho)}$ for any rotation r (to find the isometry, it is useful to recall that, for any fiber bundle, the pushforward of $\mathcal{A}^\sharp$ under the right action of r is $(r^{-1}\mathcal{A}r)^\sharp$) and therefore, $R(k^{(\rho)})$ also defines a function in shape space.

In particular, we can evaluate $k^{(\rho)}$ at the elements S_α and S_β ($\alpha, \beta = x, y, z$) in $so(3)$. A little bit of algebra reveals,

$$k^{(\rho)}_{\alpha\beta} \equiv k^{(\rho)}(S_\alpha, S_\beta) \equiv h(S^\sharp_\alpha, S^\sharp_\beta) = \Re\langle S_\alpha S_\beta\rangle - \langle S_\alpha\rangle\langle S_\beta\rangle\,, \tag{2.1.9}$$

where the expectation values are computed w.r.t. ρ. The 3×3 matrix with entries $k^{(\rho)}_{\alpha\beta}$ is the matricial representation of $k^{(\rho)}$ w.r.t. the basis $\{S_x, S_y, S_z\}$ of $so(3)$. Abusing of notation, we denote this matrix also by $k^{(\rho)}$. In terms of it, we can express $R(k^{(\rho)})$ as follows (the details of the calculation can be found in appendix C, equation (C.1.23)),

$$R(k^{(\rho)}) = \frac{\mathrm{Tr}\,(k^{(\rho)})^2 - 2\,\mathrm{Tr}\,((k^{(\rho)})^2)}{2\mathrm{Det}\;k^{(\rho)}}\,. \tag{2.1.10}$$

Next, we define the *horizontal metric* for $\mathscr{S}$, g. Just as the vertical one is defined by restricting the Fubini-Study metric to vertical vectors, the horizontal metric is defined in terms of the horizontal ones. Given $\underline{u}$ and $\underline{v}$ vectors tangent to a point $\mathcal{S}$ in $\mathscr{S}$, define their inner product $g(\underline{u}, \underline{v})$ as follows,

$$g(\underline{u}, \underline{v}) = h(u, v)\,, \tag{2.1.11}$$

where u and v are any two horizontal vectors tangent to a point in $\mathbb{P}(\mathcal{H}_s)$ that get projected to $\underline{u}$ and $\underline{v}$ respectively, $\pi_* u = \underline{u}$, $\pi_* v = \underline{v}$. Using the fact that the Fubini-Study metric is invariant w.r.t. the action of $SO(3)$, it is straightforward to prove that the particular choice of u and v is irrelevant. Denote by $R(g)$ the Ricci scalar for the metric g.

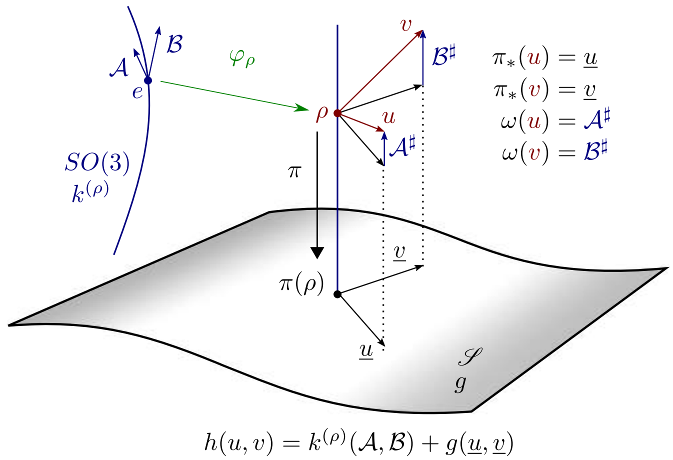


Figure 2.2: Visual representation of the formula (2.1.12). $SO(3)$ is mapped onto the fiber of ρ using φ_ρ (2.1.3). φ_ρ sends the identity rotation e to ρ. By decomposing u and v in their vertical and horizontal components (that are orthogonal), we can compute $h(u, v)$ in terms of the metrics g for $\mathscr{S}$, and $k^{(\rho)}$ for $SO(3)$.

Notice that, because vertical and horizontal vectors are orthogonal to each other by construction, the inner product h between two vectors u and v tangent at ρ can be written in terms of k and g in the following way,

$$h(u, v) = g(\pi_* u, \pi_* v) + k^{(\rho)}(\omega(u), \omega(v)) \,. \tag{2.1.12}$$

In figure 2.2, we present a representation of the previous result.

Note that, by (2.1.12), if u and v are two tangent vectors that get projected to the same vector in $\mathscr{S}$ (that is, if $\pi_* u = \pi_* v$), but u is horizontal while v is not, it is clear that the magnitude of u is smaller than the one of v, $h(u, u) < h(v, v)$ (since $\omega(u)$ is zero).

This observation has two interesting consequences. The first one is that the length of a horizontal curve in $\mathbb{P}(\mathcal{H}_s)$ is smaller than the one of a non-horizontal curve (as long as their projection in $\mathscr{S}$ is the same) . The second one is that the h-length of a horizontal curve is equal to the g-length of its projection in $\mathscr{S}$.

Finally, we introduce two extra functions in shape space. First, define Φ as follows,

$$\Phi = \ln \operatorname{Det} \, k^{(\rho)} \,, \tag{2.1.13}$$

where $k^{(\rho)}$ denotes the matrix for the vertical metric defined in (2.1.9). In equation (C.2.37), we show that $\operatorname{Det} \, k^{(\rho)}$ does define a function in shape space.

The second function we define is closely related to the *degree of coherence* $|\langle S\rangle|$ of a state $\rho = |\psi\rangle\langle\psi|$ defined in [61] as follows: given v_ψ the spin expectation value of ρ (see eq. (1.3.18)) , define $|\langle S\rangle|$ as the euclidean magnitude of the vector v_ψ. Since, under the right action (1.1.8), v_ψ only rotates, it is easy to verify that $|\langle S\rangle|$ defines a function in shape space. The quantity ℓ we introduce is very similar, but we consider the magnitude of v_ψ w.r.t. to the metric $k^{(\rho)}$ (regarding v_ψ as a vector in $so(3)$) instead of the euclidean one,

$$\ell = \left(\sum_{\alpha,\beta=1}^{3} k^{(\rho)}_{\alpha\beta} \langle S_\alpha\rangle\langle S_\beta\rangle \right)^{1/2} . \tag{2.1.14}$$

It is simple enough to prove that, just as $|\langle S\rangle|$, ℓ also defines a function in shape space.

In the previous paragraphs, we have introduced five functions for shape space, mainly, $\mathrm{Tr}\,(\Omega^2)$, $R(k^{(\rho)})$, $R(g)$, Φ and ℓ. As it turns out, these quantities are not independent. The following relationships come from a big calculation that can be found in appendix C.2. These relations are (we omit the superindex (ρ) from now on),

$$R(h) - 3(8s+2) + \frac{3\ell^2}{2e^\Phi} = R(g) - \frac{3}{4}\,\mathrm{Tr}\,(\Omega^2)\,, \tag{2.1.15}$$

and,

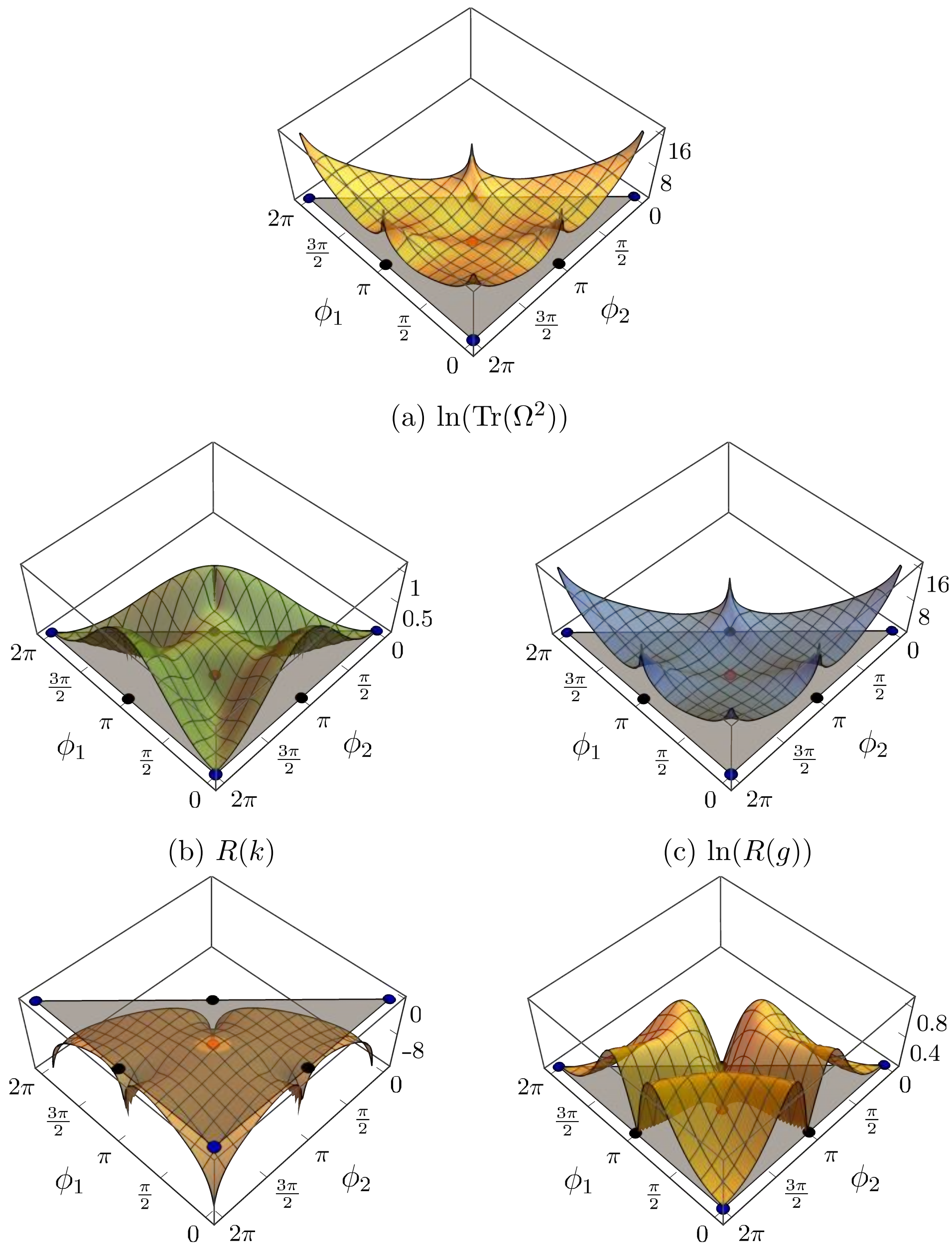
$$\mathrm{Tr}\,(\Omega^2) = \frac{3}{s} R(h) - 4R(k) + ||\nabla_{\mathscr{S}}\Phi||^2_{\mathscr{S}} + 2\nabla^2_{\mathscr{S}}\Phi\,, \tag{2.1.16}$$

where we used the metric g of $\mathscr{S}$ to define the gradient of Φ ($\nabla_{\mathscr{S}}\Phi$), its Laplacian ($\nabla^2_{\mathscr{S}}\Phi$) and the magnitude of $\nabla_{\mathscr{S}}\Phi$ ($||\nabla_{\mathscr{S}}\Phi||_{\mathscr{S}}$). Also, $R(h)$ denotes the (constant) Ricci scalar for the Fubini-Study metric h (c.f. appendix C.2.3),

$$R(h) = 8s(2s+1)\,.$$

A plot of these functions is presented in figure 2.3 for the case of $s = 3/2$. These graphics were produced in terms of the coordinates presented in figure 2.1. Since $\mathscr{S}$ is three-dimensional in this case, we fixed $\theta = \pi/2$ and made the corresponding plot in the $\phi_1\phi_2$ plane.

Notice that all the functions have special behavior at two particular kinds of shapes; the ones where all the stars are in a diameter of the sphere (the black and blue dots in the figure), and the one where they make a big equilateral triangle (the red dots), the shape of the only anticoherent state for a spin-3/2. For instance, $\mathrm{Tr}\,(\Omega^2)$ has a pole in the black and blue dots, while it attains a minimum at the red point, the only zeros of ℓ occur for these types of points, and so on. This behavior appears to continue for higher values of s, in particular, the shape of anticoherent states of the highest degree possible (for a certain value of s) appear to be local minima of $\mathrm{Tr}\,(\Omega^2)$. These facts suggest that these quantities are related with the degree of anticoherence of a state, but, as we have already mentioned, finding their precise physical meaning is a problem we will address in a future work.



(a) $\ln(\mathrm{Tr}(\Omega^2))$

(b) $R(k)$

(c) $\ln(R(g))$

(d) Φ

(e) ℓ

Figure 2.3: Plot of the functions $\ln(\mathrm{Tr}\,(\Omega^2))$, $R(k)$, $\ln(R(g))$, Φ and ℓ for $s = 3/2$, using the coordinates of figure 2.1 with $\theta = \pi/2$. The domain of the functions is shown as a gray triangle. Three type of shapes are highlighted in the domain, the red point corresponds to the shape of a big equilateral triangle; the blue point, to the one of coherent states and the black one, to the one where two stars coincide and the remaining one is antipodal to the other two.

2.1.2 A simple example, the $s = 1$ case

In what follows, we specify the definitions of the previous sections to the $s = 1$ case. This value of s is relevant because the resulting expressions are algebraically manageable. The results are written in terms of the coordinates introduced in (2.1.5). Because shape space is one dimensional in this case, the following equalities hold trivially,

$$R(g) = \mathrm{Tr}\,(\Omega^2) = 0\,.$$

The points of the section defined in (2.1.5), can be written as a matrix in the following way,

$$\rho(q) = \frac{1}{3+\cos q}\begin{pmatrix} 4\cos^4\frac{q}{4} & 0 & -\sin^2\frac{q}{2} \\ 0 & 0 & 0 \\ -\sin^2\frac{q}{2} & 0 & 4\sin^4\frac{q}{4} \end{pmatrix}\,. \tag{2.1.17}$$

By taking the derivative w.r.t. q, we can calculate the representation of the tangent vectors,

$$w = \partial_q\rho = -\frac{2}{(3+\cos q)^2}\begin{pmatrix} \sin^3\frac{q}{2} & 0 & \sin q \\ 0 & 0 & 0 \\ \sin q & 0 & -\sin^3\frac{q}{2} \end{pmatrix} \tag{2.1.18}$$

By making a direct computation, we find the following expression for the vertical vectors (2.1.2),

$$\begin{aligned} S_x^\sharp &= \frac{2i\sqrt{2}\cos\frac{q}{2}}{3+\cos q}\begin{pmatrix} 0 & -\cos^2\frac{q}{4} & 0 \\ \cos^2\frac{q}{4} & 0 & -\sin^2\frac{q}{4} \\ 0 & \sin^2\frac{q}{4} & 0 \end{pmatrix} \\ S_y^\sharp &= \frac{2\sqrt{2}}{3+\cos q}\begin{pmatrix} 0 & -\cos^2\frac{q}{4} & 0 \\ -\cos^2\frac{q}{4} & 0 & \sin^2\frac{q}{4} \\ 0 & \sin^2\frac{q}{4} & 0 \end{pmatrix} \\ S_z^\sharp &= \frac{2i\sin^2\frac{q}{2}}{3+\cos q}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.1.19}$$

A visual representation of ρ, $\partial_q\rho$ and the vertical vectors is presented in figure 2.4.

After some algebra, one can notice that the inner product h between w and the vertical vectors is zero. This implies that w is a horizontal vector (this is the main reason why we define the coordinates q as in (2.1.5)), so we can use it to compute the metric g (2.1.11),

$$g_{qq} \equiv \frac{1}{2}\,\mathrm{Tr}\,(w^2) = \frac{\sin^2\frac{q}{2}}{(3+\cos q)^2}$$

Now, we proceed to compute the vertical metric. Using the expression (2.1.9) we obtain that k^ρ is diagonal in this case, and it is given by the following expression,

$$k^{(\rho)} = \begin{pmatrix} \Delta S_x^2 & 0 & 0 \\ 0 & \Delta S_y^2 & 0 \\ 0 & 0 & \Delta S_z^2 \end{pmatrix} = \frac{2}{3+\cos q}\begin{pmatrix} \cos^2\frac{q}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2\sin^4\frac{q}{4}}{3+\cos q} \end{pmatrix} \tag{2.1.20}$$

By considering the determinant of $k^{(\rho)}$, we obtain the following expression for Φ (2.1.13),

$$\Phi = \ln\left\{\frac{4\sin^2\frac{q}{2}\sin^2 q}{(3+\cos q)^4}\right\}. \tag{2.1.21}$$

The result for the Ricci scalar (2.1.10) turns out to be, after some algebra,

$$R(k) = 2\,, \tag{2.1.22}$$

that is independent of q.

Next, we find an expression for ℓ (2.1.14). To this end, first we compute the spin expectation value v_ψ (1.3.18). The result is,

$$v_\psi = \frac{4\cos\frac{q}{2}}{3+\cos q}(0,0,1)\,.$$

The magnitude ℓ of this vector according to the metric $k^{(\rho)}$ turns out to be,

$$\ell = \frac{4\sin\frac{q}{2}\sin q}{(3+\cos q)^2}\,.$$

Note that this result can also be obtained by considering the expression for Φ (2.1.21) and equation (2.1.15).

Finally, by integrating the square root of the metric $\sqrt{g_{qq}}$ from 0 to q, we can compute the geodesic distance (in $\mathscr{S}$) between the shape of coherent states and the one of $\rho(q)$. As we see in the next subsection, this corresponds

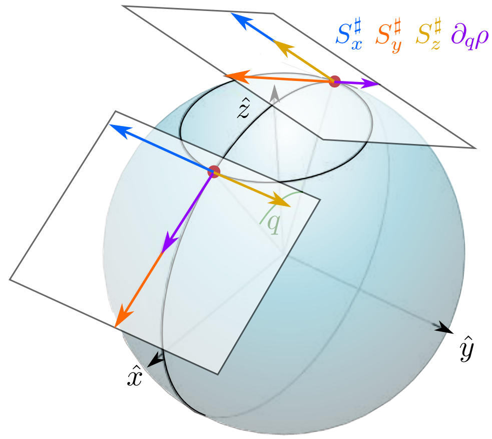


Figure 2.4: Visual representation of ρ (2.1.17), $\partial_q\rho$ (2.1.18) and of the vertical vectors of equation (2.1.19). The constellation of ρ (2.1.4) is shown in red. The small rectangles denote the tangent space of the sphere at the red points, and $\partial_q\rho$, $S_x^\sharp$, $S_y^\sharp$ and $S_z^\sharp$ are represented as vectors in these spaces. The arrows for $S_\alpha^\sharp$ were obtained by considering an infinitesimal rotation of the red dots around the axis α, while the ones for $\partial_q\rho$, by applying an infinitesimal change in the parameter q.

to the geometric entanglement of $\rho(q)$ (when regarding the spin as a system of $2s$ spin-1/2). In this case the result is,

$$E(q) = \int_0^q \sqrt{g_{qq}(t)}dt = \frac{\pi}{4} - \arctan\left(\cos\tfrac{q}{2}\right) .$$

A plot of $g_{qq}(q)$ and of $E(q)$ is presented in figure 2.5. Clearly, when $q = 0$, $E(q)$ is also zero. This is because $q = 0$ denotes the shape of coherent states, the only separable ones. The maximally entangled state corresponds to $q = \pi$, that is, to the shape where the two stars are antipodal to each other. Therefore, the diameter of $\mathscr{S}$ is $\pi/4$. Compare this with $\pi/2$, the one of projective Hilbert space.

From the same figure 2.5, notice that g_{qq} is zero when q is equal to zero. This is not a singularity of the metric, but merely an effect of the coordinates we are using — at this point the pushforward of the mapping between coordinates and shapes becomes singular. This type of effects tends to occur for shapes where two or more stars coincide. See [62] for an extended discussion (in this article, the authors consider the mapping between the coefficients of a polynomial and its roots; by reinterpreting their results in terms of the Majorana representation, we can arrive at the same conclusions for our case).

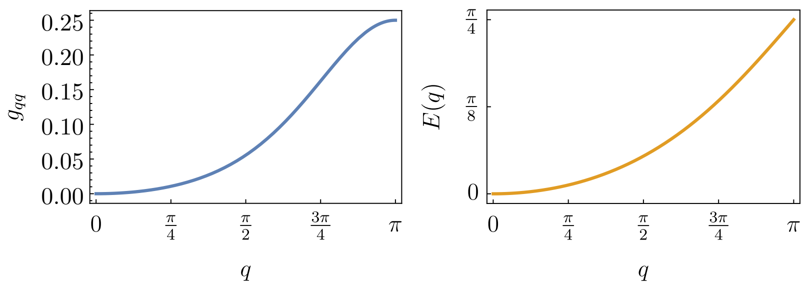


Figure 2.5: Plot of the metric g_{qq} and of the entanglement $E(q)$

2.1.3 Geodesics in $\mathbb{P}(\mathcal{H}_s)$ and in $\mathscr{S}$

In terms of the metric g (2.1.11), we can define the *geodesic distance* $d_{\mathscr{S}}$ between two shapes $\mathcal{S}_1$ and $\mathcal{S}_2$. Denote by f_1 and f_2 the fiber of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. As we prove in the following paragraph, $d_{\mathscr{S}}(\mathcal{S}_1, \mathcal{S}_2)$ has a simple geometrical interpretation; it is the shortest distance (defined as in 1.1.4)) between the elements of f_1 and f_2,

$$d_{\mathscr{S}}(\mathcal{S}_1, \mathcal{S}_2) = \min_{\rho_1 \in f_1, \rho_2 \in f_2} d_{\mathbb{P}(\mathcal{H}_s)}(\rho_1, \rho_2) = \min_{\rho_2 \in f_2} d_{\mathbb{P}(\mathcal{H}_s)}(\rho_1^0, \rho_2)\,, \tag{2.1.23}$$

where ρ_1^0 denotes any state in f_1. The fact that we can fix any element in f_1 and only minimize over f_2 comes from the invariance of the Fubini-Study metric under the right action of rotations.

Now we prove the equality (2.1.23). Let ρ_2^0 be an element in f_2 that minimizes the right side of (2.1.23). Denote by γ the shortest h-geodesic that connects ρ_1^0 with ρ_2^0. Note that its length is $d_{\mathbb{P}(\mathcal{H}_s)}(\rho_1^0, \rho_2^0)$. Since ρ_2^0 is the state in f_2 closest to ρ_1^0, among all curves that starts at ρ_1^0 and ends at any point of f_2, γ is of *minimal length*. As argued just after (2.1.12), this implies that γ is a horizontal curve and, therefore, the length of its projection, $\pi(\gamma) = \underline{\gamma}$, is the same as the one of γ, $d_{\mathbb{P}(\mathcal{H}_s)}(\rho_1^0, \rho_2^0)$. Note that, as we are about to prove, $\underline{\gamma}$ is the shortest curve in $\mathscr{S}$ that connects $\mathcal{S}_1$ with $\mathcal{S}_2$, and therefore its length is also $d_{\mathscr{S}}(\mathcal{S}_1, \mathcal{S}_2)$ by definition. To verify that $\underline{\gamma}$ is indeed the shortest one, assume a shorter curve $\underline{\tilde{\gamma}}$ exists. Consider a horizontal curve $\tilde{\gamma}$ that begins at ρ_0^1 and gets projected onto $\underline{\tilde{\gamma}}$. By construction, $\tilde{\gamma}$ begins at ρ_1, ends at a point of f_2, and its length — that is equal to the one of $\underline{\tilde{\gamma}}$ — is smaller than the one of γ, a contradiction. By equating both results for the length of $\underline{\gamma}$ just obtained, we conclude that (2.1.23) holds.

We can use this definition to give a geometrical characterization of the measure of entanglement (1.1.15) of a state. As noted in section 1.3.4, coherent states are the only separable ones. Therefore, the entanglement of ρ is the

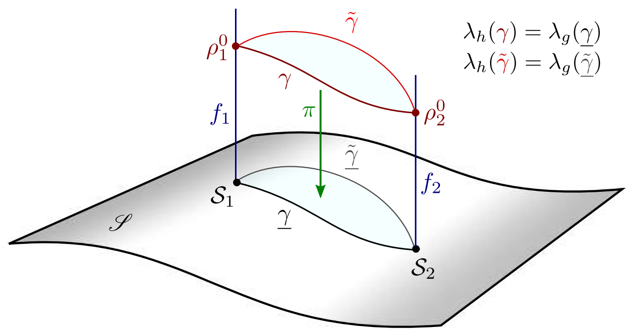


Figure 2.6: Sketch of the proof of (2.1.23). Since ρ_2^0 is the element in f_2 closest to ρ_1^0, γ, the h-geodesic that joins them, is horizontal. Therefore, its h-length, $\lambda_h(\lambda)$, is equal to both, the r.h.s. of (2.1.23) and the length of its projection $\underline{\gamma}$, $\lambda_g(\underline{\gamma}) = \lambda_h(\lambda)$. From here, we can conclude $\underline{\gamma}$ is the shortest curve connecting $\mathcal{S}_1$ with $\mathcal{S}_2$, as any other curve $\underline{\tilde{\gamma}}$ might be lifted to a horizontal curve $\tilde{\gamma}$ (preserving its length) that is longer than γ. Therefore, $\lambda_g(\underline{\gamma})$ is also equal to the l.h.s. (2.1.23).

distance $d_{\mathscr{S}}$ between the shape of ρ and the shape of coherent states. In some cases, this simplifies the calculation of the entanglement of a state, as shown in the previous subsection. In these terms, the maximally entangled states are those such that their shape is the furthest away from the one of coherent states.

As noted previously, the geodesic that connects ρ with the closest coherent is horizontal. This renders them relevant in the context of the *quantum brachistochrone problem* [63, 64], where the fastest way (under a certain set of conditions, see [64] for details) to evolve a state ρ_i to a final state ρ_f is sought. As it turns out, the answer is that the optimal evolution occurs along the geodesic connecting ρ_i with ρ_f. In this context, if we want to evolve ρ_i into *any separable state* as fast as possible, we have to follow the horizontal geodesic connecting ρ_i with the closest coherent state.

Although useful when studying rotations, the Majorana is less intuitive when working with horizontal geodesics. For example, consider the horizontal geodesic that connects the state with constellation $\{s_1, s_2, s_3\}$ (see fig. 2.7) and ends in the coherent state in the direction $\hat{x}$. One might expect that, as we advance in the geodesic, the stars would follow great circles to get to the blue point, but this is not the case. As we can see in the figure, initially s_1 *gets away* from the blue point, only to get close again (after coinciding with s_2) following a complicated trajectory.

In the previous paragraphs we proved that any horizontal geodesic gets projected by π onto a geodesic of $\mathscr{S}$. This raises the question, does this occur

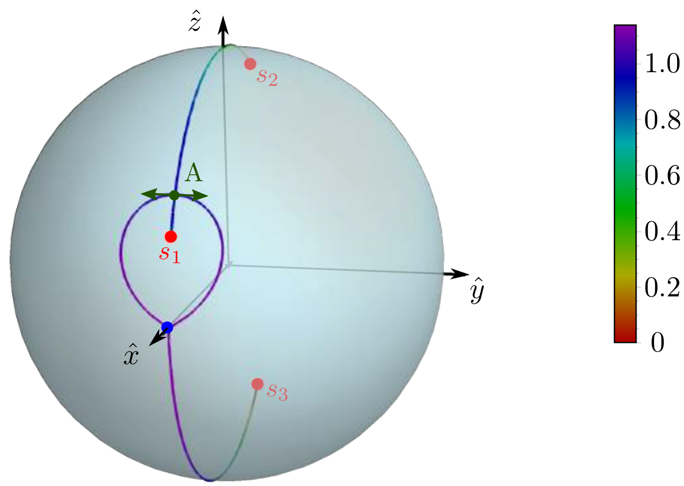


Figure 2.7: Visualization of an horizontal geodesic that begins at the state with constellation $\{s_1, s_2, s_3\}$ and ends in the coherent state associated to the direction shown in blue. As we advance in the geodesic, the stars move in the trajectory shown in the figure. The value of the arclength parameter of the geodesic is color coded according to the bar legend in the right. Notice that s_1 is almost still throughout most of the geodesic. Eventually, the stars s_1 and s_2 coincide in the green point A. Afterwards, they spread *equiangularly* in the direction of the green arrows, until the three of them coincide in the blue point. This type of spreading after two stars coincide has already been reported in [65].

for all type of geodesics (including not horizontal ones)? The answer is no, as we show in the following paragraphs. Although here we present only the final results, the intermediate calculations can be found in appendix C.3.

Consider a geodesic in $\mathbb{P}(\mathcal{H}_s)$ parametrized as $\rho(t)$. Denote by $\underline{\rho}(t)$ its projection, $\pi(\rho(t)) = \underline{\rho}(t)$. Suppose $\rho(t)$ is arclength parametrized. For any t, the vector tangent to $\rho(t)$ can be written as,

$$\dot{\rho}(t) = \sum_{\alpha=x,y,z} v^\alpha S^\sharp_\alpha + \sum_{i=1}^{4s-3} v^i E_i\,, \tag{2.1.24}$$

where E_i $(i = 1, \dots, 4s-3)$ denotes the elements of a orthonormal basis for horizontal space chosen continuously over an open set containing $\rho(t)$. For the remainder of this chapter, we use Einstein notation for sum over repeated indices. Also, we drop the superindex (ρ) in $k^{(\rho)}$ from now on. Greek letters range over x, y and z, while Latin indices, over $1, 2, \dots, 4s-3$. We also use the metrics g and $k^{(\rho)}$ to raise and lower indices. Note that we are working in an *anholonomic* basis, since the vector fields $S^\sharp_\alpha$, $S^\sharp_\beta$, E_i and E_j do not commute in general.

In terms of (2.1.24), $\dot{\underline{\rho}}(t)$ can be computed as,

$$\dot{\underline{\rho}}(t) = v^i \pi_*(E_i)\,. \tag{2.1.25}$$

where π_* denotes the pushforward of π.

After some algebra that can be found in the previously mentioned appendix, we obtain that the equations for v^α and v^i are,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(v_\alpha) &= 0\,, \\ \dot{v}^i + \Gamma(g)^i{}_{jk} v^j v^k + \Omega^\alpha{}_k{}^i v^k v_\alpha + \frac{1}{2} k^{\alpha\beta}{}_{,}{}^i v_\alpha v_\beta &= 0\,, \end{aligned} \tag{2.1.26}$$

where $\Gamma(g)^i{}_{jk}$ denotes the components of the Levi-Civita connection of the metric g (the equivalent of the Christoffel symbols for anholonomic basis) and $f_{,i}$ is a shorthand notation for the derivative of the function f in the direction of E_i, $f_{,i} = E_i \triangleright f$.

Notice that v_α (with the index lowered with the metric) is a conserved quantity. In particular, if all the components v_α are zero for a certain t, they are zero always. This implies that, if the tangent vector of a geodesic is horizontal at some particular t then the entire geodesic is horizontal.

The second equation of (2.1.26) answers our question; in general, geodesics in $\mathbb{P}(\mathcal{H}_s)$ do not get projected onto geodesics of $\mathscr{S}$. Indeed, by considering

(2.1.25), we can conclude that only when the sum of the third and fourth term in the l.h.s of (2.1.26) is zero (for example, when ρ is horizontal) $\underline{\rho}$ is a geodesic.

Note that $\underline{\rho}(t)$ is not arclength parametrized in general (w.r.t. g) . We can reparametrize everything so it is. Denote by τ the arclength parameter of $\underline{\rho}$ and by a prime derivatives w.r.t. τ. By writing $\rho'(\tau) = u^\alpha S^\sharp_\alpha + u^i E_i$, one can find that the resulting equations for u^α and u^i are,

$$\left(\frac{u_\alpha}{\sqrt{1+Q^2}}\right)' = 0\,,$$
$$u^{i'} + \Gamma(g)^i{}_{jk} v^j v^k + \Omega^\alpha{}_k{}^i u^k u_\alpha + \frac{1}{2} k^{\alpha\beta}{}_,{}^i u_\alpha u_\beta - \frac{(Q^2)'}{1+Q^2} u^i = 0\,, \qquad (2.1.27)$$

where Q is the magnitude of the vertical component of ρ', $Q^2 = u^\alpha u^\beta k_{\alpha\beta}$. If we suppose that (2.1.27) is the equation of motion of a particle living in $\mathscr{S}$, said particle would not be free, there would be a Lorentz-type force $\Omega^\alpha{}_k{}^i u^k u_\alpha$ plus another force dependent on its velocity (unless, Q^2 is zero).

Finally, we use the previous equations to characterize the coherent state closest to $\rho_0 = |\psi_0\rangle\langle\psi_0|$. Denote it by $\rho_{\hat{\boldsymbol{n}}} = |\hat{\boldsymbol{n}}\rangle\langle\hat{\boldsymbol{n}}|$. By writing $|\psi_f\rangle = |\hat{\boldsymbol{n}}\rangle$ in equation (1.1.12), we can characterize the geodesic $\rho(t)$ connecting ρ_0 with $\rho_{\hat{\boldsymbol{n}}}$. As already argued, since $\rho_{\hat{\boldsymbol{n}}}$ is the closest coherent state, $\rho(t)$ is horizontal. By the argument following (2.1.26), this curve is horizontal if and only if $\dot{\rho}$ is horizontal for a particular t, say, the initial one, $t = 0$. By applying, the horizontality condition to $\dot{\rho}(0)$, we can conclude that

$$h(\dot{\rho}(0), S^\sharp_\alpha) = 0\,,$$

for all vertical vectors $S^\sharp_\alpha$. By considering (1.1.12), it is easy to see that the previous condition reduces to,

$$\langle\psi_0|S_\alpha|\psi_0^\perp\rangle = \langle\psi_0^\perp|S_\alpha|\psi_0\rangle\,,$$

where $|\psi_0^\perp\rangle$ is given by equation (1.1.13). By substituting $|\psi_0^\perp\rangle \propto |\hat{\boldsymbol{n}}\rangle - \langle\psi_0|\hat{\boldsymbol{n}}\rangle|\psi_0\rangle$ we obtain, after some straightforward algebra, that the horizontality condition is equivalent to the following,

$$\langle\psi_0|S_\alpha|\hat{\boldsymbol{n}}\rangle = \langle\hat{\boldsymbol{n}}|S_\alpha|\psi_0\rangle$$

Note that the previous condition is linear in S_α. In particular, if we consider the *raising operator* in the direction $\hat{n}$, $S^+_{\hat{n}}$, the condition implies,

$$\langle\hat{\boldsymbol{n}}, s-1|\psi_0\rangle = 0\,,$$

that is, $|\psi_0\rangle$ is orthogonal to $|\hat{\boldsymbol{n}}, s-1\rangle$. This is the same condition we obtained in [1] but deduced in a completely different way.

2.1.4 Horizontal and vertical vectors and Berry curvature,

In this section, we find expressions for the Berry curvature $\mathcal{K}_B$ [31] when evaluated at horizontal and vector fields. These results also allow us to give a relatively simple expression for the components $\Omega_{\alpha ij}$ of the curvature Ω. Most of the explicit calculations can be found in appendices C.4 and C.6.

These results are written using the notation introduced in theorem 2. In these terms, the horizontal vectors E_i tangent at a point $\rho = |\psi\rangle\langle\psi|$ can be uniquely written as,

$$E_i = |\psi\rangle\langle\psi_i| + |\psi_i\rangle\langle\psi| \, ,$$

where $|\psi_i\rangle$ satisfy the equality $\langle\psi_i|\psi\rangle = 0$.

The Berry curvature at the point $\rho = |\psi\rangle\langle\psi|$, evaluated at two vertical vectors, $S^\sharp_\alpha$ and $S^\sharp_\beta$, can be written as

$$\mathcal{K}_B(S^\sharp_\alpha, S^\sharp_\beta) = \epsilon^\gamma{}_{\alpha\beta}\langle S_\gamma\rangle \, ,$$

where the expectation values are computed w.r.t. ρ and $\epsilon^\gamma{}_{\alpha\beta}$ denotes the components of the Levi-Civita tensor,

$$\epsilon^\gamma{}_{\alpha\beta} = \begin{cases} 0 & \text{if any two indices are the same} \\ 1 & \text{if } (\alpha,\beta,\gamma) \text{ is an even permutation of } (x,y,z) \\ -1 & \text{if } (\alpha,\beta,\gamma) \text{ is an odd permutation of } (x,y,z) \end{cases} .$$

For one horizontal vector and one vertical the result is,

$$\mathcal{K}_B(S^\sharp_\alpha, E_i) = \langle S_\alpha\rangle_{,i} \, ,$$

where $_{,i}$ denotes the derivative in the direction E_i.

Finally, we deal with the case of two horizontal vectors. The resulting expression is,

$$\mathcal{K}_B(E_i, E_j) = 2\Im\langle\psi_i|\psi_j\rangle \, .$$

The previous equation can also be written in terms of the coefficients of the curvature Ω. The result is,

$$\mathcal{K}_B(E_i, E_j) = \frac{4\Im\langle\psi_i|S_\alpha|\psi_j\rangle - \Omega_{\alpha ij}}{2\langle S_\alpha\rangle} \, .$$

By combining the previous results we obtain,

$$\Omega_{\alpha ij} = 4\Im\langle\psi_i|(S_\alpha - \langle S_\alpha\rangle\mathbb{1})|\psi_j\rangle \, .$$

This is one of the simplest expressions (from a numerical point of view) we have found that determines the coefficients of the curvature Ω. Another expression worth checking out is (C.7.3) of appendix C.7.1.

Chapter 3

Stellar representation for the Grassmannians

Given a certain k-plane through the origin contained in $\mathcal{H}_s$, what are the rotations that leave it invariant? This type of problem is relevant in the context of quantum computing, as these rotations might be used to implement quantum gates [66].

For the case $k = 1$, 1-planes are projective rays, and their symmetries can be found immediately by looking at the constellation of a state in the ray. To solve this problem for $k \geq 2$, in this chapter we define the stellar representation for $Gr_k(\mathcal{H}_s)$; a generalization of the Majorana representation that assigns to a k-plane a constellation. As we find further down, by looking at the constellation of a k-plane — and some extra ingredients — we can deduce its rotational symmetries.

The recipe to define a constellation of a k-plane Π presented in this chapter is very similar to the one used in section 1.3.4 for the constellation of a state; first, we define the coherent k-plane associated to the direction $\hat{\boldsymbol{n}}$, $\Pi_{\hat{\boldsymbol{n}}}$. Then, the constellation of Π consists of all the directions $\hat{n}$ such that the product (1.2.4) $\langle \Pi_{-\hat{\boldsymbol{n}}}, \Pi \rangle$ is zero. In the rest of this chapter, we work this recipe in detail, and mention some interesting properties we have found about this representation.

3.1 The coherent k-planes

We define a coherent plane in the direction $\hat{n}$ as follows,

$$\Pi_{\hat{\boldsymbol{n}}} = \text{span}\{|\hat{\boldsymbol{n}}, s\rangle, |\hat{\boldsymbol{n}}, s-1\rangle|\hat{\boldsymbol{n}}, s-2\rangle, \ldots, |\hat{\boldsymbol{n}}, s-k+1\rangle\},$$

where $|\hat{\boldsymbol{n}}, m\rangle$ denotes the eigenstate of $S \cdot \hat{n}$ with eigenvalue m. An explicit expression for these states can be found in (B.0.10). When $k = 1$, we recover the usual coherent states, $\Pi_{\hat{\boldsymbol{n}}} = [\hat{\boldsymbol{n}}]$.

The coherent k-planes satisfy many properties analogous to the ones satisfied by spin coherent states. Here we mention some of them.

If we take a coherent plane and *rotate it*,[1] the resulting space is also a coherent plane. Furthermore, by choosing a single coherent plane, and rotating it in all the possible ways, we obtain the whole space of coherent k-planes. This space can be shown to be topologically a sphere, and the distance (1.2.5) between two coherent planes is the following,

$$d(\Pi_{\hat{\boldsymbol{m}}}, \Pi_{\hat{\boldsymbol{n}}}) = \arccos\left(\cos^{k(2s-k+1)} \tfrac{\theta}{2}\right) , \tag{3.1.1}$$

where θ denotes the angle between $\hat{m}$ and $\hat{n}$. This result is analogous to the one for coherent states (1.3.17). A proof of this formula is given right before equation (3.5.6).

Coherent planes are also the most classical in a sense. Given a plane Π, denote by $\mathcal{P}$ the projection operator onto Π. The restriction of an operator A of Hilbert space to Π is defined as follows,

$$A^{\Pi} = \mathcal{P} A \mathcal{P} . \tag{3.1.2}$$

In these terms, we define the expected spin vector v_{Π} for Π as follows,

$$v_{\Pi} = (\mathrm{Tr}\,(S_x^{\Pi}), \mathrm{Tr}\,(S_y^{\Pi}), \mathrm{Tr}\,(S_z^{\Pi})) .$$

As it turns out, in analogy with (1.3.18), the coherent planes are those that maximize the magnitude of v_{Π}. A nice proof of this fact can be found in [66].

The final property we want to mention, is that coherent planes also provide a resolution of the identity, just like the coherent states (1.3.20). The precise statement is contained in the following theorem

Theorem 7. *Denote by $\mathcal{P}_{\hat{\boldsymbol{n}}}$ the projection operator for $\Pi_{\hat{\boldsymbol{n}}}$. Then, the following equality holds,*

$$U \equiv \int_{S^2} d\Omega\, \mathcal{P}_{\hat{\boldsymbol{n}}} = \frac{4\pi k}{2s+1} \mathbb{1} , \tag{3.1.3}$$

[1] The rotation of a k-plane Π by R is defined in the following way,

$$R\Pi \equiv \{D(R)|\psi\rangle \text{ where } |\psi\rangle \in \Pi\} .$$

Proof. Let R be any rotation. Then we have the following,

$$\begin{aligned} D(R)\mathcal{P}_{\hat{n}}D(R)^\dagger &= \sum_{\mu=1}^{k} D(R)|\hat{\boldsymbol{n}}, s+1-\mu\rangle\langle\hat{\boldsymbol{n}}, s+1-\mu|D(R)^\dagger \\ &= \sum_{\mu=1}^{k} |\boldsymbol{R}\hat{\boldsymbol{n}}, s+1-\mu\rangle\langle\boldsymbol{R}\hat{\boldsymbol{n}}, s+1-\mu| \\ &= \mathcal{P}_{\boldsymbol{R}\hat{\boldsymbol{n}}}\,. \end{aligned}$$

Then, for any rotation R,

$$D(R)UD(R)^\dagger = \int_{S^2} d\Omega\, D(R)\mathcal{P}_{\hat{\boldsymbol{n}}}D(R)^\dagger = \int_{S^2} d\Omega\, \mathcal{P}_{\boldsymbol{R}\hat{\boldsymbol{n}}} = U\,,$$

where U is defined in (3.1.3). The infinitesimal version of the previous equation states that U commutes with all the rotation generators S_x, S_y and S_z. Since the only operators that commute with all these three are multiples of the identity, we can conclude that $U = \lambda\mathbb{1}$ for a certain λ. By taking trace over both sides we obtain,

$$\begin{aligned} &\mathrm{Tr}\left\{\int_{S^2} d\Omega\, P_{\hat{\boldsymbol{n}}}\right\} = \lambda\,\mathrm{Tr}\,(\mathbb{1}) \Rightarrow \int_{S^2} d\Omega\ \mathrm{Tr}\,(\mathcal{P}_{\hat{\boldsymbol{n}}}) = \lambda(2s+1) \\ &\Rightarrow \int_{S^2} d\Omega\, k = \lambda(2s+1) \Rightarrow 4\pi k = \lambda(2s+1) \Rightarrow \lambda = \frac{4\pi k}{2s+1}\,, \end{aligned}$$

where, for the second implication, we used the fact that the trace of a projection operator is the dimension of the subspace it projects onto. This concludes the proof. Note that for the case $k = 1$, we recover (1.3.20). □

3.1.1 Writing $\Pi_{\hat{\boldsymbol{n}}}$ in the standard form

It is often useful to write the coherent planes $\Pi_{\hat{\boldsymbol{n}}}$ in the standard form w.r.t. the basis $\{|s,m\rangle, m = s,\ldots,-s\}$ (c.f. equation (1.2.7) of section 1.2). This is what we do in what follows.

Denote by ζ_0 the complex number associated to $\hat{n}$. As we prove at the end of this section, the resulting expression for the standard form of $\Pi_{\hat{\boldsymbol{n}}}$, does not depend on ζ_0^* explicitly. This might be surprising at first, as the expression for the states $|\hat{\boldsymbol{n}}, m\rangle$ (c.f. equation (B.0.10)) that defines the coherent plane $\Pi_{\hat{\boldsymbol{n}}}$ does depend of ζ_0^*. An important consequence of this result is that $\Pi_{\hat{\boldsymbol{n}}}$ can be written as an analytical function of ζ_0, and therefore, the zeros of

$\langle \Pi_{-\hat{\boldsymbol{n}}}, \Pi \rangle$ are given by an analytic function of ζ_0. This implies, for instance, that the constellation of a k-plane is a *discrete* set of points in the unitary sphere. We take advantage of this fact in the following sections.

In the same way we can map a state $|\psi\rangle$ to its Majorana polynomial p_ψ using the linear isomorphism (1.3.14), we can assign to a k-plane $\Pi = \text{span}\{|\psi_1\rangle, \dots, |\psi_k\rangle\}$ a k-plane $P\Pi$ contained in the linear space of polynomials of degree $2s$ as follows,

$$P\Pi = \text{span}\{p_{\psi_1}, \dots, p_{\psi_k}\}\,. \tag{3.1.4}$$

It is easy to prove that $P\Pi$ does not depend on the basis chosen for Π. If we specify the previous definition to $P\Pi_{\hat{\boldsymbol{n}}}$ we obtain,

$$\begin{aligned}
&P\Pi_{\hat{\boldsymbol{n}}} \\
&= \text{span}\{(\zeta-\zeta_0)^{2s}, (\zeta-\zeta_0)^{2s-1}(\zeta-\zeta_{0A}), \dots, (\zeta-\zeta_0)^{2s-k+1}(\zeta-\zeta_{0A})^{k-1}\} \\
&= (\zeta-\zeta_0)^{2s-k+1}\text{span}\{(\zeta-\zeta_0)^{k-1}, (\zeta-\zeta_0)^{k-2}(\zeta-\zeta_{0A}), \dots, (\zeta-\zeta_{0A})^{k-1}\} \\
&= (\zeta-\zeta_0)^{2s-k+1}\text{span}\{\zeta^{k-1}, \zeta^{k-2}, \dots, 1\}\,,
\end{aligned} \tag{3.1.5}$$

where ζ_{0A} denotes the complex number antipodal to ζ_0. The last equality is due to the fact that the polynomials $(\zeta-\zeta_0)^{k-1}, (\zeta-\zeta_0)^{k-2}(\zeta-\zeta_{A0}), \dots, (\zeta-\zeta_{0A})^{k-1}$ are of degree $k-1$, linearly independent (as can be easily checked) and there are k of them, so they provide a basis for the space of polynomials of degree at most $k-1$. Notice that this expression implies that ζ_0 is a root of all the elements of $P\Pi_{\hat{\boldsymbol{n}}}$ with multiplicity $2s-k+1$ or greater. From this observation, we can conclude the following characterization of the coherent plane $\Pi_{\hat{\boldsymbol{n}}}$,

Theorem 8. *The k-coherent plane $\Pi_{\hat{n}}$ is the space of all the states such that their constellation has at least $2s+1-k$ stars in the direction $\hat{n}$.*

Note that in the previous characterization, ζ_{0A} does not appear in any shape or form. From here it is more or less clear that we can write $\Pi_{\hat{\boldsymbol{n}}}$ in standard form without appealing to ζ_{0A}.

The procedure we follow, consists of two steps. As the first one, we write $P\Pi_{\hat{\boldsymbol{n}}}$ in the standard form w.r.t. the canonical basis V for the space of polynomials, $V = \{\zeta^{2s}, \zeta^{2s-1}, \dots, 1\}$. This can be done by finding a k-frame $W = (P_1, \dots, P_k)$ of $P\Pi_{\hat{\boldsymbol{n}}}$ such that, the matrix representation A of W w.r.t. V is of the form shown in (1.2.7),

$$A = \begin{pmatrix} \mathbb{1}_{k\times k} \\ \hdashline \tilde{A}_{(N-k)\times k} \end{pmatrix}\,, \tag{3.1.6}$$

where subindices indicate the size of submatrices. If A is written as alluded to above, then, it is the matrix representation of $P\Pi_{\hat{\boldsymbol{n}}}$ in standard form we seek. Algebraically, the previous condition states that the matrix A used to write the elements of W in terms of the ones of V in the following way,

$$P_j(\zeta) = \sum_{i=1}^{2s+1} \zeta^{2s+1-i} A_{ij} \,, \tag{3.1.7}$$

is such that its components satisfy the relation $A_{ij} = \delta_{ij}$ if $i \leq k$.

As the second and final step, we use the isomorphism (1.3.14) of the Majorana polynomial to map W onto a k-frame for $\Pi_{\hat{\boldsymbol{n}}}$. Note that the Majorana polynomials of the states in the basis $\{|s,s\rangle, \dots, |s,-s\rangle\}$ are (up to some rescaling) the elements of V. Because of this observation, the expression for $\Pi_{\hat{\boldsymbol{n}}}$ in the standard form we seek is equal to the matrix representation of such k-frame (after some rescaling of its elements, more details on this later on) w.r.t. $\{|s,s\rangle, \dots, |s,-s\rangle\}$.

To find the k-frame W mentioned above, first we compute the matrix representation M w.r.t. V of the frame for $P\Pi_{\hat{\boldsymbol{n}}}$ found in (3.1.5). Note that its last element, $(\zeta-\zeta_0)^{2s-k+1}$, can be written in terms of the members of V as follows,

$$(\zeta-\zeta_0)^{2s-k+1} = \zeta^{2s-k+1} + \sum_{j=1}^{2s-k+1} \binom{2s-k+1}{j} (-1)^j \zeta_0^j \zeta^{2s-k+1-j} \,.$$

Since the previous expression only involves the following elements of V, $\zeta^{2s-k+1}, \zeta^{2s-k}, \dots, 1$, and the coefficient for ζ^{2s-k+1} is exactly one, then, the representation of $(\zeta-\zeta_0)^{2s-k+1}$ w.r.t. V as a column vector is the following,

$$(\zeta-\zeta_0)^{2s-k+1} \to \begin{pmatrix} \vec{0}_{k-1} \\ 1 \\ \vec{v}_{2s-k+1} \end{pmatrix} \,, \tag{3.1.8}$$

where $\vec{0}_{k-1}$ indicates the zero column vector in k dimensions and $\vec{v}_{2s-k+1}$ denotes the vector in $2s-k+1$ dimensions whose j entry is,

$$v_j = \binom{2s-k+1}{j} (-1)^j \zeta_0^j \,. \tag{3.1.9}$$

In equation (3.1.8), subindices indicate the length of the vector.

Then, consider the next to last element for $P\Pi_{\hat{n}}$, $(\zeta - \zeta_0)^{2s-k+1}\zeta$. Using the same procedure, we see that its expression w.r.t. V is the following,

$$(\zeta - \zeta_0)^{2s-k+1}\zeta \to \begin{pmatrix} \vec{0}_{k-2} \\ 1 \\ \vec{v}_{2s-k+1} \\ 0 \end{pmatrix} .$$

By using the same argument, we can conclude the representation for the other elements of the frame found in (3.1.5),

$$(\zeta - \zeta_0)^{2s-k+1}\zeta^m \to \begin{pmatrix} \vec{0}_{k-m-1} \\ 1 \\ \vec{v}_{2s-k+1} \\ \vec{0}_m \end{pmatrix} , \tag{3.1.10}$$

expression that is valid for all $0 \le m \le k-1$.

By putting all these columns in the same matrix, we find the following expression for M,

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots \\ \vec{v}_{2s-k+1} & 1 & 0 & \dots \\ 0 & \vec{v}_{2s-k+1} & 1 & \dots \\ \vdots & 0 & \vec{v}_{2s-k+1} & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \dots \end{pmatrix}_{(2s+1)\times k}$$

Notice the ordering of the columns of the matrix, for the first one, we took the value $m = k-1$, for the second one, $m = k-2$ and so on. In particular, the last column is the one shown in (3.1.8).

Since M is not of the form shown in (3.1.6), M is not the representation of $P\Pi_{\hat{\boldsymbol{n}}}$ in the standard form. To find the actual representation, instead of considering the basis (3.1.5) of $P\Pi_{\hat{\boldsymbol{n}}}$, we define recursively new polynomials Q_i $(i = 1, \dots, k)$ using "backward substitution",

$$\begin{aligned} Q_0(\zeta) &= (\zeta - \zeta_0)^{2s-k+1} , \\ Q_m(\zeta) &= (\zeta - \zeta_0)^{2s-k+1}\zeta^m - \sum_{m'=1}^{m} v_{m'} Q_{m-m'}(\zeta) \\ &= (\zeta - \zeta_0)^{2s-k+1}\zeta^m - \sum_{m'=1}^{m} (-1)^{m'} \binom{2s-k+1}{m'} \zeta_0^{m'} Q_{m-m'}(\zeta) , \end{aligned}$$

where we used (3.1.9) to obtain the last line. By construction, if we consider the frame W for $P\Pi_{\hat{\boldsymbol{n}}}$ whose i-th element is $P_i = Q_{k-i}$, $i = 1, \dots, k$, its matrix representation A is of the form shown in (3.1.6). Denote it by A. As mentioned previously, this implies that A satisfies the properties established in (3.1.7) and the line after it.

Finally, we use this result to write $\Pi_{\hat{n}}$ as a matrix in standard form w.r.t. the basis $\{|\hat{z}, 2s\rangle, \dots, |\hat{z}, -2s\rangle\}$. Denote by T_{majorana} the mapping that takes a Majorana polynomial to its corresponding state in Hilbert space, that is, the inverse mapping of (1.3.14). By applying T_{majorana} to equation (3.1.7), we obtain,

$$
\begin{aligned}
T_{\text{majorana}}(P_j) &= \sum_{i=1}^{2s+1} A_{ij} T_{\text{majorana}}(\zeta^{2s+1-i}) \\
&= \sum_{i=1}^{2s+1} A_{ij} (-1)^{i-1} \binom{2s}{i-1}^{-1/2} |\hat{z}, s+1-i\rangle \Rightarrow
\end{aligned}
$$

$$
(-1)^{j-1} \binom{2s}{j-1}^{1/2} T_{\text{majorana}}(P_j) = \sum_{i=1}^{2s+1} B_{ij} |\hat{z}, s+1-i\rangle \, ,
$$

where the last equation defines the matrix B. Note that B satisfies the same properties as A stressed previously, mainly that B is a $(2s+1) \times k$ matrix such that $B_{ij} = \delta_{ij}$ if $i \leq k$.

This implies that, if we consider the k-frame $\tilde{W}$ for $\Pi_{\hat{\boldsymbol{n}}}$ which j-th element is $(-1)^{j-1}\binom{2s}{j-1}^{1/2} T_{\text{majorana}}(P_j)$, $j = 1, \dots, k$, then, the resulting matrix representation B of $\tilde{W}$, is the expression for $\Pi_{\hat{\boldsymbol{n}}}$ in standard form w.r.t. $\{|\hat{z}, 2s\rangle, \dots, |\hat{z}, -2s\rangle\}$. Note that the coefficients of B do not depend on of ζ_0^*, as claimed.

To conclude this section, we give the formal definition of the constellation of a k-plane. Let Π be a k-plane. Then, the constellation of Π, $\mathcal{C}_\Pi$, is the set of all the directions $\hat{n}$ such that $\langle \Pi_{-\hat{\boldsymbol{n}}}, \Pi \rangle$ is zero (the degree of the zero has to be taken into consideration; we make this statement precise in the following section).

From this definition, it is clear that if we rotate a k-plane, its corresponding constellation rotates in the same way, just like with the Majorana representation for $\mathbb{P}(\mathcal{H}_s)$. However, as we show later in this chapter, there is one important difference between the Majorana representation for $\mathbb{P}(\mathcal{H}_s)$, and the stellar one for $Gr_k(\mathcal{H}_s)$ that we want to stress; different k-planes might have the same constellation. Because of this, the rotational symmetries of a k-plane Π might not coincide with the ones of its constellation $\mathcal{C}_\Pi$; in general Π has fewer symmetries, as a rotation in the symmetric group of $\mathcal{C}_\Pi$ might map Π onto a different plane with the same constellation.

The above definition for the constellation of a k-plane involves the product (1.2.4) between k-planes and therefore becomes hard to compute for large values of k. In the following section, we present a simpler characterization of the constellation of a k-plane.

3.2 The Majorana polynomial for a k-plane Π

In the previous section, we argued that the constellation of a k-plane is given by the zeros of an analytic function. In this section, we prove that such function is a polynomial that we refer as the Majorana polynomial of the k-plane in analogy with the one for the Majorana representation.

The first step, is to prove the following theorem,

Theorem 9. *Let Π be a k-plane. Then, $\hat{n}$ is in the constellation of Π if and only if there is a state $|\psi\rangle \in \Pi$ such that the constellation of $|\psi\rangle$ has k (or more) stars in the direction $\hat{n}$.*

Proof. Suppose that the constellation of Π has a star in the direction $\hat{n}$. By definition, this happens if and only if $\langle \Pi_{-\hat{\boldsymbol{n}}}, \Pi \rangle = 0$. As noted in theorem 3, this product is zero if and only if there is a state $|\psi\rangle$ in Π that is orthogonal to all the elements of $\Pi_{-\hat{\boldsymbol{n}}}$. Since the states $|-\hat{\boldsymbol{n}}, s\rangle, \ldots, |-\hat{\boldsymbol{n}}, s+1-k\rangle$ constitute a basis for $\Pi_{-\hat{\boldsymbol{n}}}$ by definition, the orthogonality of $|\psi\rangle$ to $\Pi_{-\hat{\boldsymbol{n}}}$ is equivalent to the following equalities,

$$\langle -\hat{\boldsymbol{n}}, m|\psi\rangle = \pm\langle \hat{\boldsymbol{n}}, -m|\psi\rangle = 0, \quad m = s, \ldots, s-k+1\,.$$

The previous equality implies that, the probability of obtaining the value $-m$ when measuring the spin projection in $\hat{n}$ for a system in the state $|\psi\rangle$, is zero. By considering theorem 6, we see that this happens if and only if the constellation of $|\psi\rangle$ has at least k stars in the direction $\hat{n}$. □

The previous theorem allows us to define the Majorana polynomial for a plane. Suppose we have a k-plane Π generated by the states $|\phi_1\rangle, \ldots, |\phi_k\rangle$. Suppose that $\hat{n}$ is an element of $\mathcal{C}_\Pi$. Denote by ζ_0 its stereographic projection. Let p_μ be the Majorana polynomial associated to the state $|\phi_\mu\rangle$. Then, $P\Pi$ (3.1.4) can be written as, $P\Pi = \text{span}\{p_1, \ldots, p_k\}$. In terms of polynomials, theorem 9 guarantees that $\hat{n}$ is an element of $\mathcal{C}_\Pi$ if and only if there is a polynomial in $P\Pi$ such that ζ_0 is a root with multiplicity k of said polynomial, in other words, that a certain linear combination of the polynomials p_μ ($\mu = 1, \ldots, k$) spanning $P\Pi$ can be written in the following way,

$$\sum_\mu A_\mu p_\mu(\zeta) = q(\zeta)(\zeta - \zeta_0)^k \,, \tag{3.2.1}$$

where q is a polynomial of degree $2s-k$. As is well-known from linear algebra, a complex number ζ_0 is a root of a polynomial p with multiplicity greater or equal than k if and only if ζ_0 is a root of $p, p', ..., p^{(k-1)}$ (a prime indicates the derivative of a function w.r.t. its argument; superscripts between parenthesis indicates the number of times the polynomial was differentiated). Therefore, ζ_0 is a root of $\sum_\mu A_\mu p_\mu(\zeta)$ with multiplicity greater or equal to k if and only the following equalities hold,

$$\begin{aligned} \sum_\mu A_\mu p_\mu(\zeta_0) &= 0 \,, \\ \sum_\mu A_\mu p'_\mu(\zeta_0) &= 0 \,, \\ &\vdots \\ \sum_\mu A_\mu p_\mu^{(k-1)}(\zeta_0) &= 0 \,. \end{aligned}$$

These equations can be regarded as a linear system of k equations with k unknowns, A_μ, $\mu = 1, \ldots, k$. As we know, this system has a nontrivial solution if and only if the following determinant is zero (when evaluated at $\zeta = \zeta_0$),

$$p_\Pi(\zeta) = \mathrm{Det} \begin{vmatrix} p_1(\zeta) & p_2(\zeta) & \ldots & p_k(\zeta) \\ p'_1(\zeta) & p'_2(\zeta) & \ldots & p'_k(\zeta) \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{(k-1)}(\zeta) & p_2^{(k-1)}(\zeta) & \ldots & p_k^{(k-1)}(\zeta) \end{vmatrix} \,. \tag{3.2.2}$$

We immediately recognize the previous expression as the *Wronskian* of the polynomials $p_1, \ldots, p_k$. From this line of reasoning, we can conclude the following theorem,

Theorem 10. *Let Π denote k-plane. Define its Majorana polynomial p_Π as in (3.2.2) (that is, the Majorana polynomial p_Π is the Wronskian of any set of k polynomials that provide a basis for $P\Pi$). Then, the constellation $\mathcal{C}_\Pi$ is given by the zeros of p_Π via the stereographic projection.*

Determining the zeros of the Majorana polynomial to find the constellation of a plane is in general much simpler than the approach presented in the previous section. Note that a change of the basis chosen for $P\Pi$, only changes the Majorana polynomial by an overall factor.

From theorem 10, we can also determine the number of stars in the constellation of a k-plane, as stated in the following theorem,

Theorem 11. *The degree of the Majorana polynomial of a generic k-plane, and hence the number of stars in the constellation of Π, is $k(2s+1-k)$.*

Proof. Let $p_1, \ldots, p_k$ denote the elements of a basis for $P\Pi$. Then, the Majorana polynomial p_Π is the determinant of the $k \times k$ matrix that appears at (3.2.2). As we show in the following paragraphs, by applying elementary row transformations, we can transform it to a matrix where all the entries are polynomials of degree $2s+1-k$. Since this type of transformations leaves the determinant invariant, and the one of this new matrix is the sum of certain products of k of these polynomials, the degree of p_Π is $k(2s+1-k)$.

Finally, we prove the claim of the previous paragraph; that by applying elementary row operations, we can take the matrix of (3.2.2) into one where all the entries are polynomials of degree $2s+1-k$. Consider any polynomial p of degree $2s$. Decompose it in the following way,

$$p(\zeta) = \sum_{i=0}^{2s-k+1} A_i \zeta^i + \sum_{i=2s-k+2}^{2s} A_i \zeta^i \equiv f(\zeta) + g(\zeta)\,, \tag{3.2.3}$$

that is, f is the sum of all the terms of p of degree less or equal than $2s-k+1$ and g is the sum of the remaining ones. Since g can be written as a linear combination of the polynomials $\zeta^{2s}, \zeta^{2s-1}, \ldots, \zeta^{2s-k+2}$, the following Wronskian is zero,

$$\begin{vmatrix} \zeta^{2s} & \zeta^{2s-1} & \ldots & \zeta^{2s-k+2} & g(\zeta) \\ 2s\zeta^{2s-1} & (2s-1)\zeta^{2s-2} & \ldots & (2s-k+2)\zeta^{2s-k} & g'(\zeta) \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{(2s)!}{(2s-k+1)!}\zeta^{2s-k+1} & \frac{(2s)!}{(2s-k)!}\zeta^{2s-k} & \ldots & \frac{(2s-k+2)!}{(2s-2k+3)!}\zeta^{2s-2k+3} & g^{(k-1)}(\zeta) \end{vmatrix} = 0. \tag{3.2.4}$$

It is clear that ζ^{2s-k+1} is a common factor of the terms in the first column; ζ^{2s-k}, of the terms in the second one, and so on. By taking these factors out of the determinant, and assuming they are not zero, we obtain the following equality,

$$\begin{vmatrix} \zeta^{k-1} & \zeta^{k-1} & \dots & \zeta^{k-1} & g(\zeta) \\ 2s\zeta^{k-2} & (2s-1)\zeta^{k-2} & \dots & (2s-k+2)\zeta^{k-2} & g'(\zeta) \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{(2s)!}{(2s-k+1)!} & \frac{(2s)!}{(2s-k)!} & \dots & \frac{(2s-k+2)!}{(2s-2k+3)!} & g^{(k-1)}(\zeta) \end{vmatrix} = 0.$$

Similarly, we factorize ζ^{k-1} from the first row, ζ^{k-2} from the second one and so on. This procedure leads to,

$$\begin{vmatrix} 1 & 1 & \dots & 1 & \frac{g(\zeta)}{\zeta^{k-1}} \\ 2s & (2s-1) & \dots & (2s-k+2) & \frac{g'(\zeta)}{\zeta^{k-2}} \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{(2s)!}{(2s-k+1)!} & \frac{(2s)!}{(2s-k)!} & \dots & \frac{(2s-k+2)!}{(2s-2k+3)!} & g^{(k-1)}(\zeta) \end{vmatrix} = 0. \tag{3.2.5}$$

By computing the determinant by minors w.r.t. the last column, and multiplying the resulting equality by ζ^{k-1} we conclude,

$$M_0 g(\zeta) + M_1 g'(\zeta)\zeta + \cdots + M_{k-1} g^{k-1}\zeta^{(k-1)} = 0\,,$$

where M_i denotes the minor of the matrix of (3.2.5) corresponding to the element $g^{(i)}(\zeta)$. Note that M_i does not depend on ζ of on g. This equality, together with (3.2.3), implies that,

$$\begin{aligned} p(\zeta) + \frac{M_1}{M_0}p'(\zeta)\zeta + \cdots + \frac{M_{k-1}}{M_0}p^{k-1}\zeta^{(k-1)} = \\ f(\zeta) + \frac{M_1}{M_0}f'(\zeta)\zeta + \cdots + \frac{M_{k-1}}{M_0}f^{k-1}\zeta^{(k-1)}\,. \end{aligned} \tag{3.2.6}$$

Since f is of degree $2s+1-k$, the l.h.s. of the previous equation is of this same degree. The punchline of what we just proved is the following: given any polynomial p of degree $2s$, the l.h.s of (3.2.6) is of degree $2s+1-k$ (the fact that M_0 is not zero can be deduced by considering the function $g(\zeta) = 1$; in this case, since the polynomials $\zeta^{2s}, \zeta^{2s-1}, \ldots, \zeta^{2s-k+2}, 1$ are linearly independent, the determinant of equation (3.2.5) can not be zero, but it would be if M_0 were zero).

Denote by r_i the i-th row of (3.2.2). Because of the result of the previous paragraph, if we replace r_1 by the following row,

$$r_1 \to r_1 + \frac{M_1}{M_0}r_2 + \cdots + \frac{M_{k-1}}{M_0}r_k\,,$$

the matrix obtained is such that all the polynomials of the first row are of degree $2s+1-k$; that is, the same degree as the one of the polynomials in the last row. By applying this same procedure to the remaining rows, we can prove the claim made in the first paragraph of the proof. □

From the previous theorem, we can conclude that the space of Majorana constellations of k-planes is of complex dimension $k(2s+1-k)$. As proved at the end of section 1.2, this is also the dimension of $Gr_k(\mathcal{H}_s)$; the space we are representing. Using the Majorana polynomial for a k-plane, we present some examples of constellations in the next section.

3.2.1 Examples of constellations

The constellation for coherent k-planes

As a first example, we find the constellation of an arbitrary coherent k-plane. To find this constellation, first we find the one for a coherent plane $\Pi_{\hat{z}}$. By rotating this constellation accordingly, we can find the constellation of any coherent k-plane.

Consider the plane $\Pi_{\hat{z}}$. By definition, this space is spanned by the states $|s,m\rangle$ ($m = s, s-1, \ldots, s-k+1$). Therefore, $P\Pi_{\hat{z}}$ is spanned by the polynomials $\zeta^{2s}, \zeta^{2s-1}, \ldots, \zeta^{2s-k+2}$. By computing their Wronskian, we can compute the Majorana polynomial $p_{\Pi_{\hat{z}}}$,

$$p_{\Pi_{\hat{z}}}(\zeta) = \begin{vmatrix} \zeta^{2s} & \zeta^{2s-1} & \ldots & \zeta^{2s-k+1} \\ 2s\zeta^{2s-1} & (2s-1)\zeta^{2s-2} & \ldots & (2s-k+1)\zeta^{2s-k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(2s)!}{(2s-k+1)!}\zeta^{2s-k+1} & \frac{(2s)!}{(2s-k)!}\zeta^{2s-k} & \ldots & \frac{(2s-k+1)!}{(2s-2k+2)!}\zeta^{2s-2k+2} \end{vmatrix} .$$

This determinant is very similar to the one that appears in (3.2.4). By using a similar procedure used to compute it, we find,

$$\begin{aligned} p_{\Pi_{\hat{z}}}(\zeta) =& \zeta^{k(2s+1-k)} \\ & \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 2s & (2s-1) & \ldots & (2s-k+1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(2s)!}{(2s-k)!} & \frac{(2s)!}{(2s-k-1)!} & \ldots & \frac{(2s-k+1)!}{(2s-2k+1)!} \end{vmatrix} . \end{aligned}$$

From here, we can conclude that the Majorana polynomial for $\Pi_{\hat{z}}$ only has one root, $\zeta = 0$, so its constellation has $k(2s+1-k)$ stars in the north pole.

The constellation of a k-plane spanned by eigenstates of $S \cdot \hat{n}$

As a second example, we compute the constellation of the following k-plane,

$$\Pi = \operatorname{span}\{|\hat{\boldsymbol{n}}, m_1\rangle, |\hat{\boldsymbol{n}}, m_2\rangle, \ldots, |\hat{\boldsymbol{n}}, m_k\rangle\} ,$$

where $s \geq m_1 > m_2, \cdots > m_k \geq -s$. Just like in the previous example, it is enough to calculate the one for the case $\hat{n} = \hat{z}$. To obtain the one for the generic case, one can rotate the one computed here accordingly.

In this case, $P\Pi$ is spanned by the polynomials $\zeta^{s+m_1}, \ldots, \zeta^{s+m_k}$. A calculation very similar to the one for the coherent k-plane reveals the following expression for p_Π,

$$p_\Pi(\zeta) \propto z^{m_1+\cdots+m_k+ks-\frac{k(k-1)}{2}} .$$

By looking at the previous polynomial, we can conclude that $\mathcal{C}_\Pi$ has $m_1 + \cdots + m_k + ks - \frac{k(k-1)}{2}$ stars in the north pole. Since the number of stars of a constellation is $k(2s+1-k)$, just as in the case of the usual Majorana representation for $\mathbb{P}(\mathcal{H}_s)$, the remaining stars are in the south pole. Therefore, in the generic case, $\mathcal{C}_\Pi$ has $m_1 + \cdots + m_k + ks - \frac{k(k-1)}{2}$ stars in $\hat{n}$ and the rest in $-\hat{n}$.

Notice that, if two different collections of numbers $s \geq m_1 > m_2 > \cdots > m_k \geq -s$ and $s \geq \tilde{m}_1 > \tilde{m}_2 > \cdots > \tilde{m}_k \geq -s$ have the same sum, $m_1 + \cdots + m_k = \tilde{m}_1 + \cdots + \tilde{m}_k$, then, the constellation for the planes

$$\Pi = \operatorname{span}\{|\hat{\boldsymbol{n}}, m_1\rangle, |\hat{\boldsymbol{n}}, m_2\rangle, \ldots, |\hat{\boldsymbol{n}}, m_k\rangle\} ,$$
$$\tilde{\Pi} = \operatorname{span}\{|\hat{\boldsymbol{n}}, \tilde{m}_1\rangle, |\hat{\boldsymbol{n}}, \tilde{m}_2\rangle, \ldots, |\hat{\boldsymbol{n}}, \tilde{m}_k\rangle\} ,$$

is the same, despite Π and $\tilde{\Pi}$ being different k-planes. The fact that they are not the same, can be seen by considering a certain $\tilde{m}_i$ that is not any of the numbers m_j. In this case, $|\hat{\boldsymbol{n}}, \tilde{m}_i\rangle$ is an element of $\tilde{\Pi}$ but is orthogonal to all the elements of Π. For example, the following planes are different, but their constellation is the same,

$$\Pi = \operatorname{span}\{|\hat{\boldsymbol{n}}, 3/2\rangle, |\hat{\boldsymbol{n}}, -3/2\rangle\} , \quad \tilde{\Pi} = \operatorname{span}\{|\hat{\boldsymbol{n}}, 1/2\rangle, |\hat{\boldsymbol{n}}, -1/2\rangle\}. \quad (3.2.7)$$

The constellations for these 2-planes are represented in figure 3.1.

This is a concrete example of the previously mentioned fact, that the stellar representation for the Grassmannian is not one to one — different k-planes might have associated the same constellation. From here a natural question arises, how many k-planes share the same constellation? We give the answer to this question in section 3.3.

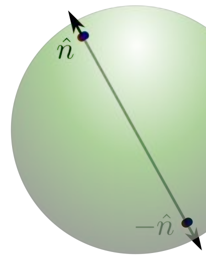


Figure 3.1: Constellation of the 2-planes Π and $\tilde{\Pi}$ of equation (3.2.7). It consists of 2 stars in $\hat{n}$ and 2 stars in $-\hat{n}$. Although both 2-planes are different, they share the same constellation.

Constellation of the plane orthogonal to a k-plane

Given a k-plane Π, we define its orthogonal complement $\Pi^{\perp}$ as the $2s+1-k$-plane containing all the states orthogonal to Π. If we know the constellation $\mathcal{C}_{\Pi}$, what can we say about $\mathcal{C}_{\Pi^{\perp}}$? The answer is contained in the following theorem.

Theorem 12. *Let Π be a k-plane and let $\Pi^{\perp}$ be its orthogonal complement. Then, the constellation of $\Pi^{\perp}$ is the* antipodal *to the one of Π, that is, if the constellation of Π is $\{\hat{n}_i, i = 1, \ldots, k(2s+1-k)\}$ then, the one of $\Pi^{\perp}$ is $\{-\hat{n}_i, i = 1, \ldots, k(2s+1-k)\}$.*

Proof. Suppose that $\hat{n}$ is an element of $\mathcal{C}_{\Pi}$. Then, by definition, the product $\langle \Pi_{-\hat{\boldsymbol{n}}}, \Pi \rangle$ is zero. Note that the orthogonal complement to $\Pi_{-\hat{\boldsymbol{n}}}$ is the coherent $2s+1-k$ plane $\Pi_{\hat{\boldsymbol{n}}}$. By this observation and theorem 3, there exists an element of Π that is also an in $\Pi_{\hat{\boldsymbol{n}}}$. Since the orthogonal complement to $\Pi^{\perp}$ is Π, the statement of the previous sentence can be formulated as follows: there is an element of $\Pi_{\hat{\boldsymbol{n}}}$ that it is orthogonal to $\Pi^{\perp}$. By considering theorem 3 again, this implies that $\langle \Pi_{\hat{\boldsymbol{n}}}, \Pi^{\perp} \rangle$ is zero. By definition, this means that there is a star in the direction $-\hat{n}$ in $\mathcal{C}_{\Pi^{\perp}}$. This concludes the proof. □

3.2.2 The multiplicity of the stars of a constellation

In theorem 9, we stated an interpretation for the direction of the stars in the constellation of a plane. In this subsection, we find a similar characterization for the case when a direction appears more than once in the constellation.

Theorem 13. *Let Π be a k-plane. The constellation of Π has at least two stars in the direction $\hat{n}$ if and only if any of the following two conditions hold,* *(i) There exists an element of* Π *which constellation has at least* $k+1$ *stars in the direction* $\hat{n}$.

(ii) There exists an element of Π *which constellation has at least* k *stars in the direction* $\hat{n}$*, and another one (linearly independent to the first) with* $k-1$ *(or more) stars in the direction* $\hat{n}$.

Proof. Suppose $\hat{n}$ appears twice or more in $\mathcal{C}_\Pi$. Denote by ζ_0 the stereographic projection of $\hat{n}$. Then, by theorem 9, there is a polynomial p in $P\Pi$ such that ζ_0 is a root of p with multiplicity greater or equal than k, that is, ζ_0 is root of p and of its first $k-1$ derivatives. Complete $\{p\}$ to a basis for $P\Pi$, $\{p, p_2, \ldots, p_k\}$. Then, p_Π is the Wronskian of these polynomials. Since $\mathcal{C}_\Pi$ contains two stars in the direction $\hat{n}$, ζ_0 is a root of p_Π and of its derivative. Because of this, the following equalities hold,

$$
\begin{aligned}
0 = p'_\Pi(\zeta_0) &= \begin{vmatrix} p(\zeta_0) & \ldots & p_k(\zeta_0) \\ p'(\zeta_0) & \ldots & p'_k(\zeta_0) \\ \vdots & \ddots & \vdots \\ p^{(k)}(\zeta_0) & \ldots & p_k^{(k)}(\zeta_0) \end{vmatrix} \\
&= (-1)^{k-1} p^{(k)}(\zeta_0) \begin{vmatrix} p_1(\zeta_0) & \ldots & p_k(\zeta_0) \\ p'_1(\zeta_0) & \ldots & p'_k(\zeta_0) \\ \vdots & \ddots & \vdots \\ p_1^{(k-2)}(\zeta_0) & \ldots & p_k^{(k-2)}(\zeta_0) \end{vmatrix},
\end{aligned}
$$

where the first line was obtained using the well-known formula for the derivative of a Wronskian, and the second one, by expanding by minors the determinant w.r.t. the first column and recalling that ζ_0 is a root of p and its first k derivatives. Note that the last product of the previous equation is zero if and only if $p^{(k)}$ is zero (this is case *(i)* of the theorem) or if the Wronskian of the polynomials $p_1, \ldots, p_k$ is zero. In the latter case, by theorem (9), there is a polynomial q in the span of $p_1, \ldots, p_k$ such that ζ_0 is a root of q with multiplicity $k-1$. This is case *(ii)*. □

Using essentially the same techniques, we can prove an analogous result for stars with multiplicity three,

Theorem 14. *Let* Π *be a* k*-plane. The constellation of* Π *has at least three stars in the direction* $\hat{n}$ *if and only if any of the following three options occur,*

(i) There exists an element of Π *which constellation has at least* $k+2$ *stars in the direction* $\hat{n}$.

(ii) There exists three linealy independent elements of Π *such that the constellation of the first one has at least* $k+1$ *stars in* $\hat{n}$*, the one of the second element has* k *(or more) stars in* $\hat{n}$ *and the one for the third element has* $k-1$ *(or more) stars in* $\hat{n}$*.*

(iii) There are two different states in Π *which constellations have at least* k *stars in the direction* $\hat{n}$*.*

By considering theorem 13 we can prove the following statement,

Theorem 15. *Let* Π *be a* k*-plane. The constellation of* Π *has exactly one star in the direction* $\hat{n}$ *if and only if all the following conditions hold,*

- *There is a unique element of* Π *whose constellation has exactly* k *stars in the direction* $\hat{n}$*,*
- *The constellation of any other state in* Π *has at most* $k-2$ *stars in the direction* $\hat{n}$*.*

Proof. Suppose there is only one star in the direction $\hat{n}$ in the constellation associated to Π. Then, by theorem 9, there is a state $|\psi\rangle$ in Π whose constellation has at least k stars in the direction $\hat{n}$. By the first condition of theorem 13, the number of stars pointing towards $\hat{n}$ in the constellation of $|\psi\rangle$ can not be $k+1$ or more, therefore, it must be exactly k. This proves the first claim of the theorem. Also, by considering the second condition of theorem 13, the constellation of any other state can not have $k-1$ stars or more in the direction $\hat{n}$. This proves the second claim. □

3.3 The number of k-planes with the same constellation

Unlike the Majorana representation for $\mathbb{P}(\mathcal{H}_s)$, the stellar representation for $Gr_k(\mathcal{H}_s)$ is not one-to-one, as different k-planes are associated to the same constellation. As we have already argued, both, $Gr_k(\mathcal{H}_s)$ and the space of constellations for k-planes have the same dimension, so only a discrete number of planes share the same constellation. From here the question is, how many k-planes share the same constellation generically? This is the question we answer in this section

In general, the computations to find how many k-planes have the same constellation are rather cumbersome. As we show explicitly in the following section, the number of 2-planes with the same constellation is two for $s=3/2$, nine for $s = 2$ and fourteen for $s = 5/2$. As it turns out, these numbers are well-known in the branch of mathematics known as *Schubert calculus* [67], as they appear in the following theorem [68],

Theorem 16 (Schubert, 1886)**.** *Let $\mathcal{H}$ denote a complex vectorial space of dimension $m + k$. Consider a collection $\{\Pi_i,\ i = 1, \ldots, k \cdot m\}$ of m-planes through the origin. Then, there are*

$$D(m,k) = \frac{1!\,2!\,3!\,\ldots(k-1)!}{m!\,(m+1)!\ldots(m+k-1)!}(m \cdot k)! \tag{3.3.1}$$

k-planes through the origin (counting multiplicity) that intersect all the m-planes Π_i non trivially (that is, the intersection contains more points besides the origin).

Taking advantage of this theorem, we can find the number of k-planes with the same constellation,

Theorem 17. *For a spin s, the number of k-planes with the same constellation is $D(m,k)$, where $m = 2s + 1 - k$*

Proof. By theorem 11, the number of stars in the constellation of a k-plane is km. Therefore, by theorem 16, it is enough to prove the following claim: *$\hat{n}$ is an element of $\mathcal{C}_\Pi$ if and only if the intersection of Π with the coherent m-plane $\Pi_{\hat{\boldsymbol{n}}}$ is not trivial.* In a sense, this means that specifying the constellation of a k-plane Π is equivalent to giving a list of all the km coherent m-planes that intersect Π non trivially.

To prove the claim of the previous paragraph, let $\hat{n}$ denote the direction of a star in the constellation of Π. Then, by theorem 9, this occurs if and only if there is a state $|\psi\rangle$ in Π whose constellation has k stars in the direction $\hat{n}$. By the characterization of coherent m-planes of theorem 8, this means that $|\psi\rangle$ is also an element of the coherent m-plane $\Pi_{\hat{\boldsymbol{n}}}$, that is, the intersection $\Pi \bigcap \Pi_{\hat{\boldsymbol{n}}}$ is not trivial. This concludes the proof. □

Since the previous theorem takes into account multiplicity, the number of *different* k-planes with the same constellation might be less than the one stated. One important example of this fact is the one for coherent k-planes; they are the only ones where the $(2s + 1 - k)k$ stars coincide. To prove this statement, first we prove two useful lemmas,

Lemma 18. *Let Π denote a k-plane such that all the stars in $\mathcal{C}_\Pi$ are in $\hat{n}$. Then, Π is invariant under all rotations around $\hat{n}$.*

Proof. Assume the contrary. Then, there is a rotation $R_{\hat{n},\theta}$ around $\hat{n}$ by an angle θ such that the rotated plane $R_{\hat{n},\theta}\Pi$ is different from Π. Because of this, the curve contained in $Gr_k(\mathcal{H}_s)$ obtained by rotating Π around $\hat{n}$,

$$\Pi(t) = R_{\hat{n},t}\Pi\,, \quad 0 \leq t \leq \theta\,,$$

contains at least two different points, $\Pi(0)$ and $\Pi(\theta)$, and therefore, and infinite number of them. Clearly, the constellation of $\Pi(t)$ also consists of a single star in $\hat{n}$, as it can be obtained by rotating $\mathcal{C}_\Pi$ around $\hat{n}$. Therefore, there is an infinite number of k-planes with the same constellation, a statement that is in contradiction with theorem 17. □

Lemma 19. *Let* Π *denote a* k*-plane that is invariant under all rotations around* $\hat{n}$*. Then,* Π *is spanned by certain eigenstates of the spin operator* $S \cdot \hat{n}$*.*

Proof. Let $|\psi\rangle$ be any element of Π. Then, $S \cdot \hat{n}|\psi\rangle$ is also in Π. Indeed, as the following curve,

$$|\psi(t)\rangle = e^{-itS\cdot\hat{n}}|\psi\rangle\,,$$

is contained in Π by hypothesis, its derivative at $t = 0$ is also an element of Π, but said derivative is proportional to $S \cdot \hat{n}|\psi\rangle$.

Let $(S \cdot \hat{n})^\Pi$ denote the restriction of $S \cdot \hat{n}$ to Π, as defined in (3.1.2). As it can be easily checked, $(S \cdot \hat{n})^\Pi$ is self-adjoint and therefore, diagonalizable. Consider a basis of Π made by eigenstates of $(S \cdot \hat{n})^\Pi$, $\{|\phi_i\rangle\,, i = 1, \ldots, k\}$. We claim that its elements are also eigenstates of $S \cdot \hat{n}$. This can be proved by considering the following implications,

$$(S \cdot \hat{n})^\Pi|\phi_i\rangle = \lambda_i|\phi_i\rangle \Rightarrow \mathcal{P}S \cdot \hat{n}|\phi_i\rangle = \lambda_i|\phi_i\rangle \Rightarrow S \cdot \hat{n}|\phi_i\rangle = \lambda_i|\phi_i\rangle\,,$$

where the first implication was obtained just by applying the definition (3.1.2), and the second one comes from the previously proved fact that $S \cdot \hat{n}|\phi_i\rangle$ is an element of Π, and therefore, $\mathcal{P}$ leaves it invariant. Since $\{|\phi_i\rangle\,, i = 1, \ldots, k\}$ is a basis for Π and only consists on eigenstates of $S \cdot \hat{n}$, the claim is proved. □

Finally, with these two lemmas, we can prove the uniqueness of the constellation of the coherent states,

Theorem 20. *Let* Π *denote a* k*-plane such that all the stars in* $\mathcal{C}_\Pi$ *are in* $\hat{n}$*. Then,* Π *is the coherent* k*-plane* $\Pi_{\hat{\boldsymbol{n}}}$*.*

Proof. By the two previous lemmas, any k-plane Π with such characteristics is spanned by certain eigenstates of $S \cdot \hat{n}$. Let $\{|\hat{\boldsymbol{n}}, m_1\rangle, \ldots, |\hat{\boldsymbol{n}}, m_k\rangle, m_1 \geq m_2 \cdots \geq m_k\}$ denote a basis for Π. As computed in section 3.2.1, the constellation of this type of planes has $m_1 + \cdots + m_k + ks - \frac{k(k-1)}{2}$ stars in $\hat{n}$ and the remaining ones in $-\hat{n}$. Since by hypothesis there are no stars in $-\hat{n}$ in $\mathcal{C}_\Pi$, the sum $m_1 + \cdots + m_k$ must attain its maximal value. This value is attained only when $m_1 = s, m_2 = s-1, \ldots, m_k = s-k+1$, that is, if Π is a coherent k-plane □

Not only different k-planes might have the same constellation, but if said constellation has a rotational symmetry, these k-planes may be permuted under the action of said rotation. We give an example of this fact in the following subsection.

3.3.1 Characterizing all the 2-planes whose constellation is a double tetrahedron

Consider the case of a spin $s = 5/2$ where, by theorem 11, the number of stars in the constellation of a 2-planes is eight. In this subsection, we find all the 2-planes for this particular value of s such that their constellation has two stars in each of the following four directions that define a tetrahedron,

$$v_1 = (0,0,1)\,, \qquad v_2 = \left(\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right)\,,$$
$$v_3 = -\left(\frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{3}\right)\,, \qquad v_4 = -\left(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{3}\right)\,. \tag{3.3.2}$$

We call this constellation a double tetrahedron and denote it by $\mathcal{C}$.

To find all the 2-planes which constellation is $\mathcal{C}$, consider a generic 2-plane $P\Pi$ in the space of polynomials of degree five written in the standard form as a matrix (w.r.t. the basis $\{\zeta^5, \zeta^4, \zeta^3, \zeta^2, \zeta^1, 1\}$) in the following way,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ A_3 & B_3 \\ A_2 & B_2 \\ A_1 & B_1 \\ A_0 & B_0 \end{pmatrix}\,. \tag{3.3.3}$$

This implies that $P\Pi = \text{span}\{p, q\}$ where

$$p(\zeta) = \zeta^5 + \sum_{i=0}^{3} A_i \zeta^i\,, \quad q(\zeta) = \zeta^4 + \sum_{i=0}^{3} B_i \zeta^i\,.$$

By imposing the condition that the zeros of the Wronskian of p and q are given by the directions of the double tetrahedron (3.3.2), we find a system of equations for the coefficients A_i and B_i. By solving it analytically in *Mathematica*, we obtain fourteen different 2-planes $P\Pi_i$, $(i = 1, \ldots, 14)$ such that their Wronskian have the zeros we want. By considering (3.1.4), we conclude there are fourteen 2-planes Π_i, $(i = 1, \ldots, 14)$ contained in $\mathcal{H}_s$ such that their constellation is $\mathcal{C}$ (we omit their explicit expression since it is not necessary for the following arguments).

Note that, the rotations that leave $\mathcal{C}$ invariant are the elements of the *rotational symmetry group of the tetrahedron*, commonly referred to as A_4. However, in general, the rotations in A_4 permute the 2-planes Π_i among themselves instead of leaving them invariant. In this sense, we say that A_4 acts on the set $\{\Pi_i,\, i = 1, \ldots, 14\}$. Under this action, some mixing between 2-plane occurs. For instance, if we take Π_1, and apply to it all the elements of A_4, the 2-planes obtained are the following,

$$\{\Pi_1, \Pi_5, \Pi_9, \Pi_{11}, \Pi_{12}, \Pi_{14}\}\,. \tag{3.3.4}$$

This set is the orbit of Π_1 and it is invariant under the action of A_4. Similarly, we can compute the orbit of Π_2. The result is,

$$\{\Pi_2, \Pi_4, \Pi_{10}, \Pi_{13}\}\,, \tag{3.3.5}$$

while the one for Π_3 is simply

$$\{\Pi_3\}\,, \tag{3.3.6}$$

that is, Π_3 is invariant under all the rotations of A_4. Finally, the last orbit is,

$$\{\Pi_6, \Pi_7, \Pi_8\}\,. \tag{3.3.7}$$

By considering these orbits we can divide the fourteen 2-planes Π_i in four different types, that we refer simply as first type (3.3.4), second type (3.3.5) and so on. In what follows, we further characterize these types. The procedure is explained in the following paragraph.

By theorem 9, for each direction $\hat{v}_j$ in $\mathcal{C}_{\Pi_i}$, there is a state $|\psi_{\hat{v}_j}\rangle$ (that in these cases turns out to be unique up to scalar multiplication) in Π_i such that its constellation has at least 2 stars in the direction $\hat{n}$. In this way, given a 2-plane Π_i, we can assign to each vertex v_j of the tetrahedron (3.3.2) a constellation, namely, the one of $|\psi_{\hat{v}_j}\rangle$. These constellations completely determine Π_i, as it can be computed as the span of the states $|\psi_{\hat{v}_j}\rangle$, $(j = 1, \ldots, 4)$. This is what we do in what follows.

First type

The planes of the first type (3.3.4) can be labeled by two vertices of the tetrahedron. Since there are six ways to chose 2 vertices, there are six planes of the first type.

For simplicity, assume we consider the 2-plane associated to the vertices v_3 and v_4. The planes labeled by another pair of vertices behave essentially in the same way. In this case, the constellations associated to v_4 and v_3 are the following,

$$v_4 \to \{v_1, v_2, v_4, v_4, v_4\},\; v_3 \to \{v_1, v_2, v_3, v_3, v_3\}\,. \tag{3.3.8}$$

Therefore, this 2-plane is the span of the states whose constellation is given in the previous equations.

For this particular plane labeled by v_3 and v_4, we can also compute the constellation associated to v_1. The result is,

$$v_1 \to \{v_1, v_1, v_2, s_1^{\mathrm{I}}, s_2^{\mathrm{I}}\}\,, \tag{3.3.9}$$

where,

$$s_1^{\mathrm{I}} = -v_1\,, \quad s_2^{\mathrm{I}} = \left(-\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right)\,.$$

Note that s_1^{I} in the point antipodal to v_1. All these stars are in a *kite* in a great circle of the sphere. These constellations are illustrated in figure 3.2. The constellation corresponding to v_2 is also a kite and can be obtained by rotating the one for v_1.

Second type

The 2-planes of the second type (3.3.5) can be labeled by one vertex of the tetrahedron. Therefore, there are four 2-planes of this type. We only give the constellation of the vertices for the 2-plane labeled by v_2. The ones for the other planes of this type can be obtained by rotating the ones shown below.

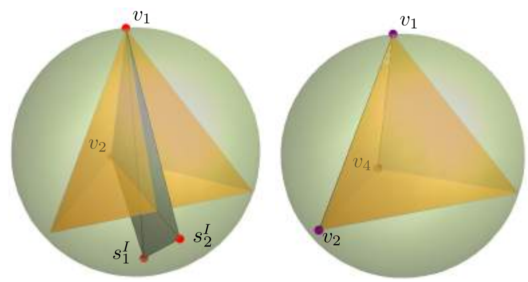


Figure 3.2: Constellations for the vertices v_1 (left) and v_4 (right) for the 2-plane labeled by v_3 and v_4 . The constellation for v_1 (3.3.9) is shown in red, while the one corresponding to v_4 (3.3.8) is shown in purple. The vertex v_1 in red is doubly degenerate, while v_4 in purple is triply degenerate. In both figures, the tetrahedron is the constellation $\mathcal{C}$ of the 2-planes in consideration.

The constellations for v_1, v_3 and v_4 are all analogous. Here we only present the one for v_1,

$$v_1 \to \{v_1, v_1, v_1, v_2, s_1^{\mathrm{II}}\} \tag{3.3.10}$$

where,

$$s_1^{\mathrm{II}} = \left(-\frac{4\sqrt{2}}{9}, 0, -\frac{7}{9}\right) .$$

The direction s_1^{II} is such that the triangle $v_2\, s_1^{\mathrm{II}}\, v_1$ is isosceles. In figure 3.3, we show this triangle.

For this plane labeled by v_2, the constellation associated to v_2, is the following,

$$v_2 \to \{v_2, v_2, s_2^{\mathrm{II}}, s_3^{\mathrm{II}}, s_4^{\mathrm{II}}\} , \tag{3.3.11}$$

where,

$$s_2^{\mathrm{II}} \approx (-0.77, 0, -0.63), \quad s_3^{\mathrm{II}} \approx (0.34, -0.74, 0.57), \quad s_4^{\mathrm{II}} \approx (0.34, 0.74, 0.57).$$

In this case, the vertices $v_2, s_2^{\mathrm{II}}, s_3^{\mathrm{II}}$ and s_4^{II} constitute a non-regular tetrahedron, where three of the faces are an isosceles triangle, and the remaining one (the one defined by s_2^{II}, s_3^{II} and s_4^{II}), is an equilateral triangle. The orientation of this tetrahedron differs from the original one (3.3.2) by a rotation of 60° around the v_2 axis. This constellation is shown in figure 3.3.

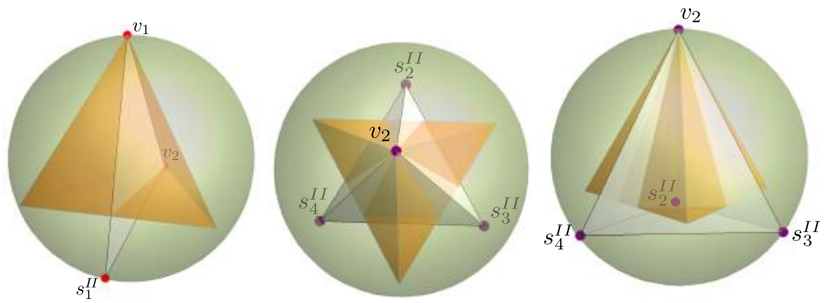


Figure 3.3: Constellations of the vertices of the tetrahedron for the second type plane labeled by the vertex v_2. Left: Constellation associated to v_1 (3.3.10). The vertex v_1 is triply degenerate. Middle and right: Constellation for v_2 (3.3.11) shown from two different viewpoints. The vertex v_2 is double degenerated

Third type

There is only one 2-plane of the third type (3.3.6). As such, the constellations for the vertices are all analogous. Here we present just the one for v_1,

$$v_1 \to \{v_1, v_1, v_2, v_3, v_4\} ,$$

that is, all the stars are in the double tetrahedron.

Fourth type

For the fourth type (3.3.7), each 2-plane is labeled by two pairs of vertices. Since there are three ways of choosing two pairs of vertices, there are three planes of this kind.

If the chosen pairs are (v_1, v_3) and (v_2, v_4), the constellation corresponding to v_1 is,

$$v_1 \to \{v_1, v_1, v_1, s_1^{\mathrm{IV}}, s_2^{\mathrm{IV}}\} \tag{3.3.12}$$

where,

$$s_1^{\mathrm{IV}} \approx (-0.72, 0.29, -0.62) , \quad s_2^{\mathrm{IV}} \approx (0.10, 0.77, -0.62) ,$$

while the one associated to v_3 is,

$$v_3 \to \{v_3, v_3, v_3, s_3^{\mathrm{IV}}, s_4^{\mathrm{IV}}\} , \tag{3.3.13}$$

where,

$$s_3^{\mathrm{IV}} \approx (-0.22, -0.57, 0.79) , \quad s_4^{\mathrm{IV}} \approx (0.60, -0.09, 0.79) .$$

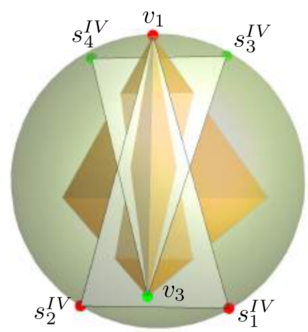


Figure 3.4: Constellations of the vertices of the double tetrahedron for the fourth-type plane labeled by the pairs of vertices (v_1, v_3) and (v_2, v_4). The constellation for v_1 (3.3.12) is shown in red, and the one for v_3, (3.3.13) in green. The vertices v_1 and v_3 are triple degenerate. The two triangles shown are equal (up to a rigid transformation) and are isosceles.

In figure 3.4, these constellations are shown. The ones corresponding to the remaining pair of vertices (v_2, v_4) are analogous to the ones presented in the figure.

As we can see, just by looking at the constellation of a k-plane, we can not find all its rotational symmetries, some extra information is needed. In the two following sections, we give two different approaches of what this extra information might be. Although the first one is simpler to compute, it only works for 2-planes, while the second one is general.

3.4 A secondary constellation for a 2-plane

In this section we explain a procedure to assign a secondary constellation to a 2-plane. In this way, we can assign to each 2-plane a pair of constellations, the one previously defined or primary constellation and the secondary one just mentioned. As we check further down, this pair of constellations has two very important properties; first of all, the procedure to build it commutes with rotations, secondly, two 2-planes are the same if and only if their pair of constellations is the same. Therefore, by looking at the rotational symmetries of the pair of constellations, we can infer the ones of the 2-plane.

The procedure to build the secondary constellation is based on the following theorem,

Theorem 21. *Consider a 2-plane* Π *and let* $\hat{n}$ *be the direction of a star in the primary constellation of* Π*. Assume that there are no other stars in* $\hat{n}$*.*

Then, there exists a unique direction $\hat{m}$ *such that the state* $|\underbrace{-\hat{n},\ldots,-\hat{n}}_{2s-1},-\hat{m}\rangle$ *is orthogonal to* Π.

Proof. Consider the $2s-1$ plane spanned by the states orthogonal to Π. Call it $\Pi^\perp$. By theorem 12, there is only one star in the direction $-\hat{n}$ in the constellation of $\Pi^\perp$. Then, by theorem 15, there is a unique state $|\psi\rangle$ in $\Pi^\perp$ such that its constellation has $2s-1$ stars in the direction $-\hat{n}$. This implies that $|\psi\rangle$ can be written as follows, $|\psi\rangle = |\underbrace{-\hat{n},\ldots,-\hat{n}}_{2s-1},-\hat{m}\rangle$, for a certain direction $\hat{m}$. The uniqueness of $|\psi\rangle$ implies the uniqueness of $\hat{m}$. This concludes the proof. □

By applying theorem 21 to each of the stars in the primary constellation of any 2-plane , we obtain a new set of $4s-2$ (c.f. theorem 11) points in the sphere, which we call the secondary constellation of Π. By definition, it is easy to check that it is *well-behaved* under rotations — for any rotation R, the secondary constellation of $R\Pi$ can be obtained by rotating the one of Π by R.

For this construction, we are assuming the generic case in which the primary constellation is not degenerate. By continuity arguments, one can extend it for all 2-planes.

As we will prove in the next theorem, this secondary constellation, together with the primary one, completely characterizes Π.

Theorem 22. *Let* $\hat{n}_i$, $i=1,\ldots,4s-2$ *be the directions of the stars in the primary constellation of a 2-plane* Π. *For each star in* $\mathcal{C}_\Pi$ *apply theorem 21 to obtain a total of* $4s-2$ *extra directions* $\hat{m}_i$, $i=1,\ldots,2s$*; the secondary constellation. Define the set* $\mathcal{S}_\Pi$ *as* $\mathcal{S}_\Pi=\{(\hat{n}_i,\hat{m}_i), i=1,\ldots,4s-2\}$. *Then* $\mathcal{S}_\Pi$ *completely characterizes the 2-plane* Π*, that is, two 2-planes* Π *and* Π' *are equal if and only if* $\mathcal{S}_\Pi=\mathcal{S}_{\Pi'}$.

Proof. Suppose we know the set $\mathcal{S}_\Pi$. To prove the theorem, it is enough to show that we can construct Π only in terms of $\mathcal{S}_\Pi$. By the definition of $\hat{m}_i$, it is clear that $|\underbrace{-\hat{n}_i,\ldots,-\hat{n}_i}_{2s-1},-\hat{m}_i\rangle$ is an element of $\Pi^\perp$. Notice that the dimension of $\Pi^\perp$ is $2s-1$. Therefore, by considering all the directions $\hat{m}_i$, we have the following,

$$\Pi^\perp = \mathrm{span}\{|\underbrace{-\hat{n}_i,\ldots,-\hat{n}_i}_{2s-1},-\hat{m}_i\rangle\,, i=1,\ldots,4s-2\}\,.$$

From here we have found $\Pi^{\perp}$. Since the orthogonal complement to $\Pi^{\perp}$ is Π, we can reconstruct Π from $\mathcal{S}_{\Pi}$. $\square$

Before going on, we want to make some remarks. The constellation of Π and $\mathcal{S}_{\Pi}$ are not independent of each other. In fact, given a constellation $\mathcal{C}$, by theorem 17, there are only $(4s-1)^{-1}\binom{4s-1}{2s}$ possible sets $\mathcal{S}_{\Pi}$ associated to $\mathcal{C}$. Also, notice that it is not enough to know both constellations to build the plane, one has to know how their elements are paired together. In this sense, $\mathcal{S}_{\Pi}$ can be described as a list of $4s-2$ oriented segments, where each one of them begins at a point of the primary constellation and ends in the corresponding star of the secondary one. By the previous theorem, the symmetries of this list of segments coincides with the one of the plane. This is the content of the following corollary,

Corollary 23. *Let Π be a 2-plane and consider the set $\mathcal{S}_{\Pi}$. Given a rotation R, define $R\mathcal{S}_{\Pi}$ as $R\mathcal{S}_{\Pi} = \{(R\hat{n}_i, R\hat{m}_i),\ i = 1, \ldots, 4s-2\}$. Then, a rotation R leaves Π invariant if and only if $R\mathcal{S}_{\Pi} = \mathcal{S}_{\Pi}$.*

The previous corollary is useful to find all the rotational symmetries of a given 2-plane. We illustrate this procedure with two examples in the following subsection.

In theorem 22, we proved that if we know $\mathcal{S}_{\Pi}$ then, we can rebuild the 2-plane Π. As a matter of a fact, we have the conjecture that the following stronger claim holds,

Conjecture 24. *Let Π be a 2-plane through the origin contained in $\mathcal{H}_s$. Then, Π can be completely characterized by knowing its primary constellation and* just one pair $(\hat{n}_i, \hat{m}_i)$ *in $\mathcal{S}_{\Pi}$.*

In section 3.4.3, we prove the conjecture for the following particular values of s, $s = 3/2, 4/2, 5/2$.

3.4.1 Examples of $\mathcal{S}_{\Pi}$

As an example, consider the case $s = 5/2$ and the 2-plane Π_1 of section 3.3.1 of the first type. Recall that its primary constellation is the double tetrahedron given in equation (3.3.2). In figure 3.5 we show the primary and secondary constellation associated to this plane. In this case, the set $\mathcal{S}_{\Pi}$ is,

$$\mathcal{S}_{\Pi} = \{(v_1, v_1), (v_1, v_1), (v_2, v_2), (v_2, v_2), (v_3, \hat{m}_3), (v_3, \hat{m}_3), (v_4, \hat{m}_4)(v_4, \hat{m}_4)\}, \tag{3.4.1}$$

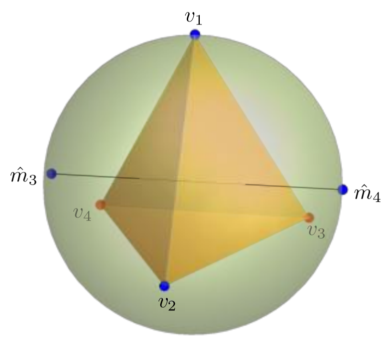


Figure 3.5: The secondary constellation of a 2-plane with a tetrahedral constellation, shown in blue. The primary constellation 2-plane is shown in red, where each vertex is double degenerate. The vertices v_1 and v_2 are in both constellations. See equation (3.4.1)

where,

$$\hat{m}_3 = \frac{1}{9}\left(\frac{\sqrt{2}}{3}, -11\sqrt{\frac{2}{3}}, \frac{1}{3}\right) , \quad \hat{m}_4 = \frac{1}{9}\left(\frac{\sqrt{2}}{3}, 11\sqrt{\frac{2}{3}}, \frac{1}{3}\right) .$$

Since the primary constellation of the plane is the double tetrahedron, all its rotational symmetries are the ones of the tetrahedron, A_4. Since the equalities $\hat{m}_1 = v_1$ and $\hat{m}_2 = v_2$ hold, any non-trivial symmetry of Π_1 must interchange these vertices; the only one in A_4 with this property is the rotation by π around the axis $(1/\sqrt{3})(\sqrt{2}, 0, 1)$. A quick computation shows that this rotation also interchanges v_3 with v_4 and $\hat{m}_3$ with $\hat{m}_4$, thus leaving $\mathcal{S}_\Pi$ invariant. By of corollary 23, this is the only symmetry rotation of Π_1.

As a second example, consider the 2-plane of the third type of section 3.3.1, Π_3. In this case, $\mathcal{S}_\Pi$ turns out to be,

$$\mathcal{S}_\Pi = \{(v_1, v_1), (v_1, v_1), (v_2, v_2), (v_2, v_2), (v_3, v_3), (v_3, v_3), (v_4, v_4), (v_4, v_4)\} ,$$

that is, the secondary constellation is equal to the primary one. From here, it is clear that all the rotational symmetries of the tetrahedron leave this 2-plane invariant.

In the following subsections we prove the conjecture 24 of the previous section for the cases $s = 3/2, 4/2, 5/2$, but before that, we prove some technical theorems.

3.4.2 Study of 2-planes: an approach based on differential equations

In this subsection, we prove some theorems that are useful to prove later results.

First we find necessary and sufficient condition for a polynomial p (which we assume has no repeated roots) to be an element of a 2-plane $P\Pi$ such that, the constellation of Π, is given by the zeros of W.

From now on, W, p and q denotes polynomials of degrees $4s-2$, $2s$ and $2s$ respectively. We remind the reader that we denote the Majorana polynomial (3.2.2) of Π by p_Π. The condition alluded to above is stated in the following theorem,

Theorem 25. *Consider W and p polynomials. Denote by ζ_i, $(i = 1, \ldots, 2s)$ the roots of p (all assumed to be different). Then, there is a 2-plane Π such that p_Π is proportional to W, and p is an element of $P\Pi$ if and only if, the following equalities hold,*

$$\frac{W'(\zeta_i)}{W(\zeta_i)} = \frac{p''(\zeta_i)}{p'(\zeta_i)}, \quad \textit{for all } i = 1, \ldots, 2s. \tag{3.4.2}$$

Proof. Suppose the polynomials p and q constitute a basis for $P\Pi$, where Π is a 2-plane such that $p_\Pi = W$. Then, the following chain of implications holds,

$$\begin{vmatrix} p & p' \\ q & q' \end{vmatrix} = W \Rightarrow pq' - p'q = W \Rightarrow p^2 \left(\frac{q}{p}\right)' = W \Rightarrow \left(\frac{q}{p}\right)' = W/p^2$$
$$\Rightarrow q = p \int \frac{W}{p^2} \ . \tag{3.4.3}$$

Since q is a polynomial, the indefinite integral $\int (W/p^2)$ must be a rational function. Note that the degree of p^2 is $4s$, while the one of W is $4s-2$. This implies that W/p^2 is a *proper rational function* and thus can be decomposed in *partial fractions* as follows (recall that p has no repeated roots by hypothesis),

$$\frac{W(\zeta)}{p^2(\zeta)} = \sum_{i=1}^{2s} \left(\frac{A_i}{\zeta - \zeta_i} + \frac{B_i}{(\zeta - \zeta_i)^2} \right) , \tag{3.4.4}$$

where A_i and B_i are complex numbers. By integrating the previous equation we obtain,

$$\int \frac{W(\zeta)}{p^2(\zeta)} d\zeta = constant + \sum_{i=1}^{2s} \left(A_i \ln(\zeta - \zeta_i) - \frac{B_i}{\zeta - \zeta_i} \right) . \tag{3.4.5}$$

From the previous expression, we can conclude that $\int(W/p^2)$ is a rational function if and only if all the coefficients A_i are zero. In this case, by multiplying (3.4.5) by p, it is clear that $p\int(W/p^2)$ is a polynomial of degree less of equal than the one of p, $2s$. By equation (3.4.3), this implies the following; there are polynomials p and q whose Wronskian is W if and only if the indefinite integral $\int(W/p^2)$ is a rational function. As previously noted, this occurs if and only if the coefficients A_i are zero.

To conclude the proof, we find a general expression for the coefficient A_i, and equal it to zero. As we check further down, (3.4.2) immediately follows from this equality.

To find an expression for A_i, we perform the summation in the right side of (3.4.4) and cancel out p^2. The result is,

$$W(\zeta) = \sum_{i=1}^{2s} \Big(A_i(\zeta - \zeta_i)p_{\hat{i}}(\zeta) + B_i p_{\hat{i}}^2(\zeta) \Big) \,, \tag{3.4.6}$$

where $p_{\hat{i}}$ denotes the polynomial obtained by omitting the factor $\zeta - \zeta_i$ from p, that is,

$$p_{\hat{i}}(\zeta) = \frac{p(\zeta)}{\zeta - \zeta_i} \,. \tag{3.4.7}$$

Note that $p_{\hat{i}}$ is zero when evaluated on the roots of p, except for ζ_i. Because of this (as it is usual when working with partial fractions) we can evaluate both sides of (3.4.6) in ζ_i to obtain B_i,

$$W(\zeta_i) = B_i p_{\hat{i}}^2(\zeta_i) \Rightarrow B_i = \frac{W(\zeta_i)}{p_{\hat{i}}^2(\zeta_i)} \,.$$

From (3.4.7), we can compute $p_{\hat{i}}(\zeta_i)$ by considering the following limit,

$$p_{\hat{i}}(\zeta_i) = \lim_{\zeta\to\zeta_i} \frac{p(\zeta)}{\zeta - \zeta_i} = p'(\zeta_i) \,, \tag{3.4.8}$$

where we used L' Hôpital rule for limits to produce the result. By substituting this expression in the equation we previously found for B_i, we obtain the following,

$$B_i = \frac{W(\zeta_i)}{(p'(\zeta_i))^2} \,. \tag{3.4.9}$$

Similarly, we can compute the coefficient A_i we seek. If we differentiate (3.4.6), and evaluate the result at $\zeta = \zeta_i$, we obtain,

$$W'(\zeta_i) = A_i p_{\hat{i}}(\zeta_i) + 2B_i p_{\hat{i}}(\zeta_i) p'_{\hat{i}}(\zeta_i) \Rightarrow A_i = \frac{W'(\zeta_i)}{p_{\hat{i}}(\zeta_i)} - 2B_i(\zeta_i) p'_{\hat{i}}(\zeta_i)\,. \tag{3.4.10}$$

From (3.4.7), we can compute $p'_{\hat{i}}(\zeta_i)$. First, we differentiate (3.4.7), and then we use L' Hôpital rule twice as follows,

$$\begin{aligned} p'_{\hat{i}}(\zeta) &= \frac{(\zeta-\zeta_i)p'(\zeta) - p(\zeta)}{(\zeta-\zeta_i)^2} \\ \Rightarrow p'_{\hat{i}}(\zeta_i) &= \lim_{\zeta\to\zeta_i} \frac{(\zeta-\zeta_i)p'(\zeta) - p'(\zeta)}{(\zeta-\zeta_i)^2} = \lim_{\zeta\to\zeta_i} \frac{(\zeta-\zeta_i)p''(\zeta)}{2(\zeta-\zeta_i)} = \frac{p''(\zeta_i)}{2}\,. \end{aligned}$$

By substituting (3.4.8), (3.4.9) and the previous expression in (3.4.10) we obtain,

$$A_i = \frac{W'(\zeta_i)}{p'(\zeta_i)} - \frac{W(\zeta_i)p''(\zeta_i)}{(p'(\zeta_i))^2}\,.$$

Since we required A_i to be zero, the Wronskian of p and q is W if and only if the following equality holds,

$$\frac{W'(\zeta_i)}{W(\zeta_i)} = \frac{p''(\zeta_i)}{p'(\zeta_i)}\,.$$

This completes the proof. □

The previous theorem can be restated in a way that is often more useful as follows,

Theorem 26. *There exist a 2-plane Π such that p_Π is proportional to W, and p is an element of $P\Pi$ if and only if there exist a polynomial m such that the following equality holds,*

$$Wp'' - W'p' + mp = 0\,. \tag{3.4.11}$$

Proof. Consider the polynomial $Q = Wp'' - W'p'$. By the result of the previous theorem, such plane Π exists if and only if $Q(\zeta_i) = 0$ for all the roots ζ_i of p. From basic algebra, we know that this occurs if and only if p divides Q, that is, if Q can be written as $Q = -m \cdot p$ for some polynomial m. □

The polynomials m defined in the previous theorem, allow us to distinguish different planes with the same constellation, The precise way is encoded in the following theorem,

Theorem 27. *The set of 2-planes whose constellation is given by the zeros of W, is in one-to-one correspondence with the set of polynomials m such that the solutions p to the differential equation (3.4.11) are all polynomials of degree (at most) $2s$.*

Proof. First, given a m such that all the solutions of (3.4.11) are polynomials of degree $2s$, we assign to it a 2-plane. To this end, denote by $P\Pi$ the solution space (in the space of polynomials) of (3.4.11). Denote by Π the corresponding 2-plane in $\mathcal{H}_s$, as in (3.1.4). Then, $\mathcal{C}_\Pi$ is given by the zeros of W. Indeed, denote by p any solution to the differential equation (3.4.11). Define q as in (3.4.3). Using essentially the same algebra used to prove theorem 25, it is easy to check that q is also a solution to (3.4.11) and that the Wronskian of p and q is W.

Reciprocally, suppose Π is such that $\mathcal{C}_\Pi$ is given by the zeros of W. We proceed to find such m. Take p_0 and q_0 elements of a certain basis for $P\Pi$. Suppose they are scaled in a way such that,

$$W = \begin{vmatrix} p_0 & p_0' \\ q_0 & q_0' \end{vmatrix} \Rightarrow W' = \begin{vmatrix} p_0 & p_0'' \\ q_0 & q_0'' \end{vmatrix}$$

Since any element p of $P\Pi$ can be written as a linear combination of p_0 and q_0, the following Wronskian needs to be zero,

$$\begin{vmatrix} p & p' & p'' \\ p_0 & p_0' & p_0'' \\ q_0 & q_0' & q_0'' \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} p_0' & p_0'' \\ q_0' & q_0'' \end{vmatrix} p - \begin{vmatrix} p_0 & p_0'' \\ q_0 & q_0'' \end{vmatrix} p' + \begin{vmatrix} p_0 & p_0' \\ q_0 & q_0' \end{vmatrix} p'' = 0$$
$$\Rightarrow W p'' - W' p' + mp = 0 \,, \tag{3.4.12}$$

where m is the following polynomial

$$m = \begin{vmatrix} p_0' & p_0'' \\ q_0' & q_0'' \end{vmatrix} .$$

By the uniqueness theorem of differential equations, any solution to (3.4.12) is a linear combination of p_0 and q_0, and hence, a polynomial of degree at most $2s$. This concludes the proof. □

Given a W, the polynomials m such that the solutions of (3.4.11) are also polynomials are known as *Van Vleck polynomials*. Therefore, the number of 2-planes that share the same constellation is equal to the number of Van Vleck polynomials for equation (3.4.11)[69].

Theorem 27 also allows us to find the number of 2-planes in certain particular cases. In what follows, we illustrate this procedure for $s = 3/2$.

An example: s=3/2

In the case of $s = 3/2$ the degree of W is four. Therefore, W can be written in the following way,

$$W(\zeta) = a\zeta^4 + b\zeta^3 + c\zeta^2 + d\zeta + f\,. \tag{3.4.13}$$

By theorem 27, to find all planes whose constellation is given by the zeros of W, it is enough find all the polynomials m such that the solutions of the differential equation (3.4.11) are all polynomials of degree $2s = 3$ or less. Because of this, given any solution p, it must also satisfy the equation $p^{(4)} = 0$. The strategy is therefore to find conditions on m so that the equation $p^{(4)} = 0$ holds for all the solutions of (3.4.11).

From (3.4.11), we can compute $p^{(4)}$. First, if we solve for p'' in (3.4.11) we obtain,

$$p'' = \frac{W'p' - mp}{W}\,.$$

By differentiating (3.4.11), we can calculate p''',

$$Wp''' - W''p' + m'p + mp' = 0 \Rightarrow p''' = \frac{(W'' - m)p' - m'p}{W}$$

Finally, by differentiating again and using the previous equations, after some algebra, we obtain the following expression involving $p^{(4)}$,

$$Wp^{(4)} + (2m' - W''')p' + \frac{1}{W}(Wm'' - (m - W'')m - W'm')p = 0\,.$$

Since $p^{(4)}$ must be zero for all the solutions p, the terms multiplying p and p' are also zero, that is,

$$2m' - W''' = 0\,, \quad Wm'' - (m - W'')m - W'm' = 0\,.$$

The first equation implies that $m = W''/2 + \kappa$, for some constant κ. After using this expression in the second equation, we obtain,

$$\frac{WW^{(4)}}{2} - \left(\frac{W''}{2} + \kappa - W''\right)\left(\frac{W''}{2} + \kappa\right) - \frac{W'W'''}{2} = 0 \Rightarrow$$
$$\Rightarrow \kappa = \pm\frac{1}{2}\sqrt{2WW^{(4)} - 2W'W''' + (W'')^2}\,.$$

By using (3.4.13) we obtain,

$$\kappa = \pm\sqrt{c^2 + 12af - 3bd}\,,$$

expression that is independent of ζ, as it was assumed. Therefore, all the 2-planes whose Majorana polynomial is proportional to W, are given by the following polynomials m,

$$m(\zeta) = \frac{W''(\zeta)}{2} \pm \sqrt{c^2 + 12af - 3bd}\,, \tag{3.4.14}$$

that is, there are generically two different planes with the same constellation when $s = 3/2$. Using the same procedure, one can prove that for $s = 2$, there are generically five possible two planes with the same constellation. The furthest we have managed to push this procedure, is for $s = 5/2$, where one can prove that there are fourteen 2-planes with the same constellation.

3.4.3 Determining a 2-plane Π by knowing $\mathcal{C}_\Pi$ and one pair in $\mathcal{S}_\Pi$

In general, by theorem 23, if we know the set $\mathcal{S}_\Pi$, we can completely determine the plane Π. In this section, we prove the conjecture 24 for the cases $s = 3/2, 4/2, 5/2$. To simplify the calculations, assume the pair $(\hat{n}, \hat{m})$ we know is such that $\hat{n} = -\hat{z}$. If this was not the case, one could simply consider a rotation that maps $\hat{n}$ to $-\hat{z}$ and the results would still hold.

First, we prove the following theorem for a general s,

Theorem 28. *Consider a 2-plane Π such that there is a unique star in $-\hat{z}$. If the pair $(-\hat{z}, \hat{m}_0)$ is an element of $\mathcal{S}_\Pi$, then ζ_0, the stereographic image of $\hat{m}_0$, is the only complex number such that $P^{(2s-1)}(\zeta_0) = 0$ for all the Majorana polynomials P in $P\Pi$.*

Proof. The strategy used to prove this theorem is as follows; first, we prove that such number ζ_0 exists. Then, we prove it is the image under the stereographic projection of $\hat{m}_0$. Let Π be a 2-plane in the space of a spin s. Suppose that there is only one star pointing towards the south pole in the constellation of Π.

Since there is only one star in $-\hat{z}$ in $\mathcal{C}_{\Pi}$, then, by theorem 15, there is exactly one state (up to multiplication by scalars) in Π whose constellation has precisely two stars in the south pole. Furthermore, the constellation of any other state has no stars in the south pole. This implies that we can find a basis $\{p, q\}$ for the space $P\Pi$ such that the degree of p and q are $2s-2$ and $2s$ respectively. By construction, any polynomial P in $P\Pi$ can be written as $P = Ap + Bq$, where A and B are complex numbers. By differentiating $(2s-1)$ times P and recalling that p is of degree $2s-2$, we can conclude the following,

$$P^{(2s-1)} = Ap^{(2s-1)} + Bq^{(2s-1)} = Bq^{(2s-1)} \,. \tag{3.4.15}$$

Since $q^{(2s-1)}$ is of degree one, it only has one root, ζ_0. The previous equation is valid for any P in $P\Pi$, so ζ_0 is a root of the $(2s-1)$ derivative of all the polynomials in $P\Pi$ and from the way we found it, it is unique.

Finally, we check that ζ_0 is the image under stereographic projection of $\hat{m}_0$. If we denote by $\tilde{\zeta}_0$ the projection of $\hat{m}_0$, then, what we want to prove is $\tilde{\zeta}_0 = \zeta_0$. We are going to prove that $P^{(2s-1)}(\tilde{\zeta}_0)$ is zero for all the polynomials in $P\Pi$ and hence, $\tilde{\zeta}_0$ is equal to ζ_0. Consider the state $|\psi\rangle$ used to find $\hat{m}_0$ of theorem 21. Recall that, by definition, $|\psi\rangle$ is in $\Pi^{\perp}$ and its constellation has $2s-1$ stars in the north pole and one in $-\hat{m}$, so its Majorana polynomial (1.3.14) is (up to a factor) the following,

$$p_{\psi}(\zeta) = \zeta^{2s-1}\left(\zeta + \frac{1}{\tilde{\zeta}_0^*}\right) = \zeta^{2s} + \frac{1}{\tilde{\zeta}_0^*}\zeta^{2s-1} \,,$$

so that $|\psi\rangle$ can be written as,

$$|\psi\rangle = |s, 2s\rangle - \frac{1}{\sqrt{2s}\tilde{\zeta}_0^*}|s, 2s-1\rangle \,.$$

Consider any state $|\phi\rangle = \sum_m C_m |s, m\rangle$ in Π and therefore orthogonal to $|\psi\rangle$. The orthogonality between $|\psi\rangle$ and $|\phi\rangle$ implies the following expression for the coefficients C_m,

$$C_{2s-1} = \sqrt{2s}\tilde{\zeta}_0 C_{2s} \,,$$

which in turn implies the following for the Majorana polynomial associated to $|\phi\rangle$,

$$p_{\phi}(\zeta) = C_{2s}\left(\zeta^{2s} - 2s\tilde{\zeta}_0\zeta^{2s-1}\right) + \sum_{m=-s}^{s-2} (-1)^{s-m} \binom{2s}{m}^{1/2} C_m \, \zeta^{s+m} \,.$$

By differentiating the previous expression $2s-1$ times we obtain that,

$$p_\phi^{(2s-1)}(\zeta) = C_{2s}\,(2s)!\left(\zeta - \tilde{\zeta}_0\right) .$$

Clearly, the only zero of p_ϕ is $\tilde{\zeta}_0$. Therefore, ζ_0 is equal to $\tilde{\zeta}_0$, as claimed. □

By virtue of the previous result, given a 2-plane Π with a star in the south pole, we can assign to it a complex number ζ_Π. As we show in the following theorems, for $s = 3/2, 4/2, 5/2$, the primary constellation of a 2-plane, together with ζ_Π, uniquely determine a 2-plane.

First we state the theorems for the cases $s = 3/2, 3/2, 5/2$. Then, we establish the proof for $s = 3/2$. The one of the other cases is similar in nature, but the calculations become much more cumbersome.

Theorem 29. *For spin $s = 3/2$, let Π be a 2-plane such that $\mathcal{C}_\Pi$ contains a star in the south pole and is given by the zeros of W (that in this case is a polynomial of degree three). Then, ζ_Π is a root of W',*

And reciprocally, given a third degree polynomial W, for each root ζ_0 of W', there exist a unique 2-plane Π such that $\zeta_\Pi = \zeta_0$ and $\mathcal{C}_\Pi$ is given by the zeros of W. In this sense, we say that the set of 2-planes Π such that $\mathcal{C}_\Pi$ is given by the zeros of W is in a one-to-one correspondence with the set of the roots of W'.

Notice that W' is a polynomial of degree two and, therefore, there are generically two 2-planes with the same constellation. This is the same conclusion that was reached in section 3.4.2.

For $s = 2$, a theorem analogous to the previous one can be stated as follows,

Theorem 30. *Consider a spin $s = 2$. Let $\mathcal{C}$ be a set of six stars, where one star is in the south pole. Suppose that $\mathcal{C}$ is given by the zeros of W, a polynomial of degree five. Then, the 2-planes whose constellation is $\mathcal{C}$ are in a one-to-one correspondence (as defined in theorem 29) with the roots of the following fifth degree polynomial,*

$$\frac{1}{5}W^{(5)}W - \frac{1}{3}W^{(4)}W' + W^{(3)}W'' .$$

From here, we can conclude that there are generically five 2-planes with the same constellation.

Finally, for the case $s = 5/2$, the corresponding theorem is the following,

Theorem 31. *Consider a spin $s = 5/2$. Let $\mathcal{C}$ be a set of eight stars, where one star is in the south pole. Suppose that $\mathcal{C}$ is given by the zeros of W, a polynomial of degree seven. Then, the set of 2-planes whose constellation is $\mathcal{C}$ is in a one-to-one correspondence with the roots of the following fourteenth degree polynomial,*

$$
\begin{aligned}
&\frac{5184}{343}W^2(W^{(7)})^3 + 36WW^{(6)}(W^{(5)})^3 - 128(W^{(3)})^3(W^{(6)})^2 - 81(W^{(5)})^4W' \\
&- \frac{648}{7}WW^{(4)}W^{(7)}(W^{(5)})^2 + \frac{1296}{7}W^{(7)}(W^{(5)})^2(W'')^2 \\
&- \frac{5184}{49}\left(W(W^{(7)})^2W^{(5)}W'' + WW^{(3)}W^{(4)}(W^{(7)})^2 - (W^{(3)})^2(W^{(7)})^2W'\right) \\
&- 648W^{(3)}(W^{(4)})^2(W^{(5)})^2 + 324W^{(4)}(W^{(5)})^3W'' \\
&+ \frac{864}{7}WW^{(3)}W^{(6)}W^{(7)}W^{(5)} + \frac{1152}{7}(W^{(3)})^2W^{(6)}W^{(7)}W'' \\
&- \frac{1296}{7}W^{(3)}W^{(7)}(W^{(5)})^2W' + 576(W^{(3)})^2W^{(4)}W^{(6)}W^{(5)} \\
&- 144W^{(3)}W^{(6)}(W^{(5)})^2W'' - \frac{2592}{7}W^{(3)}W^{(4)}W^{(7)}W^{(5)}W''\,.
\end{aligned}
$$

From here, we check that there are generically fourteen different 2-planes with the same constellation.

Finally, we present the proof of theorem 29.

Proof of theorem 29. Before making the actual proof, we give some preliminary definitions and observations.

Consider a 2-plane Π such that $\mathcal{C}_\Pi$ has a star in the south pole. Suppose that $P\Pi$ is generated by the polynomials p and q. Call W the Wronskian of these polynomials. Then, W is of degree 3.

Just like in the proof of theorem 28, assume that p and q are of degree $2s - 2 = 1$ and 3 respectively. Define the polynomial $w_{i,j}$ as follows,

$$
w_{i,j} = \begin{vmatrix} p^{(i)} & q^{(i)} \\ p^{(j)} & q^{(j)} \end{vmatrix}\,. \tag{3.4.16}
$$

Note that the polynomials $w_{i,j}$ are not independent for all i and j, as they satisfy the following relations,

$$
w'_{i,j} = w_{i+1,j} + w_{i,j+1}\,, \quad w_{i,j} = -w_{j,i}\,. \tag{3.4.17}
$$

Also, since p and q are of degree 1 and 3 respectively, $w_{i,j}$ is zero if i and j are greater than two or if either i or j is greater than four. Furthermore, for any 4 indices a, b, c, d, one can also prove with a little algebra that the following equality must hold,

$$w_{a,b}w_{c,d} = w_{a,c}w_{b,d} - w_{a,d}w_{b,c} \,. \tag{3.4.18}$$

Note that the Wronskian W of p and q is $w_{0,1}$ and that $W' = w_{0,2}$. Take any element P in the 2-plane $P\Pi$. Then, the following equality holds,

$$\begin{vmatrix} P & p & q \\ P' & p' & q' \\ P'' & p'' & q'' \end{vmatrix} = 0 \Rightarrow Pw_{1,2} - P'w_{0,2} + P''w_{0,1} = 0 \,. \tag{3.4.19}$$

By comparing this equation with the result of theorem 27, it is clear that $P\Pi$ is completely characterized by the polynomial $w_{1,2}$ (provided we also know W).

Note that it is possible to find all the polynomials $w_{1,2}$ that satisfy the relationships (3.4.17) and (3.4.18). As already noted, $w_{2,3}$ is equal to zero. Since the derivative of $w_{1,3}$ can be computed as $w'_{1,3} = w_{2,3} = 0$, then $w_{1,3}$ is a constant. A quick computation reveals the expression $W^{(3)} = 2w_{1,3}$, so that said constant is $W^{(3)}/2$ (recall that W is of degree three). Since, by equation (3.4.17), the equality $w'_{1,2} = w_{1,3}$ holds, then, $w_{1,2}$ can be written as $w_{1,2}(\zeta) = (W^{(3)}/2)(\zeta - \zeta_0)$, for a certain complex number ζ_0. But, by definition, we have the following,

$$w_{1,2} = \begin{vmatrix} p' & q' \\ p'' & q'' \end{vmatrix} = p'q'', \tag{3.4.20}$$

so that the zeros of $w_{1,2}$ are the zeros of q'' (since p' is constant). By theorem 28 this root is the complex number we denoted as ζ_Π, that is, $\zeta_0 = \zeta_\Pi$.

Now, we are ready to prove the first implication of the theorem. By using (3.4.18) for the indices 0,1,2 and 3, we obtain the following expression,

$$w_{0,1}w_{2,3} = w_{0,2}w_{1,3} - w_{0,3}w_{1,2} \Rightarrow 0 = w_{0,2}w_{1,3} - w_{0,3}w_{1,2} \,, \tag{3.4.21}$$

where we used the fact that $w_{2,3}$ is zero, as noted at the beginning of the previous paragraph. By evaluating the functions in (3.4.21) at ζ_Π we obtain (recall that $w_{1,2}(\zeta_\Pi) = 0$ and that $w_{1,3}$ is constant)

$$w_{0,2}(\zeta_\Pi) = 0 \Rightarrow W'(\zeta_\Pi) = 0 \,, \tag{3.4.22}$$

where we used the equality $W' = w'_{0,1} = w_{0,2}$ to obtain the last implication. This means that the numbers ζ_Π are roots of the polynomial W', as claimed.

Reciprocally, given ζ_0 a root of the W', define the polynomials $w_{1,2}(\zeta) = (W^{(3)}/2)(\zeta - \zeta_\Pi)$, $w_{0,1} = W$, $w_{0,2} = W'$. Define $P\Pi$ as the solution space of the differential equation (3.4.19). Our claim is that Π, the 2-plane in $\mathcal{H}_s$ corresponding to $P\Pi$, is the 2-plane we seek. To prove this, we have to show two claims; that the solutions are polynomials of degree three, and that the Wronskian of a basis of $P\Pi$ is W. The second claim is a direct application of *Abel's differential equation identity*. In the following paragraph, we prove the first claim.

Let p and q be a basis for $P\Pi$. By construction, in terms of p and q (see equation (3.4.16)), the equalities $w_{1,2}(\zeta) = (W^{(3)}/2)(\zeta - \zeta_\Pi)$, $w_{0,1} = W$, $w_{0,2} = W'$ hold. By the repeated use of equations (3.4.17) and (3.4.18), it is possible to prove that $w_{0,4} = w_{1,4} = 0$. Let P be an element of $P\Pi$. Since P can be written as a linear combination of p and q, the following determinant is zero,

$$\begin{vmatrix} P & p & q \\ P' & p' & q' \\ P^{(4)} & p^{(4)} & q^{(4)} \end{vmatrix} = 0\,.$$

By expanding by minors the previous determinant, we obtain the equality $P^{(4)}W = 0$. This implies that $P^{(4)}$ is zero, that is, P is a polynomial of degree three, as claimed. Another way to prove that $P\Pi$ is a space of polynomials, is to show that $w_{1,2}(\zeta)$ as previously defined, coincides with the polynomial m of equation (3.4.14), when $a = 0$. □

3.5 The stellar representation in terms of k-vectors

The construction of the previous section allows us to completely characterize a 2-plane in terms of two constellations. This approach only works for 2-planes. Here we present an alternative procedure to characterize a k-plane in terms of two or more constellations that it is valid for any k. The only drawback of this new approach, is that the calculations to find the new constellations are more complicated. In what follows, we work in the representation of $Gr_k(\mathcal{H}_s)$ in terms of k-vectors introduced in section 1.2.2.

In a natural way, we can define action of the rotation group $SO(3)$ on $\wedge^k(\mathcal{H}_s)$. To this end, is enough to define it for the following elements,

$$R(|\psi_1\rangle \wedge \cdots \wedge |\psi_k\rangle) \equiv (R|\psi_1\rangle) \wedge \cdots \wedge (R|\psi_k\rangle)\,. \tag{3.5.1}$$

Notice that, if $|\psi_1\rangle \wedge \cdots \wedge |\psi_k\rangle$ represents a plane Π, then, $R(|\psi_1\rangle \wedge \cdots \wedge |\psi_k\rangle)$ represents the plane $R\Pi$. Also of importance is the fact that this action leaves the inner product (1.2.9) invariant, and therefore, this action of $SO(3)$ is unitary.

By considering the infinitesimal version of (3.5.1), we can define the action of the generators of rotations S_i on $\wedge^k(\mathcal{H}_s)$ in the following way,

$$
\begin{aligned}
& S_i(|\psi_1\rangle \wedge \cdots \wedge |\psi_k\rangle) \equiv \\
& \quad (S_i|\psi_1\rangle) \wedge \cdots \wedge |\psi_k\rangle + \cdots + |\psi_1\rangle \wedge \cdots \wedge (S_i|\psi_j\rangle) \wedge \cdots \wedge |\psi_k\rangle \\
& \quad + \cdots + |\psi_1\rangle \wedge \cdots \wedge (S_i|\psi_k\rangle) \,.
\end{aligned} \tag{3.5.2}
$$

In these terms, the eigenstates of S_z are of the following form,

$$
S_z(|s, m_1\rangle \wedge \cdots \wedge |s, m_k\rangle) = (m_1 + \cdots + m_k)|s, m_1\rangle \wedge \cdots \wedge |s, m_k\rangle \,.
$$

In particular, the highest weight for S_z is $s_1 = s + s - 1 + \cdots + s - (k-1) = ks - \frac{k(k-1)}{2}$, and the highest weight vector is $|s, s\rangle \wedge \cdots \wedge |s, s-(k-1)\rangle$, a k-vector that corresponds to the coherent k-plane $\Pi_{\hat{z}}$. Since the dimension of $\wedge^k(\mathcal{H}_s)$ is $\binom{2s+1}{k}$, in general, the action (3.5.1) is not irreducible. As such, using a well-known theorem of representation theory (c.f. [70]), $\wedge^k(\mathcal{H}_s)$ can be decomposed as the following direct sum,

$$
\wedge^k(\mathcal{H}_s) = \mathcal{H}_{s_1} \oplus \mathcal{H}_{s_2} \oplus \cdots \oplus \mathcal{H}_{s_M} \,,
$$

where each subspace is an irreducible representation of $SO(3)$ and therefore, the state space of a spin s. We also assume w.l.o.g. that $s_i \geq s_{i+1}$.

In these terms, any element v in $\wedge^k(\mathcal{H}_s)$ can be written uniquely as follows,

$$
v = \bigoplus_{i=1}^{M} |\Psi_i\rangle \,, \tag{3.5.3}
$$

where $|\Psi_i\rangle$ is an element of $\mathcal{H}_{s_i}$. By considering the constellation for all the states $|\Psi_i\rangle$, we can assign to v an ordered set of M different constellations (in the case of $s_i = 0$, as a definition, we say that the constellation of $|\Psi_i\rangle$ contains no stars) $\{\mathcal{C}^1, \ldots, \mathcal{C}^M\}$, where $\mathcal{C}^i$ is the constellation of $|\Psi_i\rangle$. Note that if v is rotated, by construction, these constellations rotate by the same rotation.

However, knowing this set of M constellations is not enough to determine the k-vector v, as the following w has assigned the same set of constellations as v,

$$w = \bigoplus_{i=1}^{M} \mu_i |\Psi_i\rangle \,,$$

where μ_i $(i = 1, \ldots, M)$ is an arbitrary complex number. Because of this, one possible way to completely determine a k-vector v, is to choose a representative state for each possible constellation (for arbitrary spin). Then, besides specifying the set of M constellations $\mathcal{C}_i$, one would also need to specify a M-tuple of complex numbers, $(\nu_1, \ldots, \nu_M)$. In this case, v can be written as,

$$v = \bigoplus_{i=1}^{M} \nu_i |\tilde{\Psi}_i\rangle \,, \tag{3.5.4}$$

where $|\tilde{\Psi}_i\rangle$ is the representative state for the constellation $\mathcal{C}^i$.

If we consider a k-vector v that represents a k-plane Π as shown in equation 1.2.10, we can use the previous construction to assign an ordered set of M constellations to k-planes. Note that the particular choice of v is irrelevant, all of them produce the same set. Denote it as $\{\mathcal{C}_\Pi^1, \ldots, \mathcal{C}_\Pi^M\}$ These are the constellations mentioned at the beginning of this section. Note that this construction is very different from the one for the secondary constellation defined in 3.4. In that case, the secondary one has the same number of stars as the primary, and only two constellations were defined. On the other hand, in this case, a k-plane might be assigned more than two constellations (even if $k = 2$), and each of them has different number of stars.

As we check in the following paragraph, $\mathcal{C}_\Pi^1$ is $\mathcal{C}_\Pi$. Notice that the rotations that leave Π invariant also leave each of the constellations invariant. This fact can be used to find the rotational symmetries of a 2-plane. We do this in the following subsection.

In this paragraph, we show that the first constellation $\mathcal{C}_\Pi^1$ is $\mathcal{C}_\Pi$. Let v be a k-vector that represents Π. As already mentioned, the k-vector $|s, s\rangle \wedge \cdots \wedge |s, s-(k-1)\rangle$ is an element of $\mathcal{H}_{s_1}$. Since this space is invariant under rotations, all the states $v_{\hat{\boldsymbol{n}}} = |\hat{\boldsymbol{n}}, s\rangle \wedge \cdots \wedge |\hat{\boldsymbol{n}}, s-(k-1)\rangle$ that represent the coherent k-plane $\Pi_{\hat{\boldsymbol{n}}}$ are also elements in $\mathcal{H}_{s_1}$. As such, the inner product (1.2.9) between v, written as in (3.5.3), and $v_{\hat{\boldsymbol{n}}}$ is,

$$\langle v_{\hat{\boldsymbol{n}}}, v\rangle = \langle v_{\hat{\boldsymbol{n}}}, |\Psi_1\rangle\rangle \,, \tag{3.5.5}$$

where we used the fact that the spaces $\mathcal{H}_{s_i}$ and $\mathcal{H}_{s_j}$ are orthogonal if $i \neq j$. As already noted in (1.2.11), the product $\langle \Pi_{\hat{\boldsymbol{n}}}, \Pi\rangle$ can be computed as $\langle \Pi_{\hat{\boldsymbol{n}}}, \Pi\rangle = |\langle v_{\hat{\boldsymbol{n}}}, v\rangle|$. Therefore, the constellation of Π is given by the zeros (as a function of $\hat{n}$) of the r.h.s. (3.5.5). On the other hand, $|\Psi_1\rangle$ is a spin s_1 state, so its constellation is also given by the zeros of r.h.s. (3.5.5). Therefore, $\mathcal{C}_\Pi$ is equal to the constellation of the state $|\Psi_1\rangle$, $\mathcal{C}^1_\Pi$.

Finally, we use this line of reasoning to find the inner product between two coherent k-planes $\Pi_{\hat{\boldsymbol{m}}}$ and $\Pi_{\hat{\boldsymbol{n}}}$. Since $v_{\hat{\boldsymbol{m}}}$ and $v_{\hat{\boldsymbol{n}}}$ can be treated as states of spin s_1, by equation (1.3.17), their inner product is,

$$\langle \Pi_{\hat{\boldsymbol{m}}}, \Pi_{\hat{\boldsymbol{n}}} \rangle = |\langle v_{\hat{\boldsymbol{m}}}, v_{\hat{\boldsymbol{n}}} \rangle| = \cos^{2s_1} \tfrac{\theta}{2} \ ,$$

where θ denotes the angle between $\hat{m}$ and $\hat{n}$. Therefore, the distance (1.2.5) between $\Pi_{\hat{\boldsymbol{m}}}$ and $\Pi_{\hat{\boldsymbol{n}}}$ is,

$$d(\Pi_{\hat{\boldsymbol{m}}}, \Pi_{\hat{\boldsymbol{n}}}) = \arccos\left(\cos^{2s_1} \tfrac{\theta}{2}\right) \ , \tag{3.5.6}$$

as was previously claimed in (3.1.1).

3.5.1 Examples

In this subsection, we carry out in detail the calculations to find the set of constellations of some of the 2-planes of section 3.3.1.

In this case, we are working with a spin $s = 5/2$ and with 2-planes. Therefore, the dimension of $\wedge^k(\mathcal{H}_s)$ is $\binom{2s+1}{2} = 15$. We proceed to decompose it in terms of the invariant subspaces $\mathcal{H}_{s_i}$.

First, we begin with $\mathcal{H}_{s_1}$. As already mentioned, the 2-vector $|5/2, 5/2\rangle \wedge |5/2, 3/2\rangle$ is an element of $\mathcal{H}_{s_1}$. From here, it is clear that s_1 is equal to $s_1 = 5/2 + 3/2 = 4$. To obtain a basis for $\mathcal{H}_{s_1}$, we apply the operator S_- repeatedly to $|5/2, 5/2\rangle \wedge |5/2, 3/2\rangle$ (S_- acts in $\wedge^k(\mathcal{H}_s)$ via the usual definition, $S_- = S_x - iS_y$, where the action of S_x and S_y is defined in (3.5.2)). To simplify the notation, we denote the elements of this basis simply as $|4, m\rangle$, that is,

$$\mathcal{H}_{s_1} = \text{span}\{|4,4\rangle, |4,3\rangle, \ldots, |4,-3\rangle, |4,-4\rangle\} \, ,$$

where,

$$\begin{aligned} |4,4\rangle &= |5/2, 5/2\rangle \wedge |5/2, 3/2\rangle \, , \\ |4, m-1\rangle &= \frac{1}{\sqrt{4(4+1) - m(m-1)}} S_-(|4, m\rangle) \, . \end{aligned}$$

Then, we proceed to find $\mathcal{H}_{s_2}$. Note that $|4,3\rangle$ is the only eigenstate of S_z with eigenvalue 3 (up to a complex factor). On the other hand, the space of eigenstates of S_z with spin projection 2 is two dimensional, as it is spanned by the vectors $|5/2, 3/2\rangle \wedge |5/2, 1/2\rangle$ and $|5/2, 5/2\rangle \wedge |5/2, -1/2\rangle$. Since $|4, 2\rangle$ is the only eigenstate with eigenvalue 2 in $\mathcal{H}_{s_1}$, the eigenstate of S_z with eigenvalue 2 orthogonal to $|4, 2\rangle$, is an element of another invariant space, $\mathcal{H}_{s_2}$, with $s_2 = 2$. By making the calculations, we find the following expression for $|4, 2\rangle$,

$$|4, 2\rangle = \frac{1}{\sqrt{14}}(\sqrt{5}|5/2, 3/2\rangle \wedge |5/2, 1/2\rangle + 3|5/2, 5/2\rangle \wedge |5/2, -1/2\rangle)\,.$$

Therefore, we have the following expression for $\mathcal{H}_{s_2}$,

$$\mathcal{H}_{s_2} = \mathrm{span}\{|2, 2\rangle, |2, 1\rangle, |2, 0\rangle, |2, -1\rangle, |2, -2\rangle\}\,,$$

where,

$$|2, 2\rangle = \frac{1}{\sqrt{14}}(3|5/2, 3/2\rangle \wedge |5/2, 1/2\rangle - \sqrt{5}|5/2, 5/2\rangle \wedge |5/2, -1/2\rangle)\,,$$

$$|2, m-1\rangle = \frac{1}{\sqrt{2(2+1) - m(m-1)}} S_-(|2, m\rangle)\,.$$

Finally, we find the last invariant subspace, $\mathcal{H}_{s_3}$. Just like with the previous case, we find that the eigenvalue 0 of S_z is triply degenerate, so there is an eigenstate of S_z with eigenvalue 0 in $\mathcal{H}_{s_3}$ and s_3 is zero. This state need to be orthogonal to $|4, 0\rangle$ and $|2, 0\rangle$. Therefore, by doing the calculation we find,

$$\mathcal{H}_{s_3} = \mathrm{span}\{|0, 0\rangle\}\,,$$

where,

$$|0, 0\rangle = \frac{1}{\sqrt{3}}(|5/2, 5/2\rangle \wedge |5/2, -5/2\rangle - |5/2, 4/2\rangle \wedge |5/2, -4/2\rangle + |5/2, 1/2\rangle \wedge |5/2, -1/2\rangle)\,.$$

By adding up the dimensions of $\mathcal{H}_{s_1}$, $\mathcal{H}_{s_2}$ and $\mathcal{H}_{s_3}$, we obtain fifteen, the dimension of $\wedge^k(\mathcal{H}_s)$. Therefore, we have finished the decomposition of $\wedge^k(\mathcal{H}_s)$ in terms of invariant spaces.

In these terms, by taking advantage of the fact that states $|s_i, m\rangle$, $s_i = 4, 2, 0$; $m = -s_i, \ldots, s_i$ constitute an orthonormal basis, we can write any 2-vector v as follows,

$$v = \left(\sum_{m=-4}^{4} \langle 4, m|v\rangle |4, m\rangle\right) \oplus \left(\sum_{m=-2}^{2} \langle 2, m|v\rangle |2, m\rangle\right) \oplus (\langle 0, 0|v\rangle |0, 0\rangle)\,,$$

where $\langle s_i, m|v\rangle$ denotes the inner product $\langle |s_i, m\rangle, v\rangle$. By using equation (1.3.14), we can find the Majorana polynomials of the terms between parenthesis, and therefore, their constellation. These are the constellations $\mathcal{C}^1_\Pi$, $\mathcal{C}^2_\Pi$ and $\mathcal{C}^3_\Pi$. By definition, the constellation for the spin $s = 0$ subspace has no stars.

As a concrete example, we find the constellations of two of the 2-planes of section 3.3.1; the one we described for the first type, Π_1, and the one for the third type Π_3. Recall that the primary constellation for both planes is a double tetrahedron whose vertices are in the directions shown in equations (3.3.2), that Π_1 can be written as,

$$\Pi_1 = \text{span}\{|v_1, v_2, v_4, v_4, v_4\rangle, |v_1, v_2, v_3, v_3, v_3\rangle\}\,, \tag{3.5.7}$$

while Π_3 is,

$$\Pi_3 = \text{span}\{|v_1, v_1, v_2, v_3, v_4\rangle, |v_2, v_2, v_1, v_3, v_4\rangle\}\,. \tag{3.5.8}$$

The constellations for both 2-planes are shown in figure 3.6. For the case of Π_1, $\mathcal{C}^1_{\Pi_1}$ is a double tetrahedron (as already mentioned) and the $\mathcal{C}^2_{\Pi_1}$ is a rectangle. The only rotation that leaves both constellations invariant, is the one by π around the axis $1/\sqrt{3}(\sqrt{2}, 0, 1)$. Therefore, this is the only rotation that leaves Π_1 invariant. This is the same result as the one obtained in section 3.4.

For the case of Π_3, the constellation $\mathcal{C}^2_{\Pi_3}$ is the tetrahedron *dual* to the one of $\mathcal{C}^1_{\Pi_3}$ (the dual tetrahedron is the one such that its vertices are in the antipodal directions to the ones of the original tetrahedron). Notice that there is only one star per direction in $\mathcal{C}^2_{\Pi_3}$. The figure made by a tetrahedron and its dual is known as a *stella octangula*. Since the symmetries of a tetrahedron coincide with the ones of its dual, the rotations that leave Π' invariant are all the symmetries of tetrahedron. This also coincides with the result of section 3.4.

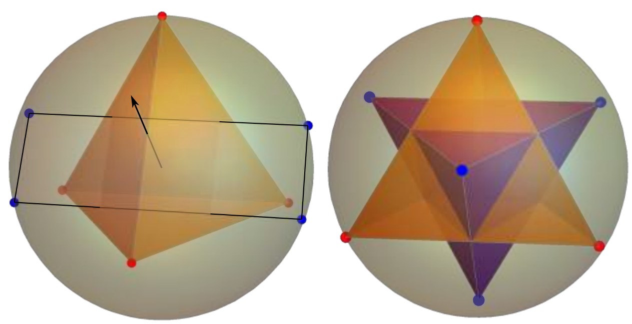


Figure 3.6: Constellation $\mathcal{C}^1_\Pi$ shown in red and $\mathcal{C}^2_\Pi$ shown in blue for the 2-plane Π_1 (3.5.7) in the left and for the 2-plane Π_3 (3.5.8) in the right. In the left figure, the vector shown in black represents the axis of the only rotation that leaves Π_1 invariant.

Chapter 4

Robustness of the Wilczek-Zee effect under external noise

Probably, most of the renewed interest in the Wilczek-Zee effect (c.f. section 1.2.3) comes from its possible applications to quantum computing, the so-called *holonomic quantum computing*. The naive main idea is that, because of their geometrical nature, *quantum gates* realized by means of the Wilczek-Zee effect are robust to external noise. There are many works in the literature that deal with the robustness of the *holonomic quantum computation*. Some of them, deal with the effects of decoherence [71–73] while some others are related with the fact that, however small, there are always errors while implementing experimentally a certain Hamiltonian [74–76]. In this chapter of the thesis, we deal with the latter. Two relevant references are [77, 78] where they studied the robustness of holonomic quantum computation for the abelian case.

First, we explain very briefly the method we used to study the robustness of *holonomic quantum computation*. To make this analysis, we work with nuclear quadrupole resonance (NQR), a simple physical system where the Wilczek-Zee effect can be studied [79, 80]. In NQR, a spin-3/2 particle is coupled to a magnetic field in such a way that the energy spectrum consists in two energies, each with degeneracy two. The 2-planes spanned by the states with the same energy depend only on the direction of the magnetic field. By varying it, appealing to the Wilczek-Zee effect, we can mix states within the same energy level or, in the language of *quantum computing*, we can implement *quantum gates* that operate on the space of a single *qubit*.

Within this framework, suppose that an experimentalist wants to produce a certain *quantum gate*, and to do this, he must manipulate the magnetic field in a certain way. Because of practical limitations, the resulting field is not *exactly* what he wants, there is always certain *noise* beyond his control that produces fluctuations. As a result, the resulting gate is not precisely the one desired. What we are going to do in what remains, is to find a way to quantify this error (the difference between the desired gate and the one obtained), to find its value in average (considering a certain ensemble of noises for the magnetic field) and to find the probability to obtain a certain gate.

4.1 The Wilczek-Zee connection and NQR

In this chapter, we study one of the simplest systems exhibiting the Wilczek-Zee effect, the one for nuclear quadrupole resonance. In this case, a spin-3/2 is coupled to an external magnetic field according to the following Hamiltonian,

$$H(B) = \mu(S \cdot B)^2 \,, \tag{4.1.1}$$

where μ is a coupling constant. The eigenstates of $H(B)$ are trivial to obtain, they are the same as the ones for the operator $S \cdot \hat{B}$. As we have been doing through this thesis, denote by $|\hat{\boldsymbol{B}}, m\rangle$ an eigenstate of $S \cdot \hat{B}$ with eigenvalue m. Clearly, the states $|\hat{\boldsymbol{B}}, \pm m\rangle$ have the same energy, therefore, the spectrum of $H(B)$ consists of two energies,

$$E^{(1/2)} = \frac{\mu}{4} \,, \quad E^{(3/2)} = \frac{9\mu}{4} \,,$$

where the degenerate 2-plane for $E^{(1/2)}$ is $\Pi^{(1/2)}(\hat{B}) = \text{span}\{|\hat{\boldsymbol{B}}, \pm 1/2\rangle\}$, and the one for $E^{(3/2)}$ is $\Pi^{(3/2)}(\hat{B}) = \text{span}\{|\hat{\boldsymbol{B}}, \pm 3/2\rangle\}$. In figure 3.1, we show the constellation for these 2-planes.

An expression for $|\hat{\boldsymbol{B}}, \pm m\rangle$ can be found in appendix , equation (B.0.10). If we denote by (Θ, Φ) the spherical coordinates defining the direction of $\hat{B}$ we have (the states are written as a row vector w.r.t. the ordered basis $|3/2, 3/2\rangle, \ldots, |3/2, -3/2\rangle$),

$$\begin{aligned}
|\hat{\boldsymbol{B}}, \ \ 3/2\rangle &= C_\Theta^3 \quad \left(1, \sqrt{3}e^{i\Phi}T_\Theta, \sqrt{3}e^{2i\Phi}T_\Theta^2, e^{3i\Phi}T_\Theta^3\right), \\
|\hat{\boldsymbol{B}}, \ \ 1/2\rangle &= C_\Theta^3 \quad \left(-\sqrt{3}e^{-i\Phi}T_\Theta, 1 - 2T_\Theta^2, e^{i\Phi}T_\Theta(2 - T_\Theta^2), \sqrt{3}e^{2i\Phi}T_\Theta^2\right), \\
|\hat{\boldsymbol{B}}, -1/2\rangle &= C_\Theta^3 e^{i\Phi}\left(\sqrt{3}e^{-2i\Phi}T_\Theta^2, e^{-i\Phi}T_\Theta(T_\Theta^2 - 2), 1 - 2T_\Theta^2, \sqrt{3}e^{i\Phi}T_\Theta\right), \\
|\hat{\boldsymbol{B}}, -3/2\rangle &= C_\Theta^3 \quad \left(-e^{-3i\Phi}T_\Theta^3, \sqrt{3}e^{-2i\Phi}T_\Theta^2, -\sqrt{3}e^{-i\Phi}T_\Theta, 1\right),
\end{aligned}$$

where T_Θ and C_Θ are defined by the equalities $T_\Theta = \tan(\Theta/2)$ and $C_\Theta = \cos(\Theta/2)$. The phase factor $e^{i\Phi}$ multiplying the r.h.s. of the third equation does not come from (B.0.10), it was added by hand so later calculations become simpler. Clearly, the previous states are ill-defined when $\hat{B}$ points toward the north or south pole. We assume this is never the case.

Now, suppose that the angles Θ and Φ are varied cyclically in time (in a rate that permits invoking adiabaticity). Denote by $\Theta(t)$ and $\Phi(t)$ $(0 \leq t \leq T)$ the time dependence of these angles. Associated to this variation, we have two closed curves in the Grassmannian, $\Pi^{(1/2)}(B(t))$ and $\Pi^{(3/2)}(B(t))$, and each of them induces a unitary transformation in $\Pi^{(1/2)}(\hat{B}(0))$ and in $\Pi^{(3/2)}(\hat{B}(0))$ respectively (via the Wilczek-Zee effect). To find them, we use (1.2.16). To this end, consider the following orthonormal 2-frames for $\pi^{(1/2)}(\hat{B})$ and $\pi^{(3/2)}(\hat{B})$,

$$|\Psi^{(3/2)}(\hat{B})\rangle = (|\hat{\boldsymbol{B}}, 3/2\rangle, |\hat{\boldsymbol{B}}, -3/2\rangle)\,, \quad |\Psi^{(1/2)}(\hat{B})\rangle = (|\hat{\boldsymbol{B}}, 1/2\rangle, |\hat{\boldsymbol{B}}, -1/2\rangle)\,.$$

For the first one, the matrix $A^{(1/2)}(t)$ of equation (1.2.15) turns out to be (after some algebra),

$$A^{(3/2)} = -3\sin^2\Theta\sigma_3\,\dot{\Phi}\,,$$

while, for the second one, the result is

$$A^{(1/2)} = \sigma_2\,\dot{\Theta}\; + \left(-\frac{1}{2}\,\sigma_0 - \sin\Theta\,\sigma_1 + \frac{\cos\Theta}{2}\,\sigma_3\right)\,\dot{\Phi}\,, \tag{4.1.2}$$

where σ_0 denotes the identity matrix and σ_i $(i = 1, .., 3)$, the Pauli matrices, that is,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.$$

In the previous equations, we did not write the functional dependence on t explicitly.

Note that $A^{(3/2)}$ is a diagonal matrix, and as such, it commutes at different times, $[A^{(3/2)}(t_1), A^{(3/2)}(t_2)] = 0$. Therefore, in this case, no matter how the magnetic field is manipulated, the problem is essentially abelian and it reduces to the one studied in [78]. On the other hand, for $A^{(1/2)}$, non-abelian effects can (and will) occur, that is why in the following sections we work only with $A^{(1/2)}$.

4.1.1 Implementing a quantum gate: the ideal case

Perhaps the simplest example that exhibits all the non-abelian features we are interested in is the following. Suppose that we want to implement a *quantum gate* that can be obtained (via the Wilczek-Zee effect) by making the magnetic field B precess around the z axis according to the following equations,

$$\Phi(t) = \omega t\,, \quad \Theta(t) = \Theta_0\,, \quad 0 \le t \le 2\pi/\omega. \tag{4.1.3}$$

with Θ_0 a constant angle. After a time $t = 2\pi/\omega$, the magnetic field comes back to its initial direction and a unitary transformation — the quantum gate we wish to implement — is induced in the initial 2-plane $\Pi^{(1/2)}(\hat{B}(0))$. Now, we find an analytical expression for this gate.

Since the result only depends on the trace of the curve $\Pi^{(1/2)}(B(t))$, we reparametrize $B(t)$ "in terms of the angle Φ itself", the result being the following,

$$\Phi(t) = t\,, \quad \Theta(t) = \Theta_0\,, \quad 0 \le t \le 2\pi. \tag{4.1.4}$$

Note that the parameter t in the previous equation is dimensionless. By using this parametrization, the expression for $A^{(1/2)}$ (4.1.2) turns out to be,

$$A^{(1/2)} = \left(-\frac{1}{2}\,\sigma_0 - \sin\Theta_0\,\sigma_1 + \frac{\cos\Theta_0}{2}\,\sigma_3\right)\,,$$

Since $A^{(1/2)}$ does not depend on t, the solution to the differential equation (1.2.16) is $U(t) = \exp(iA^{(1/2)}\,t)$. By evaluating at $t = 2\pi$, we obtain the expression we seek for the quantum gate U_0,

$$U_I \equiv U(2\pi) = e^{-i\pi} e^{2i\pi\left(-\sin\Theta_0\,\sigma_1 + \frac{\cos\Theta_0}{2}\,\sigma_3\right)} = -e^{2i\pi\left(-\sin\Theta_0\,\sigma_1 + \frac{\cos\Theta_0}{2}\,\sigma_3\right)}\,. \tag{4.1.5}$$

4.1.2 Implementing a quantum gate: the real case

From an experimental point of view, it is impossible to produce a magnetic field that precesses exactly as in (4.1.3), there are always small fluctuations due to noise effects. Because of this, the actual curve followed by the angles Θ and Φ can be written as follows,

$$\Phi(t) = \omega t + \epsilon\tilde{\phi}(t)\,, \quad \Theta(t) = \Theta_0 + \epsilon\tilde{\theta}(t)\,, \quad 0 \le t \le 2\pi/\omega, \tag{4.1.6}$$

where $\epsilon \ll 1$ and $\tilde{\phi}(t)$ and $\tilde{\theta}(t)$ are *stochastic processes* describing the noise effects. To work within the Wilczek-Zee effect framework, just like the authors of [78], we assume that the noise is periodic, $\tilde{\phi}(0) = \tilde{\phi}(2\pi/\omega)$, $\tilde{\theta}(0) = \tilde{\theta}(2\pi/\omega)$. If we do not make this assumption, we have to work with some generalization of the Wilczek-Zee effect for non-closed curves, for instance, the one defined in [66].

Since we are assuming that ϵ is small, we can invert the first equation of (4.1.6) in order to write t in terms of Φ. In this way, we can reparametrize everything in terms of Φ. The result — that is analogous to (4.1.4) — can be written as follows,

$$\Phi(t) = t\,, \quad \Theta(t) = \Theta_0 + \epsilon\theta(t)\,, \quad 0 \le t \le 2\pi\,, \tag{4.1.7}$$

where $\theta(t)$ is a periodic *stochastic process*, $\theta(0) = \theta(2\pi)$. For simplicity, we omit the expression that relates the noise θ with the functions $\tilde{\theta}$ and $\tilde{\phi}$. Instead, we work only with θ itself, ignoring the particulars of how it came to be.

After Φ varies from 0 to 2π, a unitary transformation U_R for the initial degenerate space is induced. Our goal, is to measure the *distance in average* between U_R (the produced gate) and U_I (the actual gate obtained). To this end, we need to define a distance between unitary matrices and to specify the statistics of $\theta(t)$. We do these in sections 4.1.3 and 4.1.4.

But, before that, note that it does not have any physical sense to measure the distance between U_I and U_R if $\theta(0) \neq 0$ as in this case, the quantum gates operate in different spaces. To remedy this situation, we "complete the curve" (4.1.7) so that it starts and ends in the same point as the one for the ideal case (4.1.4) in the following manner,

$$\Theta(t) = \begin{cases} \Theta_0 + \epsilon(t+1)\theta(0), & \text{if } -1 \le t \le 0 \\ \Theta_0 + \epsilon\theta(t), & \text{if } 0 \le t \le 2\pi \\ \Theta_0 + \epsilon(2\pi + 1 - t)\theta(0), & \text{if } 2\pi \le 2\pi + 1 \end{cases}, \tag{4.1.8}$$

$$\Phi(t) = \begin{cases} 0, & \text{if } -1 \le t \le 0 \\ t, & \text{if } 0 \le t \le 2\pi \\ 2\pi, & \text{if } 2\pi \le 2\pi + 1 \end{cases} \tag{4.1.9}$$

In figure 4.1, we show schematically the previous curves in the sphere. In each of them, $A^{(1/2)}$ takes the following form,

$$A^{(1/2)}(t) = \begin{cases} \epsilon\theta(0)\sigma_2, & \text{if } -1 \le t \le 0 \\ A_0 + \epsilon A_1 + \mathcal{O}(\epsilon^2), & \text{if } 0 \le t \le 2\pi \\ -\epsilon\theta(0)\sigma_2, & \text{if } 2\pi \le 2\pi + 1 \end{cases}, \tag{4.1.10}$$

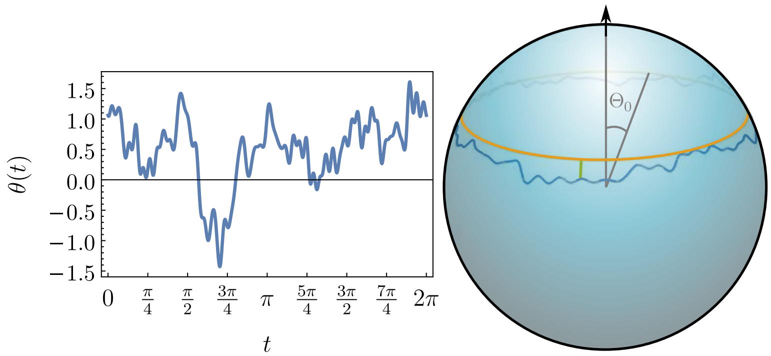


Figure 4.1: Left: An example of a particular realization of the noise $\theta(t)$. Right: The curve followed by the direction of the magnetic field whose spherical coordinates are given by (4.1.7) is shown in blue. For visual clarity, we took the value $\epsilon = 0.1$. The corresponding curve for the ideal case (4.1.4) is shown in orange. The green curve is used to "complete the blue curve" as in (4.1.9), so it beings and ends in the same point as the orange one

where,

$$\begin{aligned} A_0 &= -\frac{1}{2}\sigma_0 - \sin\Theta_0\sigma_1 + \frac{1}{2}\cos\Theta_0\sigma_3 \ , \\ A_1 &= -\cos\Theta_0\theta(t)\sigma_1 + \dot{\theta}(t)\sigma_2 - \frac{1}{2}\sin\Theta_0\theta(t)\sigma_3 \,. \end{aligned} \tag{4.1.11}$$

To solve the equation (1.2.16) for the piecewise defined matrix $A^{(1/2)}$, we can multiply the solution of each segment. Therefore, U_R can be written as,

$$U_R = e^{-i\epsilon\theta(0)\sigma_2} U_c(2\pi) e^{i\epsilon\theta(0)\sigma_2} \,, \tag{4.1.12}$$

where $U_c(t)$ is the solution to (1.2.16) for the middle segment of (4.1.10) and the remaining terms are the holonomies corresponding to the first and last segments (since $A^{(1/2)}$ is independent of t in these cases, the holonomies are trivial to compute).

To find $U_c(2\pi)$, since A_0 in (4.1.11) is time-independent, it is easier to work in the *interaction picture* to apply time-dependent perturbation theory. Suppose that $U_c(t)$ can be written as $U_c(t) = e^{iA_0 t} W(t)$. Then, W satisfies the following differential equation (dropping terms of order $\mathcal{O}(\epsilon^2)$),

$$\dot{W} = i\epsilon \mathbf{A}_1 W \,, \tag{4.1.13}$$

where $\mathbf{A}_1 = e^{-iA_0 t} A_1 e^{iA_0 t}$. Some algebra reveals the expression $\mathbf{A}_1 = v \cdot \sigma$, where $\Omega = \sqrt{1 + 3\sin^2 \Theta_0}$ and,

$$\begin{aligned} v_1(t) &= \frac{\cos\Theta_0}{\Omega^2}\left((1-\Omega^2-\cos(\Omega t))\theta(t) - \Omega\sin(\Omega t)\dot{\theta}(t)\right) , \\ v_2(t) &= \cos(\Omega t)\dot{\theta}(t) - \frac{\sin(\Omega t)}{\Omega}\theta(t) , \\ v_3(t) &= \frac{\sin\Theta_0}{2\Omega^2}\left((4-\Omega^2-4\cos(\Omega t))\theta(t) - 4\Omega\sin(\Omega t)\dot{\theta}(t)\right) . \end{aligned} \tag{4.1.14}$$

With these expressions, we can compute W of equation (4.1.13) up to second order in ϵ through a Magnus expansion [81]. The result is the following,

$$W(2\pi) = e^{i\epsilon \int_0^{2\pi} dt_1\, \mathbf{A}_1(t_1)} . \tag{4.1.15}$$

Using this result along with the definition of W, we obtain the following expression for U_R (4.1.12),

$$U_R = e^{-i\epsilon\theta(0)\sigma_2} e^{2i\pi A_0} W(2\pi) e^{i\epsilon\theta(0)\sigma_2} = U_I e^{\epsilon i\kappa\cdot\sigma} , \tag{4.1.16}$$

where we defined $e^{\epsilon i\kappa\cdot\sigma} = e^{-2i\pi A_0} e^{-i\epsilon\theta(0)\sigma_2} e^{2i\pi A_0} W(2\pi) e^{i\epsilon\theta(0)\sigma_2}$, and noted that $e^{2i\pi A_0}$ is the gate for the ideal case — what we named U_I in (4.1.5). A little bit of algebraic manipulation reveals,

$$\kappa\cdot\sigma \equiv \theta(0) w\cdot\sigma + \int_0^{2\pi} dt_1\, v(t_1)\cdot\sigma , \tag{4.1.17}$$

where,

$$w\cdot\sigma = \frac{\cos\Theta_0 \sin(2\pi\Omega)}{\Omega}\sigma_1 + (1-\cos(2\pi\Omega))\sigma_2 + \frac{2\sin\Theta_0\sin(2\pi\Omega)}{\Omega}\sigma_3 . \tag{4.1.18}$$

Note that κ is linear in θ. Now that we have an expression for the resulting gate, we proceed to find a way to measure distance between gates.

4.1.3 Quantifying the error in the implementation of a quantum gate

In this subsection, we find a way to define a distance between the gate obtained in the ideal case, U_I, and the one obtained in the real case, U_R. When working in the context of quantum computing, it is customary to use the concept of *fidelity* to quantify the distance between two states. For pure states, the fidelity essentially reduces to the Fubini-Study distance (1.1.4). Because of this, fidelity can not distinguish between states that differ by a phase and, therefore, it is too coarse for our purposes. It also has the disadvantage that it measures distance between states, and we are interested in the one between the gates themselves. As such, we need a notation of distance between unitary matrices. There are various proposals in the literature to define such concept (c.f. [82]). For this work, we use the one that we consider the simplest — we give the details in the following paragraph.

As it is well-know , the only bi-invariant metric on $SU(2)$ (unique up to scaling) assigns the following value to the distance between the matrices U and V[83],

$$d(U,V) = \arccos\left(\frac{1}{2}\operatorname{Tr}\left(UV^\dagger\right)\right).$$

Note that the invariance of d implies that the distance between matrices does not depend on the basis used to express them. We can not extend this definition to the whole unitary group $U(2)$ because, in this case, d is no longer a real number. To remedy this situation, we propose the following definition of d for any two elements of $U(2)$,

$$d(U,V) = \left|\arccos\left(\frac{1}{2}\operatorname{Tr}\left(UV^\dagger\right)\right)\right|. \tag{4.1.19}$$

In this way, as it is desirable, the result is still independent of the basis used to express the matrices in question.

Finally, to conclude this section, we compute d for a case that is of interest in the following sections: the distance between an element of the form $e^{ib\cdot\sigma}$ (being b a 3-dimensional vector with a magnitude $|b|$ between 0 and π) and the identity matrix $\mathbb{1}$ is,

$$d(e^{ib\cdot\sigma},\mathbb{1}) = \left|\arccos\left(\frac{1}{2}\operatorname{Tr}\left(e^{ib\cdot\sigma}\right)\right)\right| = |b|, \tag{4.1.20}$$

where we used the well-known relation for the exponential of the Pauli matrices, $e^{ib\cdot\sigma} = \cos|b|\,\mathbb{1} + i\sin|b|\,\hat{b}\cdot\sigma$, to conclude the last equality. With this notion of distance, we can quantify the experimental error when trying to implement the gate U_I for one realization of the noise θ. To find its average, we need to specify the statistics of the noise. We do this in the following subsection.

4.1.4 The statistics of the noise

In this subsection, we mention the statistics to model the noise θ. We try to maintain the results as general as possible. We only assume four properties for the stochastic process θ,

- For each realization, $\theta(t)$ is a periodic function of the parameter $t = \phi$, with period 2π.
- In average (over the ensemble of noises considered) $\theta(t)$ is zero for all $0 \leq t \leq 2\pi$.
- No angle ϕ is privileged, that is, the statistical properties of the system do not depend of ϕ.
- The time evolution of the total magnetic field B is slow enough so that it can be considered adiabatic.

The first property implies that the noise can be written as a Fourier series in the following way,

$$\theta(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \theta_m e^{imt} \,, \tag{4.1.21}$$

where the relation $\theta_m = \theta^*_m$ holds, so that $\theta(t)$ is a real-valued function. In these terms, giving the statistics of the stochastic process θ is equivalent to specifying the ones of the coefficients θ_m, $m \geq 0$.

If we denote the average over the ensemble of noises by an upper line, the second property implies $\overline{\theta(t)} = 0$. The third one guarantees that the correlation function $\overline{\theta(t_1)\theta(t_2)}$ only depends on the difference $t_1 - t_2$, $\overline{\theta(t_1)\theta(t_2)} \equiv R(t_1 - t_2)$. In particular, $\overline{\theta^2(t)} = R(0)$ is independent of t. By differentiating $R(t_1 - t_2)$ w.r.t. t_1 (a dot denotes the derivative of a function w.r.t. its argument), and $\dot{R}(t_1 - t_2)$ w.r.t. t_2, we can compute the following correlations,

$$\dot{R}(t_1 - t_2) = \overline{\dot{\theta}(t_1)\theta(t_2)} \,, \quad \ddot{R}(t_1 - t_2) = -\overline{\dot{\theta}(t_1)\dot{\theta}(t_2)} \,. \tag{4.1.22}$$

As we show in appendix A.1, in terms of the coefficients $\theta_m = \theta^{\Re}_m + i\theta^{\Im}_m$, these properties translate to the following,

$$\overline{\theta^{\Re}_m} = \overline{\theta^{\Im}_m} = 0,\ \overline{\theta^{\Re}_m\theta^{\Im}_n} = 0,\ \overline{(\theta^{\Re}_m)^2} = \overline{(\theta^{\Im}_m)^2} \equiv \rho^2_m,\ \overline{\theta^{\Re}_m\theta^{\Re}_n} = \delta_{m,-n}\,\rho^2_m \,. \tag{4.1.23}$$

By considering the average of these coefficients, we can write R as,

$$R(t_1 - t_2) = \frac{\rho^2_0}{2\pi} + \frac{2}{\pi} \sum_{m=1}^{\infty} \rho^2_m \cos(m(t_1 - t_2)) \,. \tag{4.1.24}$$

Finally, the fourth property implies that the modulus of the coefficients θ_m have to decay *quickly enough*. The precise details are related with the source of the noise and its relative magnitude with the magnetic field.

4.1.5 Finding the average distance between the ideal gate and the real one

Using the results from the previous sections, we can quantify the average error when trying to realize the gate U_I. We define this average error $d_{\rm rms}$ as,

$$d_{\rm rms}^2 = \overline{d^2(U_R, U_I)} = \overline{d^2(e^{i\epsilon\kappa\cdot\sigma}, \mathbb{1})} = \epsilon^2\overline{|\kappa|^2}\,, \tag{4.1.25}$$

where we used (4.1.16), (4.1.19) and (4.1.20) to obtain the last equality. What follows is a long calculation. The details can be found in appendix A.2. Here we just present the final result,

$$\begin{aligned} d_{\rm rms}^2 = \epsilon^2\frac{\Omega^2-1}{2\Omega^2}\int_0^{2\pi} dt\ R(t)\Big(t\Omega^2 + 4(2\pi - t)(\Omega^2-1)\cos(\Omega t) \\ - 4\Omega\sin(\Omega t) + 4\Omega\sin(2\pi\Omega - \Omega t) - 4t - 2\pi\Omega^2 + 8\pi\Big)\,. \end{aligned} \tag{4.1.26}$$

Note that there are no terms involving $\dot\theta$. This is because they were integrated by parts using (4.1.22). Also, note that the result is linear in R. Therefore, by using (4.1.24), we can calculate the integral explicitly. The result is,

$$d_{\rm rms}^2 = \epsilon^2\left(\frac{1}{4}d_{{\rm rms},0}^2\,\rho_0^2 + \sum_{m=1}^{\infty} d_{{\rm rms},m}^2\,\rho_m^2\right)\,,$$

where,

$$d_{{\rm rms},m}^2 = \begin{cases} \frac{2(\Omega^2-1)(4(\Omega^2-1)\sin^2(\pi\Omega)+\pi^2\Omega^2(4-\Omega^2))}{\pi\Omega^4} & \text{if } m=0 \\ \frac{8\left(\Omega^2-1\right)^2\left(m^2+\Omega^2\right)\sin^2(\pi\Omega)}{\pi\Omega^2(m^2-\Omega^2)^2} & \text{if } m>0 \end{cases}\,. \tag{4.1.27}$$

In figure 4.2, we present a plot of $d_{{\rm rms},m}$ as a function of Θ_0 for various values of m. Note that all the plots, except for the one for $m=2$, behave more or less in the same manner. Their value is zero for $\Theta_0 = 0$, $\Theta_0 = \pi/2$, and their amplitude decrease at higher frequencies. This is very similar to the abelian case, where the geometric phase is proportional to the area enclosed by the magnetic field after a full turn.

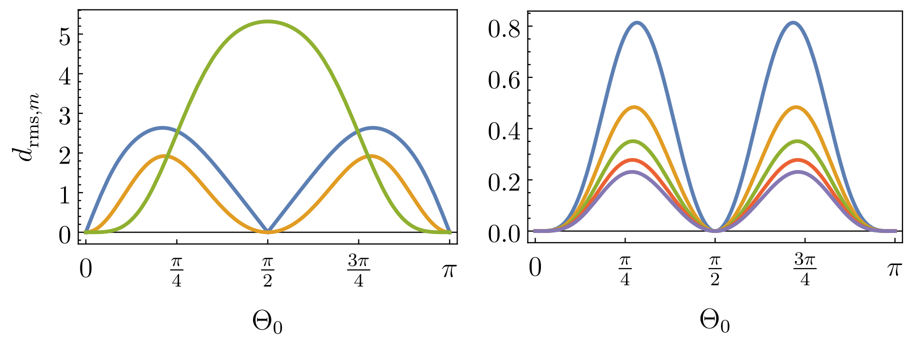


Figure 4.2: Plot of $d^2_{\mathrm{rms},m}$ vs Θ_0 for equation (4.1.27). The values considered for the left graph were $m = 0$ (blue), $m = 1$ (orange), $m = 2$ (green). For the right one, $m = 3$ (blue), $m = 4$ (orange), $m = 5$ (green), $m = 6$ (red) and $m = 7$ (purple). Note the difference in scale between the two graphs.

However, for $m = 2$, the behavior is completely different, the function attains a maximum at $\Theta_0 = \pi/2$ and the scale is bigger than the one for the other ones. This is a non-abelian effect and has to be considered when realizing quantum gates. For example, if an experimentalist wants to use the angle Θ_0 where the error is minimum, the answer for the abelian case would be to consider $\Theta_0 = \pi/2$. For the non-abelian case, this is no longer the case, as the error for this angle is maximal.

4.1.6 Probability distribution for the gates obtained

Finally, to conclude this section, we find an analytical expression for the probability to obtain a certain gate after a single realization of the noise, assuming the distribution of the coefficients $\theta_m^{\Re}$ and $\theta_m^{\Im}$ is Gaussian. As it was shown in (4.1.16), the gate obtained after one experiment is $U_I U^{i\kappa\cdot\sigma}$. If we know the vector[1] κ, we know the gate produced. We could specify κ by its three Cartesian components $(\kappa_x, \kappa_y, \kappa_z)$, but the expressions get considerably simpler if we work with the frame $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ where,

$$\hat{x}' = \cos\eta\hat{x} + \sin\eta\hat{z}\,, \quad \hat{y}' = \hat{y}\,, \quad \hat{z}' = -\sin\eta\hat{x} + \cos\eta\hat{z}\,, \tag{4.1.28}$$

with η being defined by the equality $\cos\eta = \cos\Theta_0/\Omega$. Note that, if we make the following definitions, $\sigma_{1'} \equiv \hat{x}'\cdot\sigma$, $\sigma_{2'} \equiv \hat{y}'\cdot\sigma$ and $\sigma_{3'} \equiv \hat{z}'\cdot\sigma$, then, we

[1] Note that κ does not represent a direction in physical space, it is only related with the basis $|\hat{\boldsymbol{B}}, \pm 3/2\rangle$ of the degenerate space with respect to which the Pauli matrices σ_i are defined.

can write A_0 of equation (4.1.11) simply as,

$$A_0 = -\frac{1}{2}\sigma_0 + \frac{\Omega}{2}\sigma_{3'}\,.$$

In these terms, we can find the distribution of κ for a noise $\theta(t)$ (4.1.21) of a single frequency $m \neq 0$,

$$\theta(t) = \frac{1}{\sqrt{2\pi}}(\theta_m e^{imt} + \theta_m^* e^{-imt})\,. \tag{4.1.29}$$

The details of the following calculation can be found in the appendix A.3. In this case, the resulting vector κ^m after a single realization is (the components of κ^m are written w.r.t. the frame (4.1.28)),

$$\kappa_{3'}^m = 0\,, \begin{pmatrix}\kappa_{1'}^m \\ \kappa_{2'}^m\end{pmatrix} = \frac{4\,(\Omega^2-1)\sin(\pi\Omega)}{\sqrt{2\pi}\Omega(\Omega^2-m^2)}\begin{pmatrix}\Omega\cos(\pi\Omega) & m\sin(\pi\Omega) \\ \Omega\sin(\pi\Omega) & -m\cos(\pi\Omega)\end{pmatrix}\begin{pmatrix}\theta_m^{\Re} \\ \theta_m^{\Im}\end{pmatrix}\,. \tag{4.1.30}$$

By inverting the previous equation, we can find the distribution of $\kappa_{1'}^m$ and $\kappa_{2'}^m$, provided we know the one of $\theta_m^{\Re}$ and $\theta_m^{\Im}$. As we already mentioned, we assume their probability is Gaussian. Because of the equations (4.1.23), $\theta_m^{\Re}$ and $\theta_m^{\Im}$ must be statistically independent and their corresponding probability density function is the same, namely,

$$\Pr\left[\theta_m^{\Re} = \alpha\right] = \Pr\left[\theta_m^{\Im} = \alpha\right] = \frac{1}{\sqrt{2\pi}\rho_m}e^{-\frac{\alpha^2}{2\rho_m^2}}\,. \tag{4.1.31}$$

By making the computation, the probability density function for the resulting vector κ^m turns out to be (c.f. appendix A.3),

$$\Pr\left[(\kappa_{1'}^m, \kappa_{2'}^m, \kappa_{3'}^m) = (k_{1'}, k_{2'}, k_{3'})\right] = \frac{m\Omega}{2\pi\tilde{\rho}_m^2}e^{-\frac{1}{2\tilde{\rho}_m^2}(\Omega^2p^2+q^2m^2)}\delta(k_{3'})\,, \tag{4.1.32}$$

where δ denotes the Dirac delta function and,

$$\begin{aligned}\tilde{\rho}_m &= \frac{4m\sin(\pi\Omega)(\Omega^2-1)}{\sqrt{2\pi}(\Omega^2-m^2)}\rho_m \\ p &= \sin(\pi\Omega)k_{1'} - \cos(\pi\Omega)k_{2'}\,, \\ q &= \cos(\pi\Omega)k_{1'} + \sin(\pi\Omega)k_{2'}\,.\end{aligned} \tag{4.1.33}$$

The equation (4.1.32) has been verified by generating random noise according to (4.1.31) and computing the corresponding holonomy numerically. As we

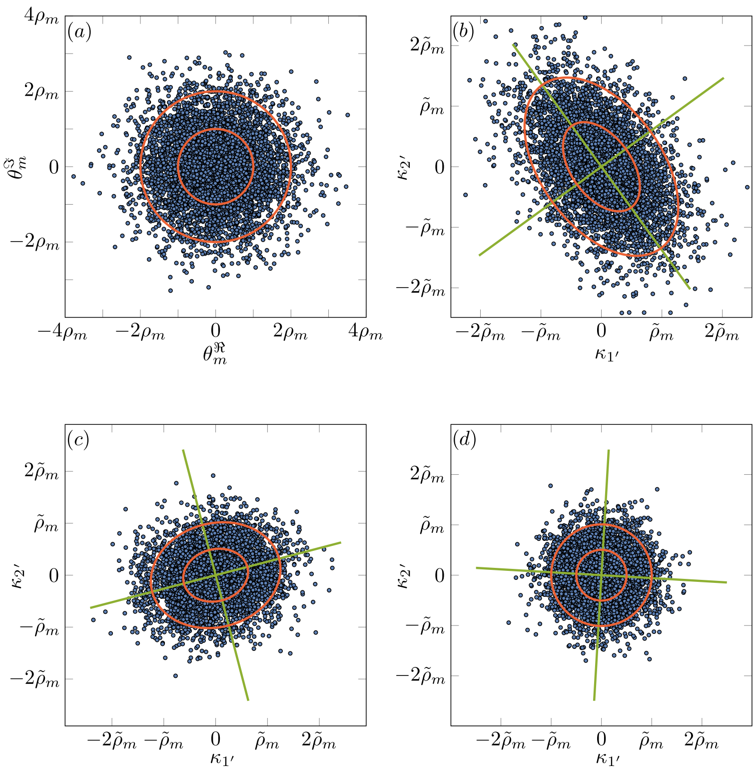


Figure 4.3: Numerical verification of equation (4.1.32). 5000 random realizations of noise of a single frequency (4.1.29) were generated according to the distribution (4.1.31). (a): Representation of the generated noise in the $\theta_m^{\Re}\theta_m^{\Im}$ plane by a blue point. Curves of constant probability density –circles of radius ρ_m and $2\rho_m$– are shown in orange. Since (4.1.31) is a 2d Gaussian distribution, roughly 68.2% of the points are inside the inner circle and 86.4%, inside the outer one. For each point, the holonomy for this noise was computed numerically taking $\epsilon = 0.001$ and $m = 2$. (b)-(d): Representation of the vector κ obtained after each realization in the $\kappa_{1'}\kappa_{2'}$ plane for different values of Θ_0. The orange ellipses are curves of constant probability density according to (4.1.32). As predicted by (4.1.32), for each graph, around 68% of the points are inside the inner ellipse and around 86.4%, inside the outer one. The values for Θ_0 considered were (b) $\Theta_0 = \pi/8$, (c) $\Theta_0 = \pi/4$ and (d) $\Theta = 9\pi/20$. For other values of m, $m \neq 0$, the plots look more or less the same.

can see in figure 4.3, there is an excellent agreement between the numerical results and the analytical expression.

Now, we consider a noise where only the zero mode is present,

$$\theta(t) = \frac{1}{\sqrt{2\pi}}\theta_0^{\Re}\,. \tag{4.1.34}$$

In this case, the resulting vector $\kappa = \kappa^0$ for a single realization is,

$$\begin{aligned}
\kappa^0_{1'} &= \frac{\sin(2\pi\Omega)(\Omega^2-1)}{\sqrt{2\pi}\Omega^2}\theta_0^{\Re}\,,\\
\kappa^0_{2'} &= \frac{2\sin^2(\pi\Omega)(\Omega^2-1)}{\sqrt{2\pi}\Omega^2}\theta_0^{\Re}\,,\\
\kappa^0_{3'} &= \frac{3\sqrt{2\pi}\sin(2\Theta_0)}{4\Omega}\theta_0^{\Re}\,.
\end{aligned} \tag{4.1.35}$$

Note that all the possible vectors κ^0 that can be obtained for different realizations of the noise lie on a line. If we assume the probability distribution for θ_0 is like in (4.1.31), the resulting density for κ^0 is given by the following expression,

$$\Pr\left[(\kappa^0_{1'},\kappa^0_{2'},\kappa^0_{3'}) = (k_{1'},k_{2'},k_{3'})\right] = \delta\left(k_{1'}-\mu k_{3'}\right)\delta\left(k_{2'}-\nu k_{3'}\right)\frac{1}{\sqrt{2\pi}\tilde{\rho}_0}e^{-\frac{k^2_{3'}}{2\tilde{\rho}_0^2}}\,, \tag{4.1.36}$$

where,

$$\mu = \frac{2\sin(2\pi\Omega)(\Omega^2-1)}{3\pi\Omega\sin(2\Theta_0)},\ \nu = \frac{4\sin^2(\pi\Omega)(\Omega^2-1)}{3\pi\Omega\sin(2\Theta_0)},\ \tilde{\rho}_0 = \frac{3\sqrt{2\pi}\sin(2\Theta_0)}{4\Omega}\rho_0,$$

Finally, by considering the previous cases, we can compute the probability distribution for a general noise (4.1.21). As noted right after (4.1.17), κ is linear in θ. Therefore, for the general case,

$$\kappa = \sum_{m=0}^{\infty}\kappa^m\,. \tag{4.1.37}$$

Since the vectors κ^m are linearly independent, we can calculate the distribution of their sum. The details are in appendix A.4. The result is,

$$\Pr\left[(\kappa_{1'},\kappa_{2'},\kappa_{3'}) = (k_{1'},k_{2'},k_{3'})\right] = \frac{1}{(2\pi)^{3/2}\lambda_1\lambda_2\tilde{\rho}_0}e^{-\frac{1}{2}\left(\frac{u^2}{\lambda_1^2}+\frac{v^2}{\lambda_2^2}+\frac{k^2_{3'}}{\tilde{\rho}_0^2}\right)}\,, \tag{4.1.38}$$

where,

$$u = \sin(\pi\Omega)(k_{1'} - \mu k_{3'}) - \cos(\pi\Omega)(k_{2'} - \nu k_{3'}) \,,$$
$$v = \cos(\pi\Omega)(k_{1'} - \mu k_{3'}) + \sin(\pi\Omega)(k_{2'} - \nu k_{3'}) \,,$$

and,

$$\lambda_1 = \frac{1}{\Omega}\sqrt{\sum_{m=1}^{\infty} \tilde{\rho}_m^2} \,, \qquad \lambda_2 = \sqrt{\sum_{m=1}^{\infty} \frac{\tilde{\rho}_m^2}{m^2}} \,,$$

that is, the probability distribution of the gates obtained is also Gaussian.

Conclusions

In this thesis, we used the Majorana representation to study three related yet different problems. Each one of them is a line of investigation that it is worth exploring further.

For chapter 2, most of the mathematical work is already done; we have already found simple enough expressions to compute the Ricci scalar of the horizontal metric, $R(g)$ and the coefficient for the curvature $\mathrm{Tr}\,(\Omega^2)$. What remains, it is to find a clearer physical meaning of these quantities. In particular, it might be interesting to study their relationship with the degree of anticoherence of a state or with their degree of entanglement. It would also be very interesting to describe quantum mechanics in terms of shape space, as it is being done for instance, in general relativity. One of the objectives that we have not been able to fulfill, is to find an action that yields the metric g as a solution to the corresponding equations.

In what concerns chapter 3, we are mostly on uncharted territory. We have just introduced the stellar representation for the Grassmannian, so there are many directions to explore. For instance, one could try to find the physical relation between two k-planes with the same constellation, or try to shed some light on why the number of k-planes with the same constellation is the one shown in theorem 17. Also of interest, is to check if the conjecture 24 holds, or to generalize theorems 13-15 for an arbitrary degeneration of a star in the constellation of a plane. In addition to this, we are currently looking for a way to describe a k-plane in terms of mathematical objects with rotational properties easy to visualize. The results of section 3.5 are very close to what we are looking for, but it has the disadvantage that the M-tuple $(\nu_1, \ldots, \nu_M)$ of equation (3.5.4) transforms in a complicated way when a k-plane is rotated.

Regarding chapter 4, one could try to extent the results of section 1.2.3 to consider the *quantum* nature of the magnetic field. We have made a similar analysis for the abelian case in [84], but the generalization to the non-abelian case is far from straightforward. In particular, difficulties arise when one tries to assign a geometric phase to the subsystem of the spin, as the full quantum system also includes the magnetic field. We are also interested in revisiting the calculations of said chapter to see if they can be done in a simpler way; in particular, it would be useful to understand why there is a sort of resonance for the frequency $m = 2$, as this kind of resonances are the ones an experimentalist would want to avoid when implementing a quantum gate.

We do not claim that we will find the answers to all the question posed above, but we think they have interesting answers. Currently, we are working mostly in the questions for chapter 3.

Appendices

Appendix A

Calculations of chapter 4

In this appendix we carry out some of the calculations of chapter 4.

A.1 Statistics of the coefficients $\theta_m^{\Re}$ and $\theta_m^{\Im}$

In this section we prove formula (4.1.23), that is, if θ denotes a stochastic process as in (4.1.21), and the correlation function $\overline{\theta(t_1)\theta(t_2)}$ only depends on the difference $t_1 - t_2$, then, the following relations hold,

$$\overline{\theta_m^{\Re}} = \overline{\theta_m^{\Im}} = 0,\ \overline{\theta_m^{\Re}\theta_n^{\Im}} = 0,\ \overline{(\theta_m^{\Re})^2} = \overline{(\theta_m^{\Im})^2} \equiv \rho_m^2,\ \overline{\theta_m^{\Re}\theta_n^{\Re}} = \delta_{m,-n}\,\rho_m^2\,. \quad (4.1.23)$$

where we wrote θ_m in terms of its real an imaginary parts, $\theta_m = \theta_m^{\Re} + i\theta_m^{\Im}$.

By making the calculation explicitly we have,

$$\overline{\theta(t_1)\theta(t_2)} = \frac{1}{2\pi}\sum_{m,n=-\infty}^{\infty}\overline{\theta_m\theta_n}e^{imt_1}e^{int_2} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{imt_1}\sum_{n=-\infty}^{\infty}\overline{\theta_m\theta_n}e^{int_2}\,. \tag{A.1.1}$$

By hypothesis, the previous result can be written as a function R of $t_1 - t_2$ solely. Clearly R can also be expanded in terms of Fourier modes,

$$\overline{\theta(t_1)\theta(t_2)} = R(t_1 - t_2) = \frac{1}{2\pi}\sum_{m=\infty}^{\infty}c_m e^{im(t_1-t_2)} = \frac{1}{2\pi}\sum_{m=\infty}^{\infty}e^{imt_1}c_m e^{-imt_2}\,.$$

By the uniqueness of the coefficients in a Fourier decomposition, by comparing the previous equation with (A.1.1) and considering the coefficient for e^{imt_1}, we can conclude the following,

$$\sum_{n=-\infty}^{\infty}\overline{\theta_m\theta_n}e^{int_2} = c_m e^{-imt_2}\,.$$

Again, by appealing to the uniqueness of the Fourier expansion, we see that the following expression holds,

$$\overline{\theta_m \theta_n} = c_m \delta_{m,-n} \,.$$

Now, suppose that m and n are such that $m \neq n$, $m \neq -n$. By writing this coefficients in terms of their real and imaginary parts, $\theta_m = \theta_m^{\Re} + i\theta_m^{\Im}$, we conclude from $\overline{\theta_m \theta_n} = 0$ the following equalities,

$$\overline{\theta_m^{\Re}\theta_n^{\Re}} - \overline{\theta_m^{\Im}\theta_n^{\Im}} = 0\,, \quad \overline{\theta_m^{\Im}\theta_n^{\Re}} + \overline{\theta_n^{\Im}\theta_m^{\Re}} = 0\,.$$

Now, by considering the expression $\overline{\theta_m \theta_{-n}} = 0$ and recalling that $\theta_{-n} = \theta_n^*$ (as noted right after (4.1.21)) we obtain,

$$\overline{\theta_m^{\Re}\theta_n^{\Re}} + \overline{\theta_m^{\Im}\theta_n^{\Im}} = 0\,, \quad \overline{\theta_m^{\Im}\theta_n^{\Re}} - \overline{\theta_n^{\Im}\theta_m^{\Re}} = 0\,.$$

By combining the previous four equalities, we can conclude that the variables involved are statistically independent,

$$\overline{\theta_m^{\Re}\theta_n^{\Re}} = \overline{\theta_m^{\Im}\theta_n^{\Im}} = \overline{\theta_m^{\Im}\theta_n^{\Re}} = \overline{\theta_n^{\Im}\theta_m^{\Re}} = 0\,.$$

Finally, the expression $\overline{\theta_m \theta_m} = 0$ implies,

$$\overline{(\theta_m^{\Re})^2} = \overline{(\theta_m^{\Im})^2}\,, \quad \overline{\theta_m^{\Re}\theta_n^{\Im}} = 0\,.$$

The previous six equalities implies (4.1.23), provided we define ρ_m as the standard deviation of the variable $\theta_m^{\Re}$, $\rho_m^2 = \overline{(\theta_m^{\Re})^2}$.

A.2 Calculation of the average distance $d^2_{\mathbf{rms}}$

In this section, we present the calculations that leads to the expression (4.1.26) for d^2_{rms}. From (4.1.25), we know we can write d^2_{rms} in terms of the vector κ as $d^2_{\text{rms}} = \epsilon^2\overline{|\kappa|^2}$. By considering (4.1.17), we can compute κ as,

$$\kappa = \theta(0)w + \int_0^{2\pi} v(t_1) \equiv \theta(0)w + P\,. \tag{A.2.1}$$

Therefore, the square of the norm of κ is, in average,

$$\overline{|\kappa|^2} = \overline{\theta(0)^2}|w|^2 + \overline{|P|^2} + 2\overline{\theta(0)P}\cdot w\,. \tag{A.2.2}$$

The calculation of $|w|^2$ is straight-forward from equation (4.1.18),

$$\overline{\theta(0)^2}|w|^2 = 4\sin^2(\pi\Omega)R(0)\,. \tag{A.2.3}$$

Now we make the one for $\overline{|p|^2}$. By using the definition of P (A.2.1) we conclude,

$$\overline{|P|^2} = \overline{\left(\int_0^{2\pi} dt\, v(t)\right) \cdot \left(\int_0^{2\pi} dt\, v(t)\right)} = \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2\, \overline{v(t_1) \cdot v(t_2)}\,. \tag{A.2.4}$$

By using the definition of v (4.1.14), we obtain after some algebra,

$$\overline{v(t_1) \cdot v(t_2)} = \left(\frac{64\cos(\Delta\Omega) - 12\Omega^4 + 63\Omega^2 - 51}{64\Omega^2}\right) R(\Delta) + \frac{2}{\Omega}\sin(\Delta\Omega)\dot{R}(\Delta) - \cos(\Delta\Omega)\ddot{R}(\Delta)\,, \tag{A.2.5}$$

where we defined $\Delta = t_1 - t_2$, and used (4.1.22) to express the average of products involving θ and $\dot{\theta}$. Note that the previous expression only depends on the difference Δ, and furthermore, it is even in Δ (because $R(t)$ (4.1.24) is even). Because of this, the value of the integral (A.2.4) is twice the one for the integral over the triangle contained in the portion of the square $[0, 2\pi] \times [0, 2\pi]$ (in the $t_1 t_2$ plane) where $t_2 \leq t_1$, that is ,

$$\int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2\, \overline{v(t_1) \cdot v(t_2)} = 2\int_{\triangle} dt_1 dt_2\, \overline{v(t_1) \cdot v(t_2)} = 2\int_0^{2\pi} d\Delta \int_{\sigma} dt_1\, \overline{v(t_1) \cdot v(t_2)}$$
$$= 2\int_0^{2\pi} d\Delta\, (2\pi - \Delta)\overline{v(t_1) \cdot v(t_2)}\,.$$

Integrating by parts, we can get rid of the derivatives of R in the integrand (A.2.5). Most of the resulting boundary terms are zero; this comes from the equalities $R(0) = R(2\pi)$ (because R is periodic) and $\dot{R}(0) = \dot{R}(2\pi)$ (since $\dot{R}$ is an even function). The result is,

$$\overline{|P|^2} = \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2\, \overline{v(t_1) \cdot v(t_2)} = 4\sin^2(\pi\Omega)R(0) + Q\,, \tag{A.2.6}$$

where Q is given by the following integral,

$$Q = \frac{(1-\Omega^2)}{2\Omega^2}\int_0^{2\pi} d\Delta \Big(8\Omega\sin(\Delta\Omega) + (2\pi - \Delta)\left(4(\Omega^2 - 1)\cos(\Delta\Omega) + 4 - \Omega^2\right)\Big)R(\Delta)\,. \tag{A.2.7}$$

Now, we compute the vector $\overline{\theta(0)p}$ of equation (A.2.2). By recalling the relations $\overline{\theta(0)\theta(t)} = R(t)$ and (4.1.14) we conclude,

$$\overline{\theta(0)P_1} = \frac{\cos\Theta_0}{\Omega^2}\left(\int_0^{2\pi} dt(1-\Omega^2-\cos(\Omega t))R(t) - \Omega\int_0^{2\pi} dt\dot{R}(t)\sin(\Omega t)\right),$$

$$\overline{\theta(0)P_2} = -\frac{1}{\Omega}\int_0^{2\pi} dt\, R(t)\sin(\Omega t) + \int_0^{2\pi} dt\,\dot{R}(t)\cos(\Omega t),$$

$$\overline{\theta(0)P_3} = \frac{\sin\Theta_0}{2\Omega^2}\left(\int_0^{2\pi} dt\,\left(4-\Omega^2-4\cos(\Omega t)\right)R(t) - 4\Omega\int_0^{2\pi} dt\,\dot{R}(t)\sin(\Omega t)\right).$$

Just as with the calculation of $\overline{|P|^2}$, we can integrate parts the terms involving $\dot{R}$. By making this, and calculating the dot product with w we obtain:

$$2w\cdot\overline{\theta(0)P} = -8\sin^2(\pi\Omega)R(0) + \frac{4\left(\Omega^2-1\right)}{\Omega}\sin(\pi\Omega)\int_0^{2\pi} dt\,\cos((\pi-t)\Omega)R(t). \tag{A.2.8}$$

Finally, by substituting (A.2.3), (A.2.6) and (A.2.8) in (A.2.2) we get the final result,

$$d_{\rm rms}^2 = \epsilon^2\frac{\Omega^2-1}{2\Omega^2}\int_0^{2\pi} dt\; R(t)\left(t\Omega^2 + 4(2\pi-t)(\Omega^2-1)\cos(\Omega t) - 4\Omega\sin(\Omega t) + 4\Omega\sin(2\pi\Omega-\Omega t) - 4t - 2\pi\Omega^2 + 8\pi\right). \tag{4.1.26}$$

Note that the terms proportional to $\sin^2(\pi\Omega)R(0)$, cancelled among each other.

A.3 Calculation of the probability density function for κ^m

In this section, we prove formulas (4.1.32) and (4.1.36). First, we compute the vector κ (4.1.17) for a single realization of a noise with a single frequency $m \neq 0$, (4.1.29). In this case, all the integrals can be computed explicitly.

After making the integration, the result for the vector P (A.2.1) is,

$$P_1 = \Lambda \cos\Theta_0(\theta_m^{\Re}(1-m^2)\Omega\cos(\pi\Omega) + \theta_m^{\Im}(1-\Omega^2)m\sin(\pi\Omega))\,,$$
$$P_2 = \Lambda\Omega(\theta_m^{\Re}(1-m^2)\Omega\sin(\pi\Omega) - \theta_m^{\Im}(1-\Omega^2)m\cos(\pi\Omega))\,,$$
$$P_3 = 2\tan\Theta_0 P_1\,,$$

where $\Lambda = \frac{2\sqrt{2}\sin(\pi\Omega)}{\sqrt{\pi}(m^2-\Omega^2)\Omega^2}$. Also, in this case, $\theta(0) = \sqrt{\frac{2}{\pi}}\theta_m^{\Re}$. By substituting these results in (A.2.1) and projecting in the $\hat{x}'\hat{y}'\hat{z}'$ frame (4.1.28), we obtain the following expression for the vector $\kappa^m = \kappa$,

$$\kappa_{3'}^m = 0\,, \begin{pmatrix}\kappa_{1'}^m \\ \kappa_{2'}^m\end{pmatrix} = \frac{4\,(\Omega^2-1)\sin(\pi\Omega)}{\sqrt{2\pi}\Omega(\Omega^2-m^2)}\begin{pmatrix}\Omega\cos(\pi\Omega) & m\sin(\pi\Omega) \\ \Omega\sin(\pi\Omega) & -m\cos(\pi\Omega)\end{pmatrix}\begin{pmatrix}\theta_m^{\Re} \\ \theta_m^{\Im}\end{pmatrix}. \tag{4.1.30}$$

The previous equation can be easily inverted to express $\theta_m^{\Re,\Im}$ in terms of $\kappa_{1',2'}^m = k_{1',2'}$ as follows,

$$\begin{pmatrix}\theta_m^{\Re} \\ \theta_m^{\Im}\end{pmatrix} = \frac{\sqrt{2\pi}(\Omega^2-m^2)}{4m\,(\Omega^2-1)\sin(\pi\Omega)}\begin{pmatrix}m\cos(\pi\Omega) & m\sin(\pi\Omega) \\ \Omega\sin(\pi\Omega) & -\Omega\cos(\pi\Omega)\end{pmatrix}\begin{pmatrix}k_{1'} \\ k_{2'}\end{pmatrix}. \tag{A.3.1}$$

In terms of p, q and $\tilde{\rho}_m$ (4.1.33), the previous equalities can be written as,

$$\theta_m^{\Re} = \frac{qm\rho_m}{\tilde{\rho}_m}\,, \quad \theta_m^{\Im} = \frac{p\Omega\rho_m}{\tilde{\rho}_m}\,. \tag{A.3.2}$$

Since $\theta_m^{\Re}$ and $\theta_m^{\Im}$ are statistically independent as shown in (4.1.23), the density probability function that the pair $(\kappa_{1'}^m, \kappa_{2'}^m)$ attains the value $(k_{1'}, k_{2'})$ is proportional to the product of the density functions that $\theta_m^{\Re,\Im}$ takes the values shown in (A.3.2), that is,

$$\Pr\left[(\kappa_{1'}^m, \kappa_{2'}^m, \kappa_{3'}^m) = (k_{1'}, k_{2'}, k_{3'})\right] = J \Pr\left[\theta_m^{\Re} = \frac{qm\rho_m}{\tilde{\rho}_m}\right] \Pr\left[\theta_m^{\Im} = \frac{p\Omega\rho_m}{\tilde{\rho}_m}\right],$$

being J the Jacobian of the transformation (A.3.1), $J = \frac{m\Omega\rho_m^2}{\tilde{\rho}_m^2}$. Finally, by substituting the probability density functions for $\theta_m^{\Re,\Im}$ (4.1.31) we obtain,

$$\Pr\left[(\kappa_{1'}^m, \kappa_{2'}^m, \kappa_{3'}^m) = (k_{1'}, k_{2'}, k_{3'})\right] = \frac{m\Omega}{2\pi\tilde{\rho}_m^2} e^{-\frac{1}{2\tilde{\rho}_m^2}(\Omega^2 p^2 + q^2 m^2)}\delta(k_{3'})\,. \tag{4.1.32}$$

The Dirac delta was added since $\kappa_{3'}^m$ is necessarily zero (4.1.30).

The calculation for a noise θ where only the mode $m = 0$ is present (like in (4.1.34)) is essentially the same as the one presented. The result is expressed in (4.1.36).

A.4 Calculation of the probability density function for κ

In this section, we compute the distribution for the vector κ. As argued just before (4.1.37), in the general case, κ can be written as a superposition of the vectors κ^m. In the previous section, we found that their probability density function is Gaussian in p and q for $m \neq 0$. For a given m, the variance for the variable p turned out to be $\tilde{\rho}_m^2/\Omega^2$ and $\tilde{\rho}_m^2/m^2$ for q. Since the probability distribution of the sum of Gaussian variables is also Gaussian, the sum $\sum_{m\neq 0} \kappa^m$ is also Gaussian in p and q, and its variances, λ_1^2 and λ_2^2 respectively, is the sum of the variances of the original Gaussians,

$$\lambda_1^2 = \frac{1}{\Omega^2}\sum_{m=1}^{\infty} \tilde{\rho}_m^2\,, \quad \lambda_2^2 = \sum_{m=1}^{\infty} \frac{\tilde{\rho}_m^2}{m^2}\,.$$

Denote by $K = \sum_{m\neq 0} \kappa^m$. Because the previous observation we have,

$$\begin{aligned}\Pr\left[(K_{1'}, K_{2'}, K_{3'}) = (k_{1'}, k_{2'}, k_{3'})\right] &= \frac{1}{2\pi\lambda_1\lambda_2} e^{-\frac{1}{2}\left(\frac{p^2}{\lambda_1^2}+\frac{q^2}{\lambda_2^2}\right)} \delta(k_{3'}) \\ &\equiv f(p)g(q)\delta(k_{3'}) \qquad \text{(A.4.1)}\end{aligned}$$

Finally, as it is well known in the theory of probability, the distribution of κ, that is the sum of κ^0 with K, is given by the following convolution,

$$\begin{aligned}&\int d^3x' \,\Pr\left[(\kappa^0_{1'}, \kappa^0_{2'}, \kappa^0_{3'}) = (x_{1'}, x_{2'}, x_{3'})\right] \cdot \\ &\qquad \Pr\left[(K_{1'}, K_{2'}, K_{3'}) = (k_{1'} - x_{1'}, k_{2'} - x_{2'}, k_{3'} - x_{3'})\right]\,.\end{aligned}$$

The integration over the variables $x_{1'}$ and $x_{2'}$ is trivial because of the Delta functions in (4.1.36). The result is,

$$\frac{1}{\sqrt{2\pi}\tilde{\rho}_0} \int dx'_3 \; f(\tilde{u})g(\tilde{v}) e^{-\frac{x_{3'}^2}{2\tilde{\rho}_0^2}} \;\delta(k_{3'} - x_{3'})\,,$$

where,

$$\begin{aligned}\tilde{u} &= \sin(\pi\Omega)(k_{1'} - \mu x_{3'}) - \cos(\pi\Omega)(k_{2'} - \nu x_{3'})\,,\\ \tilde{v} &= \cos(\pi\Omega)(k_{1'} - \mu x_{3'}) + \sin(\pi\Omega)(k_{2'} - \nu x_{3'})\,.\end{aligned}$$

By substituting f and g from (A.4.1) and making the integral thanks to the Delta function we obtain the final result,

$$\Pr\left[(\kappa_{1'}, \kappa_{2'}, \kappa_{3'}) = (k_{1'}, k_{2'}, k_{3'})\right] = \frac{1}{(2\pi)^{3/2}\lambda_1\lambda_2\tilde{\rho}_0} e^{-\frac{1}{2}\left(\frac{u^2}{\lambda_1^2}+\frac{v^2}{\lambda_2^2}+\frac{k_{3'}^2}{\tilde{\rho}_0^2}\right)}\,, \qquad \text{(4.1.38)}$$

where u and v are obtained by substituting $x_{3'}$ for $k_{3'}$ in the expressions for $\tilde{u}$ and $\tilde{v}$.

Appendix B

An algorithm to implement rotations

In this appendix, we find a closed expression for the matrix that represents a rotation in the space of a spin s, $\mathcal{H}_s$. The resulting expressions are much more efficient than computing the exponential matrix of the generators (for instance, using the built-in function of *Mathematica*) and were used for most of the thesis. Although they are not as powerful as the ones in [85, 86], they have the advantage that they are relatively easy to implement, and much more simpler to deduce.

As we have already discussed in section 1.3.2, $\mathcal{H}_s$ can be regarded as the space of symmetric states of a system of $2s$ spin-1/2. We work in this representation for the rest of this appendix. To find the matrix that represents a rotation, it is enough to compute the image of the state $|s, m\rangle$ (the standard eigenstate of S_z with eigenvalue m) under this rotation. As already mentioned in that section, in this representation, $|s, m\rangle$ can be written as the following completely symmetric state (see equation (1.3.2)),

$$|s, m\rangle = |\underbrace{\hat{z}, \ldots, \hat{z}}_{s+m}, \underbrace{-\hat{z}, \ldots, -\hat{z}}_{s-m}\rangle\,,$$

that is, $|s, m\rangle$ is the completely symmetric state associated to the one where $s+m$ of the constituent spins are in the state $|\hat{z}^{½}\rangle$, and, the remaining ones, in $|-\hat{z}^{½}\rangle$. Clearly, there are only $T = \binom{2s}{s+m}$ states (in the space of a system of $2s$ spin-1/2) with this property, that is, the sum of (1.3.2) corresponding to $|s, m\rangle$ only involves T different summands. Indeed, if we define N and k by $N = 2s$ and $k = s+m$, it is easy to see that the problem of counting such states reduces to the one of finding the number of ways to place k objects in N boxes, one object per box. Number this states and denote them by $|k; \hat{z}^{1/2}, -\hat{z}^{1/2}\rangle_i$, $i = 1, \ldots, T$. In these terms we have,

$$|s, m\rangle = \frac{1}{\sqrt{T}} \sum_{i=0}^{T} |k; \hat{z}^{1/2}, -\hat{z}^{1/2}\rangle_i \tag{B.0.1}$$

Consider a rotation R around an axis $\hat{n} = (n_1, n_2, n_3)$ by an angle θ. In this space, the matrix representation of R is,

$$D(R) = \underbrace{e^{-\frac{i\alpha}{2}\sigma\cdot\hat{n}} \otimes \cdots \otimes e^{-\frac{i\alpha}{2}\sigma\cdot\hat{n}}}_{2s \text{ times}}, \tag{B.0.2}$$

where σ_i $(i = 1, 2, 3)$ denotes the Pauli matrices. Using the well-known relation for the exponential of the Pauli matrices we have,

$$e^{-\frac{i\alpha}{2}\sigma\cdot\hat{n}} = \cos\left(\tfrac{\alpha}{2}\right) \mathbb{1} - i\sin\left(\tfrac{\alpha}{2}\right) \sigma\cdot\hat{n} \tag{B.0.3}$$

$$= \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) - i\sin\left(\frac{\alpha}{2}\right) n_3 & -i\sin\left(\frac{\alpha}{2}\right)(n_1 - in_2) \\ -i\sin\left(\frac{\alpha}{2}\right)(n_1 + in_2) & \cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right) n_3 \end{pmatrix} \tag{B.0.4}$$

$$\equiv \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}, \tag{B.0.5}$$

from where we can conclude the following equalities,

$$|\psi_+\rangle \equiv e^{-\frac{i\alpha}{2}\sigma\cdot\hat{n}}|\hat{z}^{1/2}\rangle = u|\hat{z}^{1/2}\rangle + v|-\hat{z}^{1/2}\rangle,$$
$$|\psi_-\rangle \equiv e^{-\frac{i\alpha}{2}\sigma\cdot\hat{n}}|-\hat{z}^{1/2}\rangle = -v^*|\hat{z}^{1/2}\rangle + u^*|-\hat{z}^{1/2}\rangle.$$

Now, we find an expression for $D(R)|s, m\rangle$. By applying (B.0.2) to (B.0.1) we get,

$$D(R)|s, m\rangle = \frac{1}{\sqrt{T}} \sum_{i=0}^{T} |k; \psi_+, \psi_-\rangle_i, \tag{B.0.6}$$

where $|k; \psi_+, \psi_-\rangle_i = D(R)|k; \hat{z}^{1/2}, -\hat{z}^{1/2}\rangle_i$ denotes the state obtained by changing the states $|\pm\hat{z}^{1/2}\rangle$ by $|\psi_\pm\rangle$ in the decomposition of $|k; \hat{z}^{1/2}, -\hat{z}^{1/2}\rangle_i$ in terms of its constituent spins. By computing the product between $|s, m'\rangle$ and $D(R)|s, m\rangle$ we obtain,

$$\langle s, m'|D(R)|s, m\rangle = \frac{1}{\sqrt{TT'}} \sum_{i'=0}^{T'} \sum_{i=0}^{T} {}_{i'}\langle k'; \hat{z}^{1/2}, -\hat{z}^{1/2}|k; \psi_+, \psi_-\rangle_i,$$

where k' and T' are defined analogously to k and T. Note that, because of the permutational symmetries of the problem, the result of fixing an i' and making the sum over i is independent of i'. Because of this,

$$\langle s, m'|D(R)|s, m\rangle = \sqrt{\frac{T'}{T}} \sum_{i=0}^{T} {}_1\langle k'; \hat{z}^{½}, -\hat{z}^{½}|k; \psi_+, \psi_-\rangle_i \,, \tag{B.0.7}$$

Up to now, the choice for $|k'; \hat{z}^{½}, -\hat{z}^{½}\rangle_1$ has been arbitrary. For the rest of the calculation, we consider,

$$|k'; \hat{z}^{½}, -\hat{z}^{½}\rangle_1 = |\underbrace{\hat{z}^{½}, \dots, \hat{z}^{½}}_{k'}, \underbrace{-\hat{z}^{½}, \dots, -\hat{z}^{½}}_{N-k'}\rangle \,.$$

Now, we can characterize the summands of (B.0.7). Suppose that i ($1 \le i \le T$) is such that, between the first k' spins-1/2 that constitute $|k; \psi_+, \psi_-\rangle_i$, there are r spins in the state $|\psi+\rangle$ and $k' - r$ in the state $|\psi-\rangle$. Then, for the remaining $N - k'$ spins-1/2, there must be $k - r$ spins in $|\psi+\rangle$ and $N - k - k' + r$ in $|\psi-\rangle$. Therefore, for this particular i, we have,

$$\begin{aligned} {}_1\langle k'; \hat{z}^{½}, -\hat{z}^{½}|k; \psi_+, \psi_-\rangle_i &= u^r(-v^*)^{k'-r} v^{k-r} (u^*)^{N-k-k'+r} \\ &= \eta^{k-r} \eta_A^{r-k'} u^k (u^*)^{N-k} \,, \end{aligned} \tag{B.0.8}$$

where we defined $\eta \equiv \frac{v}{u}$ and $\eta_A \equiv -\frac{1}{\eta^*}$. By finding a suitable r, all the summands of (B.0.7) can be written as in (B.0.8).

Now the question is the following: for a given r, how many times the summand (B.0.8) appears in (B.0.7)? After a little bit of thought, one can see that the answer to this question is equal to the number of ways to distribute k indistinguishable objects in N numbered boxes such that r objects are in the first k' boxes. It is easy to check that this number is $\binom{k'}{r}\binom{N-k'}{k-r}$. Since the arguments of the binomials coefficients involved must be positive, and the lower one must be smaller than the upper one, we can conclude the following inequalities, $k' \ge r$, $k - r \ge 0$ and $N - k' \ge k - r$. That is, the allowed values for r are,

$$\max\{0, k + k' - N\} \le r \le \min\{k, k'\}.$$

Now we calculate (B.0.7). By taking into account all the previous considerations, if we define $r_{\min} = \max\{0, k + k' - N\}$ and $r_{\max} = \min\{k, k'\}$ then, we have,

$$\langle s, m'|D(R)|s, m\rangle = u^k (u^*)^{N-k} \sqrt{\frac{T'}{T}} \sum_{r=r_{\min}}^{r_{\max}} \binom{N-k'}{k-r} \binom{k'}{r} \eta^{k-r} \eta_A^{r-k'} \,.$$

Equivalently, we can write everything in terms of m and s by recalling the expressions $k = m + s$ and $N = 2s$. The result is,

$$\langle s, m'|D(R)|s, m\rangle = \mathcal{N}\sqrt{T'}\sum_{r=r_{\min}}^{r_{\max}}\binom{s-m'}{s+m-r}\binom{s+m'}{r}\eta^{s+m-r}\eta_A^{r-s-m'}\ , \tag{B.0.9}$$

where, $T' = \binom{2s}{s+m'}$, $r_{\min} = \max\{0, m + m'\}$, $r_{\max} = \min\{s + m, s + m'\}$ and $\mathcal{N}$ is the following factor independent of m',

$$\mathcal{N} = u^{s+m}(u^*)^{s-m}\binom{2s}{s+m}^{-\frac{1}{2}}\ .$$

The previous expression determines $D(R)$. As benchmark, we mention that this expression can be used to find analytical expressions for a general rotation for up to spin 100 in a couple of minutes. In contrast, using *Mathematica*'s built-in exponential matrix, one has to struggle to find rotations for spin 10

As an important application of the previous formula, we can calculate the eigenstates of $S\cdot\hat{\omega}$ with projection m, $|\hat{\boldsymbol{\omega}}, m\rangle$. Denote by θ and ϕ the spherical coordinates that characterize $\hat{\omega}$. Then, $|\hat{\boldsymbol{\omega}}, m\rangle$ can be computed by rotating the state $|s, m\rangle$ by an angle $\alpha = \theta$ around the axis $\hat{n} = (-\sin\phi, \cos\phi, 0)$. By considering this observation and comparing with (B.0.5) we obtain,

$$u = u^* = \cos\tfrac{\theta}{2},\ v = \sin\tfrac{\theta}{2}e^{i\phi},\ \eta = \tan\tfrac{\theta}{2}e^{i\phi} = \zeta\ ,$$

where ζ denotes the complex number associated to $\hat{\omega}$ via the stereographic projection. Also note the following,

$$uu^* = u^2 = \frac{1}{1+\zeta\zeta^*}\ .$$

By substituting this expressions in (B.0.9) we obtain finally,

$$\langle s, m'|\hat{\boldsymbol{\omega}}, m\rangle = \mathcal{N}\sqrt{T'}\sum_{r=r_{\min}}^{r_{\max}}\binom{s-m'}{s+m-r}\binom{s+m'}{r}\zeta^{s+m-r}\zeta_A^{r-s-m'}\ , \tag{B.0.10}$$

where

$$\mathcal{N} = \frac{1}{(1+\zeta\zeta^*)^2}\binom{2s}{s+m}^{-\frac{1}{2}}\ .$$

Appendix C

Calculations of chapter 2

C.1 Curvature of a Lie group with a right invariant metric

Suppose we have a Lie group endowed with a metric k that is *right invariant* In this appendix, we compute the curvature (the Ricci scalar) in general for this type of metric. Einstein notation for sum over repeated indices is used throughout this chapter.

Since multiplication by the right is an isometry by hypothesis, the curvature is constant for all the elements of the group, so it is enough to make the calculation at the identity. Let e_α denote an orthonormal basis for the Lie algebra. Denote by $C^\alpha{}_{\beta\gamma}$ the structure constants w.r.t.
to this basis, that is, we have the following equalities,

$$[e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta} e_\gamma \,. \tag{C.1.1}$$

Denote by E_α the right invariant vector field associated with e_α. Because k is right invariant, it is clear that the following holds for all the elements of the group,

$$k(E_\alpha, E_\beta) = \pm\delta_{\alpha\beta} \,. \tag{C.1.2}$$

Denote by φ^α the dual basis of e_α. Also, denote by φ the *canonical Lie-algebra valued form*, defined in the following way,

$$\varphi(X) = R_{g^{-1}*} X \,. \tag{C.1.3}$$

where R_g denotes the action of g in the group by right multiplication (we opted for this notation in this appendix instead of $\triangleleft g$ used in the rest of the thesis so that the resulting expressions become simpler) and X is any vector tangent at g. With this definitions, it is clear the following equality holds,

$$\varphi = e_\alpha \varphi^\alpha \,. \tag{C.1.4}$$

Using the Maurer-Cartan equation [23, Remark 2.2.17] we can write the following,

$$d\varphi + \frac{1}{2}[\varphi, \varphi] = 0 \,. \tag{C.1.5}$$

By substituting the expression for φ (C.1.4) in the previous equality we obtain the following,

$$d\varphi^\alpha + \frac{1}{2} C^\alpha{}_{\beta\gamma} \varphi^\beta \wedge \varphi^\gamma = 0 \,. \tag{C.1.6}$$

From here, we can calculate the Riemann tensor. Given some orthonormal fields E_α and its dual forms φ^α, it is a well known theorem that if we find a matrix of 1-forms $\overline{\theta}^\alpha{}_\gamma$ such that the following equations hold,

$$\begin{aligned} d\varphi^\alpha &= -\overline{\theta}^\alpha{}_\gamma \wedge \varphi^\gamma \,, \\ \overline{\theta}_{\alpha\gamma} + \overline{\theta}_{\gamma\alpha} &= 0 \,. \end{aligned} \tag{C.1.7}$$

then we can calculate the Riemann tensor according to the following equation [23, theorem 6.2.6 and lemma 9.3.9],

$$\frac{1}{2} R^\alpha{}_{\beta\delta\mu} \varphi^\delta \wedge \varphi^\mu = d\overline{\theta}^\alpha{}_\beta + \overline{\theta}^\alpha{}_\gamma \wedge \overline{\theta}^\gamma{}_\beta \,. \tag{C.1.8}$$

In our particular case, by examining equation (C.1.6), it is clear that the following expression satisfies the requirements mentioned in (C.1.7),

$$\overline{\theta}(k)^\alpha{}_\gamma = \frac{1}{2}(C^\alpha_{\beta\gamma} - C_{\gamma\beta}{}^\alpha - C_{\beta\gamma}{}^\alpha)\varphi^\beta \equiv T^\alpha{}_{\beta\gamma}\varphi^\beta \,. \tag{C.1.9}$$

Indeed, if we lower the index α in the previous equation, the resulting term, $\overline{\theta}(k)_{\alpha\gamma}$, is antisymmetric in γ and α. Moreover, since the sum of the last two terms of $T^\alpha{}_{\beta\gamma}$ is symmetric in β and γ, when we contract it with $\varphi^\beta \wedge \varphi^\gamma$, the result is zero; therefore, the first equality in (C.1.7) holds.

From here, the components of the Riemann tensor can be calculated using the equality (C.1.8). By a direct computation, we obtain the following,

$$\begin{aligned} d\overline{\theta}(k)^\alpha{}_\beta &= T^\alpha{}_{\gamma\beta} d\varphi^\gamma = -T^\alpha{}_{\gamma\beta} T^\gamma{}_{\delta\mu} \varphi^\delta \wedge \varphi^\mu \,, \\ \overline{\theta}(k)^\alpha{}_\gamma \wedge \overline{\theta}(k)^\gamma{}_\beta &= T^\alpha{}_{\delta\gamma} T^\gamma{}_{\mu\beta} \varphi^\delta \wedge \varphi^\mu \,, \end{aligned}$$

so that by evaluating (C.1.8) at the fields E_δ and E_μ, we conclude the following expression,

$$R(k)^\alpha{}_{\beta\delta\mu} = -T^\alpha{}_{\gamma\beta}(T^\gamma{}_{\delta\mu} - T^\gamma{}_{\mu\delta}) + T^\alpha{}_{\delta\gamma}T^\gamma{}_{\mu\beta} - T^\alpha{}_{\mu\gamma}T^\gamma{}_{\delta\beta}\,. \tag{C.1.10}$$

Once we have the Riemann tensor, we can calculate the scalar curvature,

$$\begin{aligned} R(k) = R(k)^{\alpha\beta}{}_{\alpha\beta} &= -T^\alpha{}_\gamma{}^\beta(T^\gamma{}_{\alpha\beta} - T^\gamma{}_{\beta\alpha}) + T^\alpha{}_{\alpha\gamma}T^\gamma{}_\beta{}^\beta - T^\alpha{}_{\beta\gamma}T^\gamma{}_\alpha{}^\beta \\ &= T^\gamma{}_{\beta\alpha}T^\alpha{}_\gamma{}^\beta - T^\gamma{}_{\alpha\beta}T^\alpha{}_\gamma{}^\beta + T^\alpha{}_{\alpha\gamma}T^\gamma{}_\beta{}^\beta - T^\alpha{}_{\beta\gamma}T^\gamma{}_\alpha{}^\beta \end{aligned}$$

Note that the last term is the negative of the first one, so that they cancel each other out and we obtain the following,

$$R(k) = T^\alpha{}_{\alpha\gamma}T^\gamma{}_\beta{}^\beta - T^\gamma{}_{\alpha\beta}T^\alpha{}_\gamma{}^\beta \tag{C.1.11}$$

In the following paragraphs, we calculate each term of the previous equation. Using the definition (C.1.9), we have the following equality,

$$T^\alpha{}_{\alpha\gamma} = \frac{1}{2}(C^\alpha_{\alpha\gamma} - C_{\alpha\gamma}{}^\alpha - C_{\gamma\alpha}{}^\alpha) = \frac{1}{2}(C^\alpha_{\alpha\gamma} - C_{\alpha\gamma}{}^\alpha)\,, \tag{C.1.12}$$

where the last term was canceled out because the structure constants are asymmetric in the last two indices. In the same way we have,

$$T^\gamma{}_\beta{}^\beta = \frac{1}{2}(C^\gamma_\beta{}^\beta - C_\beta{}^{\beta\gamma} - C^\beta{}_\beta{}^\gamma) = -C_\beta{}^{\beta\gamma}\,, \tag{C.1.13}$$

where the first term was canceled by the asymmetry of the structure constants and we noted that the second term is equal to the third one. By multiplying the last two equalities, we obtain the following result,

$$T^\alpha{}_{\alpha\gamma}T^\gamma{}_\beta{}^\beta = \frac{1}{2}C_{\alpha\gamma}{}^\alpha C_\beta{}^{\beta\gamma} - \frac{1}{2}C^\alpha{}_{\alpha\gamma}C_\beta{}^{\beta\gamma}\,. \tag{C.1.14}$$

Now, we calculate the remaining term of (C.1.11). In this case we have,

$$T^\gamma{}_{\alpha\beta}T^\alpha{}_\gamma{}^\beta = T_{\gamma\alpha\beta}T^{\alpha\gamma\beta} = \frac{1}{2}T_{\gamma\alpha\beta}(C^{\alpha\gamma\beta} - C^{\gamma\beta\alpha} - C^{\beta\gamma\alpha}) = \frac{1}{2}T_{\gamma\alpha\beta}C^{\alpha\gamma\beta}\,, \tag{C.1.15}$$

where we use the fact that $T_{\gamma\alpha\beta}$ is antisymmetric in β and γ while the sum of the last two terms is symmetric.

Going on with the calculation, we obtain the following,

$$\frac{1}{2}T_{\gamma\alpha\beta}C^{\alpha\gamma\beta} = \frac{1}{4}(C_{\gamma\alpha\beta} - C_{\alpha\beta\gamma} - C_{\beta\alpha\gamma})C^{\alpha\gamma\beta}$$

$$
\begin{aligned}
&= \frac{1}{4} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma} - \frac{1}{4} C_{\gamma\alpha\beta} C^{\alpha\beta\gamma} + \frac{1}{4} C_{\beta\alpha\gamma} C^{\alpha\beta\gamma} \\
&= \frac{1}{4} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma} - \frac{1}{4} C_{\gamma\alpha\beta} C^{\alpha\beta\gamma} - \frac{1}{4} C_{\beta\alpha\gamma} C^{\alpha\gamma\beta} \\
&= \frac{1}{4} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma} - \frac{1}{2} C_{\gamma\alpha\beta} C^{\alpha\beta\gamma} \,, \qquad \text{(C.1.16)}
\end{aligned}
$$

where, again, we used the fact that the coefficients C are antisymmetric in the last two indices to conclude the last equality.

Finally, by using the previous equation and (C.1.14) in (C.1.11), we obtain the following result for the scalar curvature,

$$
\begin{aligned}
R(k) &= \frac{1}{2} C_{\alpha\gamma}{}^{\alpha} C_{\beta}{}^{\beta\gamma} - \frac{1}{2} C^{\alpha}{}_{\alpha\gamma} C_{\beta}{}^{\beta\gamma} - \frac{1}{4} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma} + \frac{1}{2} C_{\gamma\alpha\beta} C^{\alpha\beta\gamma} \\
&= C_{\alpha\gamma}{}^{\alpha} C_{\beta}{}^{\beta\gamma} - \frac{1}{4} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma} + \frac{1}{2} C_{\gamma\alpha\beta} C^{\alpha\beta\gamma} \,. \qquad \text{(C.1.17)}
\end{aligned}
$$

In the next section, we compute this scalar for the rotation group $SO(3)$.

C.1.1 The case of $SO(3)$

Consider the Lie algebra of $SO(3)$, and denote by S_i the standard generators of rotations. Suppose that, for a certain right orthonormal frame (of $\mathbb{R}^3$) $\hat{m}$, $\hat{n}$ and $\hat{p}$, the metric k has the following expression w.r.t. the basis $\{S \cdot \hat{m}$, $S \cdot \hat{n}$, $S \cdot \hat{p}\}$ (in the next section we check that this is indeed the case for the metric k we are interested in),

$$
k = 2 \begin{pmatrix} \Delta S_1^2 & 0 & 0 \\ 0 & \Delta S_2^2 & 0 \\ 0 & 0 & \Delta S_3^2 \end{pmatrix} \qquad \text{(C.1.18)}
$$

Define e_1 according to the following equation, $e_1 = S \cdot \hat{m}/(\sqrt{2}\Delta S_1)$. Also, define e_2 and e_3 in a similar fashion. Clearly, by construction, the matrix representation of k respect to the basis $\{e_1, e_2, e_3\}$ is the identity matrix. This implies that raising and lowering indices in this case is trivial.

Using the well known commutation relationships for $S \cdot \hat{m}$, $S \cdot \hat{n}$ and $S \cdot \hat{p}$, we obtain the following (no sum over α and β implied),

$$
[e_\alpha, e_\beta] = i \sum_\gamma \left(\frac{\Delta S_\gamma}{\sqrt{2}\Delta S_\alpha \Delta S_\beta} \right) \epsilon^\gamma{}_{\alpha\beta} e_\gamma \,, \qquad \text{(C.1.19)}
$$

where $\epsilon^\gamma{}_{\alpha\beta}$ denotes the components of the Levi-Civita tensor. From here we can obtain the structure constants (again, there is no sum implied),

$$
C^\gamma{}_{\alpha\beta} = \left(\frac{\Delta S_\gamma}{\sqrt{2}\Delta S_\alpha \Delta S_\beta} \right) \epsilon^\gamma{}_{\alpha\beta} \,, \qquad \text{(C.1.20)}
$$

By using the previous equation and (C.1.17), we can calculate the curvature in this case. Note that, because of the asymmetric properties of the Levi-Civita tensor, the first two terms of (C.1.17) are zero. Now, we calculate the remaining terms. We have,

$$\begin{aligned} C_{\alpha\beta\gamma}C^{\alpha\beta\gamma} &= C_{123}C^{123} + C_{132}C^{132} + C_{231}C^{231} + C_{213}C^{213} \\ &\quad + C_{321}C^{321} + C_{312}C^{312} \\ &= 2C_{123}C^{123} + 2C_{231}C^{231} + 2C_{321}C^{321} \\ &= \frac{\Delta S_1^2}{\Delta S_2^2 \Delta S_3^2} + \frac{\Delta S_2^2}{\Delta S_1^2 \Delta S_3^2} + \frac{\Delta S_3^2}{\Delta S_2^2 \Delta S_1^2}\,, \end{aligned}$$

while, for the last term, we have the following,

$$\begin{aligned} C_{\gamma\alpha\beta}C^{\alpha\beta\gamma} &= C_{123}C^{231} + C_{132}C^{321} + C_{213}C^{132} + C_{231}C^{312} + C_{312}C^{123} + C_{321}C^{213} \\ &= \frac{1}{\Delta S_3^2} + \frac{1}{\Delta S_2^2} + \frac{1}{\Delta S_1^2}\,. \end{aligned}$$

Putting everything together in (C.1.17) we obtain,

$$R(k) = \frac{1}{2\Delta S_3^2} + \frac{1}{2\Delta S_2^2} + \frac{1}{2\Delta S_1^2} - \frac{\Delta S_1^2}{4\Delta S_2^2 \Delta S_3^2} - \frac{\Delta S_2^2}{4\Delta S_1^2 \Delta S_3^2} - \frac{\Delta S_3^2}{4\Delta S_2^2 \Delta S_1^2} \tag{C.1.21}$$

After some simplification, the result is,

$$R(k) = \frac{2(\Delta S_1^2 \Delta S_2^2 + \Delta S_1^2 \Delta S_3^2 + \Delta S_2^2 \Delta S_3^2) - (\Delta S_1^4 + \Delta S_2^4 + \Delta S_3^4)}{4\Delta S_1^2 \Delta S_2^2 \Delta S_3^2}\,. \tag{C.1.22}$$

But, by looking at the expression for the matrix k (C.1.18) (the matrix written w.r.t. the basis $\{S\cdot\hat{m}, S\cdot\hat{n}, S\cdot\hat{p}\}$), we can note the following,

$$\begin{aligned} \mathrm{Tr}(k) &= 2(\Delta S_1^2 + \Delta S_2^2 + \Delta S_3^2)\,, \\ \mathrm{Tr}(k^2) &= 4(\Delta S_1^4 + \Delta S_2^4 + \Delta S_3^4)\,, \\ (\mathrm{Tr}(k))^2 - \mathrm{Tr}(k^2) &= 8(\Delta S_1^2 \Delta S_2^2 + \Delta S_1^2 \Delta S_3^2 + \Delta S_2^2 \Delta S_3^2) \\ \mathrm{Det}(k) &= 8(\Delta S_1^2 \Delta S_2^2 \Delta S_3^2)\,, \end{aligned}$$

So that the expression for R in terms of the matrix k is the following,

$$R(k) = \frac{(1/4)\{(\mathrm{Tr}(k))^2 - \mathrm{Tr}(k^2)\} - (1/4)\mathrm{Tr}(k^2)}{(1/2)\mathrm{Det}(k)} = \frac{(\mathrm{Tr}(k))^2 - 2\mathrm{Tr}(k^2)}{2\mathrm{Det}(k)} \tag{C.1.23}$$

Note that the previous equation is left invariant if k is replaced for the matrix $U^\dagger k U$ (being U an unitary transformation). Therefore, this equation allow us compute the curvature without the need to diagonalize the metric k first.

C.2 $\mathbb{P}(\mathcal{H}_s)$ as a fiber bundle

Remarks on notation

Here, we recall some of the notation introduced in the previous chapters of the thesis that is relevant for our calculation. We also introduce some new one.

Denote by $N = 2s$ the real dimension of $\mathbb{P}(\mathcal{H}_s)$. To carry out the calculations, we use the representation of$\mathbb{P}(\mathcal{H}_s)$ in terms of density matrices, as prescribed in (1.1.6). To simplify the final expressions, instead of defining the Fubini-Study metric h as in (1.1.11), we drop the factor 1/2 and define it simply as,

$$h(v_1, v_2) = \mathrm{Tr}\,(v_1 v_2)\,. \tag{C.2.1}$$

Of course, the results presented in section 2.1 were properly rescaled so they are valid for the original metric h (1.1.11).

As mentioned in 2.1, $\mathbb{P}(\mathcal{H}_s)$ can be decomposed as a fiber bundle where the base space is $\mathscr{S}$ and the fiber is generically isomorphic to $SO(3)$. We denote by π the projection operator for this bundle. Given a element r in $SO(3)$ the right action of r in $\mathbb{P}(\mathcal{H}_s)$ is defined as in (2.1.1). However, instead of denoting the action as $\rho \triangleleft r$, we opt for the more compact notation $R_r\rho$, that is,

$$R_r\rho = D(r)^\dagger \rho\, D(r)\,, \tag{C.2.2}$$

Given an element $\mathcal{A}$ in $so(3)$, we define the fundamental vector field $A^\sharp$ as in (2.1.2). Another important definition is the one of the connection ω (2.1.8); we say that a vector v tangent at ρ is horizontal if and only if it is perpendicular (w.r.t. the Fubini-Study metric) to all the vertical vectors (2.1.2). We denote by Ω the curvature form corresponding to ω.

We can readily check that this characterization of horizontal vectors indeed defines a connection. The required condition is that the right action of the group maps horizontal vectors into horizontal vectors. This is indeed the case, as the right action maps vertical vectors into vertical vectors (this property is valid for any fiber bundle) and preserves the Fubini-Study metric. To check this, consider a point ρ representing an element of projective space and two vectors v and w tangent at ρ. For any $r \in SO(3)$, compute the product of $R_{r*}v$ with $R_{r*}w$,

$$h(R_{r*}v, R_{r*}w) = \mathrm{Tr}\,(R_{r*}vR_{r*}w) = \mathrm{Tr}\,(D(r)^\dagger vwD(r)) = \mathrm{Tr}\,(vw) = h(v, w), \tag{C.2.3}$$

that is the product between v and w. This proves the claim

The horizontal metric g for $\mathscr{S}$ is defined as in (2.1.11). In a similar way, given a point ρ we can define a metric in $so(3)$ as in (2.1.9). However, because of the rescaling we made of the Fubini-Study metric, the expression for the matrix $k^{(\rho)}$ is,

$$k^{(\rho)}(S_\alpha, S_\beta) = k^{(\rho)}_{\alpha\beta} = 2\Re\langle S_\alpha S_\beta\rangle - 2\langle S_\alpha\rangle\langle S_\beta\rangle\,. \tag{C.2.4}$$

in terms of $k^{(\rho)}$ and g, Fubini-Study metric can be written as in (2.1.12).

A particular section of $\mathbb{P}(\mathcal{H}_s)$

In the following paragraphs, we briefly mention a particular section of $\mathbb{P}(\mathcal{H}_s)$ that we use throughout all the calculation. As it turns out, this section is well define except for a set of measure zero.

Consider an operator ρ representing a point in $\mathbb{P}(\mathcal{H}_s)$. At ρ, the 3×3 matrix for the vertical metric $k^{(\rho)}$ (C.2.4) is real and symmetric. Then, as guaranteed by a standard result in linear algebra, $k^(\rho)$ is diagonalizable, and it can be diagonalized by a rotation matrix in $SO(3)$. Call r to said matrix. Then, we have the following,

$$r^T k^{(\rho)} r = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3) \Rightarrow r^\alpha{}_\mu k^{(\rho)}_{\alpha\beta} r^\beta{}_\nu = \delta_{\mu\nu}\lambda_\mu\,. \tag{C.2.5}$$

where λ_i denotes the eigenvalues of $k^{(\rho)}$. By rewriting the l.h.s. of the equality after the implication we obtain,

$$\begin{aligned} r^\alpha{}_\mu k^{(\rho)}_{\alpha\beta} r^\beta{}_\nu &= r^\alpha{}_\mu k^{(\rho)}(S_\alpha, S_\beta) r^\beta{}_\nu = k^{(\rho)}(r^\alpha{}_\mu S_\alpha, r^\beta{}_\nu S_\beta) \\ &= k^{(\rho)}(S\cdot r_\mu, S\cdot r_\nu) \end{aligned} \tag{C.2.6}$$

where r_μ denotes μ-th column of r written as a 3 dimensional vector and $S\cdot r_\mu$ denotes the angular momentum operator associated with the vector r_μ. But, by using the algebra of rotations, one can check that the following equality holds„

$$S\cdot r_\sigma = D(r) S_\sigma D(r)^\dagger\,.$$

Substituting this equation in (C.2.6) we obtain,

$$\begin{aligned} r^\alpha{}_\mu k^{(\rho)}_{\alpha\beta} r^\beta{}_\nu &= k^{(\rho)}(D(r)S_\mu D(r)^\dagger, D(r)S_\nu D(r)^\dagger) \\ &= h(\{D(r)S_\mu D(r)^\dagger\}^\sharp, \{D(r)S_\nu D(r)^\dagger\}^\sharp) \\ &= h(R_{r*}\{D(r)S_\mu D(r)^\dagger\}^\sharp, R_{r*}\{D(r)S_\nu D(r)^\dagger\}^\sharp) \end{aligned}$$

$$= h(S_\mu^\sharp, S_\nu^\sharp) = k^{(R_r\rho)}(S_\mu, S_\nu) = k_{\mu\nu}^{(R_r\rho)} \,.$$

where we used the definition of the metric $k^{(\rho)}$ to obtain the second line and the fact that the metric h is invariant under the right action to obtain third one. Direct substitution of this result in equation (C.2.5) produces the following,

$$k_{\mu\nu}^{(R_r\rho)} = \delta_{\mu\nu}\lambda_\mu \,. \tag{C.2.7}$$

Mathematically speaking, this means that at the point $R_r\rho$, the metric k is diagonal w.r.t. the basis $\{S_x, S_y, S_z\}$. In turn, this implies that, given any ρ, we can always find a point in the same fiber such that in said point the metric k is diagonal w.r.t. the basis $\{S_x, S_y, S_z\}$. By choosing this point continuously for different shapes, we can build a section where the metric is diagonal. This fact also justify that the matrix k can be written as claimed in (C.1.18).

C.2.1 Relationship between $R(h)$, $R(k)$ $\mathrm{Tr}\,(\Omega^2)$ and $R(g)$: a first equation

In this section, we calculate the curvature $R(h)$ of the projective Hilbert space at a point ρ in terms of the curvature of g, of $k^{(\rho)}$ and of the curvature of the connection $\mathrm{Tr}\,(\Omega^2)$. To simplify the notation, sometimes we write k instead of $k^{(\rho)}$.

First, we make some definitions. Consider $F = N - 3$ orthonormal fields in $\mathscr{S}$, $\underline{E}_1, \cdots, \underline{E}_F$. Denote by $\underline{\varphi}^1, \cdots, \underline{\varphi}^F$ their dual one forms in $\mathscr{S}$. Furthermore, assume w.l.o.g. [23, theorem 9.3.7] that the vector fields are such that, at the point $\mathcal{S} = \pi(\rho)$, the following expressions hold,

$$d\underline{\varphi}^i|_s = 0 \,, \quad i = 1, \cdots, F \,. \tag{C.2.8}$$

Let $E_1, \cdots, E_F$ be the horizontal lifts of the vector fields $\underline{E}_1, \cdots, \underline{E}_F$. We denote the elements S_x, S_y and S_z of $so(3)$ generically as S_α. In general, we use Greek letters for the vertical indices. Denote by σ the section defined in the previous section. Furthermore, suppose that the point ρ we are considering is in the section, $\sigma(\mathcal{S}) = \rho$.

Over the points of the section, define the vector fields E_α as the fundamental vertical field associated to $l_\alpha = S_\alpha/(\sqrt{2}\Delta S_\alpha)$ (no sum implied), $E_\alpha = l_\alpha^\sharp$. In this way, by definition of σ, the vector fields E_α constitute an orthonormal set. Extend these vector fields to the rest of the fiber using the right action. Denote by ${C^\alpha}_{\beta\gamma}$ the structure constants of $so(3)$ w.r.t. the basis

l_α (see equation (C.1.20)). To simplify the notation, we make the following convention for the indices; indices such as i, j, k, l, m, n denote horizontal quantities, indices like $\alpha, \beta, \gamma, \delta, \mu, \nu$, as already mentioned, denote vertical indices and indices such as a, b, c, d, e, denote both vertical and horizontal indices. Note that by construction, the vector fields E_a are all right invariant.

We also denote by φ^a the 1-forms in projective Hilbert space dual to the orthonormal fields E_a. If we define $h_{ab} = h(E_a, E_b)$, using the equation (2.1.12) and the previous definitions, we can easily check that for any point ρ' in projective Hilbert space, we have the following,

$$\begin{aligned} h_{ij} &= g(\underline{E}_i, \underline{E}_j) \equiv g_{ij}\,, \\ h_{\alpha\beta} &= k^{(\rho')}(l_\alpha, l_\beta) \equiv \delta_{\alpha\beta}\,, \\ h_{i\beta} &= h_{\beta\, i} = 0\,. \end{aligned}$$

As we can note, the metric is block diagonal when considering Latin indices versus Greek ones. Because of this, we can lower and raise Greek indices with the metric k and the Latin indices with the metric g.

Suppose we have a point $\rho' = R_g \sigma(\mathcal{S}')$, with $\mathcal{S}'$ a point in shape space in the domain of σ. In said point, we define the following $so(3)$ valued 1-form,

$$\tilde{\omega}(\rho') = R^*_{g^{-1}}\, \omega(\sigma(\mathcal{S}'))\,, \tag{C.2.9}$$

Since $\tilde{\omega}$ is a 1-form valued in $so(3)$, we can write it as $\tilde{\omega} = \tilde{\omega}^\alpha l_\alpha$, where $\tilde{\omega}^\alpha$ is a real valued 1-form. Note that, at the section, the $\tilde{\omega}^\alpha = \varphi^\alpha$ holds. Indeed, it is easy to check that both forms are zero when evaluated at horizontal vectors and, because in the section the equality $\omega(E_\alpha) = l_\alpha$ holds by construction, $\tilde{\omega}^\alpha E_\beta = \varphi^\alpha E_\beta = \delta^\alpha{}_\beta$ also holds. In the same way, since Ω is a 2-form valued in $so(3)$, we can write it as $\Omega = \frac{l_\alpha}{2} \Omega^\alpha{}_{ij} \varphi^i \wedge \varphi^j$. In the previous sum there are no terms of the form φ^β because by definition, Ω is zero if any of the vectors it is valuated on is vertical.

Next, we proceed to find the relation we seek. The technique is to use (C.1.8) with the vector fields E_a, that are orthonormal by construction. First, we calculate the term $d\varphi^\gamma$, using the well known formula,

$$\begin{aligned} d\varphi^\gamma(E_a, E_b) &= E_a(\varphi^\gamma(E_b)) - E_a(\varphi^\gamma(E_b)) - \varphi^\gamma([E_a, E_b]) \\ &= -\varphi^\gamma([E_a, E_b])\,, \end{aligned}$$

where $E_a(f)$ denotes the action of the tangent vector E_a on the function f. Since the fields are all right invariant (and as a consequence their dual forms are also right invariant) we can easily check that the previous quantity is constant along the same fiber. Indeed, if we calculate it in the point $R_g\sigma(s)$, using the following equalities,

$$E_a|_{R_g\sigma(s)} = R_{g*}E_a|_{\sigma(s)}\,,$$
$$\varphi^a|_{R_g\sigma(s)} = R^*_{g^{-1}}\varphi^a|_{\sigma(s)}\,,$$

we can conclude that,

$$\begin{aligned} d\varphi^\gamma(E_a, E_b)|_{R_g\sigma(s)} &= -R^*_{g^{-1}}\varphi^\gamma([R_{g*}E_a, R_{g*}E_b]) = -R^*_{g^{-1}}\varphi^\gamma(R_{g*}[E_a, E_b]) \\ &= -R^*_{g^{-1}}R^*_g\varphi^\gamma([E_a, E_b]) = -\varphi^\gamma([E_a, E_b])\,, \end{aligned} \tag{C.2.10}$$

where the field and fields are evaluated at the section. Because of this, it is enough to calculate the commutators at the points in the section.

Consider the following three possible cases,

- $E_a = E_\alpha$ and $E_b = E_\beta$. Then we have,

$$d\varphi^\gamma(E_\alpha, E_\beta) = -\varphi^\gamma([E_\alpha, E_\beta])\,. \tag{C.2.11}$$

To calculate the commutators, we do the following trick. Since Ω is zero when evaluated at vertical fields, we have the following,

$$\begin{aligned} 0 = \Omega(E_\alpha, E_\beta) &= d\omega(E_\alpha, E_\beta) + [\omega(E_\alpha), \omega(E_\beta)] \\ &= E_\alpha\omega(E_\beta) - E_\beta\omega(E_\alpha) - \omega(E_\alpha, E_\beta) + [l_\alpha, l_\beta]\,, \end{aligned} \tag{C.2.12}$$

but (all the derivatives are evaluated at $t = 0$. Also recall the following expression for a fundamental field, $l^\sharp_\alpha = \mathrm{d}/\mathrm{d}t R_{e^{-il_\alpha t}}$),

$$\begin{aligned} E_\alpha\omega(E_\beta) &= \frac{\mathrm{d}}{\mathrm{d}t}\omega(E_\beta|_{R_{e^{-il_\alpha t}}}) = \frac{\mathrm{d}}{\mathrm{d}t}\omega(R_{e^{-il_\alpha t}*}E_\beta) = \frac{\mathrm{d}}{\mathrm{d}t}e^{il_\alpha t}\omega(E_\beta)e^{-il_\alpha t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}e^{il_\alpha t}l_\beta e^{-il_\alpha t} = -[l_\alpha, l_\beta]\,, \end{aligned} \tag{C.2.13}$$

where $E_a|_\rho$ is a shorthand notation for E_a evaluated at ρ. Obviously, in the same way, we have an analogous expression for $E_\beta\omega(E_\alpha)$. By substituting these expressions in (C.2.12) we obtain (recall that, in the point of the section, $\omega = \varphi^\gamma l_\gamma$),

$$\begin{aligned} 0 = [l_\beta, l_\alpha] - \omega([E_\alpha, E_\beta]) &\Rightarrow \omega([E_\alpha, E_\beta]) = [l_\beta, l_\alpha] \\ &\Rightarrow \varphi^\gamma([E_\alpha, E_\beta])l_\gamma = -C^\gamma{}_{\alpha\beta}l_\gamma\,. \\ &\Rightarrow \varphi^\gamma([E_\alpha, E_\beta]) = -C^\gamma{}_{\alpha\beta}\,. \end{aligned}$$

By considering these expression in (C.2.11), we obtain,

$$d\varphi^\gamma(E_\alpha, E_\beta) = C^\gamma{}_{\alpha\beta}\,. \tag{C.2.14}$$

- $E_a = E_i$ and $E_b = E_j$. Like in the previous case, we calculate the necessary commutator (C.2.10) using the curvature Ω. Since both fields are horizontal,

$$\Omega(E_i, E_j) = d\omega(E_i, E_j) = E_i\omega(E_j) - E_j\omega(E_i) - \omega([E_i, E_j]) \Rightarrow$$
$$\Omega^\gamma{}_{ij} l_\gamma = -\varphi^\gamma([E_i, E_j]) l_\gamma \Rightarrow \Omega^\gamma{}_{ij} = -\varphi^\gamma([E_i, E_j])\,.$$

  A similar analysis like the one done for the previous case leads to the following expression,

$$d\varphi^\gamma(E_i, E_j) = \Omega^\gamma{}_{ij}\,.$$

- $E_a = E_\alpha$ and $E_b = E_i$. Suppose $[E_\alpha, E_i] = -D^\beta{}_{\alpha\, i} E_\beta$ (as we check at the end of the section, this commutator needs to be vertical). Note that the coefficients D are constant along any fiber. In this case we have,

$$d\varphi^\gamma(E_\alpha, E_i) = D^\gamma{}_{\alpha\, i}\,. \tag{C.2.15}$$

  We define $D^\beta{}_{i\alpha}$ in an analogous way.

Considering all the previous cases, we can write the following,

$$\begin{aligned} d\varphi^\gamma &= \frac{1}{2}(C^\gamma{}_{\alpha\beta}\varphi^\alpha \wedge \varphi^\beta + \Omega^\gamma{}_{ij}\varphi^i \wedge \varphi^j + D^\gamma{}_{\alpha\, i}\varphi^\alpha \wedge \varphi^i + D^\gamma{}_{i\,\alpha}\varphi^i \wedge \varphi^\alpha) \\ &\equiv Q^\gamma{}_{ab}\varphi^a \wedge \varphi^b\,, \end{aligned} \tag{C.2.16}$$

where Q is given by the following equalities,

$$Q^\gamma{}_{\alpha\beta} = \frac{1}{2} C^\gamma{}_{\alpha\beta}\,, \quad Q^\gamma{}_{ij} = \frac{1}{2}\Omega^\gamma{}_{ij}\,,$$
$$Q^\gamma{}_{\alpha i} = \frac{1}{2} D^\gamma{}_{\alpha i}\,, \quad Q^\gamma{}_{i\alpha} = \frac{1}{2} D^\gamma{}_{i\alpha}\,.$$

Note that $Q^{n+\gamma}{}_{ab} = -Q^{n+\gamma}{}_{ba}$.

Next, we calculate $d\varphi^i$. Note that $\varphi^i = \pi^*\underline{\varphi}^i$. Indeed, it is a matter of routine to check that both forms are zero when evaluated in vertical vectors and coincide when evaluated in the horizontal field E_j. Because of this, we have the following equality, $d\varphi^i = \pi^* d\underline{\varphi}^i$. But, by considering the Levi-Civita connection for the metric g, we have, using (C.1.7)

$$d\varphi^i = -\pi^*\{\overline{\theta}(g)^i{}_j \wedge \underline{\varphi}^j\} = -\{\pi^*\overline{\theta}(g)^i{}_j\} \wedge \varphi^j = Q^i{}_{ab}\varphi^a \wedge \varphi^b\,, \tag{C.2.17}$$

where,

$$Q^i{}_{j\,\alpha} = Q^i{}_{\alpha\, j} = Q^i{}_{\alpha\,\beta} = 0\,,$$

$$Q^i{}_{kj} = \frac{1}{2}(\pi^*\overline{\theta}(g)^i{}_{kj} - \pi^*\overline{\theta}(g)^i{}_{jk})\,, \tag{C.2.18}$$

where we wrote $\pi^*\overline{\theta}(g)^i{}_j$ as $\pi^*\overline{\theta}(g)^i{}_j = \pi^*\overline{\theta}(g)^i{}_{jk}\varphi^k$. Again, we can note that Q is antisymmetric in the lower indices by construction. Just as we did in section C.1, we propose the Levi-Civita for the metric h as the following,

$$\overline{\theta}(h)^a{}_b = (-Q^a{}_{cb} + Q_{bc}{}^a + Q_{cb}{}^a)\varphi^c\,, \tag{C.2.19}$$

as we can easily check that it satisfies the requirements of (C.1.7).

Now, we compute the coefficients the Levi-Civita connection explicitly. In this case we have,

$$\begin{aligned}\overline{\theta}(h)^\alpha{}_\beta &= (-Q^\alpha{}_{c\beta} + Q_{\beta c}{}^\alpha + Q_{c\beta}{}^\alpha)\varphi^c \\ &= -\frac{1}{2}C^\alpha{}_{\gamma\beta}\varphi^\gamma - \frac{1}{2}D^\alpha{}_{i\beta}\varphi^i + \frac{1}{2}C_{\beta\gamma}{}^\alpha\varphi^\gamma + \frac{1}{2}D_{\beta i}{}^\alpha\varphi^i + \frac{1}{2}C_{\gamma\beta}{}^\alpha\varphi^\gamma \\ &= T^\alpha{}_{\gamma\beta}\varphi^\gamma + \frac{1}{2}(D_{\beta i}{}^\alpha - D^\alpha{}_{i\beta})\varphi^i \\ &= \overline{\theta}(k)^\alpha{}_\beta - \mathcal{D}^\alpha{}_{i\beta}\varphi^i\,.\end{aligned} \tag{C.2.20}$$

The last line defines $\mathcal{D}$. Note that T and $\mathcal{D}$ are constant along the fibers, but they are not in the horizontal directions. We can make exactly the same for the rest of the components,

$$\begin{aligned}\overline{\theta}(h)^\alpha{}_i &= (-Q^\alpha{}_{ci} + Q_{ic}{}^\alpha + Q_{ci}{}^\alpha)\varphi^c \\ &= -\frac{1}{2}D^\alpha{}_{\gamma i}\varphi^\gamma - \frac{1}{2}\Omega^\alpha{}_{ji}\varphi^j + 0 + \frac{1}{2}D_{\gamma i}{}^\alpha\varphi^\gamma \\ &= \frac{1}{2}(D_{\gamma i}{}^\alpha - D^\alpha{}_{\gamma i})\varphi^\gamma - \frac{1}{2}\Omega^\alpha{}_{ji}\varphi^j \\ &= M_{\gamma i}{}^\alpha\varphi^\gamma - \frac{1}{2}\Omega^\alpha{}_{ji}\varphi^j\,,\end{aligned} \tag{C.2.21}$$

where we defined M in the last line. Of course, this implies the following relation,

$$\overline{\theta}(h)^i{}_\alpha = -M_\gamma{}^i{}_\alpha\varphi^\gamma + \frac{1}{2}\Omega_{\alpha j}{}^i\varphi^j\,. \tag{C.2.22}$$

And finally, the remaining components,

$$\begin{aligned}\overline{\theta}(h)^i{}_j &= (-Q^i{}_{cj} + Q_{jc}{}^i + Q_{cj}{}^i)\varphi^c \\ &= \frac{1}{2}(-\pi^*\overline{\theta}(g)^i{}_{kj} + \pi^*\overline{\theta}(g)^i{}_{jk} + \pi^*\overline{\theta}(g)_{jk}{}^i - \pi^*\overline{\theta}(g)_j{}^i{}_k + \pi^*\overline{\theta}(g)_{kj}{}^i \\ &\qquad - \pi^*\overline{\theta}(g)_k{}^i{}_j)\varphi^k + \frac{1}{2}\Omega_{\gamma j}{}^i\varphi^\gamma\end{aligned}$$

$$
\begin{aligned}
&= \pi^*\overline{\theta}(g)^i{}_{jk}\varphi^k + \frac{1}{2}\Omega_{\gamma j}{}^i\varphi^\gamma \\
&= \pi^*\overline{\theta}(g)^i{}_j + \frac{1}{2}\Omega_{\gamma j}{}^i\varphi^\gamma\,.
\end{aligned} \tag{C.2.23}
$$

With all the components of the Levi-Civita tensor in hand, we calculate their derivatives, so we can compute the Riemann tensor afterwards. Therefore we have,

$$
\begin{aligned}
d\overline{\theta}(h)^\alpha{}_\beta &= d\overline{\theta}(k)^\alpha{}_\beta - \mathcal{D}^\alpha{}_{i\beta,j}\varphi^j\wedge\varphi^i \\
&= T^\alpha{}_{\gamma\beta,i}\varphi^i\wedge\varphi^\gamma + T^\alpha{}_{\gamma\beta}d\varphi^\gamma - \mathcal{D}^\alpha{}_{i\beta,j}\varphi^j\wedge\varphi^i \\
&= T^\alpha{}_{\gamma\beta,i}\varphi^i\wedge\varphi^\gamma - T^\alpha{}_{\gamma\beta}\overline{\theta}(h)^\gamma{}_a\wedge\varphi^a - \mathcal{D}^\alpha{}_{i\beta,j}\varphi^j\wedge\varphi^i \\
&= T^\alpha{}_{\gamma\beta,i}\varphi^i\wedge\varphi^\gamma - T^\alpha{}_{\gamma\beta}\overline{\theta}(h)^\gamma{}_i\wedge\varphi^i - T^\alpha{}_{\gamma\beta}\overline{\theta}(h)^\gamma{}_\delta\wedge\varphi^\delta - \mathcal{D}^\alpha{}_{i\beta,j}\varphi^j\wedge\varphi^i \\
&= T^\alpha{}_{\gamma\beta,i}\varphi^i\wedge\varphi^\gamma - T^\alpha{}_{\gamma\beta}M_{\mu\,i}{}^\gamma\varphi^\mu\wedge\varphi^i + \frac{1}{2}T^\alpha{}_{\gamma\beta}\Omega^\gamma{}_{ji}\varphi^j\wedge\varphi^i \\
&\quad - T^\alpha{}_{\gamma\beta}\overline{\theta}(k)^\gamma{}_\delta\wedge\varphi^\delta + T^\alpha{}_{\gamma\beta}\mathcal{D}^\gamma{}_{i\delta}\varphi^i\wedge\varphi^\delta - \mathcal{D}^\alpha{}_{i\beta,j}\varphi^j\wedge\varphi^i \\
&= (T^\alpha{}_{\mu\beta,i} + T^\alpha{}_{\gamma\beta}M_{\mu\,i}{}^\gamma + T^\alpha{}_{\gamma\beta}\mathcal{D}^\gamma{}_{i\mu})\varphi^i\wedge\varphi^\mu \\
&\quad - T^\alpha{}_{\gamma\beta}\overline{\theta}(k)^\gamma{}_\delta\wedge\varphi^\delta + \left(\frac{1}{2}T^\alpha{}_{\gamma\beta}\Omega^\gamma{}_{ji} - \mathcal{D}^\alpha{}_{i\beta,j}\right)\varphi^j\wedge\varphi^i\,.
\end{aligned}
$$

Recall that, as a matter of notation, we denote the derivative of a function $E_a(f)$ as $f_{,a}$. In the first equation we took advantage of the fact that at our point, $d\varphi^i = 0$. Going on with the other components,

$$
\begin{aligned}
d\overline{\theta}(h)^\alpha{}_i &= M_{\gamma\,i}{}^\alpha{}_{,j}\varphi^j\wedge\varphi^\gamma - \frac{1}{2}\Omega^\alpha{}_{ji,k}\varphi^k\wedge\varphi^j \\
&\quad - M_{\gamma\,i}{}^\alpha\overline{\theta}(h)^\gamma{}_\beta\wedge\varphi^\beta - M_{\gamma\,i}{}^\alpha\overline{\theta}(h)^\gamma{}_j\wedge\varphi^j \\
&= M_{\gamma\,i}{}^\alpha{}_{,j}\varphi^j\wedge\varphi^\gamma - \frac{1}{2}\Omega^\alpha{}_{ji,k}\varphi^k\wedge\varphi^j \\
&\quad - M_{\gamma\,i}{}^\alpha\overline{\theta}(k)^\gamma{}_\beta\wedge\varphi^\beta + M_{\gamma\,i}{}^\alpha\mathcal{D}^\gamma{}_{j\beta}\varphi^j\wedge\varphi^\beta \\
&\quad - M_{\gamma\,i}{}^\alpha M_{\beta\,j}{}^\gamma\varphi^\beta\wedge\varphi^j + \frac{1}{2}M_{\gamma\,i}{}^\alpha\Omega^\gamma{}_{kj}\varphi^k\wedge\varphi^j \\
&= (M_{\gamma\,i}{}^\alpha{}_{,j} + M_{\beta\,i}{}^\alpha\mathcal{D}^\beta{}_{j\gamma} + M_{\beta\,i}{}^\alpha M_{\gamma\,j}{}^\beta)\varphi^j\wedge\varphi^\gamma \\
&\quad + \frac{1}{2}(M_{\gamma\,i}{}^\alpha\Omega^\gamma{}_{kj} - \Omega^\alpha{}_{ji,k})\varphi^k\wedge\varphi^j - M_{\gamma\,i}{}^\alpha\overline{\theta}(k)^\gamma{}_\beta\wedge\varphi^\beta\,.
\end{aligned}
$$

Note that the following equality holds

$$
M_{\beta i}{}^\alpha\mathcal{D}^\beta{}_{j\gamma} + M_{\beta i}{}^\alpha M_{\gamma j}{}^\beta = M_{\beta i}{}^\alpha D^\beta{}_{j\gamma}\,. \tag{C.2.24}
$$

Because of this, we have,

$$d\overline{\theta}(h)^{\alpha}{}_{i} = (M_{\gamma\,i}{}^{\alpha}{}_{,j} + M_{\beta\,i}{}^{\alpha} D^{\beta}{}_{j\gamma})\varphi^{j} \wedge \varphi^{\gamma} + \frac{1}{2}(M_{\gamma\,i}{}^{\alpha}\Omega^{\gamma}{}_{kj} - \Omega^{\alpha}{}_{ji,k})\varphi^{k} \wedge \varphi^{j} - M_{\gamma\,i}{}^{\alpha}\overline{\theta}(k)^{\gamma}{}_{\beta} \wedge \varphi^{\beta}\,.$$

With a very similar calculation we obtain,

$$d\overline{\theta}(h)^{i}{}_{\alpha} = -(M_{\gamma}{}^{i}{}_{\alpha,j} + M_{\beta}{}^{i}{}_{\alpha} D^{\beta}{}_{j\gamma})\varphi^{j} \wedge \varphi^{\gamma} - \frac{1}{2}(M_{\gamma}{}^{i}{}_{\alpha}\Omega^{\gamma}{}_{kj} - \Omega_{\alpha j}{}^{i}{}_{,k})\varphi^{k} \wedge \varphi^{j} + M_{\gamma}{}^{i}{}_{\alpha}\overline{\theta}(k)^{\gamma}{}_{\beta} \wedge \varphi^{\beta}\,.$$

And finally, we derive the remaining component,

$$\begin{aligned}
d\overline{\theta}(h)^{i}{}_{j} &= \pi^{*}d\overline{\theta}(g)^{i}{}_{j} + \frac{1}{2}\Omega_{\alpha j}{}^{i}{}_{,k}\varphi^{k} \wedge \varphi^{\alpha} - \frac{1}{2}\Omega_{\alpha j}{}^{i}\overline{\theta}(h)^{\alpha}{}_{a} \wedge \varphi^{a} \\
&= \pi^{*}d\overline{\theta}(g)^{i}{}_{j} + \frac{1}{2}\Omega_{\alpha j}{}^{i}{}_{,k}\varphi^{k} \wedge \varphi^{\alpha} \\
&\quad - \frac{1}{2}\Omega_{\alpha j}{}^{i}\left(M_{\gamma\,k}{}^{\alpha}\varphi^{\gamma} \wedge \varphi^{k} - \frac{1}{2}\Omega^{\alpha}{}_{mk}\varphi^{m} \wedge \varphi^{k}\right) \\
&\quad - \frac{1}{2}\Omega_{\alpha j}{}^{i}\left(\overline{\theta}(k)^{\alpha}{}_{\beta} \wedge \varphi^{\beta} - \mathcal{D}^{\alpha}{}_{k\beta}\varphi^{k} \wedge \varphi^{\beta}\right) \\
&= \pi^{*}d\overline{\theta}(g)^{i}{}_{j} + \frac{1}{4}\Omega_{\alpha j}{}^{i}\Omega^{\alpha}{}_{mk}\varphi^{m} \wedge \varphi^{k} \\
&\quad + \frac{1}{2}\left(\Omega_{\alpha j}{}^{i}(\mathcal{D}^{\alpha}{}_{k\beta} + M_{\beta\,k}{}^{\alpha}) + \Omega_{\beta j}{}^{i}{}_{,k}\right)\varphi^{k} \wedge \varphi^{\beta} \\
&\quad - \frac{1}{2}\Omega_{\alpha j}{}^{i}\overline{\theta}(k)^{\alpha}{}_{\beta} \wedge \varphi^{\beta} \\
&= \pi^{*}d\overline{\theta}(g)^{i}{}_{j} + \frac{1}{4}\Omega_{\alpha j}{}^{i}\Omega^{\alpha}{}_{mk}\varphi^{m} \wedge \varphi^{k} \\
&\quad + \frac{1}{2}\left(\Omega_{\alpha j}{}^{i}D^{\alpha}{}_{k\beta} + \Omega_{\beta j}{}^{i}{}_{,k}\right)\varphi^{k} \wedge \varphi^{\beta} - \frac{1}{2}\Omega_{\alpha j}{}^{i}\overline{\theta}(k)^{\alpha}{}_{\beta} \wedge \varphi^{\beta}\,.
\end{aligned}$$

Now, we calculate the terms of the form $\overline{\theta}(h)^{a}{}_{b} \wedge \overline{\theta}(h)^{b}{}_{c}$, which are also necessary to obtain the Riemann tensor according to equation (C.1.8). First,

$$\begin{aligned}
\overline{\theta}(h)^{\alpha}{}_{b} \wedge \overline{\theta}(h)^{b}{}_{\beta} &= \overline{\theta}(h)^{\alpha}{}_{i} \wedge \overline{\theta}(h)^{i}{}_{\beta} + \overline{\theta}(h)^{\alpha}{}_{\gamma} \wedge \overline{\theta}(h)^{\gamma}{}_{\beta} \\
&= \left(M_{\gamma\,i}{}^{\alpha}\varphi^{\gamma} - \frac{1}{2}\Omega^{\alpha}{}_{mi}\varphi^{m}\right) \wedge \left(-M_{\gamma}{}^{i}{}_{\beta}\varphi^{\gamma} + \frac{1}{2}\Omega_{\beta j}{}^{i}\varphi^{j}\right) \\
&\quad + \left(\overline{\theta}(k)^{\alpha}{}_{\gamma} - \mathcal{D}^{\alpha}{}_{i\gamma}\varphi^{i}\right) \wedge \left(\overline{\theta}(k)^{\gamma}{}_{\beta} - \mathcal{D}^{\gamma}{}_{j\beta}\varphi^{j}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left(-M_{\mu j}{}^{\alpha}\Omega_{\beta i}{}^{j} + \Omega^{\alpha}{}_{im}M_{\mu}{}^{m}{}_{\beta}\right)\varphi^{i}\wedge\varphi^{\mu} \\
&\quad - M_{\gamma i}{}^{\alpha}M_{\delta}{}^{i}{}_{\beta}\varphi^{\gamma}\wedge\varphi^{\delta} + \overline{\theta}(k)^{\alpha}{}_{\gamma}\wedge\overline{\theta}(k)^{\gamma}{}_{\beta} \\
&\quad + \left(-\frac{1}{4}\Omega^{\alpha}{}_{jk}\Omega_{\beta i}{}^{k} + \mathcal{D}^{\alpha}{}_{j\gamma}\mathcal{D}^{\gamma}{}_{i\beta}\right)\varphi^{j}\wedge\varphi^{i} \\
&\quad + \mathcal{D}^{\gamma}{}_{i\beta}\varphi^{i}\wedge\overline{\theta}(k)^{\alpha}{}_{\gamma} - \mathcal{D}^{\alpha}{}_{i\gamma}\varphi^{i}\wedge\overline{\theta}(k)^{\gamma}{}_{\beta}\,.
\end{aligned}$$

Then,

$$\begin{aligned}
\overline{\theta}(h)^{\alpha}{}_{b}\wedge\overline{\theta}(h)^{b}{}_{i} &= \overline{\theta}(h)^{\alpha}{}_{j}\wedge\overline{\theta}(h)^{j}{}_{i} + \overline{\theta}(h)^{\alpha}{}_{\gamma}\wedge\overline{\theta}(h)^{\gamma}{}_{i} \\
&= \left(M_{\gamma j}{}^{\alpha}\varphi^{\gamma} - \frac{1}{2}\Omega^{\alpha}{}_{mj}\varphi^{m}\right)\wedge\left(\pi^{*}\overline{\theta}(g)^{j}{}_{i} + \frac{1}{2}\Omega_{\gamma i}{}^{j}\varphi^{\gamma}\right) \\
&\quad + \left(\overline{\theta}(k)^{\alpha}{}_{\gamma} - \mathcal{D}^{\alpha}{}_{j\gamma}\varphi^{j}\right)\wedge\left(M_{\beta i}{}^{\gamma}\varphi^{\beta} - \frac{1}{2}\Omega^{\gamma}{}_{mi}\varphi^{m}\right) \\
&= M_{\gamma j}{}^{\alpha}\varphi^{\gamma}\wedge\pi^{*}\overline{\theta}(g)^{j}{}_{i} \\
&\quad - \frac{1}{2}\Omega^{\alpha}{}_{kj}\varphi^{k}\wedge\pi^{*}\overline{\theta}(g)^{j}{}_{i} + \frac{1}{2}M_{\gamma j}{}^{\alpha}\Omega_{\beta i}{}^{j}\varphi^{\gamma}\wedge\varphi^{\beta} \\
&\quad + M_{\beta i}{}^{\gamma}\overline{\theta}(k)^{\alpha}{}_{\gamma}\wedge\varphi^{\beta} - \left(\mathcal{D}^{\alpha}{}_{j\delta}M_{\gamma i}{}^{\delta} + \frac{1}{4}\Omega^{\alpha}{}_{jk}\Omega_{\gamma i}{}^{k}\right)\varphi^{j}\wedge\varphi^{\gamma} \\
&\quad - \frac{1}{2}\Omega^{\gamma}{}_{ji}\overline{\theta}(k)^{\alpha}{}_{\gamma}\wedge\varphi^{j} + \frac{1}{2}\mathcal{D}^{\alpha}{}_{k\gamma}\Omega^{\gamma}{}_{ji}\varphi^{k}\wedge\varphi^{j} \\
&= \frac{1}{2}M_{\gamma j}{}^{\alpha}\Omega_{\beta i}{}^{j}\varphi^{\gamma}\wedge\varphi^{\beta} + M_{\beta i}{}^{\gamma}\overline{\theta}(k)^{\alpha}{}_{\gamma}\wedge\varphi^{\beta} \\
&\quad - \left(\mathcal{D}^{\alpha}{}_{j\delta}M_{\gamma i}{}^{\delta} + \frac{1}{4}\Omega^{\alpha}{}_{jk}\Omega_{\gamma i}{}^{k}\right)\varphi^{j}\wedge\varphi^{\gamma} \\
&\quad - \frac{1}{2}\Omega^{\gamma}{}_{ji}\overline{\theta}(k)^{\alpha}{}_{\gamma}\wedge\varphi^{j} + \frac{1}{2}\mathcal{D}^{\alpha}{}_{k\gamma}\Omega^{\gamma}{}_{ji}\varphi^{k}\wedge\varphi^{j}\,,
\end{aligned}$$

where in the last equality we took advantage of the fact that $\overline{\theta}(g)$ is zero at our point of interest.

For the remaining product we have,

$$\begin{aligned}
\overline{\theta}(h)^{i}{}_{b}\wedge\overline{\theta}(h)^{b}{}_{j} &= \overline{\theta}(h)^{i}{}_{k}\wedge\overline{\theta}(h)^{k}{}_{j} + \overline{\theta}(h)^{i}{}_{\alpha}\wedge\overline{\theta}(h)^{\alpha}{}_{j} \\
&= \left(\pi^{*}\overline{\theta}(g)^{i}{}_{k} + \frac{1}{2}\Omega_{\gamma k}{}^{i}\varphi^{\gamma}\right)\wedge\left(\pi^{*}\overline{\theta}(g)^{k}{}_{j} + \frac{1}{2}\Omega_{\beta j}{}^{k}\varphi^{\beta}\right) \\
&\quad + \left(-M_{\beta}{}^{i}{}_{\alpha}\varphi^{\beta} + \frac{1}{2}\Omega_{\alpha m}{}^{i}\varphi^{m}\right)\wedge\left(M_{\gamma j}{}^{\alpha}\varphi^{\gamma} - \frac{1}{2}\Omega^{\alpha}{}_{kj}\varphi^{k}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{4}\Omega_{\beta j}{}^{k}\Omega_{\gamma k}{}^{i} + M_{\gamma\, j}{}^{\alpha} M_{\beta}{}^{i}{}_{\alpha}\right)\varphi^{\gamma}\wedge\varphi^{\beta} \\
&\quad + \frac{1}{2}\left(M_{\beta\, j}{}^{\alpha}\Omega_{\alpha k}{}^{i} - \Omega^{\alpha}{}_{kj} M_{\beta}{}^{i}{}_{\alpha}\right)\varphi^{k}\wedge\varphi^{\beta} \\
&\quad - \frac{1}{4}\Omega^{\alpha}{}_{kj}\Omega_{\alpha m}{}^{i}\varphi^{m}\wedge\varphi^{k}\,.
\end{aligned}
$$

Finally we have all the necessary ingredients to calculate the full Riemann tensor. For our purposes, we only calculate the components that are relevant for the scalar curvature. We have then,

$$
\begin{aligned}
R(h)^{\alpha}{}_{\beta\gamma\delta} &= -T^{\alpha}{}_{\mu\beta}T^{\mu}{}_{\gamma\delta} + T^{\alpha}{}_{\mu\beta}T^{\mu}{}_{\delta\gamma} + T^{\alpha}{}_{\gamma\mu}T^{\mu}_{\delta\beta} - T^{\alpha}{}_{\delta\mu}T^{\mu}_{\gamma\beta} \\
&\quad - M_{\gamma\, i}{}^{\alpha}M^{i}_{\delta\beta} + M_{\delta\, i}{}^{\alpha}M^{i}_{\gamma\beta} \\
&= R(k)^{\alpha}{}_{\beta\gamma\delta} - M_{\gamma\, i}{}^{\alpha}M^{i}_{\delta\beta} + M_{\delta\, i}{}^{\alpha}M^{i}_{\gamma\beta} \\
R(h)^{i}{}_{jkl} &= \pi^{*}d\bar{\theta}(g)^{i}_{j}(E_k, E_l) + \frac{1}{2}\Omega_{\alpha j}{}^{i}\Omega^{\alpha}{}_{kl} - \frac{1}{4}\Omega^{\alpha}{}_{lj}\Omega_{\alpha k}{}^{i} + \frac{1}{4}\Omega^{\alpha}{}_{kj}\Omega_{\alpha l}{}^{i} \\
&= R(g)^{i}{}_{jkl} + \frac{1}{2}\Omega_{\alpha j}{}^{i}\Omega^{\alpha}{}_{kl} - \frac{1}{4}\Omega^{\alpha}{}_{lj}\Omega_{\alpha k}{}^{i} + \frac{1}{4}\Omega^{\alpha}{}_{kj}\Omega_{\alpha l}{}^{i} \\
R(h)^{\alpha}{}_{i\gamma j} &= -(M_{\gamma\, i}{}^{\alpha}{}_{,j} + M_{\beta\, i}{}^{\alpha}D^{\beta}{}_{j\gamma}) \\
&\quad + \left(\mathcal{D}^{\alpha}{}_{j\delta}M_{\gamma\, i}{}^{\delta} + \frac{1}{4}\Omega^{\alpha}{}_{jk}\Omega_{\gamma i}{}^{k}\right) - \frac{1}{2}\Omega^{\beta}{}_{ji}T^{\alpha}{}_{\gamma\beta}\,,
\end{aligned}
$$

so that,

$$
\begin{aligned}
R(h)^{\alpha\beta}{}_{\alpha\beta} &= R(k)^{\alpha\beta}{}_{\alpha\beta} - M_{\alpha\, i}{}^{\alpha}M_{\beta}{}^{i\,\beta} + M_{\beta\, i}{}^{\alpha}M_{\alpha}{}^{i\,\beta} \\
&= R(k) - D_{\alpha\, i}{}^{\alpha}D_{\beta}{}^{i\,\beta} \\
&\quad + \frac{1}{4}(D_{\beta\, i}{}^{\alpha} + D^{\alpha}{}_{i\,\beta})(D^{\beta\, i}{}_{\alpha} + D_{\alpha}{}^{i\,\beta}) \\
&= R(k) - D_{\alpha\, i}{}^{\alpha}D_{\beta}{}^{i\,\beta} \\
&\quad + \frac{1}{2}(D_{\beta\, i}{}^{\alpha}D^{\beta\, i}{}_{\alpha} + D_{\beta\, i}{}^{\alpha}D_{\alpha}{}^{i\,\beta}) \\
R(h)^{ij}{}_{ij} &= R(g)^{ij}{}_{ij} + \frac{1}{2}\Omega_{\alpha j}{}^{i}\Omega^{\alpha}{}_{i}{}^{j} - \frac{1}{4}\Omega^{\alpha}{}_{j}{}^{j}\Omega_{\alpha i}{}^{i} + \frac{1}{4}\Omega^{\alpha}{}_{ij}\Omega_{\alpha}{}^{ji} \\
&= R(g) - \frac{3}{4}\Omega^{\alpha}{}_{ij}\Omega_{\alpha}{}^{ij} \qquad \text{(C.2.25)} \\
R(h)^{\alpha i}{}_{\alpha i} &= -(M_{\alpha}{}^{i\alpha}{}_{,i} + M_{\beta\, i}{}^{\alpha}D^{\beta i}{}_{\alpha}) \\
&\quad + \left(\mathcal{D}^{\alpha i}{}_{\delta}M_{\alpha\, i}{}^{\delta} + \frac{1}{4}\Omega^{\alpha i}{}_{k}\Omega_{\alpha i}{}^{k}\right) - \frac{1}{2}\Omega^{\beta i}{}_{i}T^{\alpha}{}_{\alpha\beta}
\end{aligned}
$$

$$= -\frac{1}{2}(D_{\beta i}{}^{\alpha} D^{\beta i}{}_{\alpha} + D^{\alpha}{}_{i\beta} D^{\beta i}{}_{\alpha}) - M_{\alpha}{}^{i\alpha}{}_{,i} + \frac{1}{4}\Omega^{\alpha}{}_{ij}\Omega_{\alpha}{}^{ij},$$

where we used the fact that M is symmetric in the first and third index while $\mathcal{D}$ is antisymmetric. Going on with the calculation we have,

$$-M_{\beta i}{}^{\alpha} D^{\beta i}{}_{\alpha} + \mathcal{D}^{\alpha i}{}_{\delta} M_{\alpha i}{}^{\delta} = -\frac{1}{2}(D_{\beta i}{}^{\alpha} + D^{\alpha}{}_{i\beta}) D^{\beta i}{}_{\alpha}\,. \tag{C.2.26}$$

Finally, we calculate the scalar curvature,

$$\begin{aligned} R(h) &= R(h)^{ab}{}_{ab} = R(h)^{\alpha\beta}{}_{\alpha\beta} + R(h)^{\alpha i}{}_{\alpha i} + R(h)^{i\alpha}{}_{i+\alpha} + R(h)^{ij}{}_{ij} \\ &= R(h)^{\alpha\beta}{}_{\alpha\beta} + R(h)^{ij}{}_{ij} + 2R(h)^{\alpha i}{}_{\alpha i} \\ &= R(k) - D_{\alpha i}{}^{\alpha} D_{\beta}{}^{i\beta} + \frac{1}{2}(D_{\beta i}{}^{\alpha} D^{\beta i}{}_{\alpha} + D_{\beta i}{}^{\alpha} D_{\alpha}{}^{i\beta}) \\ &\quad + R(g) - \frac{3}{4}\Omega^{\alpha}{}_{ij}\Omega_{\alpha}{}^{ij} - (D_{\beta i}{}^{\alpha} D^{\beta i}{}_{\alpha} + D^{\alpha}{}_{i\beta} D^{\beta i}{}_{\alpha}) \\ &\quad - 2M_{\alpha}{}^{i\alpha}{}_{,i} + \frac{1}{2}\Omega^{\alpha}{}_{ij}\Omega_{\alpha}{}^{ij} \\ &= R(k) + R(g) - \frac{1}{4}\Omega^{\alpha}{}_{ij}\Omega_{\alpha}{}^{ij} - \frac{1}{2}(D_{\beta i}{}^{\alpha} D^{\beta i}{}_{\alpha} + D^{\alpha}{}_{i\beta} D^{\beta i}{}_{\alpha}) \\ &\quad - 2D_{\alpha}{}^{i\alpha}{}_{,i} - D_{\alpha i}{}^{\alpha} D_{\beta}{}^{i\beta} \end{aligned} \tag{C.2.27}$$

Until now, we have not specified the coefficients D. We calculate them in terms of the connection. Recall that they were defined by the equation $[E_\alpha, E_i] = -D^{\beta}{}_{\alpha i} E_\beta$, so that we need to calculate the commutator of such fields.

First we prove that it is indeed vertical, as was previously claimed. This is a consequence of the fact that the commutator of a right invariant field and a fundamental field is zero, as can be proved by noting that the right invariant field is constant along the integral lines of the fundamental field. Suppose that V is a vertical field and H_R is a right invariant vector field. We have then that,

$$[V, H_R] = [V^\alpha S^\sharp_\alpha, H_R] = -S^\sharp_\alpha H_R(V^\alpha) + V^\alpha [S^\sharp_\alpha, H_R] = -S^\sharp_\alpha H_R(V^\alpha)\,. \tag{C.2.28}$$

For the first equality, we used the fact that the fundamental vectors are a basis of the vertical space. For the second one, we use the product rule for commutators. Clearly, this implies that the commutator $[E_\alpha, E_i]$ is vertical, as previously claimed.

Now we calculate the commutator, using the trick of the curvature as usual,

$$\begin{aligned} 0 &= \Omega(E_\alpha, E_i) = d\omega(E_\alpha, E_i) + [\omega(E_\alpha), \omega(E_i)] \\ &= d\omega(E_\alpha, E_i) = E_\alpha\omega(E_i) - E_i\omega(E_\alpha) - \omega([E_\alpha, E_i]) \\ &= -E_i\omega(E_\alpha) - \omega([E_\alpha, E_i]) \Rightarrow \\ \omega([E_\alpha, E_i]) &= -E_i\omega(E_\alpha)\,. \end{aligned} \tag{C.2.29}$$

Recall that we are working over a point in the section σ. Define the coefficients $A^\alpha{}_i$ according to the following equality,

$$\sigma_*(\underline{E}_i) = E_i + A^\alpha{}_i S^\sharp_\alpha\,. \tag{C.2.30}$$

In other words, this means that the vertical component of the vector $\sigma_*(\underline{E}_i)$ is $A^\alpha{}_i S^\sharp_\alpha$. Note that the coefficients $A^\alpha{}_i$ are defined in terms of the vectors $S^\sharp$ instead of the fields E_α. This simplifies later expressions. By substituting this expression in (C.2.30), we obtain the following,

$$\omega([E_\alpha, E_i]) = -\sigma_*(\underline{E}_i)\omega(E_\alpha) + A^\beta{}_i S^\sharp_\beta \omega(E_\alpha)\,. \tag{C.2.31}$$

Now we consider each term. Recall that, in the section, $\omega(E_\alpha) = l_\alpha$. Clearly, the integral curves of $\sigma_*(\underline{E}_i)$ lie completely on the section. Because of this, we have (no sums implied in the following equations),

$$\begin{aligned} -\sigma_*(\underline{E}_i)\omega(E_\alpha) &= -\sigma_*(\underline{E}_i) l_\alpha = -\frac{1}{\sqrt{2}} S_\alpha \sigma_*(\underline{E}_i) \frac{1}{\Delta S_\alpha} \\ &= \frac{(\Delta S_\alpha)_{,i}}{\sqrt{2}\Delta S_\alpha^2} S_\alpha = \frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha} l_\alpha\,. \end{aligned} \tag{C.2.32}$$

Since ΔS_α is constant along the fibers (regarding it as one of the eigenvalues of the matrix k), we can change the action of $\sigma_*(\underline{E}_i)$ by the one of E_i.

For the second term of (C.2.31), we have the following,

$$A^\beta{}_i S^\sharp_\beta \omega(E_\alpha) = \sqrt{2}\Delta S_\beta A^\beta{}_i l^\sharp_\beta \omega(E_\alpha) = \sqrt{2}\Delta S_\beta A^\beta{}_i E_\beta \omega(E_\alpha)\,. \tag{C.2.33}$$

The last quantity was already calculated in equation (C.2.13), so, by considering it, we obtain the following expression,

$$\begin{aligned} A^\beta{}_i S^\sharp_\beta \omega(E_\alpha) &= -\sqrt{2}\Delta S_\beta A^\beta{}_i [l_\beta, l_\alpha] = -\sqrt{2}\Delta S_\beta A^\beta{}_i C^\gamma{}_{\beta\alpha} l_\gamma \\ &= \sqrt{2}\Delta S_\beta A^\beta{}_i C^\gamma{}_{\alpha\beta} l_\gamma\,. \end{aligned} \tag{C.2.34}$$

By substituting everything in (C.2.31) and writing the connection as $\omega = \varphi^\gamma l_\gamma$ we obtain the following,

$$\begin{aligned}
\varphi^\gamma([E_\alpha, E_i])l_\gamma &= \left(\frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\gamma} + \sqrt{2}\Delta S_\beta A^\beta{}_i C^\gamma{}_{\alpha\beta}\right) l_\gamma \Rightarrow \\
\varphi^\gamma([E_\alpha, E_i]) &= \left(\frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\gamma} + \sqrt{2}\Delta S_\beta A^\beta{}_i C^\gamma{}_{\alpha\beta}\right) \Rightarrow \\
-\varphi^\gamma(D^\beta{}_{\alpha i} E_\beta) &= \left(\frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\gamma} + \sqrt{2}\Delta S_\beta A^\beta{}_i C^\gamma{}_{\alpha\beta}\right) \Rightarrow \\
D^\gamma{}_{\alpha i} &= -\left(\frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\gamma} + \sqrt{2}\Delta S_\beta A^\beta{}_i C^\gamma{}_{\alpha\beta}\right) \Rightarrow \\
D^\gamma{}_{i\alpha} &= \frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\gamma} + \sqrt{2}\Delta S_\beta A^\beta{}_i C^\gamma{}_{\alpha\beta}\,.
\end{aligned}$$

Now, we make some heavy calculations involving D. Since the metric is diagonal in the basis we are working on, we can raise and lower indices without talking too much care. We have in this case,

$$\begin{aligned}
D^\alpha{}_{i\alpha} &= \sum_\alpha \frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha} = \sum_\alpha (\ln \Delta S_\alpha)_{,i} = \frac{1}{2}\sum_\alpha (\ln \Delta S_\alpha^2)_{,i} \\
&= \frac{1}{2}\sum_\alpha (\ln 2 + \ln \Delta S_\alpha^2)_{,i} = \frac{1}{2}\sum_\alpha (\ln 2\Delta S_\alpha^2)_{,i} = \frac{1}{2}\left(\sum_\alpha \ln 2\Delta S_\alpha^2\right)_{,i} \\
&= \frac{1}{2}\left(\ln \prod_\alpha 2\Delta S_\alpha^2\right)_{,i} = \frac{1}{2}\left(\ln \mathrm{Det} k\right)_{,i}\,, \qquad \text{(C.2.35)}
\end{aligned}$$

where k is given by the equation (C.1.18). The missing term is zero because the term $C^\alpha{}_{\alpha\beta}$ as noted in (C.1.20). Using the previous result we obtain,

$$D_\alpha{}^{i\alpha}{}_{,i} = \frac{1}{2}\sum_i (\ln \mathrm{Det} k)_{,ii} \quad D_{\alpha i}{}^\alpha D_\beta{}^{i\beta} = \frac{1}{4}\sum_i (\ln \mathrm{Det} k)_{,i}{}^2\,. \qquad \text{(C.2.36)}$$

Even though we calculated Det k in points in the section, we can readily check that it is constant along points in the same fiber. Indeed, consider a point ρ and a point $R_g\rho$ in the same fiber. We have in this case,

$$k^{(\rho)}_{\alpha\beta} = h_\rho(S^\sharp_\alpha, S^\sharp_\beta) = h_{R_g\rho}(R_{g*}S^\sharp_\alpha, R_{g*}S^\sharp_\beta) = h_{R_g\rho}((g^{-1}S_\alpha g)^\sharp, (g^{-1}S_\beta g)^\sharp)\,,$$

where we used the fact that the metric is right invariant to obtain the first equation and denote as $g^{-1}S_\alpha g$ as the adjoint action of g in the element S_α. Since we are working in $SO(3)$, said action is given by a rotation matrix, that is, $g^{-1}S_\alpha g = \mathcal{G}^\mu{}_\alpha S_\mu$, with $\mathcal{G}$ a rotation matrix. With this definition we have,

$$k^{(\rho)}_{\alpha\beta} = \mathcal{G}^\mu{}_\alpha \mathcal{G}^\nu{}_\beta h_{R_g\rho}(S^\sharp_\mu, S^\sharp_\nu) = \mathcal{G}^\mu{}_\alpha \mathcal{G}^\nu{}_\beta k^{(\rho)}_{\mu\nu} = \mathcal{G}^T{}_\alpha{}^\mu k^{(\rho)}_{\mu\nu} \mathcal{G}^\nu{}_\beta \,,$$

or, written as a matrix,

$$k^{(\rho)} = \mathcal{G}^T k^{(R_g\rho)} \mathcal{G} = k^{(\rho)} = \mathcal{G}^{-1} k^{(R_g\rho)} \mathcal{G} \,,$$

where we used the fact that the matrix $\mathcal{G}$ represents a rotation, so that the following holds, $\mathcal{G}^T = \mathcal{G}^{-1}$. Clearly this implies that the determinant is constant along the fibers,

$$\text{Det } k^{(\rho)} = \text{Det } k^{(R_g\rho)} \,. \tag{C.2.37}$$

Now, we have for the last term,

$$(D_{\beta i}{}^\alpha D^{\beta i}{}_\alpha + D^\alpha{}_{i\,\beta} D^{\beta i}{}_\alpha) = (D_{\beta i}{}^\alpha + D^\alpha{}_{i\,\beta}) D^{\beta i}{}_\alpha \,.$$

But,

$$(D_{\beta i}{}^\alpha + D^\alpha{}_{i\,\beta}) = \frac{2(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\beta} + \sqrt{2}\Delta S_\gamma A^\gamma{}_i (C_\beta{}^\alpha{}_\gamma + C^\alpha{}_{\beta\gamma}) \,.$$

Using the fact that the metric is diagonal, and the expression (C.1.20), we can easily check that,

$$C^\alpha{}_{\beta\gamma} = -\left(\frac{\Delta S_\alpha}{\Delta S_\beta}\right)^2 C_\beta{}^\alpha{}_\gamma \,, \tag{C.2.38}$$

so that,

$$(D_{\beta i}{}^\alpha + D^\alpha{}_{i\,\beta}) = \frac{2(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\beta} + \sqrt{2}\Delta S_\gamma A^\gamma{}_i C_\beta{}^\alpha{}_\gamma \left(1 - \left(\frac{\Delta S_\alpha}{\Delta S_\beta}\right)^2\right) \,. \tag{C.2.39}$$

Note that,

$$1 - \frac{\Delta S_\alpha^2}{\Delta S_\beta^2} = \frac{\Delta S_\alpha}{\Delta S_\beta}\left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right) \,, \tag{C.2.40}$$

$$\frac{\Delta S_\alpha}{\Delta S_\beta} C_\beta{}^\alpha{}_\gamma = \frac{\Delta S_\alpha}{\Delta S_\beta}\left(\frac{\Delta S_\beta}{\sqrt{2}\Delta S_\alpha \Delta S_\gamma}\right)\epsilon_\beta{}^\alpha{}_\gamma \,, = -\left(\frac{1}{\sqrt{2}\Delta S_\gamma}\right)\epsilon^\alpha{}_{\beta\gamma} \,, \tag{C.2.41}$$

so that,

$$({D_{\beta i}}^{\alpha} + {D^{\alpha}}_{i\,\beta}) = \frac{2(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\beta} - {A^\gamma}_i {\epsilon^\alpha}_{\beta\gamma}\left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right), \qquad \text{(C.2.42)}$$

We also rewrite the term ${D^{\beta i}}_{\alpha}$ in a more convenient fashion,

$$\begin{aligned} {D^{\beta i}}_{\alpha} &= \frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\beta} + \sqrt{2}\Delta S_\gamma {A^\gamma}_i\left(\frac{\Delta S_\beta {\epsilon^\beta}_{\alpha\gamma}}{\sqrt{2}\Delta S_\alpha \Delta S_\gamma}\right) \\ &= \frac{(\Delta S_\alpha)_{,i}}{\Delta S_\alpha}\delta_{\alpha\beta} + {A^\mu}_i {\epsilon^\beta}_{\alpha\mu}\left(\frac{\Delta S_\beta}{\Delta S_\alpha}\right) \qquad \text{(C.2.43)} \end{aligned}$$

Now we have to multiply the terms of (C.2.42) and (C.2.43) and sum over α, β and i. As we can easily check, there are no cross terms mixing $\delta_{\alpha\beta}$ with the Levi-Civita tensor ϵ. The reason is that the first is only non zero when $\alpha = \beta$, while the second one is only non zero when $\alpha \neq \beta$. Because of this, the product under consideration only has two terms. The first one being,

$$\sum_{\alpha\beta i} \frac{2(\Delta S_\alpha)_{,i}^2}{\Delta S_\alpha^2}\delta_{\alpha\beta}^2 = 2\sum_{\alpha i}\frac{{(\Delta S_\alpha)_{,i}}^2}{\Delta S_\alpha^2}\,, \qquad \text{(C.2.44)}$$

while the second one is,

$$-\sum_{\alpha\beta i}\left\{{A^\gamma}_i {\epsilon^\alpha}_{\beta\gamma}\left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right){A^\mu}_i {\epsilon^\beta}_{\alpha\mu}\right\}\left(\frac{\Delta S_\beta}{\Delta S_\alpha}\right),$$

Notice that, in the previous equation, the term inside the braces is antisymmetric in α, β since it is the product of three antisymmetric terms. Because of this, we can consider only the antisymmetric part of the last term to obtain,

$$\begin{aligned} &-\frac{1}{2}\sum_{\alpha\beta i}{A^\gamma}_i {\epsilon^\alpha}_{\beta\gamma}\left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right){A^\mu}_i {\epsilon^\beta}_{\alpha\mu}\left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right) = \\ &\qquad -\frac{1}{2}\sum_{\alpha\beta i}{A^\gamma}_i {\epsilon^\alpha}_{\beta\gamma}{A^\mu}_i {\epsilon^\beta}_{\alpha\mu}\left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right)^2, \end{aligned}$$

Putting everything together, we obtain,

$$({D_{\beta i}}^{\alpha}{D^{\beta i}}_{\alpha} + {D^{\alpha}}_{i\,\beta}{D^{\beta i}}_{\alpha}) = ({D_{\beta i}}^{\alpha} + {D^{\alpha}}_{i\,\beta}){D^{\beta i}}_{\alpha}$$

$$= 2\sum_{\alpha i} \frac{(\Delta S_\alpha)_i^2}{\Delta S_\alpha^2} - \frac{1}{2}\sum_{\alpha\beta i} A^\gamma{}_i \epsilon^\alpha{}_{\beta\gamma} A^\mu{}_i \epsilon^\beta{}_{\alpha\mu} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right)^2 , \tag{C.2.45}$$

By substituting (C.2.45) and (C.2.36) in (C.2.27), and using the fact that, for the projective Hilbert space of a spin s, the curvature is $R(h) = 4s(2s+1)$ (as it is proved in equation (C.2.67) further down) we obtain,

$$4s(2s+1) = R(k) + R(g) - \frac{1}{4}\Omega^\alpha{}_{ik}\Omega_\alpha{}^{ik} + \frac{1}{4}\sum_{\alpha\beta i} A^\gamma{}_i \epsilon^\alpha{}_{\beta\gamma} A^\mu{}_i \epsilon^\beta{}_{\alpha\mu} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right)^2 - \sum_{\alpha i} \frac{(\Delta S_\alpha)_{,i}{}^2}{\Delta S_\alpha^2} - \sum_i (\ln \mathrm{Det} k)_{,ii} - \frac{1}{4}\sum_i (\ln \mathrm{Det} k)_{,i}{}^2 . \tag{C.2.46}$$

C.2.2 Relationship between $R(h)$, $R(k)$ $\mathrm{Tr}\,(\Omega^2)$ and $R(g)$: a second equation

In this section, we rewrite the result (C.2.46) of the previous section so that the resulting one does not involve the coefficients A of the particular section σ we chose; that is, we write (C.2.46) in a covariant way.

First, consider the term $E_i(k_{\alpha\beta})$. Using equation (C.2.30) we can rewrite it as,

$$E_i(k_{\alpha\beta}) = \sigma_*(\underline{E}_i)k_{\alpha\beta} - A^\mu{}_i S^\sharp_\mu k_{\alpha\beta} . \tag{C.2.47}$$

For points over the section, the first term in the r.h.s. is non zero only if $\alpha = \beta$. In the other cases, by construction of σ, $k_{\alpha\beta}$ is zero along the points in the section, so its derivative in directions tangent to the section is of course zero.

In case that $\alpha = \beta$ we have, using the expression (C.1.18) (no sum implied over α),

$$\sigma_*(\underline{E}_i)k_{\alpha\alpha} = 2\sigma_*(\underline{E}_i)\Delta S_\alpha^2 = 4\Delta S_\alpha \sigma_*(\underline{E}_i)\Delta S_\alpha = 4\Delta S_\alpha(\Delta S_\alpha)_{,i} , \tag{C.2.48}$$

where, we used the fact that ΔS_α is constant along the fibers (just like we did in the previous section) to change the derivative $\sigma_*(\underline{E}_i)$ by the one with respect E_i to conclude the last equality.

By considering the previous observations we obtain,

$$\sigma_*(\underline{E}_i)k_{\alpha\beta} = 4\Delta S_\alpha(\Delta S_\alpha)_{,i}\delta_{\alpha\beta}\,. \tag{C.2.49}$$

Now, we calculate $S^\sharp_\mu k_{\alpha\beta}$ at our point of interest, $\rho_0 = \sigma(\mathcal{S})$. Note that an integral curve of the field $S^\sharp_\mu$ is $\rho(t) = R_{g(t)}\rho_0$, with $g(t) = e^{-iS_\mu t}$. Along this curve, the function $k_{\alpha\beta}$ has the following values,

$$\begin{aligned} k_{\alpha\beta} = h_{\rho(t)}(S^\sharp_\alpha, S^\sharp_\beta) = h_{R_{g(t)}\rho_0}(S^\sharp_\alpha, S^\sharp_\beta) &= h_{\rho_0}(R_{g(t)^{-1}*}S^\sharp_\alpha, R_{g(t)^{-1}*}S^\sharp_\beta) \\ &= h_{\rho_0}((gS_\alpha g^{-1})^\sharp, (gS_\beta g^{-1})^\sharp)\,, \end{aligned} \tag{C.2.50}$$

where we used the fact that the metric is right invariant for the third equality, and the way the fundamental vectors transform with the pushforward of the right action in any fiber bundle to conclude the last one. By taking the derivative w.r.t. t, and by remembering that, at the section σ, k is diagonal we obtain,

$$\begin{aligned} S^\sharp_\mu(k_{\alpha\beta}) &= h_\rho(-i[S_\mu, S_\alpha]^\sharp, S^\sharp_\beta) + h_\rho(S^\sharp_\alpha, -i[S_\mu, S_\beta]^\sharp) \\ &= h_\rho(\epsilon^\gamma{}_{\mu\alpha}S^\sharp_\gamma, S^\sharp_\beta) + h_\rho(S^\sharp_\alpha, \epsilon^\gamma{}_{\mu\beta}S^\sharp_\gamma) && \text{(C.2.51)} \\ &= 2\Delta S^2_\beta\epsilon^\beta{}_{\mu\alpha} + 2\Delta S^2_\alpha\epsilon^\alpha{}_{\mu\beta} = 2\epsilon^\beta{}_{\mu\alpha}(\Delta S^2_\beta - \Delta S^2_\alpha) && \text{(C.2.52)} \end{aligned}$$

From this, by considering (C.2.47), (C.2.49) and (C.2.52), we conclude following equation, (no sum over α, β or i implied),

$$k_{\alpha\beta,i} = E_i k_{\alpha\beta} = 4\Delta S_\alpha(\Delta S_\alpha)_{,i}\delta_{\alpha\beta} - 2A^\mu{}_i\epsilon^\beta{}_{\mu\alpha}(\Delta S^2_\beta - \Delta S^2_\alpha)\,. \tag{C.2.53}$$

Using this equation, we can compute the following term,

$$\begin{aligned} &\sum_{\alpha\beta i} k^{\alpha\mu}k^{\beta\nu}k_{\alpha\beta,i}k_{\mu\nu,i} \\ &\quad= \sum_{\alpha\beta i}(4\Delta S_\alpha(\Delta S_\alpha)_{,i}\delta_{\alpha\beta} - 2A^\mu{}_i\epsilon^\beta{}_{\mu\alpha}(\Delta S^2_\beta - \Delta S^2_\alpha)) \\ &\qquad\qquad \cdot (4\Delta S_\gamma(\Delta S_\gamma)_{,i}\delta_{\gamma\delta} - 2A^\nu{}_i\epsilon^\delta{}_{\nu\gamma}(\Delta S^2_\delta - \Delta S^2_\gamma))k^{\alpha\gamma}k^{\delta\beta} \\ &\quad= \sum_{\alpha\beta i}(4\Delta S_\alpha(\Delta S_\alpha)_{,i}\delta_{\alpha\beta} - 2A^\mu{}_i\epsilon^\beta{}_{\mu\alpha}(\Delta S^2_\beta - \Delta S^2_\alpha)) \\ &\qquad\qquad \cdot (4\Delta S_\gamma(\Delta S_\gamma)_{,i}\delta_{\gamma\delta} - 2A^\nu{}_i\epsilon^\delta{}_{\nu\gamma}(\Delta S^2_\delta - \Delta S^2_\gamma))\frac{\delta_{\alpha\gamma}\delta_{\gamma\delta}}{4\Delta S^2_\alpha\Delta S^2_\beta} \\ &\quad= \sum_{\alpha\beta i}\frac{1}{4\Delta S^2_\alpha\Delta S^2_\beta}(4\Delta S_\alpha(\Delta S_\alpha)_{,i}\delta_{\alpha\beta} - 2A^\mu{}_i\epsilon^\beta{}_{\mu\alpha}(\Delta S^2_\beta - \Delta S^2_\alpha)) \\ &\qquad\qquad \cdot (4\Delta S_\alpha(\Delta S_\alpha)_{,i}\delta_{\alpha\beta} - 2A^\nu{}_i\epsilon^\beta{}_{\nu\alpha}(\Delta S^2_\beta - \Delta S^2_\alpha)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha\beta i} \frac{1}{\Delta S_\alpha^2 \Delta S_\beta^2} (2\Delta S_\alpha (\Delta S_\alpha)_{,i} \delta_{\alpha\beta} - A^\mu{}_i \epsilon^\beta{}_{\mu\alpha} (\Delta S_\beta^2 - \Delta S_\alpha^2)) \\
&\qquad \cdot (2\Delta S_\alpha (\Delta S_\alpha)_{,i} \delta_{\alpha\beta} - A^\nu{}_i \epsilon^\beta{}_{\nu\alpha} (\Delta S_\beta^2 - \Delta S_\alpha^2)) \\
&= \sum_{\alpha\beta i} \left(2\frac{(\Delta S_\alpha)_{,i}}{\Delta S_\beta} \delta_{\alpha\beta} - A^\mu{}_i \epsilon^\beta{}_{\mu\alpha} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta} \right) \right) \\
&\qquad \cdot \left(2\frac{(\Delta S_\alpha)_{,i}}{\Delta S_\beta} \delta_{\alpha\beta} - A^\nu{}_i \epsilon^\beta{}_{\nu\alpha} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta} \right) \right) ,
\end{aligned}
$$

where the sums over μ and ν are left implied. Note that, as it has already happened in the previous section, the terms proportional to the Kronecker delta and the ones to the Levi-Civita tensor do not mix; when one of them is not zero the other one is. Because of this, after making the product, there are only two non zero terms. The first one is,

$$
\sum_{\alpha\beta i} 4\frac{(\Delta S_\alpha)_{,i}{}^2}{\Delta S_\beta^2} \delta_{\alpha\beta}^2 = 4\sum_{\alpha i} \frac{(\Delta S_\alpha)_{,i}{}^2}{\Delta S_\alpha^2} , \tag{C.2.54}
$$

while the second one is,

$$
\sum_{\alpha\beta i} A^\mu{}_i \epsilon^\beta{}_{\mu\alpha} A^\nu{}_i \epsilon^\beta{}_{\nu\alpha} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta} \right)^2 , \tag{C.2.55}
$$

By putting together the last three equations, we obtain,

$$
\begin{aligned}
&\sum_{\alpha\beta i} k^{\alpha\mu} k^{\beta\nu} (k_{\alpha\beta})_{,i} (k_{\mu\nu})_{,i} \\
&\qquad = 4\sum_{\alpha i} \frac{(\Delta S_\alpha)_{,i}{}^2}{\Delta S_\alpha^2} + \sum_{\alpha\beta i} A^\mu{}_i \epsilon^\beta{}_{\mu\alpha} A^\nu{}_i \epsilon^\beta{}_{\nu\alpha} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta} \right)^2 \\
&\qquad = 4\sum_{\alpha i} \frac{(\Delta S_\alpha)_{,i}{}^2}{\Delta S_\alpha^2} - \sum_{\alpha\beta i} A^\mu{}_i \epsilon^\beta{}_{\alpha\mu} A^\nu{}_i \epsilon^\alpha{}_{\beta\nu} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta} \right)^2 ,
\end{aligned} \tag{C.2.56}
$$

where the last term was computed in following way,

$$
\begin{aligned}
&\sum_{\alpha\beta i} A^\mu{}_i \epsilon^\beta{}_{\mu\alpha} A^\nu{}_i \epsilon^\beta{}_{\nu\alpha} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta} \right)^2 \\
&\qquad = \sum_{\alpha\beta i} A^\mu{}_i \epsilon^\beta{}_{\alpha\mu} A^\gamma{}_i \epsilon^\beta{}_{\alpha\gamma} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta} \right)^2
\end{aligned}
$$

$$= -\sum_{\alpha\beta i} A^{\mu}{}_{i}\epsilon^{\beta}{}_{\alpha\mu} A^{\gamma}{}_{i}\epsilon^{\alpha}{}_{\beta\gamma} \left(\frac{\Delta S_\beta}{\Delta S_\alpha} - \frac{\Delta S_\alpha}{\Delta S_\beta}\right)^2 . \tag{C.2.57}$$

By substituting (C.2.56) in (C.2.46), we conclude,

$$\begin{aligned} 4s(2s+1) = R(k) + R(g) - \frac{1}{4}\Omega^{\alpha}{}_{ik}\Omega_{\alpha}{}^{ik} - \frac{1}{4}\sum_i k^{\alpha\mu}k^{\beta\nu}k_{\alpha\beta,i}k_{\mu\nu,i} \\ - \sum_i (\ln \mathrm{Det} k)_{,ii} - \frac{1}{4}\sum_i (\ln \mathrm{Det} k)_{,i}{}^2 . \end{aligned} \tag{C.2.58}$$

Finally, recall that the basis $\{\underline{E}_i\}$ is orthonormal by definition. Therefore, to compute the gradient or Laplacian of a function f (denoted by $\nabla_{\mathscr{S}} f$ and $\nabla^2_{\mathscr{S}} f$ respectively), we can use the well known formulas valid for Euclidean spaces. Also, we can trivially raise the index i. This implies,

$$\begin{aligned} 4s(2s+1) = R(k) + R(g) - \frac{1}{4}\,\mathrm{Tr}\,(\Omega^2) - \frac{1}{4}k^{\alpha\mu}k^{\beta\nu}k_{\alpha\beta,}{}^{i}k_{\mu\nu,i} \\ - \nabla^2_{\mathscr{S}}\Phi - \frac{1}{4}||\nabla_{\mathscr{S}}\Phi||^2_{\mathscr{S}} , \end{aligned} \tag{C.2.59}$$

where we defined,

$$\Phi \equiv \ln \mathrm{Det}\ k\,, \qquad \mathrm{Tr}\,(\Omega^2) \equiv \Omega^{\alpha}{}_{ik}\Omega_{\alpha}{}^{ik} . \tag{C.2.60}$$

Checking the formulas for $s = 1$

In this subsection, we check the formula (C.2.46) for $s = 1$. This calculation also serves as an example to see how to deal with the terms appearing in said formula.

We make our calculations using the section defined in (2.1.5). In section 2.1.2, we can find the expression for the metric k along with the one of ΔS_α –note that some rescaling has to be done because of the redefinition of the Fubini-Study metric done in (C.2.1). As noted in 2.1.2, the vectors v tangent to the section are horizontal. Also, the metric k, is diagonal when evaluated at the points of the section. Because of this, the section of (2.1.5) coincides with the section σ used to obtain (C.2.46). The horizontality of v implies (see (C.2.30)),

$$A^{\alpha}{}_{i} = 0 . \tag{C.2.61}$$

Now we need to find the expression for terms of the type $f_{,i}$. Since in this case $\mathscr{S}$ is one dimensional, the index i can only take the value 1. Although the vector ∂_q is horizontal, it is not of unitary magnitude. Because of this we have to work with the normalized field $E_q \equiv E_q = \frac{1}{\sqrt{g_{qq}}}\partial_q = \frac{3+\cos q}{\sqrt{2}\sin(q/2)}\partial_q$.

From equation (2.1.20), we can read the coefficients ΔS_α^2 (no rescaling needed in this case). From (2.1.21), we can obtain Det k. After rescaling accordingly, the result is,

$$\text{Det}\ k = \frac{32\sin^2(q/2)\sin^2(q)}{(3+\cos q)^4}\,.$$

With the help of *Mathematica*, we find the following expressions,

$$\begin{aligned}
-\sum_{\alpha i}\frac{(\Delta S_\alpha)_{,i}{}^2}{\Delta S_\alpha^2} &= -\sum_{\alpha=x,y,z}^{3}\frac{(E_q(\Delta S_\alpha))^2}{\Delta S_\alpha^2}\\
&= -\frac{69+(2+\cos q+64\csc^4\frac{q}{2})\cos q}{4(1+\cos q)}\,,\\
-\sum_i(\ln\text{Det}k)_{,ii} &= -E_q(E_q(\ln\text{Det}k)) = \frac{(3+\cos q)^4}{4(\cos q-1)^2\cos^2\frac{q}{2}}\,,\\
-\frac{1}{4}\sum_i(\ln\text{Det}k)_{,i}{}^2 &= -\frac{1}{4}(E_q(\ln\text{Det}k))^2 = \frac{(\cos 2q-20\cos q-13)^2}{16(\cos q-1)\sin^2 q}\,,
\end{aligned}$$

Surprisingly enough, the sum of these three terms is 11,

$$-\sum_{\alpha i}\frac{(\Delta S_\alpha)_{,i}{}^2}{\Delta S_\alpha^2}-\sum_i(\ln\text{Det}k)_{,ii}-\frac{1}{4}\sum_i(\ln\text{Det}k)_{,i}{}^2 = 11\,. \tag{C.2.62}$$

Since $\mathscr{S}$ is one dimensional in this case, the equalities $R(g) = -(1/4)\Omega^\alpha{}_{ij}\Omega_\alpha{}^{ij} = 0$ hold. By substituting these equalities, (C.2.61), (C.2.62) and $s=1$ in (C.2.46) we obtain,

$$12 = R(k)+0+0+0+11\,, \tag{C.2.63}$$

This result implies that $R(k)$ is equal to one, just as was obtained in (2.1.22) (after properly rescaling). From here we can conclude that (C.2.46) holds ,at least, for the case of $s=1$.

C.2.3 Ricci scalar for $\mathbb{P}(\mathcal{H}_s)$

Up to now in this appendix, we have barely used that we are working with the Fubini-Study metric h. In fact, a close examination reveals that (C.2.46) and (C.2.59) are valid for any fiber bundle where the acting group is $SO(3)$. In this section, we use the fact that we are working with $\mathbb{P}(\mathcal{H}_s)$. The first step is to compute the Riemann tensor for h. Here we do not make all the calculations, since that can be found in the literature (c.f. [9, 87]). We just present them here so we can reference them later on. we present it here to later reference some partial results.

To make this calculation, we work with affine coordinates (c.f. equation (1.1.5)). Denote them by $z^1, \ldots, z^{2s}$. Suppose z^j can be written in terms of its real and imaginary parts as $z^j = x^j + iy^j$. In these terms, define the following complex tangent vectors,

$$\partial_{z^j} = \frac{1}{2}(\partial_{x^j} - i\partial_{y^j}) , \quad \partial_{\bar{z}^j} = \frac{1}{2}(\partial_{x^j} + i\partial_{y^j}) . \tag{C.2.64}$$

We can calculate the metric h in terms of ∂_j and $\partial_\imath$. The calculation is straightforward, but a little lengthy. Here we only present the final result,

$$\begin{aligned} h_{i\bar{k}} &\equiv h(\partial_{z^i}, \partial_{\bar{z}^k}) = \frac{\mathcal{N}^2\delta_{ik} - \bar{z}^i z^k}{\mathcal{N}^4} , \qquad & h_{\bar{\imath}k} &\equiv h(\partial_{\bar{z}^i}, \partial_{z^k}) = \frac{\mathcal{N}^2\delta_{ik} - z^i \bar{z}^k}{\mathcal{N}^4} , \\ h_{ik} &\equiv h(\partial_{z^i}, \partial_{z^k}) = 0 , & h_{\bar{\imath}\bar{k}} &\equiv h(\partial_{\bar{z}^i}, \partial_{\bar{z}^k}) = 0 , \end{aligned} \tag{C.2.65}$$

where $\mathcal{N}$ denotes the positive factor defined by the following equation,

$$\mathcal{N}^2 = 1 + \sum_{z=1}^{2s} |z^i|^2$$

Note that, at the point where all the coordinates z^i are zero, the metric reduces simply to $h_{i\bar{k}} = h_{\bar{\imath}k} = \delta_{ik}$. By inverting the previous equations, the inverse metric for a generic point can be written as,

$$h^{i\bar{k}} = \mathcal{N}^2(\delta_{ik} + \bar{z}^i z^k) , \quad h^{\bar{\imath}k} = \mathcal{N}^2(\delta_{ik} + z^i \bar{z}^k) , \quad h^{ik} = h^{\bar{\imath}\bar{k}} = 0 . \tag{C.2.66}$$

Since the projective Hilbert space with the Fubini-Study metric is a Kähler space, the components of the Riemann tensor R can be computed using the formulas of [9]. The result is the following,

$$R^a{}_{b\bar{f}e} = -\delta_{ab}h_{e\bar{f}} - \delta_{ae}h_{b\bar{f}} = -R^a{}_{be\bar{f}} , \quad R^{\bar{a}}{}_{\bar{b}f\bar{e}} = -\delta_{ab}h_{\bar{e}f} - \delta_{ae}h_{\bar{b}f} = -R^{\bar{a}}{}_{\bar{b}\bar{e}f} ,$$

while the rest of the components are zero. Using this expressions and the one for the metric (C.2.66), we obtain the following results (that are useful for the calculation of the curvature $R(h)$) ,

$$\begin{aligned} R^{\bar{a}b}{}_{\bar{c}d} &= h^{be}R^{\bar{a}}{}_{e\bar{c}d} + h^{b\bar{e}}R^{\bar{a}}{}_{\bar{e}\bar{c}d} = h^{b\bar{e}}R^{\bar{a}}{}_{\bar{e}\bar{c}d} = h^{b\bar{e}}(\delta_{ae}h_{\bar{c}d} + \delta_{ac}h_{\bar{e}d}) \\ &= h^{b\bar{a}}h_{\bar{c}d} + \delta_{ac}\delta_{bd} , \\ R^{\bar{a}b}{}_{\bar{a}d} &= h^{b\bar{a}}h_{\bar{a}d} + \delta_{aa}\delta_{bd} = \delta_{bd} + (2s)\delta_{bd} = (2s+1)\delta_{bd} , \\ R^{a\bar{b}}{}_{c\bar{d}} &= R^{\bar{b}a}{}_{\bar{d}c} = h^{\bar{b}a}h_{\bar{d}c} + \delta_{ac}\delta_{bd} , \end{aligned}$$

$$\begin{aligned}
&R^{a\bar{b}}{}_{a\bar{d}} = h^{\bar{b}a}h_{\bar{d}a} + \delta_{aa}\delta_{bd} \ = (2s+1)\delta_{bd}\,, \\
&R^{ab}{}_{cd} = h^{be}R^{a}{}_{ecd} + h^{b\bar{e}}R^{a}{}_{\bar{e}cd} = 0\cdot R^{a}{}_{ecd} + h^{b\bar{e}}\cdot 0 = 0\,, \\
&R^{\bar{a}\bar{b}}{}_{\bar{c}\bar{d}} = 0\,.
\end{aligned}$$

From these expressions, we can calculate the Ricci tensor, obtaining the following expressions,

$$R^{b}{}_{f} = R^{ab}{}_{af} + R^{\bar{a}b}{}_{\bar{a}f} = (2s+1)\delta_{bf}\,, \quad R^{\bar{b}}{}_{\bar{f}} = R^{a\bar{b}}{}_{a\bar{f}} + R^{\bar{a}\bar{b}}{}_{\bar{a}\bar{f}} = (2s+1)\delta_{bf}\,.$$

Finally, we can conclude,

$$R(h) = R^{a}{}_{a} + R^{\bar{a}}{}_{\bar{a}} = 2s(2s+1) + 2s(2s+1) = 4s(2s+1)\,. \tag{C.2.67}$$

To conclude this section, we write an expression that is useful for some calculations of the next section. Recall that, at the point where all the coordinates z^i are zero, the metric (C.2.65) is simpler. Therefore, in said point, we have the following expressions for the Riemann tensor,

$$R^{\bar{a}b}{}_{\bar{c}d} = \delta_{ba}\delta_{cd} + \delta_{ac}\delta_{bd}\,, \tag{C.2.68}$$

while the rest of the non zero components can be obtained using the symmetries of Riemann tensor.

C.2.4 Relationship between $R(h)$, $R(k)$ $\mathrm{Tr}\,(\Omega^2)$ and $R(g)$: a third equation

By considering equation (C.2.25), we can calculate the term $\Omega^{\alpha}{}_{ij}\Omega_{\alpha}{}^{ij}$ in terms of the curvature $R(g)$, provided we calculated previously the term $R(h)^{ij}{}_{ij}$. This what we do in this section.

First we check that $R(h)^{ij}{}_{ij}$ is independent of the basis in tangent space chosen, as long as the basis consist only on vertical and horizontal vectors. Suppose we have a new basis $\{V_a\}$ for tangent space, defined in the following way (using the notation of the previous sections),

$$\begin{aligned}
V_{\alpha} &= \Lambda^{\beta}{}_{\alpha}E_{\beta}\,, & V_i &= G^{j}{}_{i}E_j\,, \\
W^{\alpha} &= (\Lambda^{-1})^{\alpha}{}_{\beta}\varphi^{\beta}\,, & W^i &= (G^{-1})^{i}{}_{j}\varphi^{j}\,,
\end{aligned}$$

where W^a is the dual basis of V_a. Note that, by construction, V_α is a vertical vector, while V^i is horizontal. From here, it is immediate that $R(h)^{ij}{}_{ij}$ is invariant under the mentioned changes of basis; indeed, if we compute it w.r.t. the basis $\{V_a\}$, we obtain the following,

$$R_V(h)^{ij}{}_{ij} = R(h)(W^i, W^j, V_i, V_j)$$

$$
\begin{aligned}
&= (G^{-1})^i{}_k (G^{-1})^j{}_l G^m{}_i G^n{}_j R(h)(\varphi^k, \varphi^l, E_m, E_n) \\
&= \delta^k{}_m \delta^l{}_n R(h)(\varphi^k, \varphi^l, E_k, E_l) = R(h)(\varphi^k, \varphi^l, E_k, E_l) = R(h)^{kl}{}_{kl}\,,
\end{aligned}
$$

so $R(h)^{ij}{}_{ij}$ produces the same result for both basis. Using this fact, one can also prove that this term is constant along the fibers; this is immediate from the fact that the right action R_{r*} maps horizontal vectors in horizontal vectors and vertical vectors in vertical vectors and that it is an isometry.

Now, we proceed to make the actual calculation. Since it does not depend on the basis (as long as it consists only on horizontal and vertical vectors) we work in a whole new basis that simplifies the calculation. In the following paragraphs, we explain how to build said basis. The ideas is to give some special coordinates to $\mathbb{P}(\mathcal{H}_s)$ and then take the partial derivatives as basis vectors.

Consider any normalized state in Hilbert space. Call it $|\psi_0\rangle$. Now, we show how to calculate $R(h)^{ij}{}_{ij}$ in the fiber associated with the point $[\psi_0]$ (as we have already proved that this is a constant quantity along the fibers). To this end, consider the states $|\psi_\alpha\rangle = iS_\alpha|\psi_0\rangle$ Note that we are considering the state $iS_\alpha|\psi_0\rangle$ instead of $-iS_\alpha|\psi_0\rangle$ as one may expect. This because we are working with a right actions of with left one. Because of this, the state resulting of rotating $|\psi_0\rangle$ infinitesimally around the axis α is $|\psi_0\rangle + i\epsilon S_\alpha|\psi_0\rangle = |\psi_0\rangle + i\epsilon|\psi_\alpha\rangle$.

Complete the set $\{|\psi_0\rangle, \{|\psi_\alpha\rangle\}\}$ to a basis of Hilbert space

$$
\{|\psi_a\rangle\} = \{|\psi_0\rangle, |\psi_x\rangle, |\psi_y\rangle, |\psi_z\rangle, |\psi_1\rangle, \ldots, |\psi_{2s-3}\rangle\}\,,
$$

in such way that the equalities $\langle\psi_i|\psi_0\rangle = \langle\psi_i|\psi_\alpha\rangle = 0$, $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, hold for all $1 \leq i, j \leq 2s-3$ (this can be achieved, for instance, using the Gram-Schmidt procedure).

In terms of the states, we give complex coordinates to projective Hilbert space z^a $(a = 1, \ldots, 2s)$ by considering the following,

$$
|\psi\rangle = |\psi_0\rangle + z^a|\psi_a\rangle\,, \quad \{z^a\} \to [\psi] \tag{C.2.69}
$$

Note that this expression is analogous for the one for affine coordinates (1.1.5), the difference being that there the set $|\psi\rangle$ was orthonormal while the one we are considering here is not –the vectors $|\psi_0\rangle, \{|\psi_\alpha\rangle\}$ are not orthogonal among each other. Because of this, the expressions found for the Riemann tensor in C.2.3 does not hold. We could make a calculation very similar to the one made in said section to find the components of the Riemann tensor w.r.t. the basis $\{\partial_{z^a}\}$, but we have found to just transform the expressions of C.2.3 to obtain the one in our basis. We explain how to make this in the following paragraphs.

Consider some states $|\eta_\alpha\rangle$ in the linear space spanned by the vectors $\{|\psi_0\rangle, |\psi_\alpha\rangle\}$ such that the set $\{|\psi_0\rangle, \{|\eta_\alpha\rangle\}\}$ is orthonormal (again, this can be accomplished by using the Gram-Schmidt procedure). This implies that the set $\{|\psi_0\rangle, \{|\psi_i\rangle\}, \{|\eta_\alpha\rangle\}\}$ is orthonormal, and now we can use the expressions of the section C.2.3. Since $|\eta_\alpha\rangle$ can be written in terms of $|\psi_0\rangle, \{|\psi_\alpha\rangle\}$ by definition, we can define the matrix H according to the following equations,

$$|\eta_\alpha\rangle = (H^{-1})^0{}_\alpha|\psi_0\rangle + (H^{-1})^\mu{}_\alpha|\psi_\mu\rangle\,, \quad |\psi_\alpha\rangle = H^0{}_\alpha|\psi_0\rangle + H^\mu{}_\alpha|\eta_\mu\rangle\,. \tag{C.2.70}$$

Since the set $|\psi_0\rangle$, $|\eta_\alpha\rangle$ is orthonormal, the coefficients H are not arbitrary, they have to satisfy some relations. This relations can be deduced by considering the equations $\langle\psi_0|\eta_\alpha\rangle = 0$ and $\langle\eta_\alpha|\eta_\beta\rangle = \delta_{\alpha\beta}$. The first equations imply that $(H^{-1})^0{}_\alpha = -(H^{-1})^\mu{}_\alpha\langle\psi_0|\psi_\mu\rangle$. This in turn implies,

$$|\eta_\alpha\rangle = (H^{-1})^\mu{}_\alpha(|\psi_\mu\rangle - \langle\psi_0|\psi_\mu\rangle|\psi_0\rangle)\,. \tag{C.2.71}$$

By substituting this equation in $\langle\eta_\alpha|\eta_\beta\rangle = \delta_{\alpha\beta}$ we obtain the following,

$$\begin{aligned}\delta_{\alpha\beta} = \langle\eta_\alpha|\eta_\beta\rangle &= (H^{-1})^\mu{}_\beta\,\langle\eta_\alpha|(|\psi_\mu\rangle - \langle\psi_0|\psi_\mu\rangle|\psi_0\rangle) = (H^{-1})^\mu{}_\beta\,\langle\eta_\alpha|\psi_\mu\rangle \\ &= (\overline{H^{-1}})^\nu{}_\alpha(H^{-1})^\mu{}_\beta\,(\langle\psi_\nu| - \langle\psi_\nu|\psi_0\rangle\langle\psi_0|)|\psi_\mu\rangle = (\overline{H^{-1}})^\nu{}_\alpha(H^{-1})^\mu{}_\beta N_{\nu\mu}\,,\end{aligned}$$

where a bar denotes complex conjugation, and we made the following definition,

$$N_{\nu\mu} = \langle\psi_\nu|\psi_\mu\rangle - \langle\psi_\nu|\psi_0\rangle\langle\psi_0|\psi_\mu\rangle\,. \tag{C.2.72}$$

Note that, the equality $N_{\mu\nu} = \bar{N}_{\nu\mu}$ holds.

In the same way, we can check the following equality,

$$\langle\psi_0|\psi_\alpha\rangle = \langle\psi_0|(H^0{}_\alpha|\psi_0\rangle + H^\mu{}_\alpha)|\eta_\mu\rangle = H^0{}_\alpha\,,$$

so that,

$$\begin{aligned}\langle\psi_\beta|\psi_\alpha\rangle &= (\bar{H}^0{}_\beta\langle\psi_0| + \bar{H}^\nu{}_\beta\langle\eta_\nu|)(H^0{}_\alpha|\psi_0\rangle + H^\mu{}_\alpha|\eta_\mu\rangle) = \bar{H}^0{}_\beta H^0{}_\alpha + \bar{H}^\nu{}_\beta H^\mu{}_\alpha\delta_{\nu\mu} \\ &= \langle\psi_0|\psi_\alpha\rangle\langle\psi_\beta|\psi_0\rangle + \bar{H}^\nu{}_\beta H^\mu{}_\alpha\delta_{\nu\mu}\,.\end{aligned}$$

From were we can conclude the following expression,

$$\bar{H}^\nu{}_\beta H^\mu{}_\alpha\delta_{\nu\mu} = \bar{H}^\mu{}_\beta H^\mu{}_\alpha = N_{\beta\alpha}\,. \tag{C.2.73}$$

Next, we prove an equality that may seem obvious at first glance, but in fact, it is not completely trivial. The matrix H that maps the basis $\{|\psi_0\rangle, \{|\eta_\alpha\rangle\}\}$ into $\{|\psi_0\rangle, \{|\psi_\alpha\rangle\}\}$ can be written in the following way,

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ {H^0}_1 & {H^1}_1 & {H^2}_1 & {H^3}_1 \\ {H^0}_2 & {H^1}_2 & {H^2}_2 & {H^3}_2 \\ {H^0}_3 & {H^1}_2 & {H^2}_3 & {H^3}_3 \end{pmatrix},$$

while its inverse matrix H^{-1} is given by the following expression,

$$H^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ {(H^{-1})^0}_1 & {(H^{-1})^1}_1 & {(H^{-1})^2}_1 & {(H^{-1})^3}_1 \\ {(H^{-1})^0}_2 & {(H^{-1})^1}_2 & {(H^{-1})^2}_2 & {(H^{-1})^3}_2 \\ {(H^{-1})^0}_3 & {(H^{-1})^1}_2 & {(H^{-1})^2}_3 & {(H^{-1})^3}_3 \end{pmatrix}.$$

Since the product of H and H^{-1} is the identity matrix, when we multiply the α-th row of H by the β-th column of H^{-1} we obtain,

$${(H^{-1})^\beta}_\nu {H^\nu}_\alpha = {\delta^\beta}_\alpha \,.$$

In the same fashion we obtain,

$${H^\beta}_\nu {(H^{-1})^\nu}_\alpha = {\delta^\beta}_\alpha \,.$$

Finally we prove another identity for the coefficients H. Since the set $\{|\psi_0\rangle, \{|\psi_\alpha\rangle\}\}$ is orthonormal, when restricted to the linear space spanned by said set (which includes the states $|\psi_\alpha\rangle$), we have the following ($\mathbb{1}$ denotes the identity operator of this subspace),

$$\mathbb{1} = |\psi_0\rangle\langle\psi_0| + |\eta_\mu\rangle\langle\eta_\mu| \Rightarrow \langle\psi_\beta|\psi_\alpha\rangle = \langle\psi_\beta|\psi_0\rangle\langle\psi_0|\psi_\alpha\rangle + \langle\psi_\beta|\eta_\mu\rangle\langle\eta_\mu|\psi_\alpha\rangle \,.$$

By substituting the expression for $|\eta_\mu\rangle$, and remembering the definition of $N_{\alpha\beta}$ (C.2.72), we obtain the following,

$$\begin{aligned} N_{\beta\alpha} &= {(H^{-1})^\sigma}_\mu {(\overline{H^{-1}})^\nu}_\mu \langle\psi_\beta|(|\psi_\sigma\rangle - \langle\psi_0|\psi_\sigma\rangle|\psi_0\rangle)(\langle\psi_\nu| - \langle\psi_\nu|\psi_0\rangle\langle\psi_0|)|\psi_\alpha\rangle \,. \\ &= {(H^{-1})^\sigma}_\mu {(\overline{H^{-1}})^\nu}_\mu N_{\beta\sigma} N_{\nu\alpha} \,. \end{aligned}$$

If we denote the elements of the inverse matrix of N as $N^{\beta\alpha}$ the previous equality can be written as,

$$N^{\sigma\beta} N_{\beta\alpha} N^{\alpha\nu} = {(H^{-1})^\sigma}_\mu {(\overline{H^{-1}})^\nu}_\mu \Rightarrow {(H^{-1})^\sigma}_\mu {(\overline{H^{-1}})^\nu}_\mu = N^{\sigma\nu} \,.$$

In the following lines, we put together all the relevant equations in order to refer them later,

$$\begin{aligned}(H^{-1})^{\sigma}{}_{\mu}(\overline{H^{-1}})^{\nu}{}_{\mu} &= N^{\sigma\nu}\,,\\ H^{\beta}{}_{\nu}(H^{-1})^{\nu}{}_{\alpha} &= \delta^{\beta}{}_{\alpha}\,,\\ (H^{-1})^{\beta}{}_{\nu}H^{\nu}{}_{\alpha} &= \delta^{\beta}{}_{\alpha}\,,\\ \bar{H}^{\mu}{}_{\beta}H^{\mu}{}_{\alpha} &= N_{\beta\alpha}\,,\\ (\overline{H^{-1}})^{\nu}{}_{\alpha}(H^{-1})^{\mu}{}_{\beta}N_{\nu\mu} &= \delta_{\alpha\beta}\end{aligned} \tag{C.2.74}$$

Now, we compute the Riemann tensor. Given $2s$ complex numbers ζ^a, give affine coordinates to $\mathbb{P}(\mathcal{H}_s)$ by considering the following basis,

$$\{|\psi_0\rangle, |\eta_x\rangle, |\eta_y\rangle, |\eta_z\rangle, |\psi_1\rangle, \ldots, |\psi_{2s-3}\rangle\}\,.$$

w.r.t. this basis, the equations of section C.2.3 holds. What remains is to see how to transform between the coordinates z^a /C.2.69) and the coordinates ζ^a. Given some coordinates z^a, the corresponding state $|\psi\rangle$ is the following,

$$\begin{aligned}|\psi_0\rangle + z^i|\psi_i\rangle + z^\alpha|\psi_\alpha\rangle &= |\psi_0\rangle + z^i|\psi_i\rangle + z^\alpha H^0{}_\alpha|\psi_0\rangle + z^\alpha H^\mu{}_\alpha|\eta_\mu\rangle\\ &= (1+z^\alpha H^0{}_\alpha)|\psi_0\rangle + z^i|\psi_i\rangle + z^\alpha H^\mu{}_\alpha|\eta_\mu\rangle\,.\end{aligned}$$

But, projectively, this state is equivalent to the following one,

$$|\psi_0\rangle + \frac{z^i}{1+z^\alpha H^0{}_\alpha}|\psi_i\rangle + \frac{z^\beta}{1+z^\alpha H^0{}_\alpha}H^\mu{}_\beta|\eta_\mu\rangle \;= |\psi_0\rangle + \zeta^i|\psi_i\rangle + \zeta^\mu|\eta_\mu\rangle\,.$$

From here we obtain the transformation law between coordinates,

$$\zeta^i = \frac{z^i}{1+z^\alpha H^0{}_\alpha}\,,\quad \zeta^\mu = \frac{z^\beta}{1+z^\alpha H^0{}_\alpha}H^\mu{}_\beta\,.$$

By differentiating the previous equalities we obtain the following,

$$\begin{aligned}d\zeta^i &= \frac{dz^i}{1+z^\alpha H^0{}_\alpha} - \frac{z^i H^0{}_\gamma}{(1+z^\alpha H^0{}_\alpha)^2}dz^\gamma\\ d\zeta^\mu &= \frac{H^\mu{}_\beta}{1+z^\alpha H^0{}_\alpha}dz^\beta - \frac{z^\beta H^0{}_\gamma}{(1+z^\alpha H^0{}_\alpha)^2}H^\mu{}_\beta dz^\gamma\,.\end{aligned}$$

By evaluating at the origin (where all the z^a are zero) we obtain the following expressions valid at $[\psi_0]$,

$$d\zeta^i = dz^i\,,\quad d\zeta^\mu = H^\mu{}_\beta dz^\beta\,. \tag{C.2.75}$$

This equation, along with (C.2.74), implies the following set of equalities,

$$\partial_{z^i} = \partial_{\zeta^i}\,, \quad \partial_{z^\mu} = H^\beta{}_\mu \partial_{\zeta^\beta}\,, \quad dz^i = d\zeta^i\,, \quad dz^\mu = (H^{-1})^\mu{}_\beta d\zeta^\beta\,. \qquad (\text{C.2.76})$$
$$\partial_{\bar z^i} = \partial_{\bar\zeta^i}\,, \quad \partial_{\bar z^\mu} = \bar H^\beta{}_\mu \partial_{\bar\zeta^\beta}\,, \quad d\bar z^i = d\bar\zeta^i\,, \quad d\bar z^\mu = (\overline{H^{-1}})^\mu{}_\beta d\bar\zeta^\beta\,.$$

By using this results with (C.2.68), we can check that, up to symmetries, the only non zero components of the Riemann tensor are the following,

$$\begin{aligned}
R^{z^i\bar z^j}{}_{z^k\bar z^l} &= R(dz^i, d\bar z^j, \partial_{z^k}, \partial_{\bar z^l}) = R(d\zeta^i, d\bar\zeta^j, \partial_{\zeta^k}, \partial_{\bar\zeta^l}) = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\,,\\
R^{z^i\bar z^\alpha}{}_{z^j\bar z^\beta} &= R(dz^i, d\bar z^\alpha, \partial_{z^j}, \partial_{\bar z^\beta}) = (\overline{H^{-1}})^\alpha{}_\mu \bar H^\nu{}_\beta R(d\zeta^i, d\bar\zeta^\mu, \partial_{\zeta^j}, \partial_{\bar\zeta^\nu})\\
&= (\overline{H^{-1}})^\alpha{}_\mu \bar H^\nu{}_\beta (\delta_{ij}\delta_{\mu\nu} + 0) = \delta_{ij}\delta_{\alpha\beta}\,,\\
R^{z^i\bar z^\alpha}{}_{z^\beta\bar z^j} &= (\overline{H^{-1}})^\alpha{}_\mu H^\nu{}_\beta R(d\zeta^i, d\bar\zeta^\mu, \partial_{\zeta^\nu}, \partial_{\bar\zeta^j}) = 0 + 0 = 0 \qquad (\text{C.2.77})\\
R^{z^\alpha\bar z^\beta}{}_{z^\mu\bar z^\nu} &= (H^{-1})^\alpha{}_\sigma (\overline{H^{-1}})^\beta{}_\lambda H^\gamma{}_\mu \bar H^\tau{}_\nu R(d\zeta^\sigma, d\bar\zeta^\lambda, \partial_{\zeta^\gamma}, \partial_{\bar\zeta^\tau})\\
&= (H^{-1})^\alpha{}_\sigma (\overline{H^{-1}})^\beta{}_\lambda H^\gamma{}_\mu \bar H^\tau{}_\nu (\delta_{\sigma\lambda}\delta_{\gamma\tau} + \delta_{\sigma\gamma}\delta_{\lambda\tau})\\
&= (H^{-1})^\alpha{}_\lambda (\overline{H^{-1}})^\beta{}_\lambda H^\tau{}_\mu \bar H^\tau{}_\nu + (H^{-1})^\alpha{}_\sigma (\overline{H^{-1}})^\beta{}_\lambda H^\sigma{}_\mu \bar H^\lambda{}_\nu\\
&= N_{\nu\mu} N^{\alpha\beta} + \delta_{\alpha\mu}\delta_{\beta\nu}\,,
\end{aligned}$$

where we used the equations (C.2.74) to obtain the last line.

To perform the calculation of $R(h)^{ij}{}_{ij}$, we need a basis made up just of vertical and horizontal vectors. By inspection of (C.2.69) we can check that the vectors ∂_{x^α} (with $z^a = x^a + iy^a$) is a basis for vertical space. Since $|\psi_i\rangle$ is orthogonal to $|\psi_\alpha\rangle$ by construction, it is easy to check that the vectors ∂_{x^i} and ∂_{y^i} are orthogonal to ∂_{x^α}, so they are horizontal. The vectors ∂_{y^α}, however, are not perpendicular to ∂_{x^β} in general, and therefore are not horizontal. To have an horizontal basis, consider the vectors u_α defined as the horizontal component of the vector ∂_{y^α},

$$\partial_{y^\alpha} = u_\alpha + B^\mu{}_\alpha \partial_{x^\mu}\,, \qquad (\text{C.2.78})$$

where the coefficients B encodes the result of applying the connection ω to the state ∂_{y^α}, $\omega(\partial_{y^\alpha}) = B^\mu{}_\alpha \partial_{x^\mu}$.

By computing the product of ∂_{y^α} with ∂_{x^β} we obtain the following,

$$h(\partial_{x^\beta}, \partial_{y^\alpha}) = B^\mu{}_\alpha h(\partial_{x^\beta}, \partial_{x^\mu})\,. \qquad (\text{C.2.79})$$

On the other hand, we can calculate their product directly using (C.2.64) and (C.2.65). Recall that, in the origin, the metric satisfies $h_{i\bar k} = h_{\bar i k} = \delta_{ik}$. Therefore, we obtain in this case,

$$h(\partial_{x^\beta}, \partial_{y^\alpha}) = ih(\partial_{z^\beta} + \partial_{\bar z^\beta}, \partial_{z^\alpha} - \partial_{\bar z^\alpha}) = ih(\partial_{\bar z^\beta}, \partial_{z^\alpha}) - ih(\partial_{z^\beta}, \partial_{\bar z^\alpha})$$

$$
\begin{aligned}
&= i\bar{H}^{\nu}{}_{\beta}H^{\lambda}{}_{\alpha}h(\partial_{\bar{\zeta}^{\nu}}, \partial_{\zeta^{\lambda}}) - iH^{\nu}{}_{\beta}\bar{H}^{\lambda}{}_{\alpha}h(\partial_{\zeta^{\nu}}, \partial_{\bar{\zeta}^{\lambda}}) \\
&= i\bar{H}^{\nu}{}_{\beta}H^{\lambda}{}_{\alpha}\delta_{\nu\lambda} - iH^{\nu}{}_{\beta}\bar{H}^{\lambda}{}_{\alpha}\delta_{\nu\lambda} = i\bar{H}^{\nu}{}_{\beta}H^{\nu}{}_{\alpha} - iH^{\nu}{}_{\beta}\bar{H}^{\nu}{}_{\alpha} \\
&= i(N_{\beta\alpha} - N_{\alpha\beta})\,.
\end{aligned}
$$

In an analogous fashion, we have the following,

$$
\begin{aligned}
h(\partial_{x^{\beta}}, \partial_{x^{\alpha}}) &= h(\partial_{z^{\beta}} + \partial_{\bar{z}^{\beta}}, \partial_{z^{\alpha}} + \partial_{\bar{z}^{\alpha}}) = h(\partial_{\bar{z}^{\beta}}, \partial_{z^{\alpha}}) + h(\partial_{z^{\beta}}, \partial_{\bar{z}^{\alpha}}) \\
&= \bar{H}^{\nu}{}_{\beta}H^{\lambda}{}_{\alpha}h(\partial_{\bar{\zeta}^{\nu}}, \partial_{\zeta^{\lambda}}) + H^{\nu}{}_{\beta}\bar{H}^{\lambda}{}_{\alpha}h(\partial_{\zeta^{\nu}}, \partial_{\bar{\zeta}^{\lambda}}) \\
&= \bar{H}^{\nu}{}_{\beta}H^{\lambda}{}_{\alpha}\delta_{\nu\lambda} + H^{\nu}{}_{\beta}\bar{H}^{\lambda}{}_{\alpha}\delta_{\nu\lambda} = \bar{H}^{\nu}{}_{\beta}H^{\nu}{}_{\alpha} + H^{\nu}{}_{\beta}\bar{H}^{\nu}{}_{\alpha} \\
&= N_{\beta\alpha} + N_{\alpha\beta}\,.
\end{aligned}
$$

Substituting this expression in (C.2.79) produces the following result,

$$
i(N_{\beta\alpha} - N_{\alpha\beta}) = B^{\mu}{}_{\alpha}(N_{\beta\mu} + N_{\mu\beta})\,. \tag{C.2.80}
$$

At last, we are ready to make the actual calculation. Note that the basis

$$
\{\{\partial_{x^{\alpha}}\}, \{\partial_{x^{i}}\}, \{\partial_{y^{i}}\}, \{u_{\alpha}\}\}\,, \tag{C.2.81}
$$

consist only on vertical vectors and horizontal vectors, so we can calculate $R(h)^{ij}{}_{ij}$ using it. Abusing of notation, denote the dual basis as

$$
\{\{dX^{\alpha}\}, \{dX^{i}\}, \{dY^{i}\}, \{U^{\alpha}\}\}\,.
$$

Clearly, we have the following equalities,

$$
U^{\beta}(\partial_{x^{\alpha}}) = U^{\beta}(\partial_{x^{i}}) = U^{\beta}(\partial_{y^{i}}) = 0\,.
$$

By using equation (C.2.78) we can also conclude that,

$$
U^{\beta}(\partial_{y^{\alpha}}) = \delta^{\beta}{}_{\alpha}\,.
$$

The previous equalities implies that the following expression holds, $U^{\beta} = dy^{\beta}$. By using essentially the same argument, we can obtain the following relations, $dX^{i} = dx^{i}$, $dY^{i} = dy^{i}$. By evaluating dX^{β} in the basis vectors $\{\{\partial_{x^{\alpha}}\}, \{\partial_{y^{\alpha}}\}, \{\partial_{x^{i}}\}, \{\partial_{y^{i}}\}\}$, we conclude,

$$
dX^{\beta}(\partial_{x^{\alpha}}) = \delta^{\beta}{}_{\alpha}\,, \quad dX^{\beta}(\partial_{x^{i}}) = dX^{\beta}(\partial_{y^{i}}) = 0\,, \quad dX^{\beta}(\partial_{y^{\alpha}}) = B^{\beta}{}_{\alpha}\,,
$$

so that,

$$
dX^{\beta} = dx^{\beta} + B^{\beta}{}_{\alpha}dy^{\alpha}\,. \tag{C.2.82}
$$

Since the set $\{\{\partial_{x^i}\}, \{\partial_{y^i}\}, \{u_\alpha\}\}$ contains all the horizontal vectors, we have,

$$\begin{aligned}
&R(h)^{ij}{}_{ij} = \\
&R(dX^i, dX^j, \partial_{x^i}, \partial_{x^j}) + R(dY^i, dY^j, \partial_{y^i}, \partial_{y^j}) + 2R(dX^i, dY^j, \partial_{x^i}, \partial_{y^j}) \\
&\quad + 2R(U^\alpha, dX^i, u_a, \partial_{x^i}) + 2R(U^\alpha, dY^i, u_a, \partial_{y^i}) + R(U^\alpha, U^\beta, u_a, u_\beta)\,.
\end{aligned} \tag{C.2.83}$$

Now, we calculate term by term. Note that the term in the first line is,

$$R(dx^i, dx^j, \partial_{x^i}, \partial_{x^j}) + R(dy^i, dy^j, \partial_{y^i}, \partial_{y^j}) + 2R(dx^i, dy^j, \partial_{x^i}, \partial_{y^j})\,,$$

that is simply the curvature of a projective Hilbert space with complex dimension $2s-3$, –the one corresponding to a spin $s-3/2$ (the original space has complex dimension $2s$. From this space we are removing 6 real vectors, u_α and ∂_{x^α}, this gives as a result a space with complex dimension $2s-3$). Using the result of eq. (C.2.67) we obtain,

$$\begin{aligned}
&R(dX^i, dX^s, \partial_{x^i}, \partial_{x^s}) + R(dY^i, dY^s, \partial_{y^i}, \partial_{y^s}) + 2R(dX^i, dY^s, \partial_{x^i}, \partial_{y^s}) \\
&\qquad = 4(s-3/2)(2s-3+1) = (4s-6)(2s-2) = 4(2s-3)(s-1)\,.
\end{aligned} \tag{C.2.84}$$

For the next term of the (C.2.83), using (C.2.78), we have the following,

$$\begin{aligned}
2R(U^\alpha, dX^i, u_a, \partial_{x^i}) &= 2R(dy^\alpha, dx^i, \partial_{y^\alpha} - B^\mu{}_\alpha \partial_{x^\mu}, \partial_{x^i}) \\
&= 2R(dy^\alpha, dx^i, \partial_{y^\alpha}, \partial_{x^i}) - 2B^\mu{}_\alpha R(dy^\alpha, dx^i, \partial_{x^\mu}, \partial_{x^i})\,.
\end{aligned} \tag{C.2.85}$$

But, by (C.2.64),

$$2R(dy^\alpha, dx^i, \partial_{y^\alpha}, \partial_{x^i}) = \frac{1}{4} R(dz^\alpha - d\bar{z}^\alpha, dz^i + d\bar{z}^i, \partial_{z^\alpha} - \partial_{\bar{z}^\alpha}, \partial_{z^i} + \partial_{\bar{z}^i})\,.$$

By looking at (C.2.77), we note that the only non zero component of the Riemann tensor are those where in the first two indices there is one with a bar and one without a bar. Because of this we have,

$$\begin{aligned}
2R(dy^\alpha, dx^i, \partial_{y^\alpha}, \partial_{x^i}) &= \frac{1}{2}(R^{z^\alpha \bar{z}^i}{}_{z^\alpha \bar{z}^i} - R^{z^\alpha \bar{z}^i}{}_{\bar{z}^\alpha z^i} - R^{\bar{z}^\alpha z^i}{}_{z^\alpha \bar{z}^i} + R^{\bar{z}^\alpha z^i}{}_{\bar{z}^\alpha z^i}) \\
&= \frac{1}{2}(2\delta_{\alpha\alpha}\delta_{ii}) = (3)(2s-3) = 3(2s-3)\,.
\end{aligned}$$

Using the same line of reasoning we obtain,

$$R(dy^\alpha, dx^i, \partial_{x^\mu}, \partial_{x^i}) = \frac{-i}{4}(R^{z^\alpha \bar{z}^i}{}_{z^\mu \bar{z}^i} + R^{z^\alpha \bar{z}^i}{}_{\bar{z}^\mu z^i} - R^{\bar{z}^\alpha z^i}{}_{z^\mu \bar{z}^i} - R^{\bar{z}^\alpha z^i}{}_{\bar{z}^\mu z^i})$$

$$= \frac{-i}{4}(\delta_{\alpha\mu}\delta_{ii} - \delta_{\alpha\mu}\delta_{ii}) = 0\,.$$

By substituting the previous expressions in (C.2.85) we obtain,

$$2R(U^\alpha, dX^i, u_a, \partial_{x^i}) = 3(2s-3) \tag{C.2.86}$$

In a very similar way we can calculate $2R(U^\alpha, dY^i, u_a, \partial_{Y^i})$ of (C.2.83). In this case we have,

$$\begin{aligned}2R(U^\alpha, dY^i, u_a, \partial_{y^i}) &= 2R(dy^\alpha, dy^i, \partial_{y^\alpha} - B^\mu{}_\alpha \partial_{x^\mu}, \partial_{y^i})\\ &= 2R(dy^\alpha, dy^i, \partial_{y^\alpha}, \partial_{y^i}) - 2B^\mu{}_\alpha R(dy^\alpha, dy^i, \partial_{x^\mu}, \partial_{y^i})\,.\end{aligned}$$

For the first term we have,

$$\begin{aligned}2R(dy^\alpha, dy^i, \partial_{y^\alpha}, \partial_{y^i}) &= \frac{1}{2}(R^{z^\alpha \bar{z}^i}{}_{z^\alpha \bar{z}^i} - R^{z^\alpha \bar{z}^i}{}_{\bar{z}^\alpha z^i} - R^{\bar{z}^\alpha z^i}{}_{z^\alpha \bar{z}^i} + R^{\bar{z}^\alpha z^i}{}_{\bar{z}^\alpha z^i}))\\ &= \frac{1}{2}(2\delta_{\alpha\alpha}\delta_{ii}) = 3(2s-3)\end{aligned}$$

while we have for the remaining one,

$$\begin{aligned}R(dy^\alpha, dy^i, \partial_{x^\mu}, \partial_{y^i}) &= -\frac{1}{4}(R^{z^\alpha \bar{z}^i}{}_{z^\mu \bar{z}^i} - R^{z^\alpha \bar{z}^i}{}_{\bar{z}^\mu z^i} + R^{\bar{z}^\alpha z^i}{}_{z^\mu \bar{z}^i} - R^{\bar{z}^\alpha z^i}{}_{\bar{z}^\mu z^i})\\ &= -\frac{-1}{4}(\delta_{\alpha\mu}\delta_{ii} - \delta_{\alpha\mu}\delta_{ii}) = 0\,.\end{aligned}$$

This together with the previous equalities gives the following result,

$$2R(U^\alpha, dY^i, u_a, \partial_{Y^i}) = 3(2j-3)\,. \tag{C.2.87}$$

Now, we calculate the longest term of (C.2.83). The calculation goes as follows,

$$\begin{aligned}&R(U^\alpha, U^\beta, u_a, u_\beta)\\ &\quad= R(dy^\alpha, dy^\beta, \partial_{y^\alpha} - B^\mu{}_\alpha \partial_{x^\mu}, \partial_{y^\beta} - B^\nu{}_\beta \partial_{x^\nu})\\ &\quad= R(dy^\alpha, dy^\beta, \partial_{y^\alpha}, \partial_{y^\beta}) - B^\nu{}_\beta R(dy^\alpha, dy^\beta, \partial_{y^\alpha}, \partial_{x^\nu})\\ &\qquad - B^\mu{}_\alpha R(dy^\alpha, dy^\beta, \partial_{x^\mu}, \partial_{y^\beta}) + B^\mu{}_\alpha B^\nu{}_\beta R(dy^\alpha, dy^\beta, \partial_{x^\mu}, \partial_{x^\nu})\\ &\quad= R(dy^\alpha, dy^\beta, \partial_{y^\alpha}, \partial_{y^\beta}) - 2B^\nu{}_\beta R(dy^\alpha, dy^\beta, \partial_{y^\alpha}, \partial_{x^\nu})\\ &\qquad + B^\mu{}_\alpha B^\nu{}_\beta R(dy^\alpha, dy^\beta, \partial_{x^\mu}, \partial_{x^\nu})\,,\end{aligned} \tag{C.2.88}$$

where we used the fact that the third and the fourth terms are the same to obtain the last line (this can be checked using the symmetries of the Riemann tensor and renaming the indices).

As usual, we make the calculation term by term in the following paragraphs. For the first term we have,

$$\begin{aligned}
&R(dy^\alpha, dy^\beta, \partial_{y^\alpha}, \partial_{y^\beta})\\
&= \frac{1}{4}(R^{z^\alpha \bar{z}^\beta}{}_{z^\alpha \bar{z}^\beta} + R^{z^\alpha \bar{z}^\beta}{}_{\bar{z}^\alpha z^\beta} + R^{\bar{z}^\alpha z^\beta}{}_{z^\alpha \bar{z}^\beta} + R^{\bar{z}^\alpha z^\beta}{}_{\bar{z}^\alpha z^\beta})\\
&= \frac{1}{4}(N_{\beta\alpha}N^{\alpha\beta} - N_{\alpha\beta}N^{\alpha\beta} - N_{\beta\alpha}N^{\beta\alpha} + N_{\alpha\beta}N^{\beta\alpha} + \delta_{\alpha\alpha}\delta_{\beta\beta}\\
&\qquad - \delta_{\alpha\beta}\delta_{\beta\alpha} - \delta_{\alpha\beta}\delta_{\beta\alpha} + \delta_{\alpha\alpha}\delta_{\beta\beta})\\
&= \frac{1}{4}(N_{\beta\alpha}N^{\alpha\beta} - N_{\alpha\beta}N^{\alpha\beta} - N_{\beta\alpha}N^{\beta\alpha} + N_{\alpha\beta}N^{\beta\alpha}) + \frac{1}{2}(\delta_{\alpha\alpha}\delta_{\beta\beta} - \delta_{\alpha\beta}\delta_{\beta\alpha})\\
&= \frac{1}{4}(2\delta_\beta{}^\beta - 2N_{\alpha\beta}N^{\alpha\beta}) + \frac{1}{2}(9 - \delta_{\alpha\alpha}) = \frac{3}{2} + 3 - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta}\\
&= \frac{9}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta}\,.
\end{aligned} \tag{C.2.89}$$

For the second term we have,

$$\begin{aligned}
&R(dy^\alpha, dy^\beta, \partial_{y^\alpha}, \partial_{x^\nu})\\
&= -\frac{i}{4}(-R^{z^\alpha \bar{z}^\beta}{}_{z^\alpha \bar{z}^\nu} + R^{z^\alpha \bar{z}^\beta}{}_{\bar{z}^\alpha z^\nu} - R^{\bar{z}^\alpha z^\beta}{}_{z^\alpha \bar{z}^\nu} + R^{\bar{z}^\alpha z^\beta}{}_{\bar{z}^\alpha z^\nu})\\
&= -\frac{i}{4}(-N_{\nu\alpha}N^{\alpha\beta} - N_{\alpha\nu}N^{\alpha\beta} + N_{\nu\alpha}N^{\beta\alpha} + N_{\alpha\nu}N^{\beta\alpha} - \delta_{\alpha\alpha}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\alpha}\\
&\qquad + \delta_{\alpha\nu}\delta_{\beta\alpha} + \delta_{\alpha\alpha}\delta_{\beta\nu})\\
&= -\frac{i}{4}(-\delta_\nu{}^\beta - N_{\alpha\nu}N^{\alpha\beta} + N_{\nu\alpha}N^{\beta\alpha} + \delta_\nu{}^\beta) = -\frac{i}{4}(N_{\nu\alpha}N^{\beta\alpha} - N_{\alpha\nu}N^{\alpha\beta})\\
&= \frac{1}{2}\Im(N_{\nu\alpha}N^{\beta\alpha})\,,
\end{aligned} \tag{C.2.90}$$

where we use the fact that the equalities $N_{\alpha\nu} = \bar{N}_{\nu\alpha}$ and $N^{\alpha\beta} = \bar{N}^{\beta\alpha}$ hold. Note that, by contracting $N_{\nu\alpha}N^{\beta\alpha})$ with $B^\nu{}_\beta$ and adding some zeros we obtain the following,

$$\begin{aligned}
B^\nu{}_\beta N_{\nu\alpha}N^{\beta\alpha} &= B^\nu{}_\beta(N_{\nu\alpha} + N_{\alpha\nu})N^{\beta\alpha} - B^\nu{}_\beta N_{\alpha\nu}N^{\beta\alpha}\\
&= i(N_{\alpha\beta} - N_{\beta\alpha})N^{\beta\alpha} - B^\nu{}_\beta\delta_\nu{}^\beta\\
&= i(\delta_\alpha{}^\alpha - N_{\beta\alpha}N^{\beta\alpha}) - B^\nu{}_\nu = i(3 - N_{\beta\alpha}N^{\beta\alpha}) - B^\nu{}_\nu\,,
\end{aligned} \tag{C.2.91}$$

where we used (C.2.80) to conclude the second line. Since the coefficients B are real, and the term $N_{\beta\alpha}N^{\beta\alpha}$ is also real (as can be easily checked), we have the following,

$$\Re(B^{\nu}{}_{\beta}N_{\nu\alpha}N^{\beta\alpha}) = -B^{\nu}{}_{\nu}\,, \quad \Im(B^{\nu}{}_{\beta}N_{\nu\alpha}N^{\beta\alpha}) = 3 - N_{\beta\alpha}N^{\beta\alpha}\,. \qquad \text{(C.2.92)}$$

By putting together the previous equalities, we obtain the following expression,

$$-2B^{\nu}{}_{\beta}R(dy^{\alpha}, dy^{\beta}, \partial_{y^{\alpha}}, \partial_{x^{\nu}}) = N_{\beta\alpha}N^{\beta\alpha} - 3 \qquad \text{(C.2.93)}$$

For the last term of (C.2.88) we have,

$$\begin{aligned}
&R(dy^{\alpha}, dy^{\beta}, \partial_{x^{\mu}}, \partial_{x^{\nu}}) \\
&= -\frac{1}{4}(-R^{z^{\alpha}\bar{z}^{\beta}}{}_{z^{\mu}\bar{z}^{\nu}} - R^{z^{\alpha}\bar{z}^{\beta}}{}_{\bar{z}^{\mu}z^{\nu}} - R^{\bar{z}^{\alpha}z^{\beta}}{}_{z^{\mu}\bar{z}^{\nu}} - R^{\bar{z}^{\alpha}z^{\beta}}{}_{\bar{z}^{\mu}z^{\nu}}) \\
&= \frac{1}{4}(R^{z^{\alpha}\bar{z}^{\beta}}{}_{z^{\mu}\bar{z}^{\nu}} + R^{z^{\alpha}\bar{z}^{\beta}}{}_{\bar{z}^{\mu}z^{\nu}} + R^{\bar{z}^{\alpha}z^{\beta}}{}_{z^{\mu}\bar{z}^{\nu}} + R^{\bar{z}^{\alpha}z^{\beta}}{}_{\bar{z}^{\mu}z^{\nu}}) \\
&= \frac{1}{4}(N_{\nu\mu}N^{\alpha\beta} - N_{\mu\nu}N^{\alpha\beta} - N_{\nu\mu}N^{\beta\alpha} + N_{\mu\nu}N^{\beta\alpha} + \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} \\
&\qquad\quad - \delta_{\alpha\nu}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\nu}) \\
&= \frac{1}{4}(N_{\nu\mu}N^{\alpha\beta} - N_{\mu\nu}N^{\alpha\beta} - N_{\nu\mu}N^{\beta\alpha} + N_{\mu\nu}N^{\beta\alpha}) + \frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}) \\
&= \frac{1}{2}(\Re(N_{\nu\mu}N^{\alpha\beta}) - \Re(N_{\mu\nu}N^{\alpha\beta})) + \frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu})\,. \qquad \text{(C.2.94)}
\end{aligned}$$

To continue with the calculation, we have to contract the previous term with $B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}$. The result of contracting it with $N_{\nu\mu}N^{\alpha\beta}$ is,

$$\begin{aligned}
B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\nu\mu}N^{\alpha\beta} &= B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}(N_{\nu\mu} - N_{\mu\nu})N^{\alpha\beta} + B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\mu\nu}N^{\alpha\beta} \\
&= -iB^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}B^{\lambda}{}_{\mu}(N_{\nu\lambda} + N_{\lambda\nu})N^{\alpha\beta} + B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\mu\nu}N^{\alpha\beta} \\
&= -iB^{\mu}{}_{\alpha}B^{\lambda}{}_{\mu}B^{\nu}{}_{\beta}(N_{\nu\lambda} + N_{\lambda\nu})N^{\alpha\beta} + B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\mu\nu}N^{\alpha\beta} \\
&= B^{\mu}{}_{\alpha}B^{\lambda}{}_{\mu}(N_{\lambda\beta} - N_{\beta\lambda})N^{\alpha\beta} + B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\mu\nu}N^{\alpha\beta} \\
&= B^{\mu}{}_{\alpha}B^{\lambda}{}_{\mu}(N^{\alpha\beta}N_{\lambda\beta} - \delta^{\alpha}{}_{\lambda}) + B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\mu\nu}N^{\alpha\beta} \\
&= B^{\mu}{}_{\alpha}B^{\lambda}{}_{\mu}N^{\alpha\beta}N_{\lambda\beta} - B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} + B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\mu\nu}N^{\alpha\beta}\,.
\end{aligned}$$

But, the first term of last line of the previous equation, can be simplified in the following way,

$$\begin{aligned}
B^{\mu}{}_{\alpha}B^{\lambda}{}_{\mu}N^{\alpha\beta}N_{\lambda\beta} &= B^{\mu}{}_{\alpha}N^{\alpha\beta}B^{\lambda}{}_{\mu}(N_{\lambda\beta} + N_{\beta\lambda}) - B^{\mu}{}_{\alpha}N^{\alpha\beta}B^{\lambda}{}_{\mu}N_{\beta\lambda} \\
&= iB^{\mu}{}_{\alpha}N^{\alpha\beta}(N_{\beta\mu} - N_{\mu\beta}) - B^{\mu}{}_{\alpha}B^{\lambda}{}_{\mu}\delta^{\alpha}{}_{\lambda} \\
&= iB^{\mu}{}_{\alpha}(\delta^{\alpha}{}_{\mu} - N^{\alpha\beta}N_{\mu\beta}) - B^{\mu}{}_{\alpha}B^{\lambda}{}_{\mu}\delta^{\alpha}{}_{\lambda}
\end{aligned}$$

$$
\begin{aligned}
&= iB^{\mu}{}_{\mu} - iB^{\mu}{}_{\alpha}N^{\alpha\beta}N_{\mu\beta} - B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} \\
&= iB^{\mu}{}_{\mu} - i(i(3 - N_{\alpha\beta}N^{\alpha\beta}) - B^{\mu}{}_{\mu}) - B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} \\
&= 2iB^{\mu}{}_{\mu} + 3 - N_{\alpha\beta}N^{\alpha\beta} - B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\,,
\end{aligned}
$$

where we used (C.2.91) to get the fifth line. Using this expression in the previous one we obtain,

$$
B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\nu\mu}N^{\alpha\beta} = 2iB^{\mu}{}_{\mu} + 3 - N_{\alpha\beta}N^{\alpha\beta} - 2B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} + B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}N_{\mu\nu}N^{\alpha\beta}\,.
$$

By remembering that the coefficients B are real, and taking the real part in the previous equation we obtain,

$$
B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}(\Re(N_{\nu\mu}N^{\alpha\beta}) - \Re(N_{\mu\nu}N^{\alpha\beta})) = \frac{3}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta} - B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\,.
$$

Combining this expression with (C.2.94) we obtain,

$$
\begin{aligned}
&B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}R(dy^{\alpha}, dy^{\beta}, \partial_{x^{\mu}}, \partial_{x^{\nu}}) \\
&\quad = \frac{3}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta} - B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} + \frac{1}{2}B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}) \\
&\quad = \frac{3}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta} - B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{1}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} \\
&\quad = \frac{3}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta} + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\,.
\end{aligned}
\tag{C.2.95}
$$

Direct substitution of (C.2.89), (C.2.93) and (C.2.95) in (C.2.88), produces the following result,

$$
\begin{aligned}
R(U^{\alpha}, U^{\beta}, u_{\alpha}, u_{\beta}) &= \left(\frac{9}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta}\right) + \left(N_{\beta\alpha}N^{\beta\alpha} - 3\right) \\
&\qquad + \left(\frac{3}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta} + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\right) \\
&= 3 + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\,.
\end{aligned}
\tag{C.2.96}
$$

By substituting (C.2.84), (C.2.87), (C.2.86) and (C.2.96) in (C.2.83) we obtain the following result,

$$
\begin{aligned}
R(h)^{ij}{}_{ij} &= 4(2s-3)(s-1) + 3(2s-3) + 3(2s-3) \\
&\qquad + \left(3 + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\right) \\
&= 4(2s-3)(s-1) + 12s - 15 + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}
\end{aligned}
$$

$$= 4(2s-3)(s-1) + 12s - 12 - 3 + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}$$
$$= 4(2s-3)(s-1) + 12(s-1) - 3 + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}$$
$$= (s-1)(8s-12+12) - 3 + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}$$
$$= 8s(s-1) - 3 + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\,. \tag{C.2.97}$$

Up to now, the calculation has been completely general in the sense that we have not assumed any specifics about the states $\{|\psi_\alpha\rangle\}$. The next step, is to simplify the result using the structure of said states.

By remembering that $|\psi_\alpha\rangle = iS_\alpha|\psi_0\rangle$, we have the following expression for $N_{\mu\nu}$ (C.2.72)

$$N_{\nu\mu} = \langle S_\nu S_\mu\rangle - \langle S_\nu\rangle\langle S_\mu\rangle\,,$$

(the expectation values are referred the state $|\psi_0\rangle$) so that the expression (C.2.80) involving B turns out to be the following,

$$-2\Im\langle S_\beta S_\alpha\rangle = B^{\mu}{}_{\alpha}\cdot 2(\Re\langle S_\beta S_\mu\rangle - \langle S_\beta\rangle\langle S_\mu\rangle) = B^{\mu}{}_{\alpha}k_{\mu\beta}\,, \tag{C.2.98}$$

where k denotes the metric in the vertical space (C.2.4). By using the commutation relationships of $so(3)$, we can simplify the left side,

$$[S_\beta, S_\alpha] = i\epsilon^{\gamma}{}_{\beta\alpha}S_\gamma \Rightarrow \langle S_\beta S_\alpha\rangle - \langle S_\alpha S_\beta\rangle = i\epsilon^{\gamma}{}_{\beta\alpha}\langle S_\gamma\rangle$$
$$\Rightarrow 2i\Im\langle S_\beta S_\alpha\rangle = i\epsilon^{\gamma}{}_{\beta\alpha}\langle S_\gamma\rangle \Rightarrow 2\Im\langle S_\beta S_\alpha\rangle = \epsilon^{\gamma}{}_{\beta\alpha}\langle S_\gamma\rangle\,.$$

By substituting the last equality in (C.2.98) we obtain,

$$\epsilon^{\gamma}{}_{\alpha\beta}\langle S_\gamma\rangle = B^{\mu}{}_{\alpha}k_{\mu\beta} \Rightarrow B^{\mu}{}_{\alpha} = k^{\beta\mu}\epsilon^{\gamma}{}_{\alpha\beta}\langle S_\gamma\rangle\,. \tag{C.2.99}$$

In particular, for the term $B^{\mu}{}_{\mu}$ we have, $B^{\mu}{}_{\mu} = k^{\beta\mu}\epsilon^{\gamma}{}_{\mu\beta}\langle S_\gamma\rangle$. Since $k^{\beta\mu}$ is symmetric in $\beta\mu$ while $\epsilon^{\gamma}{}_{\mu\beta}$ is antisymmetric, we can conclude that $B^{\mu}{}_{\mu}$ is zero. Another consequence of (C.2.99) is the following,

$$B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} = k^{\beta\mu}\epsilon^{\gamma}{}_{\alpha\beta}\langle S_\gamma\rangle k^{\nu\alpha}\epsilon^{\lambda}{}_{\mu\nu}\langle S_\lambda\rangle\,.$$

If we work in a point in the section σ, where the vertical metric k is diagonal and can be written as in (C.1.18), the previous equation reduces to,

$$B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} = \left(\delta_{\beta\mu}\frac{1}{2\Delta S_\mu^2}\epsilon^{\gamma}{}_{\alpha\beta}\langle S_\gamma\rangle\right)\left(\delta_{\nu\alpha}\frac{1}{2\Delta S_\nu^2}\epsilon^{\lambda}{}_{\mu\nu}\langle S_\lambda\rangle\right)$$

$$
\begin{aligned}
&= \frac{1}{4\Delta S_\mu^2 \Delta S_\nu^2} \epsilon^\gamma{}_{\nu\mu} \epsilon^\lambda{}_{\mu\nu} \langle S_\lambda \rangle \langle S_\gamma \rangle \\
&= -\left(\frac{\langle S_x \rangle^2}{2\Delta S_y^2 \Delta S_z^2} + \frac{\langle S_y \rangle^2}{2\Delta S_x^2 \Delta S_z^2} + \frac{\langle S_z \rangle^2}{2\Delta S_x^2 \Delta S_y^2} \right) \\
&= -\frac{\Delta S_x^2 \langle S_x \rangle^2 + \Delta S_y^2 \langle S_y \rangle^2 + \Delta S_z^2 \langle S_z \rangle^2}{2\Delta S_x^2 \Delta S_y^2 \Delta S_z^2} \\
&= -4 \frac{\Delta S_x^2 \langle S_x \rangle^2 + \Delta S_y^2 \langle S_y \rangle^2 + \Delta S_z^2 \langle S_z \rangle^2}{\text{Det } k} = -\frac{2 \sum_{\alpha,\beta} k_{\alpha\beta} \langle S_\alpha \rangle \langle S_\beta \rangle}{\text{Det } k} \, , \\
&= -\frac{2\ell^2}{e^\Phi} \, ,
\end{aligned} \tag{C.2.100}
$$

where Φ is defined as in (C.2.60), and ℓ is defined according to the following equation,

$$
\ell = \sum_{\alpha,\beta} k_{\alpha\beta} \langle S_\alpha \rangle \langle S_\beta \rangle \, .
$$

After considering the previous expression, equation (C.2.97) becomes,

$$
R(h)^{ij}{}_{ij} = 8s(s-1) - 3 + \frac{3\ell^2}{e^\Phi} \, .
$$

Although we obtained this result considering a point in the section, since the terms involved are constant over the fibers, the result holds for any point of the fiber. Using this result in (C.2.25), we finally obtain,

$$
8s(s-1) - 3 - \frac{3}{2} B^\mu{}_\alpha B^\alpha{}_\mu = 8s(s-1) - 3 + \frac{3\ell^2}{e^\Phi} = R(g) - \frac{3}{4} \operatorname{Tr}(\Omega^2) \, . \tag{C.2.101}
$$

For the ease of the mind of the reader, we mention that this formula has been verified numerically for various random points in the case of $s = 3/2$ and $s = 2$.

C.2.5 Relationship between $R(h)$, $R(k)$ $\operatorname{Tr}(\Omega^2)$ and $R(g)$: a fourth equation

In the same way we calculated $R(h)^{ij}{}_{ij}$ in the previous section, we can also calculate $R(h)^{\alpha\beta}{}_{\alpha\beta}$ and substitute it in (C.2.25) to obtain a fourth equation. For this calculation, we work with the basis $\{\{\partial_{x^\alpha}\}, \{\partial_{x^i}\}, \{\partial_{y^i}\}, \{u_\alpha\}\}$ introduced in (C.2.81) of the previous section. Just as for $R(h)^{ij}{}_{ij}$, it is easy to check that $R(h)^{\alpha\beta}{}_{\alpha\beta}$ is base independent and constant along the fibers.

To compute $R(h)^{\alpha\beta}{}_{\alpha\beta}$, we are only contracting vertical indices, that is, the ones corresponding to the vectors $\{\partial_{x^\alpha}\}$. Because of this we have,

$$\begin{aligned}
R(h)^{\alpha\beta}{}_{\alpha\beta} &= R(dX^\alpha, dX^\beta, \partial_{x^\alpha}, \partial_{x^\beta}) \\
&= R(dx^\alpha + B^\alpha{}_\mu dy^\mu, dx^\beta + B^\beta{}_\nu dy^\nu, \partial_{x^\alpha}, \partial_{x^\beta}) \\
&= R(dx^\alpha, dx^\beta, \partial_{x^\alpha}, \partial_{x^\beta}) + B^\alpha{}_\mu R(dy^\mu, dx^\beta, \partial_{x^\alpha}, \partial_{x^\beta}) \\
&\quad B^\beta{}_\nu R(dx^\alpha, dy^\nu, \partial_{x^\alpha}, \partial_{x^\beta}) + B^\alpha{}_\mu B^\beta{}_\nu R(dy^\mu, dy^\nu, \partial_{x^\alpha}, \partial_{x^\beta}) \\
&= R(dx^\alpha, dx^\beta, \partial_{x^\alpha}, \partial_{x^\beta}) + 2B^\alpha{}_\mu R(dy^\mu, dx^\beta, \partial_{x^\alpha}, \partial_{x^\beta}) \\
&\qquad + B^\alpha{}_\mu B^\beta{}_\nu R(dy^\mu, dy^\nu, \partial_{x^\alpha}, \partial_{x^\beta}) \,, \qquad \text{(C.2.102)}
\end{aligned}$$

where we used equation (C.2.82) to get to the second line and the fact that the second and third terms of the third line are equal (as can be easily verified) to obtain the last one.

We calculate each term separately,

$$\begin{aligned}
R(dx^\alpha, dx^\beta, \partial_{x^\alpha}, \partial_{x^\beta}) &= \frac{1}{4}(R^{z^\alpha \bar{z}^\beta}{}_{z^\alpha \bar{z}^\beta} + R^{z^\alpha \bar{z}^\beta}{}_{\bar{z}^\alpha z^\beta} + R^{\bar{z}^\alpha z^\beta}{}_{z^\alpha \bar{z}^\beta} + R^{\bar{z}^\alpha z^\beta}{}_{\bar{z}^\alpha z^\beta}) \\
&= \frac{9}{2} - \frac{1}{2} N_{\alpha\beta} N^{\alpha\beta} \,, \qquad \text{(C.2.103)}
\end{aligned}$$

where in the last line we used the fact that the term in the second line is equal to the one in eq. (C.2.89).

For the second term of (C.2.102), we see that after some renaming of the indices, it can be written as follows,

$$\begin{aligned}
2B^\alpha{}_\mu R(dy^\mu, dx^\beta, \partial_{x^\alpha}, \partial_{x^\beta}) &= 2B^\nu{}_\beta R(dy^\beta, dx^\alpha, \partial_{x^\nu}, \partial_{x^\alpha}) \\
&= 2B^\nu{}_\beta R(dx^\alpha, dy^\beta, \partial_{x^\alpha}, \partial_{x^\nu}) \,.
\end{aligned}$$

Going on with the calculation,

$$\begin{aligned}
& R(dx^\alpha, dy^\beta, \partial_{x^\alpha}, \partial_{x^\nu}) \\
&= -\frac{i}{4}(- R^{z^\alpha \bar{z}^\beta}{}_{z^\alpha \bar{z}^\nu} - R^{z^\alpha \bar{z}^\beta}{}_{\bar{z}^\alpha z^\nu} + R^{\bar{z}^\alpha z^\beta}{}_{z^\alpha \bar{z}^\nu} + R^{\bar{z}^\alpha z^\beta}{}_{\bar{z}^\alpha z^\nu}) \\
&\quad - \frac{i}{4}(- N_{\nu\alpha} N^{\alpha\beta} + N_{\alpha\nu} N^{\alpha\beta} - N_{\nu\alpha} N^{\beta\alpha} + N_{\alpha\nu} N^{\beta\alpha} - \delta_{\alpha\alpha}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\alpha} \\
&\quad - \delta_{\alpha\nu}\delta_{\beta\alpha} + \delta_{\alpha\alpha}\delta_{\beta\nu}) \\
&= -\frac{i}{4}(- \delta_\nu{}^\beta + N_{\alpha\nu} N^{\alpha\beta} - N_{\nu\alpha} N^{\beta\alpha} + \delta_\nu{}^\beta) = -\frac{i}{4}(N_{\alpha\nu} N^{\alpha\beta} - N_{\nu\alpha} N^{\beta\alpha}) \\
&= \frac{i}{4}(N_{\nu\alpha} N^{\beta\alpha} - N_{\alpha\nu} N^{\alpha\beta}) = -\frac{1}{2}\Im(N_{\nu\alpha} N^{\beta\alpha}) \,.
\end{aligned}$$

But, by using (C.2.92) in the previous expressions we obtain,

$$2B^{\alpha}{}_{\mu}R(dy^{\mu}, dx^{\beta}, \partial_{x^{\alpha}}, \partial_{x^{\beta}}) = -2B^{\nu}{}_{\beta} \cdot \frac{1}{2}\Im(N_{\nu\alpha}N^{\beta\alpha}) = N_{\beta\alpha}N^{\beta\alpha} - 3\,. \tag{C.2.104}$$

Finally, for the last term of (C.2.102) we have, after some renaming,

$$\begin{aligned} B^{\alpha}{}_{\mu}B^{\beta}{}_{\nu}R(dy^{\mu}, dy^{\nu}, \partial_{x^{\alpha}}, \partial_{x^{\beta}}) &= B^{\mu}{}_{\alpha}B^{\nu}{}_{\beta}R(dy^{\alpha}, dy^{\beta}, \partial_{x^{\mu}}, \partial_{x^{\nu}}) \\ &= \frac{3}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta} + \frac{1}{2}B^{\mu}{}_{\mu}B^{\nu}{}_{\nu} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\,. \\ &= \frac{3}{2} - \frac{1}{2}N_{\alpha\beta}N^{\alpha\beta} - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\,, \end{aligned} \tag{C.2.105}$$

where we used (C.2.95) to get the second line and the fact that $B^{\mu}{}_{\mu}$ is zero (as was proven in the previous section). By using the equations (C.2.103), (C.2.104) and (C.2.105) in (C.2.102), we obtain the following result,

$$R(h)^{\alpha\beta}{}_{\alpha\beta} = 3 - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu}\,.$$

Now, consider the first equation of (C.2.25) and eq. (C.2.27). If we add them up together, we can conclude the following result,

$$\begin{aligned} R(h) + R(h)^{\alpha\beta}{}_{\alpha\beta} = R(k) - D_{\alpha i}{}^{\alpha}D_{\beta}{}^{i\beta} + R(k) + R(g) - \frac{1}{4}\Omega^{\alpha}{}_{ij}\Omega_{\alpha}{}^{ij} \\ - 2D_{\alpha}{}^{i\alpha}{}_{,i} - D_{\alpha\,i}{}^{\alpha}D_{\beta}{}^{i\,\beta} \\ = 2R(k) + R(g) - \frac{1}{4}\,\mathrm{Tr}\,(\Omega^2) - 2D_{\alpha i}{}^{\alpha}D_{\beta}{}^{i\beta} - 2D_{\alpha}{}^{i\alpha}{}_{,i}\,. \end{aligned}$$

In (C.2.67), we obtained the value of $R(h)$, $R(h) = 4s(2s+1)$. Substituting this along with the expression we just obtained for $R(h)^{\alpha\beta}{}_{\alpha\beta}$ in the previous equation, produces the following result,

$$\begin{aligned} 4s(2s+1) + 3 - \frac{3}{2}B^{\mu}{}_{\alpha}B^{\alpha}{}_{\mu} = 2R(k) + R(g) - \frac{1}{4}\,\mathrm{Tr}\,(\Omega^2) \\ - 2D_{\alpha i}{}^{\alpha}D_{\beta}{}^{i\beta} - 2D_{\alpha}{}^{i\alpha}{}_{,i}\,. \end{aligned} \tag{C.2.106}$$

This expression can be rewritten in a more convenient fashion. Using equation (C.2.35), we have the following,

$$-2D_{\alpha i}{}^{\alpha}D_{\beta}{}^{i\beta} = -\frac{1}{2}\sum_i(\ln\,\mathrm{Det}\;k)_{,i} = -\frac{1}{2}||\nabla_{\mathscr{S}}\Phi||^2_{\mathscr{S}}\,, \tag{C.2.107}$$

where Φ is defined as in (C.2.60). By using equation (C.2.36) we also have,

$$-2D_\alpha{}^{i\alpha}{}_{,i} = -\sum_i (\ln \text{ Det } k)_{ii} = -\nabla^2_{\mathscr{S}}\Phi\,. \tag{C.2.108}$$

Finally, we also have the expression (C.2.100) for the term $B^\mu{}_\alpha B^\alpha{}_\mu$. After using all this equalities in (C.2.106), the result is,

$$4s(2s+1) + 3 + \frac{3\ell^2}{e^\Phi} = 2R(k) + R(g) - \frac{1}{4}\,\text{Tr}\,(\Omega^2) - \frac{1}{2}||\nabla_{\mathscr{S}}\Phi||^2_{\mathscr{S}} - \nabla^2_{\mathscr{S}}\Phi\,, \tag{C.2.109}$$

By subtracting this equation from (C.2.101) and solving for Tr (Ω^2), we obtain the following expression, quantities. By subtracting the previous expressions we obtain the main result of this section,

$$\text{Tr}\,(\Omega^2) = 12(2s+1) - 4R(k) + ||\nabla_{\mathscr{S}}\Phi||^2_{\mathscr{S}} + 2\nabla^2_{\mathscr{S}}\Phi\,. \tag{C.2.110}$$

C.3 Projection of geodesics of $\mathbb{P}(\mathcal{H}_s)$ in $\mathscr{S}$

In this section, we study the projection of geodesics of $\mathbb{P}(\mathcal{H}_s)$ into $\mathscr{S}$. To this end, first we give some mathematical preliminaries. Suppose that we have $2s$ vectorial fields X_a that are linearly independent. In general, the coefficients of the Levi-Civita connection for the metric h (the generalization of the Christoffel symbols for non coordinate basis) w.r.t. this fields can be written as,

$$\Gamma(h)^a{}_{bc} = \frac{1}{2}(\mathcal{C}^a{}_{cb} - \mathcal{C}_{bc}{}^a - \mathcal{C}_{cb}{}^a + h^{ad}h_{db,c} + h^{ad}h_{dc,b} - h^{ad}h_{bc,d})\,, \tag{C.3.1}$$

where the subindex $_{,b}$ (like in the previous sections) denotes the derivative along the field X_b and the coefficients $\mathcal{C}$ are defined according to the following equations,

$$[X_b, X_c] = \mathcal{C}^a{}_{bc}X_a\,.$$

For this calculation, we work with the fields defined in section C.2.1,

$$X_i = E_i\,, \quad X_\alpha = S^\sharp_\alpha\,.$$

Now, we calculate the coefficients $\Gamma(h)$. To this end, we need the commutators between the fields. We have already calculated some of them in section (C.2.1). The results are the following,

$$[S^\sharp_\alpha, S^\sharp_\beta] = \epsilon^\gamma{}_{\alpha\beta}S^\sharp_\gamma\,, \quad [S^\sharp_\alpha, E_i] = 0\,, \quad [E_i, E_j] = -2Q^l{}_{ij}E_l - \Omega^\alpha{}_{ij}S^\sharp_\alpha\,. \tag{C.3.2}$$

where Q is defined in equation (C.2.18). In these term, the expressions for the metric h, are the following,

$$h_{ij} = \delta_{ij}\,, \quad h_{\alpha j} = 0\,, \quad h_{\alpha\beta} = k_{\alpha\beta}\,.$$

Using this results in the expression for the Christoffel symbols (C.3.1), produces the following results,

$$\begin{aligned} \Gamma(h)^i{}_{jk} = \Gamma(g)^i{}_{jk},\, \Gamma(h)^i{}_{\alpha k} = \frac{1}{2}k_{\alpha\beta}\Omega^\beta{}_{ki},\, \Gamma(h)^i{}_{k\alpha} = \frac{1}{2}k_{\alpha\beta}\Omega^\beta{}_{ki}, \\ \Gamma(h)^i{}_{\alpha\beta} = -\frac{1}{2}k_{\alpha\beta,i},\, \Gamma(h)^\alpha{}_{ij} = \frac{1}{2}\Omega^\alpha{}_{ij}, \Gamma(h)^\alpha{}_{i\beta} = \frac{1}{2}k^{\alpha\gamma}k_{\gamma\beta,i} = \Gamma(h)^\alpha{}_{\beta i}, \\ \Gamma(h)^\alpha{}_{\beta\gamma} = \frac{1}{2}(\epsilon^\alpha{}_{\gamma\beta} + \epsilon_{\gamma\beta}{}^\alpha + \epsilon_{\beta\gamma}{}^\alpha). \end{aligned} \tag{C.3.3}$$

Most of the calculations are straightforward, except for the one for the last Christoffel symbol. We briefly mentioned how to compute it. By the first line of result (C.2.52), $k_{\alpha\beta,\gamma}$ is given by the following expression,

$$k_{\alpha\beta,\gamma} = \epsilon^\mu{}_{\gamma\alpha}k_{\mu\beta} + \epsilon^\mu{}_{\gamma\beta}k_{\mu\alpha} = \epsilon_{\beta\gamma\alpha} + \epsilon_{\alpha\gamma\beta}\,. \tag{C.3.4}$$

By using this expression in the one for the Christoffel symbols (C.3.1), we have the following,

$$\begin{aligned} \Gamma(h)^\alpha{}_{\beta\gamma} &= \frac{1}{2}(\epsilon^\alpha{}_{\gamma\beta} - \epsilon_{\beta\gamma}{}^\alpha - \epsilon_{\gamma\beta}{}^\alpha + k^{\alpha\delta}k_{\delta\beta,\gamma} + k^{\alpha\delta}k_{\delta\gamma,\beta} - k^{\alpha\delta}k_{\beta\gamma,\delta}) \\ &= \frac{1}{2}(\epsilon^\alpha{}_{\gamma\beta} - \epsilon_{\beta\gamma}{}^\alpha - \epsilon_{\gamma\beta}{}^\alpha + k^{\alpha\delta}(\epsilon_{\delta\gamma\beta} + \epsilon_{\beta\gamma\delta}) + k^{\alpha\delta}(\epsilon_{\delta\beta\gamma} + \epsilon_{\gamma\beta\delta}) \\ &\quad - k^{\alpha\delta}(\epsilon_{\gamma\delta\beta} + \epsilon_{\beta\delta\gamma})) \\ &= \frac{1}{2}(\epsilon^\alpha{}_{\gamma\beta} - \epsilon_{\beta\gamma}{}^\alpha - \epsilon_{\gamma\beta}{}^\alpha + \epsilon^\alpha{}_{\gamma\beta} + \epsilon_{\beta\gamma}{}^\alpha + \epsilon^\alpha{}_{\beta\gamma} \\ &\quad + \epsilon_{\gamma\beta}{}^\alpha - \epsilon_\gamma{}^\alpha{}_\beta - \epsilon_\beta{}^\alpha{}_\gamma) \\ &= \frac{1}{2}(\epsilon^\alpha{}_{\gamma\beta} - \epsilon_\gamma{}^\alpha{}_\beta - \epsilon_\beta{}^\alpha{}_\gamma) = \frac{1}{2}(\epsilon^\alpha{}_{\gamma\beta} + \epsilon_{\gamma\beta}{}^\alpha + \epsilon_{\beta\gamma}{}^\alpha)\,, \end{aligned}$$

as claimed. Finally, we have all the necessary ingredients for the geodesic equation. Suppose the tangent vector to a geodesic can be written as in (2.1.24)

$$\dot\rho(t) = \sum_{\alpha=x,y,z} v^\alpha S^\sharp_\alpha + \sum_{i=1}^{4s-3} v^i E_i\,. \tag{2.1.24}$$

Then, if we consider a non vertical index i, the corresponding geodesic equation for the component v^i reads as follows,

$$\dot{v}^i + \Gamma(g)^i{}_{jk} v^j v^k + k_{\alpha\beta}\Omega^\beta{}_{ki} v^k v^\alpha - \frac{1}{2} k_{\alpha\beta,i} v^\alpha v^\beta = 0\,, \tag{C.3.5}$$

while the equation for a vertical one v^α is,

$$\begin{aligned}\dot{v}^\alpha + \frac{1}{2}\Omega^\alpha{}_{ij} v^j v^k + k^{\alpha\gamma} k_{\gamma\beta,i} v^i v^\beta + \frac{1}{2}(\epsilon^\alpha{}_{\gamma\beta} + \epsilon_{\gamma\beta}{}^\alpha + \epsilon_{\beta\gamma}{}^\alpha) v^\gamma v^\beta = 0 \Rightarrow \\ \dot{v}^\alpha + k^{\alpha\gamma} k_{\gamma\beta,i} v^i v^\beta + \frac{1}{2}(\epsilon_{\gamma\beta}{}^\alpha + \epsilon_{\beta\gamma}{}^\alpha) v^\gamma v^\beta = 0 \Rightarrow \\ \dot{v}^\alpha + k^{\alpha\gamma} k_{\gamma\beta,i} v^i v^\beta + \epsilon_{\gamma\beta}{}^\alpha v^\gamma v^\beta = 0\,. \end{aligned} \tag{C.3.6}$$

Note that the vertical components, *the charge*, in general is not conserved. This is unlike the case of [23], where it is. This is because in the case of [23] they assumed that the vertical metric was bi-invariant, this implies that the third term of the previous equation is zero (c.f. lemma 9.3.8 of said book). In fact, in our case, if we assume that k is bi-invariant, we would have that k is a multiple of the identity, which would implies that $\epsilon_{\gamma\beta}{}^\alpha$ is antisymmetric in $\gamma\beta$. In our case is not. In [23], the author also assume that the vertical metric is "constant" (see definition 9.3.4 of the book and compare it with (2.1.12), where k depends on the point ρ in projective Hilbert space). In his case, this implies that $k_{\alpha\beta,i} = 0$. If we also impose this two conditions, we also obtain that the charge is conserved.

C.3.1 Parametrizing with respect to length in shape space

The geodesics in the previous section are parametrized by arclength w.r.t. the Fubini-Study metric. Because of this, its projection in $\mathscr{S}$ is not parametrized by arclength. In this subsection, we parametrize them by arclength in shape space.

By considering the projection of (2.1.24) into shape space, one can see that the projected curve has as tangent vector $v^i \underline{E}_i$. Since the vector fields $\underline{E}_i$ constitute an orthonormal set, the squared size of the tangent vector at an arbitrary t is $\sum_i (v^i)^2$. Because of this, if we call τ to the arclength parameter in shape space and t the one in the total space ($\mathbb{P}(\mathcal{H}_s)$), it is well known that the following relationship holds,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{\sum_i (v^i)^2}}\,. \tag{C.3.7}$$

Let μ be the curve parametrized using τ as a parameter, $\mu(\tau) = \rho(t(\tau))$, and denote its tangent vector as $u^a X_a$. In general, we use a dot to denote derivatives w.r.t. t and a prime for derivatives w.r.t. τ. Using the chain rule, we have the following,

$$u^a X_a = \frac{\mathrm{d}t}{\mathrm{d}\tau}\dot{\rho} = \frac{1}{\sqrt{\sum_i (v^i)^2}} v^a X_a \Rightarrow u^a = \frac{v^a}{\sqrt{\sum_i (v^i)^2}} \tag{C.3.8}$$

Since the geodesic is parametrized w.r.t. arclength in total space, the following relationship holds,

$$\sum_i (v^i)^2 + k_{\alpha\beta} v^\alpha v^\beta = 1\,.$$

By substituting (C.3.8) in the previous equation we obtain,

$$\sum_i (v^i)^2 (1 + k_{\alpha\beta} u^\alpha u^\beta) = 1 \;\Rightarrow \sum_i (v^i)^2 = \frac{1}{1 + k_{\alpha\beta} u^\alpha u^\beta} \equiv \frac{1}{1+Q^2}\,,$$

where we defined Q as $Q^2 = k_{\alpha\beta} u^\alpha u^\beta$. By using this expression in (C.3.7) and (C.3.8) we obtain,

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \sqrt{1+Q^2}\,, \quad v^a = \frac{u^a}{\sqrt{1+Q^2}}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{\sqrt{1+Q^2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\,.$$

If we use this expressions in the geodesic equation for the component v^a, we obtain the following,

$$\begin{aligned}
\frac{1}{\sqrt{1+Q^2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{u^a}{\sqrt{1+Q^2}}\right) + \frac{1}{1+Q^2}\Gamma(h)^a{}_{bc} u^b u^c = 0 \Rightarrow \\
\frac{1}{1+Q^2} u'^a - \frac{u^a}{2(1+Q^2)^2}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau} + \frac{1}{1+Q^2}\Gamma(h)^a{}_{bc} u^b u^c = 0 \Rightarrow \\
u'^a + \Gamma(h)^a{}_{bc} u^b u^c - \frac{u^a}{2(1+Q^2)}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau} = 0\,.
\end{aligned} \tag{C.3.9}$$

The previous expression imply that, the equations for the movement w.r.t. the parameter τ, can be obtained from (C.3.5) and (C.3.6) by substituting u for v and adding the extra term $-\frac{u^a}{2(1+Q^2)}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau}$. We can further simply this expression by calculating the derivative of Q^2 w.r.t. τ. Using the product rule we have

$$\frac{\mathrm{d}Q^2}{\mathrm{d}\tau} = \frac{\mathrm{d}k_{\alpha\beta}}{\mathrm{d}\tau} u^\alpha u^\beta + 2k_{\alpha\beta}\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} u^\beta \tag{C.3.10}$$

but, by using the chain rule we have,

$$\frac{\mathrm{d}k_{\alpha\beta}}{\mathrm{d}\tau} = k_{\alpha\beta,i}u^i + k_{\alpha\beta,\gamma}u^\gamma \,.$$

Note that the term $\epsilon_{\beta\gamma\alpha}u^\gamma u^\alpha$ is zero, because $\epsilon_{\beta\gamma\alpha}$ is antisymmetric in $\alpha\gamma$ while $u^\gamma u^\alpha$ is symmetric. This, together with (C.3.4), implies that,

$$k_{\alpha\beta,\gamma}u^\gamma u^\alpha u^\beta = 0 \Rightarrow \frac{\mathrm{d}k_{\alpha\beta}}{\mathrm{d}\tau}u^\alpha u^\beta = k_{\alpha\beta,i}u^i u^\alpha u^\beta \,.$$

Using this result in (C.3.10) produces the following expression,

$$\begin{aligned}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau} &= k_{\alpha\beta,i}u^i u^\alpha u^\beta + 2k_{\alpha\beta}\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau}u^\beta \\ &= k_{\alpha\beta,i}u^i u^\alpha u^\beta - 2k_{\alpha\beta}\left(k^{\alpha\gamma}k_{\gamma\mu,i}u^i u^\mu + \epsilon_{\gamma\lambda}{}^{\alpha}u^\gamma u^\lambda - \frac{u^\alpha}{2(1+Q^2)}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau}\right)u^\beta \\ &= k_{\alpha\beta,i}u^i u^\alpha u^\beta - 2k_{\beta\mu,i}u^i u^\mu u^\beta - 2\epsilon_{\gamma\lambda\beta}u^\gamma u^\lambda u^\beta + \frac{k_{\alpha\beta}u^\alpha u^\beta}{1+Q^2}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau} \\ &= -k_{\alpha\beta,i}u^i u^\alpha u^\beta + \frac{Q^2}{1+Q^2}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau} = -k_{\alpha\beta,i}u^i u^\alpha u^\beta + \left(1 - \frac{1}{1+Q^2}\right)\frac{\mathrm{d}Q^2}{\mathrm{d}\tau}\,,\end{aligned}$$

where we used the equation (C.3.9) and (C.3.6) to obtain the second line, and noticed that $\epsilon_{\gamma\lambda\beta}u^\gamma u^\lambda u^\beta$ is zero because the symmetric properties of the indices involved to get the last line. This implies that,

$$\frac{1}{1+Q^2}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau} = -k_{\alpha\beta,i}u^i u^\alpha u^\beta \,. \tag{C.3.11}$$

Substitution of this result in (C.3.9) produces the following result,

$$u'^a + \Gamma(h)^a{}_{bc}u^b u^c + \frac{1}{2}k_{\alpha\beta,i}u^i u^a u^\alpha u^\beta = 0\,.$$

Finally, by using the expressions for the Christoffel symbols (C.3.3), we obtain the following equations; in shape space,

$$u'^i + \Gamma(g)^i{}_{jk}u^j u^k + \Omega_{\alpha ki}u^k u^\alpha - \frac{1}{2}k_{\alpha\beta,i}u^\alpha u^\beta + \frac{1}{2}k_{\alpha\beta,j}u^j u^i u^\alpha u^\beta = 0\,, \tag{C.3.12}$$

and the equation for the charge,

$$u'^\alpha + k^{\alpha\gamma}k_{\gamma\beta,i}u^i u^\beta + \epsilon_{\gamma\beta}{}^{\alpha}u^\gamma u^\beta + \frac{1}{2}k_{\gamma\beta,i}u^i u^\gamma u^\alpha u^\beta = 0\,. \tag{C.3.13}$$

If we contract the previous equation with $k_{\alpha\mu}$, we obtain the following,

$$\begin{aligned}
k_{\alpha\mu}u'^{\alpha} + k_{\alpha\mu}k^{\alpha\gamma}k_{\gamma\beta,i}u^iu^\beta + k_{\alpha\mu}\epsilon_{\gamma\beta}{}^{\alpha}u^\gamma u^\beta + \frac{1}{2}k_{\alpha\mu}k_{\gamma\beta,i}u^iu^\gamma u^\alpha u^\beta = 0 \Rightarrow \\
k_{\alpha\mu}u'^{\alpha} + k_{\mu\beta,i}u^iu^\beta + \epsilon_{\gamma\beta\mu}u^\gamma u^\beta + \frac{1}{2}k_{\gamma\beta,i}u^iu^\gamma u_\mu u^\beta = 0 \Rightarrow \\
k_{\alpha\mu}u'^{\alpha} + k_{\mu\beta,i}u^iu^\beta + \epsilon_{\gamma\beta\mu}u^\gamma u^\beta + \epsilon_{\mu\beta\gamma}u^\gamma u^\beta + \frac{1}{2}k_{\gamma\beta,i}u^iu^\gamma u_\mu u^\beta = 0 \Rightarrow \\
k_{\alpha\mu}u'^{\alpha} + k_{\mu\alpha,i}u^iu^\alpha + \epsilon_{\gamma\alpha\mu}u^\gamma u^\alpha + \epsilon_{\mu\alpha\gamma}u^\gamma u^\alpha + \frac{1}{2}k_{\gamma\beta,i}u^iu^\gamma u_\mu u^\beta = 0 \Rightarrow \\
k_{\alpha\mu}u'^{\alpha} + k_{\mu\alpha,i}u^iu^\alpha + k_{\mu\alpha,\gamma}u^\gamma u^\alpha + \frac{1}{2}k_{\gamma\beta,i}u^iu^\gamma u_\mu u^\beta = 0 \Rightarrow \\
\frac{\mathrm{d}}{\mathrm{d}\tau}(k_{\alpha\mu}u^\alpha) + \frac{1}{2}k_{\gamma\beta,i}u^iu^\gamma u_\mu u^\beta = 0 \Rightarrow \\
\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} + \frac{1}{2}k_{\gamma\beta,i}u^iu^\gamma u_\mu u^\beta = 0 \Rightarrow \\
\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} - \frac{1}{2(1+Q^2)}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau}u_\mu = 0 \Rightarrow \\
\frac{1}{\sqrt{1+Q^2}}\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} - \frac{1}{2(1+Q^2)^{3/2}}\frac{\mathrm{d}Q^2}{\mathrm{d}\tau}u_\mu = 0 \Rightarrow \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{u_\mu}{\sqrt{1+Q^2}}\right) = 0
\end{aligned}$$

where we used the equation (C.3.4) to obtain the fifth line and (C.3.11) to obtain the sixth. By using equation (C.3.7), the previous equality implies that v_μ is conserved. Because of this, it is more convenient to write everything in terms of u_μ instead of u^μ, since its movement equation is simpler. This produces the following set of equations,

$$\begin{aligned}
\left(\frac{u_\mu}{\sqrt{1+Q^2}}\right)' = 0\,, \\
u'^i + \Gamma(g)^i{}_{jk}u^ju^k + \Omega^\alpha{}_k{}^iu^ku_\alpha + \frac{1}{2}k^{\alpha\beta}{}_,{}^iu_\alpha u_\beta - \frac{(Q^2)'}{2(1+Q^2)}u^i = 0\,, \\
\frac{(Q^2)'}{1+Q^2} = k^{\alpha\beta}{}_{,j}u^ju_\alpha u_\beta\,,
\end{aligned}$$

where we used the following equalities in (C.3.12) to obtain the second line,

$$\begin{aligned}
k_{\alpha\beta}k^{\alpha\mu}k^{\beta\nu} = k^{\mu\nu} \Rightarrow k_{\alpha\beta,}{}^ik^{\alpha\mu}k^{\beta\nu} + k_{\alpha\beta}k^{\alpha\mu}{}_,{}^ik^{\beta\nu} + k_{\alpha\beta}k^{\alpha\mu}k^{\beta\nu}{}_,{}^i = k^{\mu\nu}{}_,{}^i \Rightarrow \\
k_{\alpha\beta,}{}^ik^{\alpha\mu}k^{\beta\nu} + k^{\nu\mu}{}_,{}^i + k^{\mu\nu}{}_,{}^i = k^{\mu\nu}{}_,{}^i \Rightarrow k_{\alpha\beta,}{}^ik^{\alpha\mu}k^{\beta\nu} = -k^{\mu\nu}{}_,{}^i \Rightarrow
\end{aligned}$$

$$k_{\alpha\beta,}{}^{i}k^{\alpha\mu}k^{\beta\nu}u_{\mu}u_{\nu} = -k^{\mu\nu}{}_{,}{}^{i}u_{\mu}u_{\nu}\,, \quad .$$

If the curve is parametrized w.r.t. arclength in the total space, from the previous results we can see the equations are the following,

$$\dot{v}_{\mu} = 0\,,$$

$$\dot{v}^{i} + \Gamma(g)^{i}{}_{jk}v^{j}v^{k} + \Omega^{\alpha}{}_{k}{}^{i}v^{k}v_{\alpha} + \frac{1}{2}k^{\alpha\beta}{}_{,}{}^{i}v_{\alpha}v_{\beta} = 0\,.$$

C.4 The Berry curvature and the connection ω

As it is well known, the Berry connection allows us to define geometric phases for closed curves in $\mathbb{P}(\mathcal{H}_s)$. By talking closed curves infinitesimally small, we can define a curvature form, $\mathcal{K}_B$. In this section, we study the behavior of $\mathcal{K}_B$ when evaluated at horizontal and vertical vectors according to the connection ω introduced in the previous chapters. First, we give some preliminaries.

The sphere of normalized states in $\mathcal{H}_s$ can be regarded as a fiber bundle, where the acting group is $U(1)$, the fiber is $\mathbb{P}(\mathcal{H}_s)$ and the projection operators maps $|\psi\rangle$ to $[\psi]$. Given a normalized state $|\psi\rangle$, the space tangent to it can be represented in the following way,

$$T_{|\psi\rangle} = \{|\mu\rangle \in \mathcal{H} \text{ such that } \Re\langle\psi|\mu\rangle = 0\}\,.$$

The Berry connection is defined for this fiber bundle. Given a normalized state $|\psi\rangle$ and a vector $|\mu\rangle$ tangent to it, the Berry connection evaluated at $|\phi\rangle$ produces the following element of $u(1)$,

$$\omega_B(|\mu\rangle; |\psi\rangle) = -i\langle\psi|\mu\rangle\,. \tag{C.4.1}$$

By integrating the previous connection, we assign geometric phases to closed curves in $\mathbb{P}(\mathcal{H}_s)$. If we calculate the exterior derivative of ω_B, and project the result to $\mathbb{P}(\mathcal{H}_s)$, we can compute the curvature form $\mathcal{K}_B$, defined for two tangent vectors of $\mathbb{P}(\mathcal{H}_s)$. This is what we do in the following paragraphs

Consider a point $|\psi\rangle$ in the unitary sphere and two tangent vectors $|\mu\rangle$ and $|\nu\rangle$. Let A and B be Hermitian operators such that $-iA|\psi\rangle = |\mu\rangle$ and $-iB|\psi\rangle = |\nu\rangle$ (finding such operators is always possible, as unitary operators acts transitively in the sphere of normalized states), and consider the vectorial fields tangent to the sphere: $V(|\phi\rangle) = -iA|\phi\rangle$ and $W(|\phi\rangle) = -iB|\phi\rangle$. In these terms, the integral line of V and W through the point $|\phi\rangle$ can be parametrized as $e^{-iAt}|\phi\rangle$ and $e^{-iBt}|\phi\rangle$ respectively.

We know that we can compute $d\omega_B$ in a coordinate free manner as follows,

$$d\omega_B(V, W) = V(\omega_B(W)) - W(\omega_B(V)) - \omega_B([V, W])\,. \tag{C.4.2}$$

We calculate each term. By considering (C.4.1), it is easy to check that at an arbitrary point $|\phi\rangle$, the following equality holds, $\omega_B(W) = -i\langle\phi|(-iB|\phi\rangle) = -\langle B\rangle$. Using this fact, we have (evaluating the derivative at $t = 0$),

$$V(\omega_B(W)) = -\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi|e^{iAt}Be^{-iAt}|\phi\rangle = -i\langle[A, B]\rangle\,.$$

In the same way, the second term can be written as,

$$W(\omega_B(V)) = -i\langle[B, A]\rangle\,.$$

Now, we treat the last term of (C.4.2). Since V and W are the vectorial fields associated to A and B, it is a well known result that their commutator is the field associated to $[A, B]$. Therefore,

$$[V, W](|\phi\rangle) = [A, B]|\phi\rangle \Rightarrow \omega_B([V, W]) = -i\langle[A, B]\rangle\,.$$

By putting the previous results in (C.4.2), we obtain,

$$d\omega_B(V, W) = i\langle[B, A]\rangle = -i\langle[A, B]\rangle\,.$$

By considering our original point $|\psi\rangle$ where $A|\psi\rangle = i|\mu\rangle$ and $B|\psi\rangle = i|\nu\rangle$, we obtain,

$$\begin{aligned} d\omega_B(|\mu\rangle, |\nu\rangle) &= -i(\langle AB\rangle - \langle BA\rangle) = -i(\langle\mu|\nu\rangle - \langle\nu|\mu\rangle) = -i2i\Im\langle\mu|\nu\rangle \\ &= 2\Im\langle\mu|\nu\rangle\,. \end{aligned}$$

Finally, we project everything to $\mathbb{P}(\mathcal{H}_s)$ to obtain $\mathcal{K}_B$. Consider a point $\rho = |\psi\rangle\langle\psi|$ and take two tangent vectors $v_1 = |\psi\rangle\langle\mu| + |\mu\rangle\langle\psi|$ and $v_2 = |\psi\rangle\langle\nu| + |\nu\rangle\langle\psi|$ as stated in theorem 2. By using the previous results we have,

$$B(v_1, v_2) = 2\Im\langle\mu|\nu\rangle = -i\mathrm{Tr}\{\rho[v_1, v_2]\}\,.$$

The last equation was obtained by simple inspection. We can check that it indeed holds in the following way,

$$-i\mathrm{Tr}\{\rho[v_1, v_2]\} = -i\langle[v_1, v_2]\rangle$$

But,

$$\langle v_1 v_2\rangle = \langle(|\psi\rangle\langle\mu| + |\mu\rangle\langle\psi|)(|\psi\rangle\langle\nu| + |\nu\rangle\langle\psi|)\rangle = \langle\mu|\nu\rangle\,,$$

where we used the fact that $\langle v_1|\psi\rangle$ and $\langle v_2|\psi\rangle = 0$ are zero. In the same way, $\langle v_2 v_1\rangle = \langle\nu|\mu\rangle$. Because of these, we have simply,

$$-i\langle[v_1, v_2]\rangle = -i(\langle\mu|\nu\rangle - \langle\nu|\mu\rangle) = 2\mathrm{Im}\langle\mu|\nu\rangle = \mathcal{K}_B(v_1, v_2)\,,$$

as claimed.

The previous result allow us to study the behavior of the curvature $\mathcal{K}_B$ when evaluated in horizontal and vertical vectors. By taking $|\mu\rangle = iS_\alpha|\psi\rangle$ and $|\nu\rangle = iS_\beta|\psi\rangle$ we can study the behavior for two vertical vectors,

$$\mathcal{K}_B(S^\sharp_\alpha, S^\sharp_\beta) = 2\Im\langle\psi|S_\alpha S_\beta|\psi\rangle = -i\langle[S_\alpha, S_\beta]\rangle = -i\epsilon^\gamma{}_{\alpha\beta}\langle S_\gamma\rangle\,.$$

If we consider $|\mu\rangle = iS^\sharp_\alpha|\psi\rangle$ and $|\nu\rangle = |\psi_i\rangle$, where $E_i = |\psi\rangle\langle\psi_i| + |\psi_i\rangle\langle\psi|$ we have the following expression for one horizontal vector and one vertical,

$$\begin{aligned}\mathcal{K}_B(S^\sharp_\alpha, E_i) = & -2\Im i\langle\psi|S_\alpha|\psi_i\rangle = 2\Re\langle\psi|S_\alpha|\psi_i\rangle = \langle\psi|S_\alpha|\psi_i\rangle + \langle\psi_i|S_\alpha|\psi\rangle \\ = & \langle S_\alpha\rangle_{,i}\,.\end{aligned}$$

Finally, if we consider $|\mu\rangle = |\psi_i\rangle$ and $|\nu\rangle = |\psi_j\rangle$ we obtain the expression for two horizontal vectors,

$$\mathcal{K}_B(E_i, E_j) = 2\Im\langle\psi_i|\psi_j\rangle\,. \tag{C.4.3}$$

The previous expression can be written in terms of the coefficient for the connection Ω. Before computing this relation, we give some preliminaries in the following sections.

C.5 The little group of a point in $\mathbb{P}(\mathcal{H}_s)$

As it is well known, the unitary group $U(2s+1)$ acts on $\mathcal{H}_s$ by the left, and has (real) dimension $(2s+1)^2$. Given a state $|\psi\rangle$, we can define the little group given by $|\psi\rangle$ –the subgroup of $U(2s+1)$ consistent on all the operators that fixes $|\psi\rangle$. Simple counting argument show that this space has real dimension $(2s)^2 = 4s^2$, and that the corresponding little algebra is given by all the self adjoint operators H such that $H|\psi\rangle = 0$. In this section, we find a basis for this little algebra. The result of this section are useful for later sections.

We can build a basis for this little algebra. To this end, consider an orthonormal basis that includes $|\psi\rangle = |\psi_1\rangle$, $\{|\psi_i\rangle,\, i = 1, \dots, 2s+1\}$. Then, a possible basis for the little algebra is,

$$\begin{aligned}\rho_i &= |\psi_i\rangle\langle\psi_i|\,, \quad (2 \le i \le 2s+1)\,, \\ X_{ik} &= |\psi_i\rangle\langle\psi_k| + |\psi_k\rangle\langle\psi_i|\,, \quad (2 \le i \le 2s+1, 2 \le k < i)\,, \\ Y_{ik} &= -i(|\psi_i\rangle\langle\psi_k| - |\psi_k\rangle\langle\psi_i|)\,, \quad (2 \le i \le 2s+1, 2 \le k < i)\,.\end{aligned} \tag{C.5.1}$$

Note that X is symmetric in its indices while Y is antisymmetric. We can make a similar analysis for $\mathbb{P}(\mathcal{H}_s)$. Consider a point $\rho = |\psi\rangle\langle\psi|$. As $U(2s+1)$ also acts in $\mathbb{P}(\mathcal{H}_s)$ by conjugation (c.f. (1.1.7)), we can define a little group for ρ

As it is easy to check, a possible basis for the little algebra consists on the operators of equation (C.5.1) and additionally the operator ρ_1 (it is clear that $e^{-it\rho}$ fixes ρ). For completeness, here we list some of the commutator relationships, (in all cases, $2 \leq i, k, l \leq 2s+1$, and different indices have different values)

$$
\begin{aligned}
[\rho_i, X_{ik}] &= |\psi_i\rangle\langle\psi_k| - |\psi_k\rangle\langle\psi_i| = iY_{ik} \,, \\
[\rho_i, Y_{ik}] &= -i|\psi_i\rangle\langle\psi_k| - i|\psi_k\rangle\langle\psi_i| = -iX_{ik} \,, \\
[X_{ik}, X_{il}] &= |\psi_k\rangle\langle\psi_l| - |\psi_l\rangle\langle\psi_k| = iY_{kl} \,, \\
[X_{ik}, Y_{il}] &= -i|\psi_k\rangle\langle\psi_l| - i|\psi_l\rangle\langle\psi_k| = -iX_{kl} \,, \\
[X_{ik}, Y_{ik}] &= 2i\rho_i - 2i\rho_k \,, \\
[Y_{ik}, Y_{il}] &= |\psi_k\rangle\langle\psi_l| - |\psi_l\rangle\langle\psi_k| = iY_{kl} \,,
\end{aligned}
$$

while the missing relationships are either zero or can be obtained from the previous commutators using the symmetric properties of X_{ij} and Y_{ij}.

Finally, we make some useful observations for later reference. Consider an arbitrary element A of $u(2s+1)$. By the right action, we can consider the tangent vector $A^\sharp$ at ρ_1 as in (2.1.2). A quick computation reveals the following expression for the interior product h (C.2.1) between this type of vectors,

$$
h(A^\sharp, B^\sharp) = 2\Re\langle AB\rangle - 2\langle A\rangle\langle B\rangle \,. \tag{C.5.2}
$$

Clearly, A is in the little algebra of ρ if and only if $A^\sharp$ is zero. Because of this, we have that $(A+B)^\sharp = B^\sharp$ for all B in $u(2s+1)$ and A in the little algebra. In particular, the equality $(B - \langle B\rangle\rho)^\sharp = B^\sharp$ holds for any B. But clearly, the expectation value (computed w.r.t. ρ) $\langle B - \langle B\rangle\rho\rangle$ is zero. Since the right action of $u(2s+1)$ is transitive, we can conclude the following observation. Given any vector v tangent at ρ, there exist an (not unique) element B in $u(2s+1)$ such that

$$
B^\sharp = v \,, \quad \langle B\rangle = 0 \,. \tag{C.5.3}
$$

C.6 The Berry curvature evaluated at two horizontal fields

In this section, we come back to subject of expressing the Berry curvature evaluated at two horizontal vectors in terms of ${\Omega^\alpha}_{ij}$ presented in C.4.

Suppose that the horizontal vector fields E_i can be written as $E_i(\rho) = H_i(\rho)^\sharp$, for certain operators $H_i(\rho)$ in $U(2s+1)$. Because of the results of the previous section, we can assume w.l.o.g. that $\langle H(\rho)\rangle$ is zero for each ρ in projective Hilbert space. We stress that this operators depend on ρ. In fact, one can prove that the operators H_i must depend on ρ –it is impossible to find an operator H_i such that $H_i^\sharp$ is horizontal for each ρ. A simple computation reveals the following

It is possible to compute the commutator $[E_i, E_j]$ in terms of $H_i(\rho)$ and $H_j(\rho)$. Because of their dependence on ρ, the result is not simply the field $-i[H_i(\rho), H_j(\rho)]^\sharp$, additional terms appear. For the calculation, we use the following expression for the commutator of two fields,

$$[H_i(\rho)^\sharp, H_j(\rho)^\sharp] = \lim_{t\to 0} \frac{\chi_{t*}^{-1} \circ H_j(\rho)^\sharp \circ \chi_t(\rho) - H_j^\sharp(\rho)}{t}\,, \qquad \text{(C.6.1)}$$

where $\chi_t(\rho)$ denotes the evolution operator associated with the flow of the field $H_i^\sharp(\rho)$.

We can calculate the flow operator $\chi_t(\rho)$ up to first order in t as follows,

$$\chi_t(\rho) = \rho + tH_i(\rho)^\sharp\,, \quad \chi_t^{-1}(\rho) = \rho - tH_i(\rho)^\sharp\,,$$

From previous equation, we can calculate the pushforward of the mapping χ_t^{-1} by considering the following relation,

$$\chi_t^{-1}(\rho + \tau H_j(\rho)^\sharp) = \rho + sH_j(\rho)^\sharp - tH_i(\rho + \tau H_j(\rho)^\sharp)^\sharp\,.$$

By taking the derivative w.r.t. τ and evaluating at $\tau = 0$ we obtain the following,

$$\chi_{t*}^{-1}(H_j(\rho)^\sharp) = H_j(\rho)^\sharp - t\frac{\mathrm{d}}{\mathrm{d}\tau}H_i(\rho + \tau H_j(\rho)^\sharp)^\sharp\,, \qquad \text{(C.6.2)}$$

but (ignoring the terms of higher order in τ),

$$\begin{aligned} H_i(\rho + \tau H_j(\rho)^\sharp)^\sharp &= i[H_i(\rho + \tau H_j(\rho)^\sharp), \rho + \tau H_j(\rho)^\sharp] \\ &= i[H_i(\rho), \rho] + i\tau[H_i(\rho), H_j(\rho)^\sharp] + i\tau[dH_i(H_j(\rho)^\sharp), \rho] \end{aligned}$$

$$= i[H_i(\rho),\rho] + i\tau[H_i(\rho),H_j(\rho)^\sharp] + \tau(dH_i(H_j(\rho)^\sharp))^\sharp\,, \tag{C.6.3}$$

so that,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}H_i(\rho+\tau H_j(\rho)^\sharp)^\sharp = i[H_i(\rho),H_j(\rho)^\sharp] + (dH_i(H_j(\rho)^\sharp))^\sharp\,,$$

which implies (by substituting the previous equation in (C.6.2)),

$$\chi_{t*}^{-1}(H_j(\rho)^\sharp) = H_j(\rho)^\sharp - it[H_i(\rho),H_j(\rho)^\sharp] - t(dH_i(H_j(\rho)^\sharp))^\sharp\,.$$

By evaluating this expression at $\chi_t(\rho) = \rho + tH_i(\rho)^\sharp$ and only considering the terms of lower order we obtain,

$$\begin{aligned}
&\chi_{t*}^{-1}(H_j(\rho+tH_i(\rho)^\sharp)^\sharp)\\
&= H_j(\rho+tH_i(\rho)^\sharp)^\sharp - it[H_i(\rho),H_j(\rho)^\sharp] - t(dH_i(H_j(\rho)^\sharp))^\sharp\\
&= i[H_j(\rho),\rho] + it[H_j(\rho),H_i(\rho)^\sharp] + t(dH_j(H_i(\rho)^\sharp))^\sharp\\
&\qquad - it[H_i(\rho),H_j(\rho)^\sharp] - t(dH_i(H_j(\rho)^\sharp))^\sharp\\
&= H_j(\rho)^\sharp + it([H_j(\rho),H_i(\rho)^\sharp] - [H_i(\rho),H_j(\rho)^\sharp]) + t(dH_j(H_i(\rho)^\sharp)\\
&\qquad - dH_i(H_j(\rho)^\sharp))^\sharp\,,
\end{aligned}$$

where we interchanged i with j and changed τ to t in equation (C.6.3) to obtain the second line. We can simplify the result a little bit more by noticing the following,

$$\begin{aligned}
[H_j(\rho),H_i(\rho)^\sharp] &= [H_j(\rho),[H_i(\rho),\rho]] = i[H_j(\rho),H_i(\rho)],\rho] + [H_i(\rho),i[H_j(\rho),\rho]]\\
&= [H_j(\rho),H_i(\rho)]^\sharp + [H_i(\rho),H_j(\rho)^\sharp]
\end{aligned}$$

By using this equality in the one previous to it, we obtain,

$$\begin{aligned}
\chi_{t*}^{-1}(H_j(\rho+tH_i(\rho)^\sharp)^\sharp) = H_j(\rho)^\sharp &+ it[H_j(\rho),H_i(\rho)]^\sharp\\
&+ t(dH_j(H_i(\rho)^\sharp) - dH_i(H_j(\rho)^\sharp))^\sharp\,.
\end{aligned}$$

Finally, by using this result in (C.6.1), we get the following expression for the commutator of the two fields,

$$\begin{aligned}
[H_i(\rho)^\sharp,H_j(\rho)^\sharp] &= i[H_j(\rho),H_i(\rho)]^\sharp + t(dH_j(H_i(\rho)^\sharp) - dH_i(H_j(\rho)^\sharp))^\sharp\\
&= i[H_j(\rho),H_i(\rho)]^\sharp + t(dH_j(H_i(\rho)^\sharp) - dH_i(H_j(\rho)^\sharp))^\sharp\\
&= (i[H_j(\rho),H_i(\rho)] + dH_j(H_i(\rho)^\sharp) - dH_i(H_j(\rho)^\sharp))^\sharp\,.
\end{aligned}$$

By comparing the previous equation with (C.3.2), we conclude

$$[H_i(\rho)^\sharp, H_j(\rho)^\sharp] = [E_i, E_j] = -2Q^l{}_{ij}E_l - \Omega^\alpha{}_{ij}S^\sharp_\alpha = (-2Q^l{}_{ij}H_l - \Omega^\alpha{}_{ij}S_\alpha)^\sharp ,$$

so that,

$$(-2Q^l{}_{ij}H_l - \Omega^\alpha{}_{ij}S_\alpha)^\sharp = (i[H_j(\rho), H_i(\rho)] + dH_j(H_i(\rho)^\sharp) - dH_i(H_j(\rho)^\sharp))^\sharp .$$

If two operators in $u(2s+1)$ satisfy the equality $A^\sharp = B^\sharp$, we can not conclude that they are equal; the most general statement we can do is that they differ by an element of the little algebra of the point in consideration. Because of this we have,

$$-2Q^l{}_{ij}H_l - \Omega^\alpha{}_{ij}S_\alpha + a = i[H_j(\rho), H_i(\rho)] + dH_j(H_i(\rho)^\sharp) - dH_i(H_j(\rho)^\sharp) , \tag{C.6.4}$$

where a is an element of the little algebra for ρ. Consider the basis for the little algebra that consists on the operators of equation (C.5.1) and ρ. Note that $|\psi\rangle$ (where $\rho = |\psi\rangle\langle\psi|$) is annihilated for all the elements of the basis except for ρ. Because of this, if we write a as $a = \Delta\rho + \Theta^i\rho_i + G^{ij}X_{ij} + \Lambda^{ij}Y_{ij}$, we have the following equality,

$$a|\psi\rangle = \Delta|\psi\rangle .$$

Because of this, by applying the l.h.s. and r.h.s. of (C.6.4) to $|\psi\rangle$ we obtain the following (from now on, we stop writing the explicit dependence of H_i on ρ)

$$(-2Q^l{}_{ij}H_l - \Omega^\alpha{}_{ij}S_\alpha + \Delta)|\psi\rangle = (i[H_j, H_i] + dH_j(H_i^\sharp) - dH_i(H_j^\sharp))|\psi\rangle . \tag{C.6.5}$$

Before going on manipulating (C.6.5), we find an useful identity. Recall that, by construction, the expected value of H_j is always zero, $\langle H_j\rangle = 0$. Because of this, its derivative in the direction $H_i^\sharp$, that $\langle H_j\rangle_{,i}$ is zero, but,

$$\begin{aligned} &\langle H_j\rangle_{,i} \\ &\quad = \mathrm{Tr}\,(\rho H_j)_{,i} = \mathrm{Tr}\,(\rho_{,i}H_j) + \mathrm{Tr}\,(\rho H_{j,i}) = \mathrm{Tr}\,(H_i^\sharp H_j) + \mathrm{Tr}\,(\rho dH_j(H_i^\sharp)) \\ &\quad = i\,\mathrm{Tr}\,([H_i, \rho]H_j) + \langle dH_j(H_i^\sharp)\rangle = i\,\mathrm{Tr}\,(H_i\rho H_j - \rho H_i H_j)) + \langle dH_j(H_i^\sharp)\rangle \\ &\quad = i\,\mathrm{Tr}\,(\rho H_j H_i - \rho H_i H_j) + \langle dH_j(H_i^\sharp)\rangle = i\langle [H_j, H_i]\rangle + \langle dH_j(H_i^\sharp)\rangle . \end{aligned}$$

Since this quantity has to be zero, we can conclude that,

$$\langle dH_j(H_i^\sharp)\rangle = -i\langle [H_j, H_i]\rangle .$$

By changing i with j, we also obtain the following,

$$-\langle dH_i(H_j^\sharp)\rangle = i\langle [H_i, H_j]\rangle = -i\langle [H_j, H_i]\rangle \,.$$

Because of this we have,

$$\langle dH_j(H_i^\sharp) - dH_i(H_j^\sharp)\rangle = \langle dH_j(H_i^\sharp)\rangle - \langle dH_i(H_j^\sharp)\rangle = -2i\langle [H_j, H_i]\rangle \,. \quad \text{(C.6.6)}$$

Now we get back to (C.6.5). By projecting in $\langle\psi|$ in both sides of (C.6.5), remembering that $\langle H_l\rangle = 0$, and using (C.6.6) we obtain,

$$-\Omega^\alpha{}_{ij}\langle S_\alpha\rangle + \Delta = -i\langle [H_j, H_i]\rangle \,. \quad \text{(C.6.7)}$$

In this way, if we find the valued of Δ, we can write the expected value $\langle [H_j, H_i]\rangle$ in terms of the curvature Ω. To find Δ, we calculate the projection of (C.6.5) into $\langle\psi|S_\mu$. The result is,

$$\begin{aligned}\langle -2Q^l{}_{ij}S_\mu H_l - \Omega^\alpha{}_{ij}S_\mu S_\alpha + \Delta S_\mu\rangle \\ = \langle S_\mu[H_j, H_i] + S_\mu dH_j(H_i^\sharp) - S_\mu dH_i(H_j^\sharp)\rangle \,. \quad \text{(C.6.8)}\end{aligned}$$

Our claim is that the real part of the first term is zero. Indeed, since $H_i^\sharp$ is horizontal, $\Re\langle S_\mu H_i\rangle$ is is zero (recall that $\langle H_i\rangle$ is zero and equation (C.5.2)). Also, by considering the derivative of $\Re\langle S_\mu H_i\rangle$ in the direction $H_j(\rho)^\sharp$, we can simplify the r.h.s. of (C.6.8). The reasoning is as follows. First, we compute said derivative,

$$\begin{aligned}\langle S_\mu H_i\rangle_{,j} &= \operatorname{Tr}(H_j^\sharp S_\mu H_i) + \operatorname{Tr}(\rho S_\mu dH_i(H_j^\sharp)) \\ &= \operatorname{Tr}(i[H_j, \rho]S_\mu H_i) + \langle S_\mu dH_i(H_j^\sharp)\rangle \\ &= i\operatorname{Tr}((H_j\rho - \rho H_j)S_\mu H_i)) + \langle S_\mu dH_i(H_j^\sharp)\rangle \\ &= i\operatorname{Tr}(\rho S_\mu H_i H_j) - i\operatorname{Tr}(\rho H_j S_\mu H_i) + \langle S_\mu dH_i(H_j^\sharp)\rangle \\ &= i\langle S_\mu H_i H_j\rangle - i\langle H_j S_\mu H_i\rangle + \langle S_\mu dH_i(H_j^\sharp)\rangle \,.\end{aligned}$$

Since the real part of the previous expression needs to be zero, we can conclude the following,

$$\begin{aligned}0 = \Re\langle S_\mu H_i\rangle_{,j} &= \Re(\langle S_\mu dH_i(H_j^\sharp)\rangle + i\langle S_\mu H_i H_j\rangle - i\langle H_j S_\mu H_i\rangle) \\ &\Rightarrow \Re\langle S_\mu dH_i(H_j^\sharp)\rangle = \Re\, i\langle H_j S_\mu H_i - S_\mu H_i H_j\rangle \,.\end{aligned}$$

By interchanging i with j from the previous equation and subtracting it from the original one, we obtain,

$$\Re\langle S_\mu dH_i(H_j^\sharp) - S_\mu dH_j(H_i^\sharp)\rangle = \Re\, i\langle H_j S_\mu H_i - H_i S_\mu H_j - S_\mu[H_i, H_j]\rangle$$

$$= 2\Re\, i\langle H_j S_\mu H_i\rangle - \Re\, i\langle S_\mu [H_i, H_j]\rangle\,,$$

where we used the fact that $-i\langle H_i S_\mu H_j\rangle$ is the complex conjugate of $i\langle H_j S_\mu H_i\rangle$ so that they have the same real part. By taking into consideration these equalities in (C.6.8), we conclude,

$$-\Omega^{\alpha}{}_{ij}\Re\langle S_\mu S_\alpha\rangle + \Delta\langle S_\mu\rangle = 2\Re\, i\langle H_i S_\mu H_j\rangle\,. \tag{C.6.9}$$

The previous equation is valid for any value of μ. By considering (C.6.7) and (C.2.4) in (C.6.9) we can conclude,

$$\begin{aligned}
-\Omega^{\alpha}{}_{ij}\left(\frac{k_{\mu\alpha}}{2} + \langle S_\alpha\rangle\langle S_\mu\rangle\right) + \Delta\langle S_\mu\rangle &= 2\Re\, i\langle H_i S_\mu H_j\rangle \Rightarrow \\
-\frac{\Omega_{\mu ij}}{2} + \langle S_\mu\rangle(-\Omega^{\alpha}{}_{ij}\langle S_\alpha\rangle + \Delta) &= 2\Re\, i\langle H_i S_\mu H_j\rangle \Rightarrow \\
-\frac{\Omega_{\mu ij}}{2} + \langle S_\mu\rangle(-i\langle [H_j, H_i]\rangle) &= 2\Re\, i\langle H_i S_\mu H_j\rangle \Rightarrow \\
i\langle [H_j, H_i]\rangle &= -\frac{1}{\langle S_\mu\rangle}\left(\frac{\Omega_{\mu ij}}{2} + 2\Re i\langle H_i S_\mu H_j\rangle\right)\,.
\end{aligned} \tag{C.6.10}$$

Define $|\psi_j\rangle = iH_j|\psi\rangle$ and $|\psi_i\rangle = iH_i|\psi\rangle$. Then, $H_j^\sharp$ can be written as,

$$H_j^\sharp = i[H_j, \rho] = |\psi_j\rangle\langle\psi| + |\psi\rangle\langle\psi_j|\,,$$

while

$$\langle\psi|\psi_j\rangle = i\langle H_j\rangle = 0\,.$$

Because of this, we see that $|\psi_j\rangle$ is the vector defined in theorem 2. The same can be said for $|\psi_i\rangle$. In terms of this vectors,

$$\begin{aligned}
i\langle [H_j, H_i]\rangle &= i(\langle\psi_j|\psi_i\rangle - \langle\psi_i|\psi_j\rangle) = 2\Im\langle\psi_i|\psi_j\rangle = \mathcal{K}_B(E_i, E_j)\,, \\
2\Re i\langle H_i S_\mu H_j]\rangle &= 2\Re i\langle\psi_i|S_\mu|\psi_j\rangle = -2\Im\langle\psi_i|S_\mu|\psi_j\rangle\,,
\end{aligned} \tag{C.6.11}$$

where we used equation (C.4.3) for the first line. Direct substitution of this equalities in (C.6.10), produces the following result,

$$\mathcal{K}_B(E_i, E_j) = \frac{2\Im\langle\psi_i|S_\mu|\psi_j\rangle - \Omega_{\mu ij}}{2\langle S_\mu\rangle}\,. \tag{C.6.12}$$

In principle, we can use this equation (with any value of μ) along with (C.6.7) to write the geometric phase (C.4.3) in terms of Ω. By using (C.2.4) in (C.6.9) we obtain the following,

The previous equation has been checked numerically for various points in the case of $j = 3/2$ and $j = 2$.

C.7 Applications of the little algebra

C.7.1 Another expression for the coefficients of the connection Ω

In this section, we introduce some new mathematical concepts that allows us to find a simpler expression for $\Omega_{\mu ij}$.

Given a generic point ρ in $\mathbb{P}(\mathcal{H}_s)$, consider the basis for $u(2j+1)$ that consists on the angular momentum operators S_α, the horizontal operators $H_i(\rho)$ introduced in the previous section, ρ and the little algebra (C.5.1). Denote the elements of this basis generically as O_A, where Greek indices denotes vertical quantities, Latin, horizontal and primed Latin indices refer to the members of the little group. Denote the structure functions w.r.t. this basis as $\mathscr{C}$; $[O_A, O_B] = i\mathscr{C}^D{}_{AB} O_D$.

To find an alternative expression for $\Omega_{\mu ij}$, consider the following term of (C.6.10),

$$
\begin{aligned}
2\Re\, i\langle H_i S_\mu H_j\rangle &= i\langle H_i S_\mu H_j\rangle - i\langle H_j S_\mu H_i\rangle \\
&= i(\langle H_i S_\mu H_j\rangle - \langle S_\mu H_i H_j\rangle + \langle S_\mu H_i H_j\rangle - \langle S_\mu H_j H_i\rangle \\
&\quad + \langle S_\mu H_j H_i\rangle - \langle H_j S_\mu H_i\rangle) \\
&= i(\langle [H_i, S_\mu] H_j\rangle + \langle S_\mu [H_i, H_j]\rangle + \langle [S_\mu, H_j] H_i\rangle) \\
&= i(i\mathscr{C}^A{}_{i\mu}\langle O_A H_j\rangle + i\mathscr{C}^A{}_{ij}\langle S_\mu O_A\rangle + i\mathscr{C}^A{}_{\mu j}\langle O_A H_i\rangle)\,.
\end{aligned}
$$

By taking the real part in both sides of the previous equation, we obtain the following result,

$$
2\Re\, i\langle H_i S_\mu H_j\rangle = -\mathscr{C}^A{}_{i\mu}\Re\langle O_A H_j\rangle - \mathscr{C}^A{}_{ij}\Re\langle S_\mu O_A\rangle - \mathscr{C}^A{}_{\mu j}\Re\langle O_A H_i\rangle\,. \tag{C.7.1}
$$

We simplify (C.7.1) by considering each term separately. To compute the first one, note the following,

$$
\begin{aligned}
\Re\langle O_{1'} H_j\rangle &= \langle H_j\rangle = 0\,, \\
\Re\langle O_{A'} H_j\rangle &= 0\,, \\
\Re\langle O_\nu H_j\rangle &= \Re\langle S_\nu H_j\rangle = \frac{h_{\nu j}}{2} = 0\,, \\
\Re\langle H_k H_j\rangle &= \frac{g_{kj}}{2}\,.
\end{aligned} \tag{C.7.2}
$$

The index $1'$ corresponds to ρ, the index A', denote the rest of the elements of the little algebra. The first two equalities can be obtained by a direct computation. The third one, can be deduced by recalling (C.5.2) and the fact that horizontal vectors are perpendicular to verticals. Finally, the last one can also be obtained from (C.5.2). These equalities implies that $\mathscr{C}^{A}{}_{i\mu}\Re\langle O_A H_j\rangle = (1/2)\mathscr{C}^{k}{}_{i\mu}g_{kj} = (1/2)\mathscr{C}_{ji\mu}$. In the same way, the third term of (C.7.1) is $\mathscr{C}^{A}{}_{\mu j}\Re\langle O_A H_i\rangle = (1/2)\mathscr{C}_{i\mu j}$. For the remaining term, $\mathscr{C}^{A}{}_{ij}\Re\langle S_\mu O_A\rangle$, by a similar procedure used to obtain (C.7.2), we have the following,

$$\begin{aligned}
&\Re\langle S_\mu O_{1'}\rangle = \langle S_\mu\rangle\,,\\
&\Re\langle S_\mu O_A\rangle = 0\,,\\
&\Re\langle O_\mu O_\nu\rangle = \Re\langle S_\mu S_\nu\rangle = \frac{k_{\mu\nu}}{2} + \langle S_\mu\rangle\langle S_\nu\rangle\,,\\
&\Re\langle S_\mu H_k\rangle = \frac{h_{\mu k}}{2} = 0\,.
\end{aligned}$$

so,

$$\begin{aligned}
\mathscr{C}^{A}{}_{ij}\Re\langle S_\mu O_A\rangle &= \mathscr{C}^{1'}{}_{ij}\langle S_\mu\rangle + \mathscr{C}^{\nu}{}_{ij}\left(\frac{k_{\mu\nu}}{2} + \langle S_\mu\rangle\langle S_\nu\rangle\right)\\
&= (\mathscr{C}^{1'}{}_{ij} + \mathscr{C}^{\nu}{}_{ij}\langle S_\nu\rangle)\langle S_\mu\rangle + \frac{\mathscr{C}_{\mu ij}}{2}\,.
\end{aligned}$$

On the other hand, by considering (C.6.11), we have the following,

$$\begin{aligned}
\mathcal{K}_B(E_i,E_j) &= -i\langle[H_i,H_j]\rangle = -i(i\mathscr{C}^{A}{}_{ij}\langle O_A\rangle) = \mathscr{C}^{A}{}_{ij}\langle O_A\rangle\\
&= \mathscr{C}^{1'}{}_{ij} + \mathscr{C}^{\nu}{}_{ij}\langle S_\nu\rangle\,,
\end{aligned}$$

where we used the fact that all the operators O_A have zero expectation value except for ρ and S_α. By using this expression in the previous one, we obtain the following expression,

$$\mathscr{C}^{A}{}_{ij}\Re\langle S_\mu O_A\rangle = \mathcal{K}_B(E_i,E_j)\langle S_\mu\rangle + \frac{\mathscr{C}_{\mu ij}}{2}\,.$$

Substituting the terms we just obtained in (C.7.1), we obtain,

$$\begin{aligned}
2\Re\, i\langle H_i S_\mu H_j\rangle &= -\frac{1}{2}(\mathscr{C}_{ji\mu} + \mathscr{C}_{i\mu j} + \mathscr{C}_{\mu ij}) - \mathcal{K}_B(E_i,E_j)\langle S_\mu\rangle \Rightarrow\\
-4\Re\, i\langle H_i S_\mu H_j\rangle &= \mathscr{C}_{ji\mu} + \mathscr{C}_{i\mu j} + \mathscr{C}_{\mu ij} + 2\langle S_\mu\rangle\mathcal{K}_B(E_i,E_j)\,.
\end{aligned}$$

Finally, the usage of this result in expression (C.6.10), together with (C.6.11), produces the following equation after some algebra,

$$\Omega_{\mu ij} = \mathscr{C}_{ji\mu} + \mathscr{C}_{i\mu j} + \mathscr{C}_{\mu ij}\,. \tag{C.7.3}$$

C.8 Relationship with the Schrödinger equation

Suppose that the temporal evolution of a spin s is given by a Hamiltonian H. Call $\rho(t)$ the evolution of the system in projective Hilbert space. As it is well known, the time derivative of ρ is given by the following equation,

$$\dot{\rho} = -i[H, \rho] = -H^{\sharp} \equiv v^{\alpha} S^{\sharp}_{\alpha} + v^{i} E_{i} \,,$$

The components v^{α} can be regarded as a speed in the fibers, and the coefficients v^{i} as a speed in $\mathscr{S}$. We can find an expression for this components. By calculating the product of $\dot{\rho}$ and $S^{\sharp}_{\beta}$ on one hand, we have the following,

$$h(\dot{\rho}, S^{\sharp}_{\beta}) = v^{\alpha} k_{\alpha\beta} = v_{\beta} \,.$$

On the other hand, by equation (C.5.2),

$$h(\dot{\rho}, S^{\sharp}_{\beta}) = -h(H^{\sharp}, S^{\sharp}_{\beta}) = -2(\Re\langle H S_{\beta}\rangle - \langle H\rangle\langle S_{\beta}\rangle) = -2\,\mathrm{Corr}(H, S_{\beta}) \,,$$

where we defined the correlation of two operators $\mathrm{Corr}(A, B)$ as $\mathrm{Corr}(A, B) = \langle AB\rangle + \langle BA\rangle - \langle A\rangle\langle B\rangle$. Note that $\mathrm{Corr}(A, B)$ is a function defined over projective Hilbert space. By considering both expressions for $h(\dot{\rho}, S^{\sharp}_{\beta})$, we can write,

$$v_{\beta} = -2\,\mathrm{Corr}(H, S_{\beta}) \,.$$

We can make exactly the same for horizontal vectors. The result is

$$v_{i} = -2\,\mathrm{Corr}(H, H_{i}(\rho)) \,.$$

The punchline of this results is the following, the speed v_{a} is related with the correlation between the operator O_{a} and the Hamiltonian H.

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