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Contents

Introduction

This thesis consists of two different topics that are not related. The thesis has two different and independent parts that can be read in any order.

The purpose of this work is to study two topics in singularity theory:

1. The McKay correspondence for Gorenstein surface singularities.

2. Classification and properties of some classes of real singularities.

The McKay correspondence for Gorenstein surface singularity

Let $\Gamma \leq SL(2,\mathbb{C})$ be a finite subgroup. Denote by $X_\Gamma := \mathbb{C}^2/\Gamma$ the singular variety obtained from the natural action of Γ in $\mathbb{C}^2$, and let $\pi\colon \tilde{X}_\Gamma \to X_\Gamma$ be the minimal resolution with exceptional divisor E. The classical McKay's correspondence gives a one to one correspondence between the irreducible components of E and the irreducible representations of Γ.

The construction of the correspondence is as follows: Let $\Gamma \leq SL(2,\mathbb{C})$ be a finite subgroup and denote by

$$\mathrm{Irr}\Gamma := \{\rho_0, \rho_1, \dots, \rho_r\},$$

the set of irreducible representations of Γ, where ρ_0 is the trivial representation.

Denote by Q the natural representation given by the inclusion $\Gamma \subset SL(2,\mathbb{C})$. Therefore by Maschke's theorem we get

$$\rho_i \otimes Q = \bigoplus_j a_{ij}\rho_j. \tag{1}$$

Using representation theory, McKay constructed a directed graph $\mathfrak{G}_\Gamma$ as follows:

1. One vertex for each irreducible representation.

2. Given the decomposition (1), we draw a_{ij} arrows from the vertex v_i to the vertex v_j.

It is easy to see that for each finite subgroup $\Gamma \subset SL(2,\mathbb{C})$, we have $a_{ij} = a_{ji}$ and a_{ij} must be 0 or 1, therefore the directed graph constructed by McKay is a graph [38, Chapter 10].

Theorem 1 ([23, McKay])**.** *Let Γ be a finite subgroup of $SL(2,\mathbb{C})$ and denote by $X_\Gamma := \mathbb{C}^2/\Gamma$ the singular variety obtained from the natural action of Γ in $\mathbb{C}^2$. Then the dual graph of the minimal resolution of X_Γ and the graph $\mathfrak{G}_\Gamma$ without the trivial representation coincide.*

The first studies of McKay's correspondence in more geometrical terms were done by Gonzales-Sprinberg and Verdier [16]. As before let Γ be a finite subgroup of $SL(2,\mathbb{C})$, $X_\Gamma := \mathbb{C}^2/\Gamma$ and denote by $\tilde{X}_\Gamma$ the minimal resolution of X_Γ, Gonzales-Sprinberg and Verdier constructed for each irreducible, non-trivial representation of Γ a locally free sheaf $\mathcal{M}$ over the minimal resolution, such that the first Chern class of $\mathcal{M}$ intersects only one component of the exceptional divisor. The work of Gonzales-Sprinberg and Verdier was done case by case.

Later Artin and Verdier [4] gave a proof of the McKay correspondence using only properties of double rational point singularities. From now denote by (X, x) the germ of a normal surface singularity with structure sheaf $\mathcal{O}_X$, $\pi\colon \tilde{X} \to X$ the minimal resolution of X and by E the exceptional divisor.

Artin and Verdier considered the case when X is the spectrum of a rational double point singularity, they gave a correspondence between:

1. Isomorphism classes of indecomposable reflexive modules.

2. Vertices of the dual graph of the minimal resolution of X.

This correspondence is called the geometrical McKay correspondence.

A natural task is extend this correspondence to another kind of singularities, this was done by the following people:

1. Hélène Esnault [13] studied the case of rational surface singularities, she defined the concept of a full sheaf: We say that a locally free sheaf $\tilde{M}$ over $\tilde{X}$ is full if $\tilde{M} = \pi^* M/\text{torsion}$, where M is a reflexive $\mathcal{O}_X$-module.

   Esnault used the ideas of Artin and Verdier and she constructed a correspondence between full sheaves and the irreducible components of the exceptional divisor. It is important to recall that this correspondence is not a bijection, in fact, Esnault constructed an example of a singularity and two non-isomorphic full sheaves with the same first Chern class.

2. Later Jurgen Wunram [37] studied the case of quotient singularities. Using the concept of full sheaf he obtained a one to one correspondence between irreducible components of E and full sheaves $\tilde{M}$ such that $R^1\pi_*\tilde{M}^\vee$ equal to zero. This kind of full sheaves are called special and will be important in our setting.

3. Constantin P. Kahn [21] studied the case of a normal surface singularity. In his work he defined a full sheaf as follows: A locally free sheaf $\mathcal{M}$ is called full if and only if there is a reflexive $\mathcal{O}_X$-module M such that $\mathcal{M} \cong (\pi^* M)^{\vee\vee}$.

   The aim of Kahn was to study the problem of classification of reflexive modules and in order to do that he introduces the notion of reduction cycles which allowed him to relate his problem to the task of classification of reflexive modules over certain projective curves.

   Kahn was able to obtain a complete classification of the indecomposable reflexive modules when the singularity is a simply elliptic singularity.

In this thesis we use the ideas given by Artin-Verdier [4] and Esnault [13] in order to study the reflexive modules over a Gorenstein singularity. From now on (X, x) denotes a complex analytic germ of a normal two-dimensional singularity, p_g the geometric genus and $\pi\colon \tilde{X} \to (X, x)$ denotes a resolution.

In Chapter 1 we introduce the idea of the full sheaf associated to a reflexive module (Definition 1.6) and we generalize the notion of special full sheaf as follows (Definition 1.12).

Definition. Let M be a reflexive $\mathcal{O}_X$-module and denote by $\mathcal{M} = (\pi^* M)^{\vee\vee}$. The full sheaf $\mathcal{M}$ is *the full sheaf associated to* M.

Definition. A full sheaf $\mathcal{M}$ on $\tilde{X}$ of rank r is called *special* if $\dim_{\mathbb{C}} \left(R^1\pi_* \left(\mathcal{M}^{\vee} \right) \right) = rp_g$.

Notice that our definition is a generalization of the concept given by Wunram and both definitions coincide in the case of a rational singularity. In the same chapter we define the specialty defect (Definition 1.13) and the notion of a special module (Definition 1.14).

Definition. Let $\mathcal{M}$ be a full sheaf on $\tilde{X}$ of rank r. *The defect of specialty of* $\mathcal{M}$ is the number $\dim_{\mathbb{C}} \left(R^1\pi_* \left(\mathcal{M}^{\vee} \right) \right) - rp_g$.

Definition. Let M be a reflexive $\mathcal{O}_X$-module. We say that M is *a special module* if for any resolution $\pi \colon \tilde{X} \to X$ the full sheaf $\mathcal{M} = (\pi^* M)^{\vee\vee}$ is special.

Both notions of specialty will be very important in this work.

Our first results given in Chapter 2 are two generalizations of the construction given by Artin-Verdier [4] and Esnault [13]. First in Section 2.1 we study the construction at the singularity, in this case we give a correspondence between reflexive modules of rank r with r sections, and Cohen-Macaulay modules of dimension one with r generators. In this section the main results are Corollary 2.2 and Proposition 2.5.

Corollary 2 (Direct correspondence at the singularity)**.** *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Given a reflexive $\mathcal{O}_X$-module of rank r with r generic sections, we can associate a Cohen-Macaulay $\mathcal{O}_X$-module of dimension one and r generators.*

Proposition 3 (Inverse correspondence at the singularity)**.** *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Given a Cohen-Macaulay $\mathcal{O}_X$-module of dimension one and r generators, we can associate a reflexive $\mathcal{O}_X$-module and r sections.*

At the end of the Section 2.1 we study both correspondences. It is easy to verify that (up to a natural isomorphism) the direct and the inverse correspondence are inverse to each other (Remark 2.6).

Later in Section 2.2 we study the construction at the resolution, in this case we give a correspondence between full sheaves of rank r with r sections, and Cohen-Macaulay shaves of dimension one such that its support is not contained in the exceptional divisor, together with r generators. This case is analogous to the correspondence at singularity but it is necessary to do some preliminary work. In this section the main results are Proposition 2.10 and Proposition 2.16.

Proposition 4 (Direct correspondence at the resolution)**.** *Let (X, x) be a complex analytic germ of a normal two-dimensional singularity and $\pi \colon \tilde{X} \to X$ be a resolution of X. Given a full sheaf $\mathcal{M}$ of rank r with r generic sections, we associate three things:*

1. *A Cohen-Macaulay sheaf $\mathcal{A}$ of dimension one such that its support D intersects the exceptional divisor in a finite set.*
2. *An $\mathcal{O}_X$-module $\mathcal{C}$ contained in $\pi_*\mathcal{A}$.*
3. *A collection of r generators of $\mathcal{C}$ as $\mathcal{O}_X$-module.*

The module $\mathcal{C}$ has the following property: Denote by $\mathcal{O}_{\pi_* D}$ the structure sheaf of $\pi(D)$ and $n\colon \tilde{D} \to D$ the normalization of D. Then we have the following inclusions

$$\mathcal{O}_{\pi_* D} \subset \mathcal{C} \subset \pi_* n_* \mathcal{O}_{\tilde{D}}.$$

Proposition 5 (Inverse correspondence at the resolution)**.** *Let (X, x) be a complex analytic germ of a normal two-dimensional singularity, $\pi\colon \tilde{X} \to X$ be a resolution of X and D be any curve on $\tilde{X}$ such that $D \cap E = \{p_1, \ldots, p_k\}$. Let $n\colon \tilde{D} \to D$ be its normalization and $\pi|_D\colon D \to \pi(D)$ the restriction of the resolution map to D. Following Esnault [13] denote by $\mathcal{O}_{\pi_* D}$ the structure sheaf of $\pi(D)$, notice that it coincides with the image of $\mathcal{O}_X$ in $\pi_* \mathcal{O}_D$. Let $\mathcal{C}$ be an $\mathcal{O}_X$-submodule of $\pi_* n_* \mathcal{O}_{\tilde{D}}$ verifying $\mathcal{O}_{\pi_* D} \subset \mathcal{C}$ and define $\mathcal{A} := (\pi|_D)^* \mathcal{C}$.*

Consider $\{\phi_1, \ldots, \phi_r\}$ a minimal set of generators of $\mathcal{C}$ as $\mathcal{O}_X$-module, therefore we get the following exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{A} \to 0, \tag{2.2.11}$$

and dualizing this sequence we get

$$0 \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Also consider the following diagram of exact sequences obtained by applying the functor $\omega_{\tilde{X}} \otimes -$ to the exact sequence (2.2.11) *and taking cohomology*

$$\begin{tikzcd}
 & 0 \arrow[d] & H^1_E(\mathcal{N} \otimes \omega_{\tilde{X}}) \arrow[d, "\theta"] & \\
H^0\left(\omega^r_{\tilde{X}}\right) \arrow[r, "\beta"] \arrow[d] & H^0(\mathcal{A} \otimes \omega_{\tilde{X}}) \arrow[r, "\alpha"] \arrow[d, "\gamma_1"] & H^1(\mathcal{N} \otimes \omega_{\tilde{X}}) \arrow[r] \arrow[d, "\varphi"] & 0 \\
H^0\left(U; \omega^r_{\tilde{X}}\right) \arrow[r, "\delta"] & H^0(U; \mathcal{A} \otimes \omega_{\tilde{X}}) \arrow[r, "\gamma_2"] & H^1(U; \mathcal{N} \otimes \omega_{\tilde{X}}) &
\end{tikzcd}$$

Then the sheaf $\mathcal{M}$ is full if and only if $\operatorname{Im}\gamma_1 \subset \operatorname{Im}\delta$.

In the same section we use our results in order to construct special full sheaves.

Corollary 6. *Assume (X, x) is Gorenstein and that the Gorenstein form does not have zeros over the exceptional divisor. Let D be a smooth curve such that $D \cap E = \{p_1, \ldots, p_k\}$ and $\mathcal{C} = \pi_* \mathcal{O}_D$. Then the sheaf $\mathcal{M}$ given by the Proposition 2.16 is full and special.*

In Chapter 3 we start to work in a resolution $\pi\colon \tilde{X} \to X$ where the coefficients of the canonical cycle are non-positive, we call to this kind of resolution a non-positive resolution with respect to the canonical cycle (Definition 3.1). In this case we prove that if the sheaf $\mathcal{M}$ obtained by the Inverse Correspondence at the resolution satisfies that $\dim_{\mathbb{C}}\left(H^1(\tilde{X}, \mathcal{M}^\vee)\right)$ is equal to rp_g, where r is the rank of $\mathcal{M}$ and p_g is the geometric genus, then the sheaf $\mathcal{M}$ is a special full sheaf. The main proposition of this section is Proposition 3.2

Proposition 7. *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity and $\pi\colon \tilde{X} \to (X, x)$ be a non-positive resolution with respect to the canonical cycle. Under the hypothesis of Proposition 5, if the dimension of $R^1\pi_*\mathcal{N}$ as $\mathbb{C}$-vector space is equal to rp_g, then the condition* $\operatorname{Im}\gamma_1 \subset \operatorname{Im}\delta$ *given by the Proposition 2.16 is fulfilled.*

In Chapter 4 we compute a formula for the dimension of the first cohomology group of a full sheaf in a non-positive resolution with respect to the canonical cycle. This formula will be very important in the following chapters. In this case the main theorem of this chapter is Theorem 4.1

Theorem 8. *Let (X,x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Let M be a reflexive $\mathcal{O}_X$-module and $\pi\colon \tilde{X} \to X$ be a resolution where the Gorenstein form does not have zeros over the exceptional divisor. Denote by $\mathcal{M}$ the full sheaf associated to M and suppose that $\mathcal{M}$ has rank r and specialty defect equal to d. Then*

$$\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right) = rp_g - [c_1(\mathcal{M})]\cdot[Z_k] + d.$$

In Chapter 5 we use the previous formula in order to construct a special type of resolution where the full sheaf associated to M is generated by global sections, where M is a reflexive $\mathcal{O}_X$-module. We call to this resolution the minimal resolution adapted to M (Definition 5.5). This resolution depends of the reflexive module M and it captures the information about the dimension of the first cohomology group of the full sheaf $\mathcal{M}$ associated to M and the failure of $\mathcal{M}$ to be globally generated.

The construction of the resolution is given in Proposition 5.3.

Proposition 9. *Let (X,x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. If M is a reflexive $\mathcal{O}_X$-module, then there exists a unique minimal resolution $\rho\colon \tilde{X}' \to X$ such that $\mathcal{M}' := (\rho^*M)^{\vee\vee}$ is generated by global sections.*

Using this resolution we prove some facts about special modules. First we prove that if M is a reflexive $\mathcal{O}_X$-module and the full sheaf associated to M is special in the minimal resolution adapted to M, then M is a special module. This is done in Theorem 5.8.

Theorem 10. *Let (X,x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Let M be a reflexive $\mathcal{O}_X$-module and consider $\pi\colon \tilde{X} \to X$ the minimal resolution adapted to M and $\mathcal{M} = (\pi^*M)^{\vee\vee}$ the full sheaf associated to M. If $\mathcal{M}$ is special, then M is a special reflexive module.*

As a corollary of this theorem we prove the existence of special modules in Corollary 5.9.

Corollary 11. *Let (X,x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Then there exist special reflexives modules.*

Later we take a special module M and we compute a formula for the dimension of the first cohomology group of $\mathcal{M}\otimes\mathcal{M}^{\vee}$ where $\mathcal{M}$ is the full sheaf associated to M in its minimal adapted resolution. In this case the main theorem is Theorem 5.11

Theorem 12. *Let (X,x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Let M be a reflexive $\mathcal{O}_X$-module and consider $\pi\colon (\tilde{X},E) \to (X,x)$ the minimal resolution adapted to M. Assume that the full sheaf associated to M is special. Denote by $\mathcal{M}$ the full sheaf and assume that $\mathcal{M}$ has rank r and denote by $\mathcal{N} = \mathcal{M}^{\vee}$. Then*

$$\dim_{\mathbb{C}}\left(R^1\pi_*\left(\mathcal{M}\otimes\mathcal{N}\right)\right) = r\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right).$$

It is important to notice that the sheaf $\mathcal{M} \otimes \mathcal{M}^{\vee}$ is isomorphic to the sheaf $\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{M}, \mathcal{M})$, hence our formula is giving us the dimension of the tangent space of the deformation functor of $\mathcal{M}$ as locally free sheaf (see for example [19, Section 19]).

In Chapter 6 we use all the previous ideas and work done by Artin-Verdier [4] and Esnault [13] in order to prove that given a special module M and taking the minimal resolution adapted to M, then the full sheaf associated to M is determined by its first Chern class in the Picard group of the resolution, this is done in Proposition 6.6

Theorem 13. *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Let M be a reflexive $\mathcal{O}_R$-module. Denote by $\pi\colon \tilde{X} \to X$ the minimal resolution adapted to M and $\mathcal{M}$ the full sheaf associated to M. If $\mathcal{M}$ is a special full sheaf, then $\mathcal{M}$ is determined by its first Chern class in $Pic(\tilde{X})$.*

At the end of this chapter we define the combinatorial type of a special module as a graph $\mathfrak{G}$ and we prove the principal theorem of this thesis (Theorem 6.13). This theorem gives a complete classification of special reflexive modules over any complex analytic germ of a normal two-dimensional Gorenstein singularity.

Theorem 14. *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Then there exists a bijection between the following sets:*

1. *The set of special $\mathcal{O}_X$-modules up to isomorphism.*
2. *The set of finite pairs $(E'_1, n_1), \dots, (E'_l, n_l)$ where each E'_i is a divisor over X and n_i is a positive integer, such the minimal resolution given by Lemma 6.12 is a non-positive resolution with respect to the canonical cycle and the Gorenstein form does not have any pole in the components $E'_1, \dots, E'_l$.*

Classification and properties for some classes of real singularities

In this part of the thesis we study two aspects about real singularities.

Polar weighted homogeneous polynomials

Milnor's fibration theorem [24] is very important in singularity theory, this result give us information about the topology of the fibers of analytic functions near their critical points.

Consider the germ of an analytic map $f\colon (\mathbb{R}^{m+k}, \mathbf{0}) \to (\mathbb{R}^k, 0)$ and suppose that the origin of $\mathbb{R}^{m+k}$ is an isolated critical point, denote by $V = f^{-1}(0)$ and by $K = V \cap \mathbb{S}_{\epsilon}^{m+k-1}$ the link of the analytic map, with ϵ positive small enough.

Milnor's fibration theorem says that if $k \geq 2$, then the map $\varphi\colon \mathbb{S}^{m+k-1} \setminus K \to \mathbb{S}^{k-1}$ is a fiber bundle.

The condition of having an isolated critical point is a very strong condition and even if f satisfy the hypothesis, we can not guarantee that $\varphi = \frac{f}{\|f\|}$ as in the complex case [24].

A natural question is: What kind of real analytic maps $f\colon (\mathbb{R}^{m+k}, \mathbf{0}) \to (\mathbb{R}^k, 0)$ possess a Milnor fibration under a weaker hypothesis?

The first family of real singularities with isolated critical point and with a Milnor fibration appear in the work of José Seade [35] and was later studied in detail by Seade-Ruas-Verjovsky [33]. Inspired by these examples polar weighted homogeneous were introduced by Cisneros-Molina [9] and studied by Oka [26]. The definition is the following.

Definition. A polynomial function $f\colon \mathbb{C}^n \to \mathbb{C}$ in the variables $z_1, \ldots, z_n$ and $\bar{z}_1, \ldots, \bar{z}_n$ is *polar weighted homogeneous polynomial* if there exists $p_1, \ldots, p_n$ positive integers, $q_1, \ldots, q_n$ non-zero integers, a, c positive integers, and an action given by

$$t\tau \bullet \mathbf{z} = (t^{p_1}\tau^{q_1} z_1, \ldots, t^{p_n}\tau^{q_n} z_n),$$

such that f satisfies the following functional equation

$$f(t\tau \bullet \mathbf{z}) = t^a \tau^c f(\mathbf{z}) \ .$$

We say that the polar weighted homogeneous function f has *radial weight type* $(p_1, \ldots, p_n; a)$ and *angular weight type* $(q_1, \ldots, q_n; c)$.

Later Oka use the idea given by polar weighted homogeneous polynomials to define the mixed functions ([28]).

Definition. A complex valued function $f\colon \mathbb{C}^n \to \mathbb{C}$ expanded in a convergent power series of variables $z_1, \ldots, z_n$ and $\bar{z}_1, \ldots, \bar{z}_n$,

$$f(\mathbf{z}) = \sum_{\mu,\nu} c_{\mu,\nu} \mathbf{z}^{\mu} \bar{\mathbf{z}}^{\nu} \ ,$$

is called a *mixed analytic function* (or a *mixed polynomial*, if f is a polynomial).

It is clear by definition that any holomorphic weighted homogeneous polynomial is a polar weighted homogeneous polynomial. It is easy to see that if $f\colon \mathbb{C}^3 \to \mathbb{C}$ is a polar weighted homogeneous with isolated critical point, then the link is a Seifert manifold (Aguilar-Cabrera [2]).

The case when f is a holomorphic weighted homogeneous with an isolated critical point was studied by Orlik and Wagreich [30], they proved that the link of such polynomial is equivariantly diffeomorphic to the link of a polynomial in one of six classes given explicitly in the aforementioned paper. They also computed the Seifert invariants of the link.

Our first result in the second part of this thesis is the stability of the critical set of a polar weighted homogeneous polynomial with an isolated critical point. This is done in Corollary 8.7.

Corollary 15. *If f is a polar weighted homogeneous polynomial with isolated critical point at the origin, then under a small perturbation of their coefficients the critical point remains isolated.*

Later we generaliza the construction given by Orlik and Wagreich [30]. We construct in a natural way the following classes (Definition 8.8)

Definition. A mixed function $f(\mathbf{z})$ is said to be of class **I** (respectively **II**,..., **V**) if there are non-zero complex numbers $\alpha_1, \alpha_2, \alpha_3$ such that $f(z_1, z_2, z_3)$ is equal to

I. $\alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3}$,

II. $\alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} g_2$,

III. $\alpha_1 z_1^{a_1}\bar{z}_1^{b_1} + \alpha_2 z_2^{a_2}\bar{z}_2^{b_2} g_3 + \alpha_3 z_3^{a_3}\bar{z}_3^{b_3} g_2$,

IV. $\alpha_1 z_1^{a_1}\bar{z}_1^{b_1} + \alpha_2 z_2^{a_2}\bar{z}_2^{b_2} g_1 + \alpha_3 z_3^{a_3}\bar{z}_3^{b_3} g_2$,

V. $\alpha_1 z_1^{a_1}\bar{z}_1^{b_1} g_2 + \alpha_2 z_2^{a_2}\bar{z}_2^{b_2} g_3 + \alpha_3 z_3^{a_3}\bar{z}_3^{b_3} g_1$,

where $g_j \in \{z_j, \bar{z}_j\}$.

If we take a mixed function of some of the previous classes, in general the mixed function is not a polar weighted homogeneous polynomial. Our first result is a list of conditions that the numbers a_1, a_2, a_3 and b_1, b_2, b_3 must satisfy in order to guarantee that the mixed function is a polar weighted homogeneous polynomial. This is done in Theorem 8.12.

Theorem 16. *Let f be a mixed function of one of the classes of Definition 8.8. Then the following conditions must be satisfied in order to f be polar weighted homogeneous:*

Class I $a_j - b_j \neq 0$ *with* $j = 1, 2, 3$.

Class II

- **a)** $a_j - b_j \neq 0$ *with* $j = 1, 2, 3$ *and* $a_2 \pm b_2 \neq 1$.
- **b)** $a_1 - b_1 \neq 0$, $a_2 - b_2 = 1$, $b_2 \neq 0$ *and* $a_3 = b_3$.

Class III $a_1 - b_1 \neq 0$ *and* $a_2 - b_2$, $a_3 - b_3$ *are not both* -1. *Also:*

- **a)** $a_2 \pm b_2$ *and* $a_3 \pm b_3$ *are not* 1.
- **b)** $(a_2 + b_2)(a_3 + b_3) > 1$, $a_2 - b_2 = 1$ *and* $a_3 - b_3 = 1$.
- **c)** $a_2 = a_3 = 1$ *and* $b_2 = b_3 = 0$.

Class IV

- **a)** $a_i - b_i \neq 0$ *for* $i = 1, 2, 3$, $a_1 \pm b_1 \neq 1$ *and* $(a_1, a_2) \neq (b_1 - 1, b_2 + 2)$.
- **b)** $a_2 = b_2$, $a_1 - b_1 = 1$ *and* $b_1 \neq 0$.
- **c)** $a_3 = b_3$, $a_1 - b_1 \neq 0$, $a_1 + b_1 > 1$ *and* $(a_1, a_2) = (b_1 - 1, b_2 + 2)$.

Class V $(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) \neq -1$ *and*

$$\begin{cases} (a_{i-1}, a_{i+1}) \neq (b_{i-1} + 1, b_{i+1}), \\ (a_{i-1}, a_{i+1}) \neq (b_{i-1} - 1, b_{i+1} + 2), \end{cases} \qquad i = 1, 2, 3.$$

Later we use the previous theorem in order to compute all the weights of the actions. This computation and the action associated to the polar weighted homogeneous polynomial allow us to simplify our families. This is done in Corollary 8.15.

Corollary 17. *Let $f\colon \mathbb{C}^3 \to \mathbb{C}$ be a polar weighted homogeneous polynomial belonging to some class of Theorem 8.12. Then there exists a change of coordinates such that we get:*

Class I

$$z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2} + z_3^{a_3}\bar{z}_3^{b_3}.$$

Class II.a

$$z_1^{a_1}\bar{z}_1^{b_1}+z_2^{a_2}\bar{z}_2^{b_2}+z_3^{a_3}\bar{z}_3^{b_3}z_2.$$

Class II.b

$$z_1^{a_1}\bar{z}_1^{b_1}+z_2^{a_2}\bar{z}_2^{b_2}+\tau z_3^{a_3}\bar{z}_3^{b_3}z_2,\quad \tau\in\mathbb{S}^1.$$

Class III.a

$$z_1^{a_1}\bar{z}_1^{b_1}+z_2^{a_2}\bar{z}_2^{b_2}z_3+z_3^{a_3}\bar{z}_3^{b_3}z_2.$$

Class III.b

$$z_1^{a_1}\bar{z}_1^{b_1}+z_2^{a_2}\bar{z}_2^{b_2}z_3+\tau z_3^{a_3}\bar{z}_3^{b_3}z_2,\quad \tau\in\mathbb{S}^1.$$

Class III.c

$$z_1^{a_1}\bar{z}_1^{b_1}+z_2z_3.$$

Class IV.a

$$z_1^{a_1}\bar{z}_1^{b_1}+z_2^{a_2}\bar{z}_2^{b_2}z_1+z_3^{a_3}\bar{z}_3^{b_3}z_2.$$

Class IV.b

$$z_1^{a_1}\bar{z}_1^{b_1}+\tau z_2^{a_2}\bar{z}_2^{b_2}z_1+z_3^{a_3}\bar{z}_3^{b_3}z_2,\quad \tau\in\mathbb{S}^1.$$

Class IV.c

$$z_1^{a_1}\bar{z}_1^{b_1}+\tau z_2^{a_2}\bar{z}_2^{b_2}z_1+z_3^{a_3}\bar{z}_3^{b_3}z_2,\quad \tau\in\mathbb{S}^1$$

Class V

$$z_1^{a_1}\bar{z}_1^{b_1}z_2+z_2^{a_2}\bar{z}_2^{b_2}z_3+z_3^{a_3}\bar{z}_3^{b_3}z_1.$$

Later we use all the information in order to guarantee that the polar weighted homogeneous polynomial has an isolated critical point. This is donde in Theorem 8.17.

Theorem 18. *Let $f\colon\mathbb{C}^3\to\mathbb{C}$ be a polar weighted homogeneous polynomial belonging to some class of Corollary 8.15. Then*

1. *If f is of one of the classes* ***I****,* ***II.a****,* ***III.a****,* ***III.c****,* ***IV.a*** *or* ***V****, then f has an unique singularity at the origin.*
2. *If f is of one of the classes* ***II.b****,* ***III.b*** *or* ***IV.b****, then f has an unique singularity at the origin if and only if $\tau\neq-1$.*
3. *If f is of the classe* ***IV.c*** *then f has an unique singularity if and only if $\tau\neq 1$.*

It is important to notice that our approach is exhaustive, we have computed all conditions that a polar weighted homogeneous polynomial must satisfy in order to have an isolated critical point. In our classification we get some known families for example: the twisted Brieskorn-Pham polynomials [34] and the family studied by Haydée Aguilar [2].

Finally in this section we prove that the diffeomorphism type of the link of a polar weighted homogeneous polynomial with isolated singularity at the origin does not change under small perturbation of the coefficients of the polynomial (Theorem 8.20). The proof of this theorem is generalization of the proof of [31, Theorem 3.1.4] by Orlik and Wagreich.

The embedding method and the mixed GSV index

In Chapter 9 we present the embedding method, this method arise as a natural way to embed a real singularity in a complex singularity. In Section 9.1 first we present the embedding method and we give some basics properties, and later we use this method to give a new demonstration of the Isotopy Theorem given by Oka [29], [20].

Theorem 19 (Isotopy Theorem)**.** *Denote by*

$$f_A(\mathbf{z}) = z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2} + z_3^{a_3}\bar{z}_3^{b_3}, \tag{0.0.1}$$
$$f_B(\mathbf{z}) = z_1^{a_1}\bar{z}_1^{b_1}z_2 + z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}, \tag{0.0.2}$$
$$f_C(\mathbf{z}) = z_1^{a_1}\bar{z}_1^{b_1}z_2 + z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}z_1, \tag{0.0.3}$$

and

$$h_j(z_j) = \begin{cases} z_j & \text{if} \quad a_j - b_j > 0 \\ \bar{z}_j & \text{if} \quad a_j - b_j < 0. \end{cases}$$

for $j = 1, 2, 3$.

Consider the maps

$$\begin{aligned} g_A(\mathbf{z}) &= h_1(z_1)^{|a_1-b_1|} + h_2(z_2)^{|a_2-b_2|} + h_3(z)_3^{|a_3-b_3|}, \\ g_B(\mathbf{z}) &= h_1(z_1)^{|a_1-b_1|}z_2 + h_2(z_2)^{|a_2-b_2|}z_3 + h_3(z_3)^{|a_3-b_3|}, \\ g_C(\mathbf{z}) &= h_1(z_1)^{|a_1-b_1|}z_2 + h_2(z_2)^{|a_2-b_2|}z_3 + h_3(z_3)^{|a_3-b_3|}z_1. \end{aligned}$$

Then the Milnor fibrations of $f_l(\mathbf{z})$ *and* $g_l(\mathbf{z})$ *are* C^∞ *equivalent for* $l = A, B, C$.

Our second application is a generalization of the GSV index. Recall that this index was defined by Gómez-Mont, Seade and Verjovsky for vector fields on an isolated complex hypersurface [15]. This index capture some information about the topology of the singularity and has an analogue for real singularities. The GSV index for real singularities gives us information over the integers modulo two, our mixed GSV index gives us information over the integers. At this moment we do not know yet if our index is a lifting of the classical GSV index for real singularities.

At the end of this work we present a generalization of the classical GSV index for real singularities under certain hypothesis using the embedding method.

Organization

This thesis has two parts which are not related, therefore we decided to organize the material as follows:

Part I

Chapter 1. In this chapter we introduce all the notation that will be used in the first part of the thesis.

Chapter 2. In this chapter we generalize the construction given by Artin-Verdier and Esnault at the singularity and at the resolution.

Chapter 3. In this chapter we study the construction at the resolution given in the previous chapter in some particular type of resolution. In particular we give a condition in order to guarantee that the full sheaf that we construct is a full sheaf.

Chapter 4. In this chapter we compute a formula for the dimension of the first cohomology group of a full sheaf in a particular type of resolution.

Chapter 5. In this chapter given a reflexive module M we use the previous formula to construct a resolution where the full sheaf associated to M is generated by global section. This resolution will be very important and it allows us to obtain new information about the full sheaf. As an application of this resolution we study how the specialty defect behaves by taking the blow up of a point and we compute a formula for the sheaf of endomorphism of the full sheaf associated to M in this new resolution.

Chapter 6. In this chapter we study the special modules in more detail. In particular we prove that in the minimal adapted resolution the full sheaf associated to a special module is determined by its first Chern class in the Picard group of the resolution. At the end of the chapter we define the combinatorial type of a special module and we give their classification.

Part II

Chapter 7. In this chapter we introduce all the notation that will be used in the second part of the thesis.

Chapter 8. In this chapter we study the family of polar weighted homogeneous polynomials. We generalize the classical classification of Orlik and Wagreich and compute all the weights of the actions.

Chapter 9. In this chapter we introduce "The embedding method", that will allow us to give a new proof of the isotopy theorem. Finally in the last section we will use "The embedding method" in order to generalize the classical GSV-index, defined by Gómez-Mont, Seade and Verjovsky in [15], to certain class of real analytic map.

Part I

The McKay correspondence for Gorenstein surface singularity

Chapter 1

Background

1.1 Cohen-Macaulay modules and reflexive modules

In this section we present all the definitions and properties of Cohen-Macaulay modules and reflexive modules that will be used later. Let R be a commutative Noetherian local ring of dimension d, with maximal ideal $\mathfrak{m}$ and $R/\mathfrak{m} = \mathbb{C}$.

Definition 1.1. A module M over R is called *reflexive* if the natural homomorphism from M to its double dual $M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is an isomorphism.

Definition 1.2. A module M over R is called *Cohen-Macaulay* if the depth of M is equal to the dimension of the module. If the depth of M is equal to the dimension of the ring, then the module is called *maximal Cohen-Macaulay.*

By [38, Proposition 1.5] some basic properties of maximal Cohen-Macaulay modules are:

1. If R is a regular local ring, then any maximal Cohen-Macaulay module over R is free.
2. If R is a reduced local ring of dimension one, then an R-module M is maximal Cohen-Macaulay only when it is torsion free, that is, when the natural homomorphism $M \to M^{\vee\vee}$ is a monomorphism.
3. If R is a normal local domain of dimension two, then an R-module M is maximal Cohen-Macaulay only when it is reflexive.

The canonical module is an important module that will allow us to simplify computations. The definition is the following.

Definition 1.3. A module C over R is called *the canonical module* of R if and only if

$$\dim_{\mathbb{C}}\left(\mathrm{Ext}^{\,i}_R(\mathbb{C}, C)\right) = \delta_{id},$$

where d is the dimension of R.

The canonical module and Cohen-Macaulay modules have the following properties.

Theorem 1.4 ([7, Theorem 3.3.10]). *If C is the canonical R-module, then*

1. *For all integers $t = 0, 1, \dots, d$ and all Cohen-Macaulay R-modules M of dimension t one has*
   (a) $Ext_R^{d-t}(M, C)$ *is a Cohen-Macaulay R-module of dimension t,*
   (b) $Ext_R^i(M, C) = 0$ *for all $i \neq d - t$,*
   (c) *there exists an isomorphism $M \to Ext_R^{d-t}(Ext_R^{d-t}(M, C), C)$ which in the case $d = t$ is just the natural homomorphism from M into the bidual of M with respect to C.*
2. *For all maximal Cohen-Macaulay R-modules M one has*
   (a) $Hom_R(M, C)$ *is a maximal Cohen-Macaulay R-module,*
   (b) $Ext_R^i(M, C) = 0$ *for $i > 0$,*
   (c) *the natural homomorphism $M \to Hom_R(Hom_R(M, C), C)$ is an isomorphism.*

1.2 Full sheaves

Throughout the thesis, (X, x) will denote a complex analytic germ of a normal two-dimensional singularity, $\pi\colon (\tilde{X}, E) \to (X, x)$ a resolution with exceptional set $E = \bigcup_{i=1}^n E_i$, where $E_1, E_2, \dots, E_n$ are the irreducible components. We are going to denote by $p_g := \dim_{\mathbb{C}} \left(R^1\pi_*\mathcal{O}_{\tilde{X}}\right)$ the geometric genus of (X, x), $\omega_{\tilde{X}}$ the canonical sheaf of $\tilde{X}$, Z_K the canonical cycle, $U = X \setminus \{x\}$ and by $i\colon U \to X$ the inclusion.

As before, a module M over $\mathcal{O}_X$ is called reflexive if the natural homomorphism from M to its double dual $M^{\vee\vee} = \mathrm{Hom}_{\mathcal{O}_X}(\mathrm{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X), \mathcal{O}_X)$ is an isomorphism. By [18, Proposition 1.6] a torsion-free module M is reflexive if and only if $M \cong i_*i^*M$.

We are interested in studying the reflexive modules over the ring $\mathcal{O}_X$ and we will use the notion of full sheaves that allows us to study the problem over the resolution $\tilde{X}$.

Definition 1.5 ([21, Definition 1.1]). A locally free sheaf $\mathcal{M}$ on $\tilde{X}$ is called *full* if there is a reflexive $\mathcal{O}_X$-module M such that $\mathcal{M} \cong (\pi^*M)^{\vee\vee}$.

As we can see given a reflexive module we can associate a full sheaf. This idea give us the following definition.

Definition 1.6. Let M be a reflexive $\mathcal{O}_X$-module and denote by $\mathcal{M} = (\pi^*M)^{\vee\vee}$. The full sheaf $\mathcal{M}$ is *the full sheaf associated to M*.

Following the work of C. P. Kahn we have the following definition and proposition.

Definition 1.7 ([21, Page 144]). Let $\mathcal{M}$ be a locally free sheaf on $\tilde{X}$. We say that $\mathcal{M}$ is *generically generated by global sections or almost generated by its global sections* if the global sections of $\mathcal{M}$ generate it everywhere except (possibly) over discrete points.

Proposition 1.8 ([21, Proposition 1.2]). *A locally free sheaf $\mathcal{M}$ on $\tilde{X}$ is full if and only if*

1. *$\mathcal{M}$ is generically generated by global sections.*
2. *The natural map $H_E^1(\tilde{X}, \mathcal{M}) \to H^1(\tilde{X}, \mathcal{M})$ is injective.*

Implicitly in the proof of the last proposition Kahn gave the following result.

Proposition 1.9 ([21])**.** *If $\mathcal{M}$ is the full sheaf associated to M, then $\pi_*\mathcal{M} = M$.*

The last proposition gives us a natural way to recover the reflexive module associated to a full sheaf.

Now we have the following two lemmas that will be used later.

Lemma 1.10. *If $\mathcal{M}$ is a full sheaf, then $R^1\pi_*(\mathcal{M} \otimes \omega_{\tilde{X}}) = 0$.*

Proof. If $\mathcal{M}$ is generated by global sections consider $\{\psi_1, \ldots, \psi_k\}$ global sections that generates $\mathcal{M}$. Therefore we have the exact sequence generated by the sections

$$0 \to \mathcal{K} \to \mathcal{O}_{\tilde{X}}^k \to \mathcal{M} \to 0.$$

Applying the functor $- \otimes \omega_{\tilde{X}}$ to the previous exact sequence we obtain

$$0 \to \mathcal{K} \otimes \omega_{\tilde{X}} \to \omega_{\tilde{X}}^k \to \mathcal{M} \otimes \omega_{\tilde{X}} \to 0,$$

finally applying the functor π_* we have

$$\begin{tikzcd} 0 \arrow[r] & \pi_*\left(\mathcal{K} \otimes \omega_{\tilde{X}}\right) \arrow[r] & \pi_*\omega_{\tilde{X}} \arrow[r] & \pi_*\left(\mathcal{M} \otimes \omega_{\tilde{X}}\right) \arrow[dll, dotted] & \\ & R^1\pi_*\left(\mathcal{K} \otimes \omega_{\tilde{X}}\right) \arrow[r] & R^1\pi_*\omega_{\tilde{X}} \arrow[r] & R^1\pi_*\left(\mathcal{M} \otimes \omega_{\tilde{X}}\right) \arrow[r] & 0 \end{tikzcd}$$

By Grauert-Riemenschneider Vanishing Theorem we have $R^1\pi_*\left(\omega_{\tilde{X}}\right) = 0$, therefore $R^1\pi_*\left(\mathcal{M} \otimes \omega_{\tilde{X}}\right)$ is equal to zero.

If $\mathcal{M}$ is almost generated by its global sections, consider $\mathcal{M}'$ the subsheaf of $\mathcal{M}$ generated by global sections, therefore we get the following exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{G} \to 0,$$

with $\mathrm{Supp}(\mathcal{G})$ zero dimensional.

Applying the functor $- \otimes \omega_{\tilde{X}}$ to the previous exact sequence and later the functor π_*- we get

$$\begin{tikzcd} 0 \arrow[r] & \pi_*\left(\mathcal{M}' \otimes \omega_{\tilde{X}}\right) \arrow[r] & \pi_*\left(\mathcal{M} \otimes \omega_{\tilde{X}}\right) \arrow[r] & \pi_*\left(\mathcal{G} \otimes \omega_{\tilde{X}}\right) \arrow[dll, dotted] \\ & R^1\pi_*\left(\mathcal{M}' \otimes \omega_{\tilde{X}}\right) \arrow[r] & R^1\pi_*\left(\mathcal{M} \otimes \omega_{\tilde{X}}\right) \arrow[r] & R^1\pi_*\left(\mathcal{G} \otimes \omega_{\tilde{X}}\right) \end{tikzcd}$$

Since $\mathrm{Supp}(\mathcal{G} \otimes \omega_{\tilde{X}})$ is a finite set we have $R^1\pi_*\left(\mathcal{G} \otimes \omega_{\tilde{X}}\right) = 0$. Now since $\mathcal{M}'$ is generated by global sections, by the previous case we have $R^1\pi_*\left(\mathcal{M}' \otimes \omega_{\tilde{X}}\right) = 0$, therefore $R^1\pi_*\left(\mathcal{M} \otimes \omega_{\tilde{X}}\right) = 0$. □

Lemma 1.11. *If $\mathcal{M}$ is a full sheaf, then $\pi_*\left(\mathcal{M}^{\vee}\right) = \left(\pi_*\mathcal{M}\right)^{\vee}$.*

PROOF. Consider the following cohomology exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & H^0_E\left(\mathcal{M}^{\vee}\right) \arrow[r] & H^0\left(\mathcal{M}^{\vee}\right) \arrow[r] & H^0\left(U, \mathcal{M}^{\vee}\right) \arrow[dll, dotted] \\
& H^1_E\left(\mathcal{M}^{\vee}\right) \arrow[r] & H^1\left(\mathcal{M}^{\vee}\right) \arrow[r] & H^1\left(U, \mathcal{M}^{\vee}\right) \arrow[r] & \dots
\end{tikzcd}
$$

Since $\mathcal{M}$ is locally free we have that

$$
\begin{aligned}
H^0_E\left(\mathcal{M}^{\vee}\right) &= 0, \\
H^1_E\left(\mathcal{M}^{\vee}\right) &\cong H^1\left(\mathcal{M} \otimes \omega_{\tilde{X}}\right), \quad \text{by Serre duality,}
\end{aligned}
$$

and by Lemma 1.10 we get

$$H^1\left(\mathcal{M} \otimes \omega_{\tilde{X}}\right) = 0,$$

hence $H^0\left(\mathcal{M}^{\vee}\right) \cong H^0\left(U, \mathcal{M}^{\vee}\right)$.

Now denote by $M := \pi_* \mathcal{M}$. Since $M^{\vee}$ is reflexive we get

$$M^{\vee} = i_* i^*\left(M^{\vee}\right) = i_*\left(\mathcal{M}^{\vee}_{|_U}\right) = H^0\left(U, \mathcal{M}^{\vee}\right),$$

therefore $\pi_*\left(\mathcal{M}^{\vee}\right) \cong M^{\vee}$. □

Another notion that will be important in this work is the concept of specialty. Previously Wunram [37] and Riemenschneider [32] defined a special full sheaf as a full sheaf which its dual has the first cohomology group is equal to zero. Using this definition Wunram proved that in the case of a quotient surface singularity and taking the minimal resolution, there is a bijection between isomorphism classes of special full sheaves and irreducible components of the exceptional divisor.

For us the definition of special is as follows.

Definition 1.12. A full sheaf $\mathcal{M}$ on $\tilde{X}$ of rank r is called *special* if $\dim_{\mathbb{C}}\left(R^1\pi_*\left(\mathcal{M}^{\vee}\right)\right) = rp_g$.

Notice that this definition is a generalization of the concept given by Wunram and Riemenschneider and both definitions coincide in the case of a rational singularity.

Related to the specialty we have another concept.

Definition 1.13. Let $\mathcal{M}$ be a full sheaf on $\tilde{X}$ of rank r. *The defect of specialty of* $\mathcal{M}$ is the number $\dim_{\mathbb{C}}\left(R^1\pi_*\left(\mathcal{M}^{\vee}\right)\right) - rp_g$.

It is trivial to see that a full sheaf with defect of specialty equal to zero is a full special sheaf.

Since the definition of being special depends on the resolution, we have a another related notion.

Definition 1.14. Let M be a reflexive $\mathcal{O}_X$-module. We say that M is *a special module* if for any resolution the full sheaf associated to M is special.

Later will be clear the importance of these concepts.

Chapter 2

Artin-Verdier/Esnault correspondence

2.1 The correspondence at the singularity

In the following subsections we generalize the construction given by Artin-Verdier [4] and Esnault [13] at the singularity, this is done in two correspondences:

Direct correspondence. In Subsection 2.1.1 we work with reflexive modules over X. In this part we associate to each reflexive module of rank r with r generic sections, a Cohen-Macaulay module of dimension one with r generators.

Inverse correspondence. In Subsection 2.1.2 we work with Cohen-Macaulay modules over X of dimension one. In this part we associate to each Cohen-Macaulay module of dimension one with r generators, a reflexive module with r sections.

These correspondences are a generalization of the ideas of Artin-Verdier and Esnault. Later we will use this ideas to give a correspondence in the resolution.

In this section the singularity (X, x) will be Gorenstein, therefore $\mathcal{O}_X$ will be a dualizing module for the singularity ([12, Section 21.3]).

2.1.1 Direct correspondence

In this subsection we associate to each reflexive module of rank r with r generic sections, a Cohen-Macaulay module of dimension one with r generators. First we need to understand the cokernel generated by some generic sections of the module.

Proposition 2.1. *Let M be a reflexive $\mathcal{O}_X$-module. Suppose that rank$(M) = r$ and take $\phi_1, \dots, \phi_r$ generic sections, hence we obtain the exact sequence given by the sections*

$$0 \to \mathcal{O}_X^r \to M \to \mathcal{C}' \to 0. \tag{2.1.2}$$

Then $\mathcal{C}'$ is a Cohen-Macaulay module of dimension one.

Proof. Since the sections are generic we have that the dimension of the support of $\mathcal{C}'$ is one, therefore dualizing the exact sequence (2.1.2) we get

1. $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X) = 0$.
2. $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X) = 0$ for $i > 1$.

Now by [7, Corollary 3.5.11] the module $\mathcal{C}'$ is Cohen-Macaulay of dimension one. □

By the last proposition we can make the first correspondence.

Corollary 2.2. *Given a reflexive module of rank r with r generic sections, we can associate a Cohen-Macaulay module of dimension one and r generators.*

Proof. Let M be a reflexive $\mathcal{O}_X$-module of rank r and take r generic sections, hence we obtain the exact sequence given by the sections

$$0 \to \mathcal{O}_X^r \to M \to \mathcal{C}' \to 0. \tag{2.1.2}$$

By Proposition 2.1 the module $\mathcal{C}'$ is Cohen-Macaulay of dimension one. Dualizing the exact sequence (2.1.2) we get

$$0 \to N \to \mathcal{O}_X^r \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X) \to 0,$$

where N is the dual of M.

By Theorem 1.4 the module $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X)$ is Cohen-Macaulay of dimension one. Therefore we associate to the reflexive module M with the sections, the module $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X)$ with the generators given by the previous exact sequence. □

2.1.2 Inverse correspondence

In this part we construct the inverse of the previous correspondence: we associate to each Cohen-Macaulay module of dimension one with r generators, a reflexive module of rank r with r sections. As in the previous section first we need to understand the module of relations of the generators of the Cohen-Macaulay module.

Proposition 2.3. *Let $\mathcal{C}$ be an Cohen-Macaulay $\mathcal{O}_X$-module of dimension one, $\{\phi_1, \ldots, \phi_r\}$ a set of generators of $\mathcal{C}$ as $\mathcal{O}_X$-module and consider the exact sequence obtained by the generators*

$$0 \to N \to \mathcal{O}_X^r \to \mathcal{C} \to 0. \tag{2.1.1}$$

Then the module, N is reflexive.

Proof. Dualizing the exact sequence (2.1.1) and denoting by $M := N^{\vee}$, we obtain the exact sequence

$$0 \to \mathcal{O}_X^r \to M \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X) \to 0. \tag{2.1.2}$$

Since $\mathcal{C}$ is Cohen-Macaulay of dimension one then by Theorem 1.4 the module $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X)$ is Cohen-Macaulay of dimension one and $\mathcal{C} \cong \operatorname{Ext}^1_{\mathcal{O}_X}\left(\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X), \mathcal{O}_X\right)$. Now dualizing (2.1.2) and using the previous identification we obtain the exact sequence

$$0 \to N^{\vee\vee} \to \mathcal{O}_X^r \to \mathcal{C} \to 0,$$

hence N is reflexive. □

Remark 2.4. The last proposition give us a natural way to associate to each Cohen-Macaulay module of dimension one with r generators, a reflexive module. Notice that the last proposition does not give us sections of the reflexive module.

Our inverse correspondence is as follows.

Proposition 2.5. *Given a Cohen-Macaulay $\mathcal{O}_X$-module of dimension one and r generators, we can associate a reflexive module and r sections.*

PROOF. Let $\mathcal{C}$ be an Cohen-Macaulay $\mathcal{O}_X$-module of dimension one, $\{\phi_1, \ldots, \phi_r\}$ a set of generators of $\mathcal{C}$ as $\mathcal{O}_X$-module and consider the exact sequence given by the generators

$$0 \to N \to \mathcal{O}_X^r \to \mathcal{C} \to 0. \tag{2.1.1}$$

Dualizing the exact sequence (2.1.1) we get

$$0 \to \mathcal{O}_X^r \to M \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X) \to 0, \tag{2.1.1}$$

where M is the dual of N and this implies that M is reflexive.

Therefore to the module $\mathcal{C}$ we associate the reflexive module M with the sections given by the map from $\mathcal{O}_X^r$ to M given in the exact sequence (2.1.1). □

Following Esnault [13] we can see that in Proposition 2.3 if the system of generators it is not minimal then the reflexive module that we obtain is decomposable and one factor is a free module.

Remark 2.6. Let us say something about both correspondences.

Let M be a reflexive $\mathcal{O}_X$-module of rank r and take r generic sections, hence we obtain the exact sequence given by the sections

$$0 \to \mathcal{O}_X^r \to M \to \mathcal{C}' \to 0. \tag{2.1.2}$$

Dualizing the exact sequence (2.1.2) we get

$$0 \to N \to \mathcal{O}_X^r \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X) \to 0, \tag{2.1.3}$$

where N is the dual of M.

By Corollary 2.2 we associate to the reflexive module M and the r generic sections, the Cohen-Macaulay module $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X)$ and the generators given by the previous exact sequence.

Now dualizing the exact sequence (2.1.3) we get

$$0 \to \mathcal{O}_X^r \to M^{\vee\vee} \to \operatorname{Ext}^1_{\mathcal{O}_X}\left(\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X), \mathcal{O}_X\right) \to 0.$$

By Proposition 2.5 we associate to the Cohen-Macaulay module $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X)$ and the r generators given by the exact sequence (2.1.3), the reflexive module $M^{\vee\vee}$ and the sections given by the previous exact sequence.

Since M is reflexive and $\mathcal{C}'$ is Cohen-Macaulay of dimension one, we have that $M \cong M^{\vee\vee}$ and $\mathcal{C}' \cong \operatorname{Ext}^1_{\mathcal{O}_X}\left(\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}', \mathcal{O}_X), \mathcal{O}_X\right)$ by Theorem 1.4. Therefore the composition of both correspondences give us (up to isomorphism) the initial reflexive module and sections.

The same happens if we star with a Cohen-Macualay module of dimension one and r generators.

In the following section we will work at the resolution. In this case the correspondence will be more difficult but the procedure will be analogous.

2.2 The correspondence at the resolution

In the following subsections we generalize the correspondence given by Artin-Verdier [4] and Esnault [13] at the resolution, this is done in two constructions:

Direct correspondence. In Subsection 2.2.1 we work with full sheaves over $\tilde{X}$. In this subsection we prove that for each full sheaf of rank r with a collection of r generic sections, we can assign a Cohen-Macaulay sheaf of dimension one such that its support is not contained in the exceptional divisor, and r generators.

Inverse correspondence. In Subsection 2.2.2 we work with Cohen-Macaulay sheaves of dimension one over $\tilde{X}$ such that their support is not contained in the exceptional divisor. In this part we will give conditions in order to guarantee that each selection of a Cohen-Macaulay sheaf with a system of r generators gives us a full sheaf with r sections.

Both correspondences are similar to the correspondences of Section 2.1. These constructions are a generalization of the ideas of Artin-Verdier and Esnault.

2.2.1 Direct correspondence

In this subsection we associate to each full sheaf of rank r with a collection of r generic sections, a Cohen-Macaulay sheaf of dimension one such that its support is not contained in the exceptional divisor, together with r generators. As in the previous section first we need to understand what is the cokernel generated by some generic sections of the sheaf.

Let M be a reflexive $\mathcal{O}_X$-module and denote by $\mathcal{M} := (\pi^* M)^{\vee\vee}$ its associated full sheaf. Suppose that $\operatorname{rank}(\mathcal{M}) = r$ and take $\phi_1, \dots, \phi_r$ generic sections, hence we obtain the exact sequence given by the sections

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

We want to prove that $\mathcal{A}'$ is a Cohen-Macaulay sheaf of dimension one such that its support is not contained in the exceptional divisor. In order to prove it we need some previous work.

Consider the natural map from $\pi^* M$ to its double dual $\mathcal{M}$. This map gives us the following exact sequence

$$0 \to T \to \pi^* M \to \mathcal{M} \to \mathcal{S} \to 0 \tag{2.2.2}$$

where T is the kernel, $\mathcal{S}$ is the cokernel and the support of $\mathcal{S}$ is a finite set $\{p_1, \dots, p_k\}$ as we will see later. Let us denote by S the support of $\mathcal{S}$.

For any point p in S, we have

$$\phi_j(p) = 0, \quad \text{for any } j = 1, \dots, r. \tag{2.2.3}$$

Now for any point p outside of S, by (2.2.2) we have that

$$\mathcal{M}_p \cong (\pi^* M/T)_p\,. \tag{2.2.4}$$

Since the sheaf $(\pi^* M/T)$ is generated by global sections, then the set of points where $\mathcal{M}$ fails to be globally generated is equal to S, hence S is a finite set.

Now let $\psi_1, \dots, \psi_k$ be global sections of $\mathcal{M}$ such that they almost generate it. Denote by E the vector bundle over $\tilde{X}$ such that the sheaf of sections of E is $\mathcal{M}$ and by $\tilde{X} \times \mathbb{C}^k$ the trivial vector bundle of rank k over $\tilde{X}$.

The global sections define a morphism of vector bundles

$$\begin{tikzcd} \tilde{X} \times \mathbb{C}^k \arrow[rr, "\Psi"] \arrow[dr] & & E \arrow[dl] \\ & \tilde{X} & \end{tikzcd}$$

where $\Psi(x,(c_1,\dots,c_k)) = \sum_{j=1}^k \psi_j(x)c_j$.

Since the sections $\psi_1,\dots,\psi_k$ generate $\mathcal{M}$ outside of S, we get that the restriction

$$\Psi|_{\tilde{X}\setminus S}\colon \left(\tilde{X}\setminus S\right)\times\mathbb{C}^k \to E|_{\tilde{X}\setminus S},$$

is a surjection.

We want to prove that we can take r generic sections such that the support of the sheaf $\mathcal{A}'$ (see (2.2.1)) is not contained in the exceptional divisor. By our previous discussion we know that the support of $\mathcal{A}'$ will always contain the set S.

Let U be an open set such that the vector bundle E is trivial over U. Consider the local trivialization

$$\begin{tikzcd} E|_U \arrow[r] \arrow[d] & U \times \mathbb{C}^r \arrow[dl] \\ U & \end{tikzcd}$$

In the open set U the global sections $\psi_1,\dots,\psi_k$ can be written as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{12} & \dots & a_{rk} \end{pmatrix},$$

where the column

$$\begin{pmatrix} a_{1i} \\ \vdots \\ a_{ri} \end{pmatrix},$$

are the coordinates of ψ_i. Notice that the matrix A is an element of the set $\operatorname{Mat}(r\times k, \mathcal{O}_{\tilde{X}}(U))$.

In the open set U the restriction of the map Ψ is

$$\begin{tikzcd} U \times \mathbb{C}^k \arrow[rr, "\Psi_U"] \arrow[dr] & & U \times \mathbb{C}^r \arrow[dl] \\ & U & \end{tikzcd}$$

where $\Psi_U(x,(c_1,\dots,c_k)) = (x, A(x)(c_1,\dots,c_k)^{\intercal})$.

Now for each matrix B in $\operatorname{Mat}(k\times r,\mathbb{C})$, we get sections $\phi_1,\dots,\phi_r$ of $\mathcal{M}$ by the formula

$$(\phi,\dots,\phi_r) = (\psi_1,\dots,\psi_k)B.$$

In the open set U the sections $\phi_1, \dots, \phi_r$ are obtained just multiplying the matrices A and B,

$$(\phi, \dots, \phi_r) = AB.$$

Take a matrix B in $\mathrm{Mat}\,(k \times r, \mathbb{C})$ and consider the sections $\phi_1, \dots, \phi_r$ of $\mathcal{M}$ given as before. In the open set U the exact sequence (2.2.1) is

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_U^r \arrow[r, "{(\phi_1,\dots,\phi_r)}"] \arrow[d, "B"] & \mathcal{O}_U^r \arrow[r] & \mathcal{A}'|_U \arrow[r] & 0 \\ & \mathcal{O}_U^k \arrow[ur, "A"'] & & & \end{tikzcd}$$

Consider the stratification by rank in the set $\mathrm{Mat}\,(r \times r, \mathbb{C})$ and denote by

$$\mathrm{Mat}\,(r \times r, \mathbb{C})^i := \{c \in \mathrm{Mat}\,(r \times r, \mathbb{C}) \mid \mathrm{corank}(c) \geq i\}.$$

Now consider the map

$$\begin{aligned} \Theta\colon\ (U \setminus S) \times \mathrm{Mat}(k \times r, \mathbb{C}) &\to \mathrm{Mat}(r \times r, \mathbb{C}) \\ (x, B) &\mapsto A(x)B \end{aligned}$$

We have that

$$\begin{aligned} \mathrm{codim}(\mathrm{Mat}\,(r \times r, \mathbb{C})^i) &= i^2, \\ \dim(\tilde{X} \setminus S) &= 2. \end{aligned}$$

Since the sections $\{\psi_1, \dots, \psi_k\}$ generate $\mathcal{M}$ over the set $U \setminus S$, we get that the map Θ is a submersion and therefore it is transverse to the rank stratification.

By the parametric transversality theorem for almost every B in $\mathrm{Mat}(k \times r, \mathbb{C})$, the map

$$\begin{aligned} \hat{\Theta}\colon U \setminus S &\to \mathrm{Mat}(r \times r, \mathbb{C}) \\ x &\mapsto A(x)B \end{aligned}$$

is transverse to the rank stratification.

Hence we can choose a matrix B generic such that in each trivialization the map $\hat{\Theta}$ is transverse to the rank stratification.

Since B is generic and Ψ_U is a submersion over $\tilde{X} \setminus S$ we have that

$$\mathrm{Supp}(A) = \{x \in \tilde{X} \setminus S \,|\, \det(AB) = 0\}$$

has dimension equal to one, is smooth over $\tilde{X} \setminus S$ and it is not contained in the exceptional divisor.

Now consider the exact sequence (2.2.1)

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Dualizing the previous exact sequence we get

$$0 \to \mathcal{M}^{\vee} \to \mathcal{O}_{\tilde{X}}^r \to \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}', \mathcal{O}_{\tilde{X}}) \to 0, \tag{2.2.5}$$

dualizing the previous exact sequence we get

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M}^{\vee\vee} \to \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}} \left(\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}} (\mathcal{A}', \mathcal{O}_{\tilde{X}}), \mathcal{O}_{\tilde{X}} \right) \to 0. \tag{2.2.6}$$

Since $\mathcal{M}$ is locally free we have that $\mathcal{M}$ is isomorphic to $\mathcal{M}^{\vee\vee}$. Therefore by the exact sequences (2.2.1) and (2.2.6) we get that $\mathcal{A}'$ is isomorphic to $\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}} \left(\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}} (\mathcal{A}', \mathcal{O}_{\tilde{X}}), \mathcal{O}_{\tilde{X}} \right)$. Finally since the support of $\mathcal{A}'$ has dimension one we conclude that $\mathcal{A}'$ is a Cohen-Macaulay sheaf of dimension one.

We have proved the following proposition.

Proposition 2.7. *Let M be a reflexive $\mathcal{O}_X$-module and denote by $\mathcal{M} := (\pi^* M)^{\vee\vee}$ its associated full sheaf. Suppose that rank$(\mathcal{M}) = r$ and take $\phi_1, \ldots, \phi_r$ generic sections, hence we obtain the exact sequence given by the sections*

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Denote by D the support of $\mathcal{A}'$.

Under these assumptions we get:

1. *The support of $\mathcal{A}'$ intersects E in a finite set.*

2. *The sheaf $\mathcal{A}'$ is Cohen-Macaulay of dimension one.*

3. $\mathcal{A}'|_{\tilde{X} \setminus S} \cong \mathcal{O}_D|_{\tilde{X} \setminus S}$.

It is important to completely understand the sheaf $\mathcal{A}'$. Later all the information about the sheaf $\mathcal{A}'$ will be used in the construction of the inverse correspondence. We have two more properties about the sheaf $\mathcal{A}'$.

Proposition 2.8. *Let M be a reflexive $\mathcal{O}_X$-module and denote by $\mathcal{M} := (\pi^* M)^{\vee\vee}$ its associated full sheaf. Suppose that rank$(\mathcal{M}) = r$ and take $\phi_1, \ldots, \phi_r$ generic sections, hence we obtain the exact sequence given by the sections*

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Denote by D the support of $\mathcal{A}'$.

The sheaf $\mathcal{A}'$ is an $\mathcal{O}_D$-module. That is, the annihilator of D is contained in the annihilator of $\mathcal{A}'$.

Proof. Since the support of $\mathcal{A}'$ has dimension one and $\tilde{X}$ is smooth of dimension two, we have that locally the support of $\mathcal{A}'$ is defined by the zero set of a function. This fact and Proposition 2.7 tell us that for any point x in $\tilde{X} \setminus S$ we have that $\mathcal{A}'_x$ is isomorphic to $\mathcal{O}_{\tilde{X},x}/(f)$.

Assume that there exists a section a of $\mathcal{A}'$ such that $f \cdot a$ is different from zero. The previous remark and the assumption implies that the support of $f \cdot a$ is contained in the finite set S.

Let us denote by $(f \cdot a)$ the sheaf generated by $f \cdot a$. This sheaf give us the following exact sequence

$$0 \to (f \cdot a) \to \mathcal{A}' \to \mathcal{A}'/(f \cdot a) \to 0.$$

Dualizing the previous exact sequence we get

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}'/(fa),\mathcal{O}_{\tilde{X}}) \arrow[r] & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}',\mathcal{O}_{\tilde{X}}) \arrow[r] & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}((fa),\mathcal{O}_{\tilde{X}}) \arrow[dll, dotted] & \\ & \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}'/(fa),\mathcal{O}_{\tilde{X}}) \arrow[r] & \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}',\mathcal{O}_{\tilde{X}}) \arrow[r] & \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}((fa),\mathcal{O}_{\tilde{X}}) \arrow[r] & 0 \end{tikzcd} \tag{2.2.7}$$

By Proposition 2.7 we have that $\mathcal{A}'$ is Cohen-Macaulay of dimension one, this implies that $\mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}',\mathcal{O}_{\tilde{X}})$ is equal to zero. By the exact sequence (2.2.7) and the last identification we get that $\mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}((f\cdot a),\mathcal{O}_{\tilde{X}})$ is equal to zero but this can not happen because the support of $f\cdot a$ is finite.

Therefore $f\cdot a$ must be equal to zero. □

Now we have general proposition about reduced curves. In our situation this proposition gives us information about the sheaf $\mathcal{A}'$.

Proposition 2.9. *Let $(D,0)$ be a complex analytic germ of a reduced curve. Let $\mathcal{A}'$ be a Cohen-Macaulay sheaf of dimension one such that $\mathcal{A}'|_{D\setminus\{0\}} \cong \mathcal{O}_D|_{D\setminus\{0\}}$. Denote by $n\colon \tilde{D}\to D$ the normalization of D. Then there exists β a $\mathcal{O}_D$-module such that we have the inclusions*

$$\mathcal{O}_D \subset \beta \subset n_*\mathcal{O}_{\tilde{D}},$$

and $\mathcal{A}'$ is isomorphic to β.

Proof. Consider the following map

$$h\colon \mathcal{A}' \to \mathcal{A}'\otimes_{\mathcal{O}_D}\mathcal{O}_{\tilde{D}},$$
$$a \mapsto a\otimes 1.$$

By hypothesis we have that $\mathcal{A}'|_{D\setminus\{0\}} \cong \mathcal{O}_D|_{D\setminus\{0\}}$ and the support of $\mathcal{A}'$ is smooth over $D\setminus\{0\}$, therefore the map h is injective over $D\setminus\{0\}$. This implies that the support of the kernel of h is the set $\{0\}$ and since $\mathcal{A}'$ does not have any section supported in the set $\{0\}$ we get that the map h is injective.

Now notice that $n^*\mathcal{A}' = \mathcal{A}'\otimes_{\mathcal{O}_D}\mathcal{O}_{\tilde{D}}$. Again since $\mathcal{A}'$ does not have any section supported in the set $\{0\}$, we get that $n^*\mathcal{A}'$ is a torsion free $\mathcal{O}_{\tilde{D}}$-module of rank one and this implies that $n^*\mathcal{A}'$ is isomorphic to $\mathcal{O}_{\tilde{D}}$.

This tells us that the map

$$h\colon \mathcal{A}' \to \mathcal{O}_{\tilde{D}},$$

is injective.

Now consider the irreducible decomposition of $\tilde{D}$,

$$\tilde{D} = \tilde{D}_1 \coprod \cdots \coprod \tilde{D}_l,$$

where $\mathcal{O}_{\tilde{D}_j} = \mathbb{C}\{t_j\}$. This allows us to identify $\mathcal{O}_{\tilde{D}} = \oplus\mathbb{C}\{t_j\}$.

Therefore for any section a of $\mathcal{A}'$ we get

$$h(a) = \left(\ldots,\sum_i c_{i,j}t_j^{n_{i,j}},\ldots\right).$$

Let us denote by $h(a)_j := \sum_i c_{i,j} t_j^{n_{i,j}}$.

For any section a of $\mathcal{A}'$ we define

$$\mathrm{ord}(h(a)) := \left(\dots, \mathrm{ord}_{t_j}\left(h(a)_j\right), \dots\right),$$

where ord_{t_j} denotes the order with respect to the variable t_j. Notice that $\mathrm{ord}(h(a))$ belongs to the set $\mathbb{N}^l$.

Now for any generic λ and μ in $\mathbb{C}$ and a and a' in $\mathcal{A}'$ we have that

$$\mathrm{ord}(h(\lambda a + \mu a')) = \min\{\mathrm{ord}(h(a)), \mathrm{ord}(h(a'))\},$$

where

$$\min\{\mathrm{ord}(h(a)), \mathrm{ord}(h(a'))\} := (\dots, \min\{\mathrm{ord}_{t_j}(h(a)_j), \mathrm{ord}_{t_j}(h(a')_j)\}, \dots).$$

For any section a of $\mathcal{A}'$, we have that the vector $\mathrm{ord}(h(a))$ belongs to the set $\mathbb{N}^l$, therefore there exists a section a_0 of $\mathcal{A}'$ where $\mathrm{ord}(h(a_0))$ is the minimum. Denote by $(n_1, \dots, n_l) = \mathrm{ord}(h(a_0))$.

Consider the following maps

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "h"] \arrow[dr, "g"'] & \oplus\mathbb{C}\{t_j\} \arrow[d, "\iota"] \\ & \oplus\mathbb{C}\{t_j\}[t_j^{-1}] \end{tikzcd}$$

where $\iota(\kappa_1, \dots, \kappa_l) = (t_1^{-n_1}\kappa_1, \dots, t_l^{-n_l}\kappa_l)$ and $g = \iota \circ h$.

By construction the map g is an $\mathcal{O}_D$-monomorphism and the image of g is contained in $\mathcal{O}_{\tilde{D}}$ because

$$\min\{\mathrm{ord}(g(a)) \,|\, a \text{ is a section of } \mathcal{A}'\} = \mathrm{ord}(g(a_0)) = (0, \dots, 0).$$

Denote by $(\xi_1, \dots, \xi_l) := g(a_0)$, by the previous discussion we have that each ξ_j belongs to $\mathcal{O}^*_{\tilde{D}_j}$, where as usual $\mathcal{O}^*_{\tilde{D}_j}$ denotes the units of the local ring $\mathcal{O}_{\tilde{D}_j}$.

Finally consider the maps

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "g"] \arrow[dr, "f"'] & \mathcal{O}_{\tilde{D}} \arrow[d, "\sigma"] \\ & \mathcal{O}_{\tilde{D}} \end{tikzcd}$$

where $\sigma(\kappa_1, \dots, \kappa_l) = (\xi_1^{-1}\kappa_1, \dots, \xi_l^{-1}\kappa_l)$ and $f = \sigma \circ g$.

By construction we have that $f(a_0) = 1$, this implies that $\beta = f(\mathcal{A}')$ satisfy all the properties. $\square$

The propositions 2.8 and 2.9 tell us the structure of the sheaf $\mathcal{A}'$. In the following subsection we will use this ideas in order to construct the inverse correspondence.

Now we can construct the direct correspondence as follows: let $\mathcal{M}$ be a full sheaf of rank r. Taking r generic sections we get the exact sequence

$$0 \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{M} \to \mathcal{A}' \to 0, \tag{2.2.1}$$

and its dual is

$$0 \to \mathcal{N} \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{A} \to 0, \tag{2.2.8}$$

where $\mathcal{N} := \mathcal{M}^{\vee}$ and $\mathcal{A} := \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}', \mathcal{O}_{\tilde{X}})$.

As a consequence of Proposition 2.7 we have that $\mathcal{A}$ is Cohen-Macaulay of dimension one and its support intersects the exceptional divisor in a finite set.

Applying the functor π_*- to the exact sequence (2.2.8) we get

$$0 \to N \to \mathcal{O}_X^r \to \pi_*\mathcal{A} \to R^1\pi_*\mathcal{N} \to R^1\pi_*\mathcal{O}_{\tilde{X}}^r \to 0. \tag{2.2.9}$$

The exact sequence (2.2.9) give us the exact sequence

$$0 \to N \to \mathcal{O}_X^r \to \mathcal{C} \to 0, \tag{2.2.10}$$

where $\mathcal{C}$ is the image of $\mathcal{O}_X^r$.

This exact exact sequence give us r generators of $\mathcal{C}$ as $\mathcal{O}_X$-module.

This give us the direct correspondence.

Proposition 2.10. *Given a full sheaf $\mathcal{M}$ of rank r with r generic sections, we associate three things:*

1. *A Cohen-Macaulay sheaf $\mathcal{A}$ of dimension one such that its support D intersects the exceptional divisor in a finite set.*
2. *An $\mathcal{O}_X$-module $\mathcal{C}$ contained in $\pi_*\mathcal{A}$.*
3. *A collection of r generators of $\mathcal{C}$ as $\mathcal{O}_X$-module.*

The module $\mathcal{C}$ satisfies the following property: Denote by $\mathcal{O}_{\pi_*D}$ the structure sheaf of $\pi(D)$ and $n\colon \tilde{D} \to D$ the normalization of D. Then we have the following inclusions

$$\mathcal{O}_{\pi_*D} \subset \mathcal{C} \subset \pi_*n_*\mathcal{O}_{\tilde{D}}.$$

Proof. By the previous discussion we have proved almost everything, we just need to prove the property about the module $\mathcal{C}$.

By definition $\mathcal{C}$ is contained in $\pi_*\mathcal{A}$ and by Proposition 2.9 we know that $\mathcal{A}$ is contained in $n_*\pi_*\mathcal{O}_{\tilde{D}}$, this give us the inclusion of $\mathcal{C}$ in $\pi_*n_*\mathcal{O}_{\tilde{D}}$.

Now 1 belongs to $\pi_*\mathcal{A}$ and the image of $\mathcal{O}_X$ obtained by multiplying by 1 is exactly $\mathcal{O}_{\pi_*D}$. Since $\dim_{\mathbb{C}}(\pi_*\mathcal{A}/\mathcal{C})$ is finite, we have that 1 belongs to $\mathcal{C}$, hence $\mathcal{O}_{\pi_*D}$ is contained in $\mathcal{C}$. □

Remark 2.11. Let $\mathcal{A}$ and $\mathcal{C}$ be as in the previous discussion. Denote by D the support of $\mathcal{A}$ and $n\colon \tilde{D} \to D$ its normalization. By Proposition 2.9 we know that $\mathcal{C}$ is contained in $n_*\pi_*\mathcal{O}_{\tilde{D}}$.

In our correspondence at the singularity, the Proposition 2.9 allows us to identify the Cohen-Macaulay modules as submodules of a normalization.

2.2.2 Inverse correspondence

In this subsection we study the inverse of the previous correspondence: we want to associate to each Cohen-Macaulay sheaf of dimension one such that its support is not contained in the exceptional divisor, together with a system of r generators, a full sheaf with r sections. We will see that this construction is obstructed but we understand all the conditions.

Following Esnault [13] we have the following definition.

Definition 2.12. Let D be any curve on $\tilde{X}$. We denote by $\mathcal{O}_{\pi_* D}$ the reduced induced sub-scheme structure given to the set $\pi(D)$.

Notice that given D any curve on $\tilde{X}$, we have that $\mathcal{O}_{\pi_* D}$ it coincides with the image of $\mathcal{O}_X$ in $\pi_* \mathcal{O}_D$ (for example [14, Lemma 2.5]).

Let D be any curve on $\tilde{X}$ such that $D \cap E = \{p_1, \dots, p_k\}$. Let $n \colon \tilde{D} \to D$ be its normalization and $\pi|_D \colon D \to \pi(D)$ the restriction of the resolution map to D. Let $\mathcal{C}$ be an $\mathcal{O}_X$-submodule of $\pi_* n_* \mathcal{O}_{\tilde{D}}$ verifying $\mathcal{O}_{\pi_* D} \subset \mathcal{C}$ and define $\mathcal{A} := (\pi|_D)^* \mathcal{C}$.

Consider $\{\phi_1, \dots, \phi_r\}$ a minimal set of generators of $\mathcal{C}$ as $\mathcal{O}_X$-module. We get the following exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{A} \to 0. \tag{2.2.11}$$

Dualizing (2.2.11) and denoting by $\mathcal{M} := \mathcal{N}^{\vee}$ and $\mathcal{A}' := \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}})$, we obtain the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

In this subsection we want to give conditions for the module $\mathcal{C}$ in order to guarantee that $\mathcal{M}$ is a full sheaf. We will use Proposition 1.8 in order to establish these conditions.

Since $\mathcal{A}$ is Cohen-Macaulay of dimension one and $\tilde{X}$ is smooth, by Theorem 1.4 we have

$$\mathcal{A} \cong \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}), \mathcal{O}_{\tilde{X}}\right),$$

therefore dualizing (2.2.1) we obtain the exact sequence

$$0 \to \mathcal{N}^{\vee\vee} \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{A} \to 0,$$

hence $\mathcal{N}$ is locally free.

Remark 2.13. It is important to notice that the previous discussion says that given the sheaf $\mathcal{A}$ and r generators as $\mathcal{O}_X$-module, then the sheaf of relations of the generators is locally free. We can think this remark as the analogous of the Remark 2.4.

Since $\mathcal{M}$ is the dual of $\mathcal{N}$, we get that $\mathcal{M}$ is locally free. In order to use Proposition 1.8 we need to check two more conditions. First, the property to be almost generated by its global sections is always true.

Proposition 2.14. *The sheaf $\mathcal{M}$ given by* (2.2.1) *is almost generated by its global sections.*

Proof. Applying the functor $\pi_* -$ to the exact sequence (2.2.1) we get

$$0 \to \mathcal{O}_X^r \to \pi_* \mathcal{M} \to \pi_* \mathcal{A}' \to R^1 \pi_* \mathcal{O}_{\tilde{X}}^r \to R^1 \pi_* \mathcal{M} \to 0.$$

Denote by $\mathcal{G}$ the image of $\pi_* \mathcal{M}$ in $\pi_* \mathcal{A}'$, so we obtain the following two exact sequences

$$0 \to \mathcal{O}_X^r \to \pi_* \mathcal{M} \to \mathcal{G} \to 0,$$

$$0 \to \mathcal{G} \to \pi_* \mathcal{A}' \to R^1 \pi_* \mathcal{O}_{\tilde{X}}^r \to R^1 \pi_* \mathcal{M} \to 0. \tag{2.2.12}$$

Since the support of $\mathcal{A}'$ intersects the exceptional divisor in a finite collection of points, then we can identify $\pi_* \mathcal{A}'$ with $\mathcal{A}'$. Therefore we can consider $\mathcal{G}$ as a subsheaf of $\mathcal{A}'$.

Denote by $\mathcal{M}'$ the subsheaf of $\mathcal{M}$ generated by its global sections, therefore we have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}^r_{\tilde{X}} \arrow[r] \arrow[d] & \mathcal{M}' \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}^r_{\tilde{X}} \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{A}' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Hence it is enough to prove that the support of $\mathcal{F}'$ is a finite set. This follows from (2.2.12) and the fact that $\dim_{\mathbb{C}}(\operatorname{coker}\{\mathcal{G} \hookrightarrow \mathcal{A}'\})$ is finite. $\square$

We want to guarantee that $\mathcal{M}$ is a full sheaf, therefore we need to proof that it satisfies the cohomology condition: the natural map $H^1_E(\mathcal{M}) \to H^1(\mathcal{M})$ is injective. By Serre duality this condition is equivalent to the surjection of the natural map $H^1_E\left(\mathcal{M}^{\vee} \otimes \omega_{\tilde{X}}\right) \to H^1\left(\mathcal{M}^{\vee} \otimes \omega_{\tilde{X}}\right)$, hence we will study this map.

Apply the functor $- \otimes \omega_{\tilde{X}}$ to the exact sequence (2.2.11),

$$0 \to \mathcal{N} \otimes \omega_{\tilde{X}} \to \omega^r_{\tilde{X}} \to \mathcal{A} \otimes \omega_{\tilde{X}} \to 0. \tag{2.2.13}$$

Consider (2.2.13) and take the long exact sequence of cohomology and local cohomology

$$\begin{tikzcd}
H^0_E(\mathcal{N} \otimes \omega_{\tilde{X}}) \arrow[r, hook] \arrow[d, hook] & H^0_E\left(\omega^r_{\tilde{X}}\right) \arrow[r] \arrow[d, hook] & H^0_E(\mathcal{A} \otimes \omega_{\tilde{X}}) \arrow[r] \arrow[d, hook] & H^1_E(\mathcal{N} \otimes \omega_{\tilde{X}}) \arrow[r] \arrow[d] & H^1_E\left(\omega^r_{\tilde{X}}\right) \arrow[d] \\
H^0(\mathcal{N} \otimes \omega_{\tilde{X}}) \arrow[r, hook] \arrow[d] & H^0\left(\omega^r_{\tilde{X}}\right) \arrow[r] \arrow[d] & H^0(\mathcal{A} \otimes \omega_{\tilde{X}}) \arrow[r] \arrow[d] & H^1(\mathcal{N} \otimes \omega_{\tilde{X}}) \arrow[r] \arrow[d] & H^1\left(\omega^r_{\tilde{X}}\right) \arrow[d] \\
H^0(U; \mathcal{N} \otimes \omega_{\tilde{X}}) \arrow[r, hook] \arrow[uurrr, dotted, bend right] & H^0\left(U; \omega^r_{\tilde{X}}\right) \arrow[r] \arrow[uurrr, dotted, bend right] & H^0(U; \mathcal{A} \otimes \omega_{\tilde{X}}) \arrow[r] & H^1(U; \mathcal{N} \otimes \omega_{\tilde{X}}) \arrow[r] & H^1\left(U; \omega^r_{\tilde{X}}\right)
\end{tikzcd}$$

We have $H^1(\omega_{\tilde{X}}) = 0$ by Grauert-Riemenschneider Vanishing Theorem and $H^0_E(\mathcal{A} \otimes \omega_{\tilde{X}}) = 0$ because $\mathcal{A} \otimes \omega_{\tilde{X}}$ has depth one and its support intersects the exceptional divisor in a finite set. Therefore we have the following diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & H^1_E\left(\mathcal{N} \otimes \omega_{\tilde{X}}\right) \arrow[d, "\theta"] & \\
H^0\left(\omega^r_{\tilde{X}}\right) \arrow[r, "\beta"] \arrow[d] & H^0\left(\mathcal{A} \otimes \omega_{\tilde{X}}\right) \arrow[r, "\alpha"] \arrow[d, "\gamma_1"] & H^1\left(\mathcal{N} \otimes \omega_{\tilde{X}}\right) \arrow[r] \arrow[d, "\varphi"] & 0 \\
H^0\left(U; \omega^r_{\tilde{X}}\right) \arrow[r, "\delta"] & H^0\left(U; \mathcal{A} \otimes \omega_{\tilde{X}}\right) \arrow[r, "\gamma_2"] & H^1\left(U; \mathcal{N} \otimes \omega_{\tilde{X}}\right) &
\end{tikzcd} \tag{2.2.14}$$

Lemma 2.15. *The morphism*

$$H^1_E(\mathcal{N} \otimes \omega_{\tilde{X}}) \xrightarrow{\theta} H^1(\mathcal{N} \otimes \omega_{\tilde{X}}), \tag{2.2.15}$$

is an epimorphism if and only if

$$\operatorname{Im} \gamma_1 \subset \operatorname{Im} \delta.$$

PROOF.

Assume that θ is an epimorphism: Let $f \in H^0(\mathcal{A} \otimes \omega_{\tilde{X}})$. Since θ is a surjection, we have

$$\ker \varphi = \operatorname{Im} \theta = H^1(\mathcal{N} \otimes \omega_{\tilde{X}}),$$

therefore

$$\gamma_2(\gamma_1(f)) = \varphi(\alpha(f)) = 0.$$

Hence $\gamma_1(f) \in \ker \gamma_2$ and the kernel of γ_2 is equal to the image of δ.

Assume that $\operatorname{Im} \gamma_1 \subset \operatorname{Im} \delta$: Let $f \in H^1(\mathcal{N} \otimes \omega_{\tilde{X}})$ and $g \in H^0(\mathcal{A} \otimes \omega_{\tilde{X}})$ such that $\alpha(g) = f$. By the hypothesis and since the image of δ is equal to the kernel of γ_2, we get

$$\varphi(f) = \varphi(\alpha(g)) = \gamma_2(\gamma_1(g)) = 0,$$

hence the image of θ is equal to $H^1(\mathcal{N} \otimes \omega_{\tilde{X}})$.

□

We can summarize all the previous results in the following proposition which gives us all the conditions in order to guarantee that the sheaf $\mathcal{M}$ is full.

Proposition 2.16. *Let D be any curve on $\tilde{X}$ such that $D \cap E = \{p_1, \dots, p_k\}$. Let $n\colon \tilde{D} \to D$ be its normalization and $\pi|_D\colon D \to \pi(D)$ the restriction of the resolution map to D. Let $\mathcal{C}$ be an $\mathcal{O}_X$-submodule of $\pi_* n_* \mathcal{O}_{\tilde{D}}$ verifying $\mathcal{O}_{\pi_* D} \subset \mathcal{C}$ and define $\mathcal{A} := (\pi|_D)^* \mathcal{C}$.*

Consider $\{\phi_1, \dots, \phi_r\}$ a minimal set of generators of $\mathcal{C}$ as $\mathcal{O}_X$-module, therefore we get the following exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{A} \to 0, \tag{2.2.11}$$

and dualizing this sequence we get

$$0 \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Also consider the diagram (2.2.14)*. Then the sheaf $\mathcal{M}$ is full if and only if* $\operatorname{Im} \gamma_1 \subset \operatorname{Im} \delta$.

Now we want to give an easier description of the last proposition. First notice that

$$H^0(U, \mathcal{A} \otimes \omega_{\tilde{X}}) \cong \bigoplus \mathcal{O}_{\tilde{D}_j|_{\tilde{D}_j \setminus \{0\}}} \cong \bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}[t_j^{-1}],$$

therefore $H^0(\mathcal{A} \otimes \omega_{\tilde{X}})$ is contained in $\bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}[t_j^{-1}]$. Notice that the sections of $H^0(\mathcal{A} \otimes \omega_{\tilde{X}})$ are globally defined, therefore $H^0(\mathcal{A} \otimes \omega_{\tilde{X}})$ is contained in $\bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}$.

Now assume that the singularity is Gorenstein and denote by Ω the Gorenstein form. The divisor of Ω is

$$\operatorname{div}(\Omega) = \sum q_i E_i,$$

where each q_i is a integer.

The morphism δ is induced by the generators $\{\phi_1, \ldots, \phi_r\}$ of $\mathcal{C}$ and recall that $\mathcal{C}$ is contained in the normalization $\bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}$ (see Proposition 2.16). Therefore the sections can be written as

$$\phi_i = \left(\ldots, \sum_k c_{k,j} t_j^{n_{k,j}}, \ldots \right), \quad \text{for } i = 1, \ldots, r.$$

By the previous identifications we have that

$$\Omega\phi_i = \left(\ldots, t_j^{\sum_r (E_r \cdot D_j) q_r} \sum_k c_{k,j} t_j^{n_{k,j}}, \ldots \right), \quad \text{for } i = 1, \ldots, r.$$

Hence we can identify the image of δ with $\delta(\Omega)\mathcal{C}$ where

$$\delta(\Omega) = \left(\ldots, t_j^{\sum_r (E_r \cdot D_j) q_r}, \ldots \right).$$

Now recall that $\mathcal{C}$ is contained in $\bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}$ and it verifies that $\mathcal{O}_{\pi_* D} \subset \mathcal{C}$. Since $\mathcal{O}_{\pi_* D}$ is the structure sheaf of the curve $\pi(D)$, it is well know that we can associate to the sheaf $\mathcal{O}_{\pi_* D}$ a semigroup contained in $\bigoplus_{j=1}^{l} \mathbb{N}$. Following the same idea we can associate to the sheaf $\mathcal{C}$ a subset $\mathfrak{C}$ of $\bigoplus_{j=1}^{l} \mathbb{N}$ as follows: since $\mathcal{C}$ it is a submodule of $\bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}$, then any section s of $\mathcal{C}$ can be written as

$$s = \left(\ldots, \sum_k c_{k,j} t_j^{n_{k,j}}, \ldots \right),$$

where $\sum_k c_{k,j} t_j^{n_{k,j}}$ belongs to $\mathbb{C}\{t_j\}$.

Therefore to the section s we associate the vector give by the function

$$\begin{aligned} \operatorname{ord}\colon \mathcal{C} &\to \bigoplus_{j=1}^{l} \mathbb{N} \\ s &\mapsto \left(\ldots, \operatorname{ord}_{t_j}\left(\sum_k c_{k,j} t_j^{n_{k,j}} \right), \ldots \right), \end{aligned}$$

where ord_{t_j} denotes the order with respect to the variable t_j.

Since $\mathcal{C}$ is just an $\mathcal{O}_X$-submodule, the set $\mathfrak{C}$ is not a semigroup. Notice that by construction the set $\mathfrak{C}$ contains the semigroup associated to $\mathcal{O}_{\pi_*D}$. Now since $\mathcal{O}_{\pi_*D}$ is the image of $\mathcal{O}_X$ in $\pi_*\mathcal{O}_D$ and $\mathcal{C}$ is an $\mathcal{O}_X$-module, then the semigroup associated to $\mathcal{O}_{\pi_*D}$ acts on the set $\mathfrak{C}$.

Also it is well known that $\mathcal{O}_{\pi_*D}$ has a conductor (see for example [1, (19.21)]). In our case we define the conductor of $\mathcal{C}$ as follows (compare with [1, (19.21)]).

Definition 2.17. Let D be a curve over X and denote by $n\colon \tilde{D} \to D$ its normalization. Let $\mathcal{C}$ be an $\mathcal{O}_X$-module such that

$$\mathcal{O}_D \subset \mathcal{C} \subset n_*\mathcal{O}_{\tilde{D}}.$$

The set

$$\{s \in \mathcal{C} \mid s \cdot n_*\mathcal{O}_{\tilde{D}} \subset \mathcal{C}\},$$

is called *the conductor of* $\mathcal{C}$.

Since $\mathcal{O}_{\pi_*D}$ is contained in $\mathcal{C}$, we get that the conductor of $\mathcal{C}$ is a non-empty set. Applying the function ord to the conductor of $\mathcal{C}$ we obtain a subset of the graph $\mathfrak{C}$, which we called the conductor set of $\mathfrak{C}$. Any element of the conductor set of $\mathfrak{C}$ is called a conductor of $\mathfrak{C}$.

Notice that the conductor set of $\mathfrak{C}$ satisfies the following property: if $c = (c_1, \dots, c_l)$ is a conductor of $\mathfrak{C}$ then for any vector w in $\bigoplus_{j=1}^{l} \mathbb{N}$ we have that $c + w$ belongs to the set $\mathfrak{C}$.

Notice that the multiplication $\delta(\Omega)\mathcal{C}$ is just the translation of the set $\mathfrak{C}$ given by the vector $(\dots, \sum_r (E_r \cdot D_j)\, q_r, \dots)$.

This discussion gives us the following corollaries.

Corollary 2.18. *Assume (X, x) is Gorenstein. Let $\mathcal{C}$ as in Propostion 2.16 and $\mathfrak{C}$ as in the previous discussion. Denote by $c = (c_1, \dots, c_l)$ a conductor of $\mathfrak{C}$.*

If $c_j \leq -\sum_r (E_r \cdot D_j)\, q_r$ for each $j = 1, \dots, l$, then the sheaf $\mathcal{M}$ given by the Proposition 2.16 is full.

Proof. Let $n\colon \tilde{D} \to D$ be the normalization of the curve D. We can identify

$$\mathcal{O}_{\tilde{D}} \cong \bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}.$$

By our previous discussion we have

$$\delta(\Omega)\mathcal{C} = \left(\dots, t_j^{\sum_r (E_r \cdot D_j) q_r}, \dots\right)\mathcal{C}.$$

Since $c_j \leq -\sum_r (E_r \cdot D_j)\, q_r$ for each $j = 1, \dots, l$, then $\delta(\Omega)\mathcal{C}$ contains the ring $\bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}$, therefore it also contains $\gamma_1\left(H^1\left(\mathcal{A} \otimes \omega_{\tilde{X}}\right)\right)$. $\square$

Corollary 2.19. *Assume (X, x) is Gorenstein and that the Gorenstein form does not have zeros over the exceptional divisor. Let D be a smooth curve such that $D \cap E = \{p_1, \dots, p_k\}$ and $\mathcal{C} = \pi_*\mathcal{O}_D$. Then the sheaf $\mathcal{M}$ given by the Proposition 2.16 is full and special.*

Proof. Since D is smooth, the set $\mathfrak{C}$ is a semigroup and the zero vector is a conductor. Now since the Gorenstein form does not have zeros over the exceptional divisor we have that

$$\delta(\Omega) = \left(\dots, t_j^{\sum_r (E_r \cdot D_j) q_r}, \dots\right),$$

where each q_r is a non-positive integer.

Therefore $0 \leq -\sum_r (E_r \cdot D_j)\, q_r$ for each $j = 1, \ldots, l$. Hence by Corollary 2.18 the sheaf $\mathcal{M}$ is full.

Now consider the exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{O}_D \to 0 \tag{2.2.11}$$

Applying the functor $\pi_* -$ we obtain

$$0 \to \pi_*\mathcal{N} \to \pi_*\mathcal{O}^r_{\tilde{X}} \to \pi_*\mathcal{O}_D \to R^1\pi_*\mathcal{N} \to R^1\pi_*\mathcal{O}^r_{\tilde{X}} \to 0$$

and since we take generators of $\pi_*\mathcal{O}_D$ as $\mathcal{O}_X$-module, the last sequence says that $R^1\pi_*\mathcal{N} \cong R^1\pi_*\mathcal{O}^r_{\tilde{X}}$. □

Corollary 2.20. *Assume (X, x) is Gorenstein and that the Gorenstein form does not have zeros over the exceptional divisor. Let D be a curve such that $D \cap E = \{p_1, \ldots, p_k\}$, $n\colon \tilde{D} \to D$ the normalization and $\mathcal{C} = \pi_* n_* \mathcal{O}_D$. Then the $\mathcal{M}$ given by the Proposition 2.16 is full and special.*

Proof. The proof is analogous to the last corollary, just identify $\pi_* n_* \mathcal{O}_{\tilde{D}} \cong \bigoplus \mathbb{C}\{t_j\}$ and use the same argument. □

As we see the Gorenstein form allows us to construct many examples. Now we have two lemmas about the Gorenstein form that we will use in Chapter 5.

Lemma 2.21. *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity and $\pi\colon \tilde{X} \to X$ be a resolution with exceptional divisor $E = \bigcup_{i=1}^n E_i$. Then for any component E_j where the Gorenstein form has a pole and for any component E_k where the Gorenstein form has a zero, we have that $E_j \cap E_k = \emptyset$.*

Proof. We use induction on the number of blow ups that are necessary in order to obtain the resolution $\pi\colon \tilde{X} \to X$ from the minimal resolution.

If the singularity is a rational double point, then in the minimal resolution the Gorenstein form does not have any zero or pole. If the singularity is not a rational double point, then in the minimal resolution the Gorenstein form has a pole in every component of the exceptional divisor.

Now assume that the proposition is true for some resolution $\pi\colon \tilde{X} \to X$ with exceptional divisor $E = \bigcup_{i=1}^n E_i$. Take any point p in E and denote by $\sigma\colon \tilde{X}' \to \tilde{X}$ the blow up of the point p and by E' the new exceptional divisor.

In order to complete the proof it is enough to study the following two cases:

1. The point p belongs to a unique component E_i. In this case locally the Gorenstein form over $\tilde{X}$ is

   $$\Omega = f^{q_i}(x, y)dx \wedge dy$$

   In the chart (x, xY) of the blow up, we have that locally the Gorenstein form is

   $$\pi^*\Omega = x^{1+q_i \mathrm{mult}_p(f)} \hat{f}^{q_i}(x, Y)dx \wedge dy,$$

   where $\hat{f}^{q_i}$ is the strict transform.

   Hence in this chart we have that

(a) Near to the exceptional divisor E' the Gorenstein form is $x^{1+q_i \mathrm{mult}_p(f)} dx \wedge dy$.

(b) Near to the exceptional divisor E_i the Gorenstein form is $\hat{f}^{q_i}(x, Y) dx \wedge dy$.

This tells us that

(a) If q_i is positive, then the Gorenstein form has zeros over both components.

(b) If q_i is zero, then the Gorenstein form has zeros over the component E' and is zero over E_i.

(c) If q_i is negative, then the Gorenstein form has poles over E' and has poles over E_i.

The other chart is analogous.

2. The point p belongs to E_i and E_j. The idea is the same as before but in this case we use the induction hypothesis. Locally the Gorenstein form is

$$\Omega = f_i^{q_i}(x, y) f_j^{q_j}(x, y) dx \wedge dy.$$

In the chart (x, xY) of the blow up, we have that locally the Gorenstein form is

$$\pi^* \Omega = x^{1+q_i \mathrm{mult}_p(f_i)+q_j \mathrm{mult}_p(f_j)} \hat{f_i}^{q_i}(x, Y) \hat{f_j}^{q_j}(x, Y) dx \wedge dy,$$

where $\hat{f_i}^{q_i}$ and $\hat{f_j}^{q_j}$ are the strict transform.

By induction hypothesis we know that the number $q_i \cdot q_j$ can not be negative. Hence in this chart we have that

(a) If q_i and q_j are positive, then the Gorenstein form has zeros over all the components.

(b) If q_i is positive and q_j is zero, then the Gorenstein form has zeros over all the components.

(c) If q_i and q_j are zero, then the Gorenstein form has zeros over E' and is zero in the components E_i and E_j.

(d) If q_i is negative and q_j is zero, then the Gorenstein form has poles or is zero over the component E', the Gorenstein form is zero over E_j and has poles in the component E_i.

(e) If q_i and q_j are negative, then the Gorenstein form has poles over all the components.

In general the point p belongs to a finite number of exceptional divisors, using the same ideas we can finish the induction. □

Corollary 2.22. *Assume (X, x) that is Gorenstein and let $\pi \colon \tilde{X} \to X$ be a resolution with exceptional divisor E. Assume that the Gorenstein form has zeros in the component E_1 of the exceptional divisor. Let D be a curve such that $D \cap E = \{p_1, \ldots, p_k\}$ and at least one point belong to E_1. Denote by $n \colon \tilde{D} \to D$ the normalization of D and let $\mathcal{C}$ be any an $\mathcal{O}_X$-submodule of $\pi_* n_* \mathcal{O}_{\tilde{D}}$ verifying $\mathcal{O}_{\pi_* D} \subset \mathcal{C}$ and define $\mathcal{A} := (\pi|_D)^* \mathcal{C}$.*

Then the sheaf $\mathcal{M}$ given by the Proposition 2.16 is not full.

Proof. Let $\mathcal{C}$ be any an $\mathcal{O}_X$-submodule of $\pi_* n_* \mathcal{O}_{\tilde{D}}$ verifying $\mathcal{O}_{\pi_* D} \subset \mathcal{C}$ and define $\mathcal{A} := (\pi|_D)^* \mathcal{C}$.

Consider $\{\phi_1, \ldots, \phi_r\}$ a minimal set of generators of $\mathcal{C}$ as $\mathcal{O}_X$-module and the exact sequence given by the generators

$$0 \to \mathcal{N} \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{A} \to 0. \qquad (2.2.11)$$

Applying the functor $\pi_* -$ to the previous exact sequence we get the exact sequence

$$0 \to \pi_* \mathcal{N} \to \mathcal{O}_X^r \to \pi_* \mathcal{A} \to R^1 \pi_* \mathcal{N} \to R^1 \pi_* \mathcal{O}_{\tilde{X}}^r \to 0. \qquad (2.2.11)$$

Since we took generators of $\mathcal{C}$ as $\mathcal{O}_X$-module, we get that the image of $\mathcal{O}_X^r$ is $\mathcal{C}$. Therefore $\mathcal{C}$ is contained in $\pi_* \mathcal{A}$.

Let $D = D_1 \coprod \cdots \coprod D_l$ be the irreducible decomposition of D, hence we can identify

$$\mathcal{O}_{\tilde{D}} \cong \bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}.$$

Without loss of generality we can assume that the point p_1 belongs to D_1. This tells us that

$$\delta(\Omega)\mathcal{C} = \left(t_1^{\sum_r (E_r \cdot D_1) q_r}, \ldots, t_j^{\sum_r (E_r \cdot D_j) q_r}, \ldots \right) \mathcal{C}.$$

where q_1 is a positive integer.

Since $\mathcal{C}$ is a submodule of $\bigoplus_{j=1}^{l} \mathbb{C}\{t_j\}$, then any section s of $\mathcal{C}$ can be written as

$$s = \left(\ldots, \sum_k c_{k,j} t_j^{n_{k,j}}, \ldots \right),$$

where $\sum_k c_{k,j} t_j^{n_{k,j}}$ belongs to $\mathbb{C}\{t_j\}$.

Consider the following function

$$\operatorname{ord}_1 \colon \mathcal{C} \to \mathbb{N},$$
$$s \mapsto \operatorname{ord}_{t_1} \left(\sum_k c_{k,1} t_1^{n_{k,1}} \right),$$

where ord_{t_1} denotes the order with respect to the variable t_1.

Since $\mathcal{O}_{\pi_* D}$ is contained in $\mathcal{C}$, we have that 1 belongs to $\mathcal{C}$. Now notice that $\operatorname{ord}_1(1) = 0$, this tells us that the function has a minimum in 1.

Since D intersect the exceptional divisor in a finite set, we get

$$\pi_* \mathcal{A} \cong \pi_* (\mathcal{A} \otimes \omega_{\tilde{X}}).$$

We have that $\mathcal{C}$ is contained in $\pi_* \mathcal{A}$ and $\pi_* \mathcal{A} \cong \pi_* (\mathcal{A} \otimes \omega_{\tilde{X}})$, this tells us that 1 belongs to $\pi_* (\mathcal{A} \otimes \omega_{\tilde{X}})$.

Now assume that the sheaf $\mathcal{M} = \mathcal{N}^{\vee}$ given by the Proposition 2.16 is full.

Since $\mathcal{M}$ is full we get that $\delta(\Omega)\mathcal{C}$ contains $\gamma_1 \left(H^1 \left(\mathcal{A} \otimes \omega_{\tilde{X}} \right) \right)$, this tells us that

$$1 = \left(t_1^{\sum_r (E_r \cdot D_1) q_r}, \ldots, t_j^{\sum_r (E_r \cdot D_j) q_r}, \ldots \right) v, \qquad (2.2.16)$$

for some section v of $\mathcal{C}$.

By Lemma 2.21 we have that $\sum_r (E_r \cdot D_1) q_r$ is a positive integer, this and the equality (2.2.16) tell us that $\operatorname{ord}_1(v)$ is a negative integer, which is a contradiction.

Therefore $\mathcal{M}$ is not a full sheaf. $\square$

Chapter 3

Specialty and the cohomological condition

In this chapter we study the relation between the inverse correspondence given in the Subsection 2.2.2 and the condition of being special. In particular we prove that working in a particular kind of resolution, our inverse correspondence always gives us a full sheaf if the dimension as $\mathbb{C}$-vector space of the first cohomology group of the dual of the sheaf that we construct is equal to rp_g, where r is the rank of the sheaf and p_g is the geometric genus. This will allow us to construct special full sheaves.

In order to continue we need the following definition.

Definition 3.1. Let (X,x) denote a complex analytic germ of a normal two-dimensional Gorenstein singularity and $\pi\colon (\tilde{X},E) \to (X,x)$ denotes a resolution with exceptional divisor $E = \bigcup_{i=1}^{n} E_i$. We will say that *the resolution is non-positive with respect to the canonical cycle* if the coefficients of the canonical cycle are non-positive.

From now on (X,x) denotes a complex analytic germ of a normal two-dimensional Gorenstein singularity, $\pi\colon (\tilde{X},E) \to (X,x)$ denotes a non-positive resolution with respect to the canonical cycle with exceptional divisor $E = \bigcup_{i=1}^{n} E_i$ and Z_K denotes the canonical cycle.

Proposition 3.2. *Let D be any curve on $\tilde{X}$ such that $D \cap E = \{p_1, \ldots, p_k\}$. Let $n\colon \tilde{D} \to D$ be its normalization and $\pi|_D\colon D \to \pi(D)$ the restriction of the resolution map to D. Let $\mathcal{C}$ be an $\mathcal{O}_X$-submodule of $\pi_* n_* \mathcal{O}_{\tilde{D}}$ verifying $\mathcal{O}_{\pi_* D} \subset \mathcal{C}$ and define $\mathcal{A} := (\pi|_D)^* \mathcal{C}$.*

Consider $\{\phi_1, \ldots, \phi_r\}$ a minimal set of generators of $\mathcal{C}$ as $\mathcal{O}_X$-modules, therefore we get the following exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{A} \to 0, \tag{2.2.11}$$

and its dual

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

If the dimension of $R^1\pi_*\mathcal{N}$ as $\mathbb{C}$-vector space is equal to rp_g, then $\mathcal{M}$ is a full sheaf.

Proof. As in Proposition 2.16 consider

$$\begin{tikzcd}
 & 0 \arrow[d] \\
H^0(\omega^r_{\tilde{X}}) \arrow[r] \arrow[d] & H^0(\mathcal{A}\otimes\omega_{\tilde{X}}) \arrow[d, "\gamma_1"] \\
H^0(U;\omega^r_{\tilde{X}}) \arrow[r, "\delta"] & H^0(U;\mathcal{A}\otimes\omega_{\tilde{X}})
\end{tikzcd}$$

We are going to prove that the image of γ_1 is contained in the image of δ.

Since we are working on a non-positive resolution with respect to the canonical cycle we can consider the exact sequence

$$0 \to \omega_{\tilde{X}} \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{Z_K} \to 0. \tag{3.0.1}$$

Now apply the functor $-\otimes-$ to the sequences (2.2.11) and (3.0.1),

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{N}\otimes\omega_{\tilde{X}} \arrow[r] \arrow[d] & \omega^r_{\tilde{X}} \arrow[r] \arrow[d] & \mathcal{A}\otimes\omega_{\tilde{X}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{O}^r_{\tilde{X}} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{N}\otimes\mathcal{O}_{Z_k} \arrow[r] \arrow[d] & \mathcal{O}^r_{Z_k} \arrow[r] \arrow[d] & \mathcal{A}\otimes\mathcal{O}_{Z_k} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By the last diagram we get the following diagram of exact sequences

$$\begin{tikzcd}
 & H^0\left(\tilde{X},\omega^r_{\tilde{X}}\right) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & H^0\left(\tilde{X},\mathcal{A}\otimes\omega_{\tilde{X}}\right) \arrow[dl, "\beta"] \arrow[dd, "\gamma_1"] \\
H^0\left(\tilde{X},\mathcal{O}^r_{\tilde{X}}\right) \arrow[rr, "\rho"] \arrow[dd, "\nu"] & & H^0\left(\tilde{X},\mathcal{A}\right) \arrow[dd, "\gamma_1'"] & \\
 & H^0\left(U,\omega^r_{\tilde{X}}\right) \arrow[rr, dotted, "\delta"] \arrow[dl, dotted, "\theta"] & & H^0\left(U,\mathcal{A}\otimes\omega_{\tilde{X}}\right) \arrow[dl, "\alpha"] \\
H^0\left(U,\mathcal{O}^r_{\tilde{X}}\right) \arrow[rr, "\delta'"] & & H^0\left(U,\mathcal{A}\right) &
\end{tikzcd}$$

We need to prove that

$$\operatorname{im}(\gamma_1) \subset \operatorname{im}(\delta) \tag{3.0.2}$$

Notice that the maps α and θ are isomorphisms because the support of $\mathcal{O}_{Z_K}$ does not intersect U. Since α is injective, the condition (3.0.2) is equivalent to

$$\mathrm{Im}(\alpha\gamma_1) \subset \mathrm{Im}(\alpha\delta).$$

Since the diagram is commutative and θ is onto we get

$$\mathrm{im}(\alpha\delta) = \mathrm{im}(\delta'\theta) = \mathrm{im}(\delta')$$

Hence it is enough to prove that the image of $(\alpha\gamma_1)$ is contained in the image of δ'. Using again that the diagram is commutative and ρ is onto because $\mathcal{M}$ is special, we get

$$\mathrm{im}(\alpha\gamma_1) = \mathrm{im}(\gamma_1'\beta) \subset \mathrm{im}(\gamma_1') = \mathrm{im}(\gamma_1'\rho) = \mathrm{im}(\delta'\nu) \subset \mathrm{im}(\delta')$$

Therefore $\mathcal{M}$ is a full sheaf. $\square$

Chapter 4

Computation of the dimension of $R^1\pi_*\mathcal{M}$

Let M be a reflexive module and $\pi\colon \tilde{X} \to X$ be a non-positive resolution with respect to the canonical cycle. Denote by $\mathcal{M}$ the full sheaf associated to M. In this chapter our objective is to compute the dimension as $\mathbb{C}$-vector space of the group $R^1\pi_*\mathcal{M}$.

The formula that we obtain depends on the resolution. In next chapter we will use this formula to construct a special type of resolution where the full sheaf is generated by global sections, later we will see that this property will allow us to obtain new results.

4.1 Dimension of $R^1\pi_*\mathcal{M}$

In this section we work with a non-positive resolution with respect to the canonical cycle and we get a formula for the dimension of the first cohomology group of a full sheaf. We will see that this formula depends on the resolution.

Recall (X,x) denotes a complex analytic germ of a normal two-dimensional Gorenstein singularity, $\pi\colon (\tilde{X},E) \to (X,x)$ denotes a non-positive resolution with respect to the canonical cycle with exceptional divisor $E = \bigcup_{i=1}^{n} E_i$.

4.1.1 Grothendieck's duality

Since the singularity (X,x) is Gorenstein, the ring $\mathcal{O}_X$ is the dualizing module for the singularity ([12, Section 21.3]). In this case the Grothendieck duality for the map π [17, Ch. VII] establish the isomorphism

$$R\pi_* R\,\mathscr{H}om\,(-,\omega_{\tilde{X}}) \cong R\,\mathrm{Hom}_{\mathcal{O}_X}\,(R\pi_*-,\mathcal{O}_X)\,. \tag{4.1.1}$$

In this subsection we use this isomorphism and the Grothendieck spectral sequence in order to get some results about full sheaves and Cohen-Macaulay sheaves of dimension one such that their support is not contained in the exceptional divisor. In the following subsection we will use this computations in a crucial way.

Recall that the dualizing complexes $\omega_{\tilde{X}}$ and $\mathcal{O}_X$ are concetrated in degree -2, and this is reflected in the degrees that we use.

Full sheaves

Let M be a reflexive $\mathcal{O}_X$-module and $\mathcal{M}$ be the full sheaf associated to M. Denote by

$$N = M^{\vee},$$
$$\mathcal{N} = \mathcal{M}^{\vee}.$$

First consider

$$R\pi_* R\,\mathscr{H}om\,(\mathcal{M}, \omega_{\tilde{X}})\,.$$

In order to obtain some information we use the Grothendieck spectral sequence. In this case $E_2^{(p,q)}$ of the spectral sequence is

$$\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \pi_*\left(\mathcal{N} \otimes \omega_{\tilde{X}}\right) & R^1\pi_*\left(\mathcal{N} \otimes \omega_{\tilde{X}}\right) & 0 \\ \hline & -2 & -1 & 0 \end{array} \tag{4.1.2}$$

Notice that this spectral sequence degenerates at the E_2^{**}-term.

Now consider

$$R\,\mathrm{Hom}_{\mathcal{O}_X}\left(R\pi_*\mathcal{M}, \mathcal{O}_X\right)$$

As before we use the Grothendieck spectral sequence to obtain some results. In this case $E_2^{(p,q)}$ of the spectral sequence is

$$\begin{tikzcd} & 0 & 1 & 2 \\ -2 & \mathrm{Hom}_{\mathcal{O}_X}\left(\pi_*\mathcal{M}, \mathcal{O}_X\right) \arrow[drr] & 0 & 0 \\ -3 & 0 & 0 & \mathrm{Ext}^2_{\mathcal{O}_X}\left(R^1\pi_*\mathcal{M}, \mathcal{O}_X\right) \\ -4 & 0 & 0 & 0 \end{tikzcd}$$

In this case we have a differential, therefore we have the exact sequence

$$0 \longrightarrow E_3^{(0,-2)} \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}\left(\pi_*\mathcal{M}, \mathcal{O}_X\right) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}\left(R^1\pi_*\mathcal{M}, \mathcal{O}_X\right) \longrightarrow E_3^{(2,-3)} \longrightarrow 0. \tag{4.1.3}$$

This spectral sequence degenerates at the E_3^{**}-term, therefore we have

$$E_3^{(0,-2)} \cong E_\infty^{(0,-2)},$$
$$E_3^{(2,-3)} \cong E_\infty^{(2,-3)}.$$

Now by Grothendieck duality (4.1.1) and the spectral sequence 4.1.2 we get

$$E_\infty^{(0,-2)} \cong \pi_*\left(\mathcal{N} \otimes \omega_{\tilde{X}}\right),$$
$$E_\infty^{(2,-3)} \cong R^1\pi_*\left(\mathcal{N} \otimes \omega_{\tilde{X}}\right).$$

Since N is isomorphic to $\operatorname{Hom}_{\mathcal{O}_X}\left(\pi_*\mathcal{M}, \mathcal{O}_X\right)$ (Lemma 1.11) and by the previous identifications, we can rewrite the exact sequence (4.1.3) as

$$0 \longrightarrow \pi_*\left(\mathcal{N} \otimes \omega_{\tilde{X}}\right) \longrightarrow N \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}\left(R^1\pi_*\mathcal{M}, \mathcal{O}_X\right) \longrightarrow R^1\pi_*\left(\mathcal{N} \otimes \omega_{\tilde{X}}\right) \longrightarrow 0. \tag{4.1.4}$$

Cohen-Macaulay sheaves of dimension one

Let $\mathcal{A}$ be a Cohen-Maculay sheaf of dimension one such that its support it is not contained in the exceptional divisor.

As before first consider

$$R\pi_* R\,\mathscr{H}\!om\left(\mathcal{A}, \omega_{\tilde{X}}\right).$$

We use the Grothendieck spectal squence in order to obtain some information. In this case $E_2^{(p,q)}$ of the spectral sequence is

$$\begin{array}{c|ccc}
2 & 0 & 0 & 0 \\
1 & \pi_*\mathscr{E}\!xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}, \omega_{\tilde{X}}\right) & R^1\pi_*\mathscr{E}\!xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}, \omega_{\tilde{X}}\right) & 0 \\
0 & 0 & 0 & 0 \\
\hline
 & -2 & -1 & 0
\end{array} \tag{4.1.5}$$

Notice that in this case the spectral sequence degenerates in the second page.

Now consider

$$R\operatorname{Hom}_{\mathcal{O}_X}\left(R\pi_*\mathcal{A}, \mathcal{O}_X\right)$$

As before we use the Grothendieck spectral sequence to obtain some results. In this case $E_2^{(p,q)}$ of the spectral sequence is

$$\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
-2 & 0 & \operatorname{Ext}^1_{\mathcal{O}_X}\left(\pi_*\mathcal{A}, \mathcal{O}_X\right) & \operatorname{Ext}^2_{\mathcal{O}_X}\left(\pi_*\mathcal{A}, \mathcal{O}_X\right) \\
-3 & 0 & 0 & \operatorname{Ext}^2_{\mathcal{O}_X}\left(R^1\pi_*\mathcal{A}, \mathcal{O}_X\right) \\
-4 & 0 & 0 & 0
\end{array} \tag{4.1.6}$$

In this case the spectral sequence again degenerates in the second page. This spectral sequence give us the exact sequence

$$0 \longrightarrow \operatorname{Ext}^2_{\mathcal{O}_X}\left(R^1\pi_*\mathcal{A}, \mathcal{O}_X\right) \longrightarrow H^{-1} \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}\left(\pi_*\mathcal{A}, \mathcal{O}_X\right) \longrightarrow 0. \tag{4.1.7}$$

The Grothendieck duality (4.1.1) allow us to compare the spectral sequences (4.1.5) and (4.1.6). In this case taking in account the degrees we get

$$\begin{aligned} \operatorname{Ext}^2_{\mathcal{O}_X}\left(\pi_*\mathcal{A}, \mathcal{O}_X\right) &= 0, \\ R^1\pi_* \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}, \omega_{\tilde{X}}\right) &= 0, \\ \pi_* \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}, \omega_{\tilde{X}}\right) &\cong H^{-1}. \end{aligned}$$

Using the previous identification we can rewrite the exact sequence (4.1.7) as follows

$$0 \longrightarrow \operatorname{Ext}^2_{\mathcal{O}_X}\left(R^1\pi_*\mathcal{A}, \mathcal{O}_X\right) \longrightarrow \pi_* \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}, \omega_{\tilde{X}}\right) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}\left(\pi_*\mathcal{A}, \mathcal{O}_X\right) \longrightarrow 0. \tag{4.1.8}$$

4.1.2 Computation of $R^1\pi_*\mathcal{M}$

Recall (X, x) denotes a complex analytic germ of a normal two-dimensional Gorenstein singularity, $\pi\colon (\tilde{X}, E) \to (X, x)$ denotes a non-positive resolution with respect to the canonical cycle with exceptional divisor $E = \bigcup_{i=1}^n E_i$.

The main objective of this section is to prove to following theorem.

Theorem 4.1. *Let M be a reflexive $\mathcal{O}_X$-module and $\mathcal{M}$ be the full sheaf associated to M. Assume that $\mathcal{M}$ has rank r and specialty defect equal to d. Then*

$$\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right) = rp_g - [c_1(\mathcal{M})] \cdot [Z_k] + d.$$

This theorem will be very important in the following section. It will allow us to prove that a full special sheaf is determined by its first Chern class in a specific resolution.

In order to prove the Theorem 4.1 we need to use the direct correspondence given in Subsection 2.2.1, the results of the previous section and some preliminary work.

Let M be a reflexive $\mathcal{O}_X$-module and $\mathcal{M}$ be the full sheaf associated to M. Assume that $\mathcal{M}$ has rank r and specialty defect equal to d. Take r generic sections and consider the exact sequence obtained by the sections

$$0 \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

and its dual

$$0 \to \mathcal{N} \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{A} \to 0, \tag{2.2.11}$$

where $\mathcal{A} = \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}', \mathcal{O}_{\tilde{X}}\right)$.

By the direct correspondence we know that $\mathcal{A}'$ is a Cohen-Macaulay sheaf of dimension one such that its support it is not contained in the exceptional divisor. By Theorem 1.4 $\mathcal{A}'$ is isomorphic to $\mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}', \mathcal{O}_{\tilde{X}}\right), \mathcal{O}_{\tilde{X}}\right)$, hence we can write

$$\mathcal{A}' \cong \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}, \mathcal{O}_{\tilde{X}}\right). \tag{4.1.9}$$

Applying the functor π_*- to the exact sequence (2.2.11) we get

$$0 \longrightarrow N \longrightarrow \mathcal{O}_X^r \longrightarrow \pi_*\mathcal{A} \longrightarrow R^1\pi_*\mathcal{N} \longrightarrow R^1\pi_*\mathcal{O}_{\tilde{X}}^r \longrightarrow 0,$$

where N is $\pi_*\mathcal{N}$ and by Lemma 1.11 the module N is equal to the module $M^{\vee}$.

The last exact sequence can be split as follows

$$\begin{aligned}
0 &\longrightarrow N \longrightarrow \mathcal{O}_X^r \longrightarrow \mathcal{C} \longrightarrow 0,\\
0 &\longrightarrow \mathcal{C} \longrightarrow \pi_*\mathcal{A} \longrightarrow \mathcal{D} \longrightarrow 0,\\
0 &\longrightarrow \mathcal{D} \longrightarrow R^1\pi_*\mathcal{N} \longrightarrow R^1\pi_*\mathcal{O}_{\tilde{X}}^r \longrightarrow 0,
\end{aligned}$$

where lenght$(\mathcal{D}) = d$.

Dualizing the first and second exact sequence we obtain

$$0 \longrightarrow \mathcal{O}_X^r \longrightarrow M \xrightarrow{h} \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X) \longrightarrow 0, \tag{4.1.10}$$

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(\pi_*\mathcal{A}, \mathcal{O}_X) \xrightarrow{i} \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X) \longrightarrow \operatorname{Ext}^2_{\mathcal{O}_X}(\mathcal{D}, \mathcal{O}_X) \longrightarrow 0. \tag{4.1.11}$$

Applying the functor π_*- to the exact sequence (2.2.1) we obtain

$$0 \longrightarrow \mathcal{O}_X^r \longrightarrow M \longrightarrow \pi_*\mathcal{A}' \longrightarrow R^1\pi_*\mathcal{O}_{\tilde{X}}^r \longrightarrow R^1\pi_*\mathcal{M} \longrightarrow 0.$$

Use the identification (4.1.9) and compare the previous exact sequence with the exact sequence (4.1.10)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^r \arrow[r] \arrow[d, "Id"] & M \arrow[r, "h"] \arrow[d, "Id"] & \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X) \arrow[r] \arrow[d, "\theta"] & 0 & & \\
0 \arrow[r] & \mathcal{O}_X^r \arrow[r] & M \arrow[r, "h"] & \pi_* \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) \arrow[r] & R^1\pi_*\mathcal{O}_{\tilde{X}}^r \arrow[r] & R^1\pi_*\mathcal{M} \arrow[r] & 0
\end{tikzcd} \tag{4.1.12}$$

where Id is the identity and θ is the map that makes the diagram commute.

Since $\operatorname{Ext}^2_{\mathcal{O}_X}\left(R^1\pi_*\mathcal{A}, \mathcal{O}_X\right)$ is zero, we have that the exact sequence (4.1.8) give us

$$\pi_* \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \omega_{\tilde{X}}) \cong \operatorname{Ext}^1_{\mathcal{O}_X}(\pi_*\mathcal{A}, \mathcal{O}_X), \tag{4.1.13}$$

denote by $\mathfrak{g}$ the last isomorphism.

Now consider the exact sequence

$$0 \longrightarrow \omega_{\tilde{X}} \xrightarrow{c} \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{Z_K} \longrightarrow 0. \tag{3.0.1}$$

Applying the functor $\pi\left(\mathscr{E}xt_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},-)\right)$ to map

$$\omega_{\tilde{X}} \xrightarrow{c} \mathcal{O}_{\tilde{X}},$$

give us the map

$$\pi\left(\mathscr{E}xt_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},-)\right)(c)\colon \pi\left(\mathscr{E}xt_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}})\right) \to \pi\left(\mathscr{E}xt_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{\tilde{X}})\right). \tag{4.1.14}$$

Let us denote by c to the previous map.

Now using (4.1.14),(4.1.11), (4.1.12) and (4.1.13) we have the following maps

$$\begin{tikzcd}
\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}}) \arrow[r, "\mathfrak{g}"] \arrow[d, "Id"] & \operatorname{Ext}^1_{\mathcal{O}_X}(\pi_*\mathcal{A},\mathcal{O}_X) \arrow[r, "i"] & \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C},\mathcal{O}_X) \arrow[r, "\theta"] & \pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{\tilde{X}}) \arrow[d, "Id"] \\
\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}}) \arrow[rrr, "c"] & & & \pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{\tilde{X}})
\end{tikzcd} \tag{4.1.15}$$

Denote by $\mathfrak{c}$ the map given by the composition $\theta \circ i \circ \mathfrak{g}$.

Lemma 4.2. *The maps c and $\mathfrak{c}$ coincide.*

Proof. Consider the following map

$$f := (c - \mathfrak{c})\colon \pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}}) \to \pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{\tilde{X}})$$

Since the map $\pi\colon \tilde{X} \to X$ is an isomorphism outside the exceptional divisor, we have that for any section s of $\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}})$, the section $f(s)$ is supported in the exceptional divisor, hence $f(s) \in H^0_E\left(\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{\tilde{X}})\right)$ but this cohomology group is zero.

Therefore for any section s of $\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}})$ we have that $f(s) = 0$ which it is equivalent to say that the maps c and $\mathfrak{c}$ coincide. □

By the previous lemma we get that the diagram (4.1.15) commutes.

The previous work allows us to prove the following proposition.

Proposition 4.3. *Let $\mathcal{M}$ be a full sheaf of rank r with specialty defect equal to d. Take r generic sections and consider the exact sequence obtained by the sections*

$$0 \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{M} \to \mathcal{A}' \to 0, \tag{2.2.1}$$

and its dual

$$0 \to \mathcal{N} \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{A} \to 0, \tag{2.2.11}$$

where $\mathcal{A} = \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}',\mathcal{O}_{\tilde{X}})$. Then

$$\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right) = rp_g - \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{Z_k})\right) + d$$

Proof.

Let $\mathcal{M}$ be a full sheaf of rank r with specialty defect equal to d. Take r generic sections and consider the exact sequences (2.2.1) and (2.2.11).

Applying the functor $\mathscr{H}om(-,-)$ to the exact sequences (2.2.11) and (3.0.1) we get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \omega^r_{\tilde{X}} \arrow[r] \arrow[d] & \mathcal{M}\otimes\omega_{\tilde{X}} \arrow[r] \arrow[d] & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}}) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathcal{O}^r_{\tilde{X}} \arrow[r] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d] & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{\tilde{X}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}^r_{Z_k} \arrow[r] \arrow[d] & \mathcal{M}\otimes\mathcal{O}_{Z_k} \arrow[r] \arrow[d] & \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{Z_k}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Applying the functor π_*- to the last commutative diagram we get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & \pi_*\omega^r_{\tilde{X}} \arrow[r] \arrow[d] & \pi_*\left(\mathcal{M}\otimes\omega_{\tilde{X}}\right) \arrow[r] \arrow[d] & \pi_*\,\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}}) \arrow[r] \arrow[d, "c"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{O}^r_X \arrow[r] \arrow[d] & M \arrow[r, "h"] \arrow[d] & \pi_*\,\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{\tilde{X}}) \arrow[r, "\alpha"] \arrow[d] & R^1\pi_*\mathcal{O}^r_{\tilde{X}} \arrow[r] \arrow[d] & R^1\pi_*\mathcal{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_*\mathcal{O}^r_{Z_k} \arrow[r] \arrow[uurrr, dotted, bend right] & \pi_*\left(\mathcal{M}\otimes\mathcal{O}_{Z_k}\right) \arrow[r] \arrow[uurrr, dotted, bend right] & \pi_*\,\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{Z_k}) \arrow[r] \arrow[uurrr, dotted, bend right] & 0 & 0 &
\end{tikzcd}$$

By this diagram we get

$$\begin{aligned}
\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right) &= rp_g - \dim_{\mathbb{C}}\left(\mathrm{Im}(\alpha)\right), \\
\dim_{\mathbb{C}}\left(\pi_*\,\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{Z_k})\right) &= \dim_{\mathbb{C}}\left(\pi_*\,\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\mathcal{O}_{\tilde{X}})/\pi_*\,\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A},\omega_{\tilde{X}})\right),
\end{aligned} \tag{4.1.16}$$

and

$$\dim_{\mathbb{C}}(\operatorname{Im}\alpha) = \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \ker\alpha\right) = \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \operatorname{Im} h\right). \qquad (4.1.17)$$

Now by (4.1.12) we have

$$\operatorname{Im} h = \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X).$$

Hence by the previous equality, (4.1.17), (4.1.11) and (4.1.13) we get

$$\begin{aligned} \dim_{\mathbb{C}}(\operatorname{Im}\alpha) &= \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \operatorname{Im} h\right) \\ &= \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_X)\right) \\ &= \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \operatorname{Ext}^1_{\mathcal{O}_X}(\pi_*\mathcal{A}, \mathcal{O}_X)\right) - d \\ &= \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \omega_{\tilde{X}})\right) - d. \end{aligned} \qquad (4.1.18)$$

Let c and $\mathfrak{c}$ be the morphisms given in the diagram (4.1.15). Now by (4.1.18) and by Lemma 4.2 we have

$$\begin{aligned} \dim_{\mathbb{C}}(\operatorname{Im}\alpha) &= \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \omega_{\tilde{X}})\right) - d \\ &= \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \operatorname{Im}\mathfrak{c}\right) - d \\ &= \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \operatorname{Im} c\right) - d \\ &= \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{\tilde{X}}) / \pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \omega_{\tilde{X}})\right) - d. \end{aligned} \qquad (4.1.19)$$

Therefore by (4.1.16) and (4.1.19) we get

$$\dim_{\mathbb{C}}(\mathcal{M}) = rp_g - \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{Z_k})\right) + d.$$

□

By the Proposition 4.3 to prove Theorem 4.1 it is enough to prove that

$$\dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}, \mathcal{O}_{Z_k})\right) = [c_1(\mathcal{M})] \cdot [Z_k].$$

The following lemma gives us the previous equality.

Lemma 4.4. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two Cohen-Macaulay sheaves of dimension one such that $\mathcal{A}_1$ is contained in $\mathcal{A}_2$, the dimension as $\mathbb{C}$-vector space of $\mathcal{A}_2/\mathcal{A}_1$ is finite and the support of each sheaf intersects the exceptional divisor in the same point p. Then*

$$\dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_1, \mathcal{O}_{Z_K})\right) = \dim_{\mathbb{C}}\left(\pi_* \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2, \mathcal{O}_{Z_K})\right).$$

Proof. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ as in the statement.

By the hypothesis we have the exact sequence

$$0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_2/\mathcal{A}_1 \to 0.$$

Applying the functor $\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(-,\mathcal{O}_{Z_K})$ to the last exact sequence we get

$$\begin{aligned} 0 &\longrightarrow \mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K}) \longrightarrow \mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2,\mathcal{O}_{Z_K}) \longrightarrow \mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_1,\mathcal{O}_{Z_K}) \\ &\longrightarrow \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K}) \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2,\mathcal{O}_{Z_K}) \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_1,\mathcal{O}_{Z_K}) \\ &\longrightarrow \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K}) \longrightarrow \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2,\mathcal{O}_{Z_K}) \longrightarrow \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_1,\mathcal{O}_{Z_K}) \longrightarrow 0 \end{aligned} \tag{4.1.20}$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are Cohen-Macaulay sheaves of dimension one and the support of each sheaf intersects the exceptional divisor in a point p we have

$$\begin{aligned} \mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2,\mathcal{O}_{Z_K}) = \mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_1,\mathcal{O}_{Z_K}) = 0, \\ \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2,\mathcal{O}_{Z_K}) = \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_1,\mathcal{O}_{Z_K}) = 0. \end{aligned} \tag{4.1.21}$$

Since all the sheaves in (4.1.20) are supported in the point p we can work locally, therefore we assume that $\mathcal{O}_{\tilde{X}}$ is $\mathbb{C}[x,y]$ and Z_K is $\mathcal{O}_{\tilde{X}}/(f)$ for some function f.

Now by (4.1.20) and (4.1.21) we just need to prove the following equality

$$\dim_{\mathbb{C}}\left(\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K})\right) = \dim_{\mathbb{C}}\left(\mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K})\right). \tag{4.1.22}$$

Consider the following resolution of $\mathcal{O}_{Z_K}$

$$0 \longrightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{\cdot f} \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{Z_K} \longrightarrow 0. \tag{4.1.23}$$

Applying the functor $\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,-)$ to the last exact sequence we get

$$\begin{aligned} 0 &\longrightarrow \mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) \longrightarrow \mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) \longrightarrow \mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K}) \\ &\longrightarrow \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K}) \\ &\longrightarrow \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) \longrightarrow \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) \longrightarrow \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K}) \longrightarrow 0 \end{aligned} \tag{4.1.24}$$

Now since the support of $\mathcal{A}_2/\mathcal{A}_1$ is zero dimensional, we have by Theorem 1.4

$$\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) = 0$$

By the previous equality and the exact sequence (4.1.24) we get

$$0 \to \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K}) \to \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) \to \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}}) \to \mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K}) \to 0$$

By this exact sequence we get

$$\dim_{\mathbb{C}}\left(\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K})\right) - \dim_{\mathbb{C}}\left(\mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}})\right) + \dim_{\mathbb{C}}\left(\mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{\tilde{X}})\right) - \dim_{\mathbb{C}}\left(\mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}_2/\mathcal{A}_1,\mathcal{O}_{Z_K})\right) = 0,$$

therefore

$$\dim_{\mathbb{C}}\left(\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}_2/\mathcal{A}_1, \mathcal{O}_{Z_K}\right)\right) = \dim_{\mathbb{C}}\left(\mathscr{E}xt^2_{\mathcal{O}_{\tilde{X}}}\left(\mathcal{A}_2/\mathcal{A}_1, \mathcal{O}_{Z_K}\right)\right).$$

$\square$

Finally Theorem 4.1 follows from Proposition 4.3 and Lemma 4.4.

Chapter 5

The minimal adapted resolution

Let M be a reflexive $\mathcal{O}_X$-module. In this chapter we use the Theorem 4.1 in order to construct a special resolution where the full sheaf associated to M is generated by global sections. Working in this resolution we can study how the specialty defect behaves under the blow up of a point in the resolution, also this resolution allow us to start to study the deformation theory of the full sheaf as a locally free sheaf.

It is important to say that this resolution will be very important in following chapter.

5.1 The minimal adapted resolution to M

Let M be a reflexive $\mathcal{O}_X$-module. In this subsection we give a resolution $\pi\colon \tilde{X} \to X$, where the full sheaf $\mathcal{M}$ associated to M is generated by global sections. This resolution captures the information about the dimension of $R^1\pi_*\mathcal{M}$ and the failure of $\mathcal{M}$ of being globally generated.

Recall that (X,x) is a complex analytic germ of a normal two-dimensional Gorenstein singularity, $\pi\colon (\tilde{X},E) \to (X,x)$ is a non-positive resolution with respect to the canonical cycle with exceptional divisor $E = \bigcup_{i=1}^n E_i$ and the geometric genus is p_g.

By Theorem 4.1 it is clear that the dimension of $R^1\pi_*\mathcal{M}$ has a relation with the number $[c_1(\mathcal{M})]\cdot Z_K$. In the following lemma we make clear this relation and how it behaves with the property of being globally generated.

Lemma 5.1. *Let M be a reflexive $\mathcal{O}_X$-module and $\pi\colon \tilde{X} \to X$ be a non-positive resolution with respect to the canonical cycle. Denote by $\mathcal{M}$ the full sheaf associated to M and suppose that $\mathcal{M}$ has rank equal to r and specialty defect equal to d. Denote by $c_1(\mathcal{M})$ the first Chern class of $\mathcal{M}$. Then the following two conditions are equivalent:*

1. $\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right) = rp_g$.
2. $[c_1(\mathcal{M})]\cdot[Z_K] = d$.

Moreover, any of the previous conditions implies the following

3. *The sheaf $\mathcal{M}$ is generated by global sections.*

Proof. Let $\mathcal{M}$ be a full sheaf of rank r with specialty defect equal to d. By Theorem 4.1 we have

$$\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right) = rp_g - [c_1(\mathcal{M})]\cdot[Z_k] + d,$$

therefore the first two conditions are equivalent.

Now assume that the dimension as $\mathbb{C}$-vector space of $R^1\pi_*\mathcal{M}$ is rp_g. Take r generic sections of $\mathcal{M}$ and consider the exact sequence obtained by the sections

$$0 \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Applying the functor π_*- to the last exact sequence, we get

$$0 \longrightarrow \mathcal{O}^r_X \longrightarrow M \longrightarrow \pi_*\mathcal{A}' \longrightarrow R^1\pi_*\mathcal{O}^r_{\tilde{X}} \longrightarrow R^1\pi_*\mathcal{M} \longrightarrow 0.$$

By the assumption and the last exact sequence we get

$$0 \longrightarrow \mathcal{O}^r_X \longrightarrow M \longrightarrow \pi_*\mathcal{A}' \longrightarrow 0. \tag{5.1.1}$$

Since the support of $\mathcal{A}'$ intersects the exceptional divisor in a finite set, we get that $\mathcal{A}'$ is generated by global sections. Now by the exact sequence (5.1.1) and since $\mathcal{A}'$ is generated by global sections, we get that $\mathcal{M}$ is generated by global sections. $\square$

Let $\mathcal{M}$ be a full sheaf over $\tilde{X}$. By Theorem 4.1 we know that the specialty defect of $\mathcal{M}$ plays an important role when we study the dimension of $\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right)$. The following lemma gives us some information about how the specialty defect behaves under the blow up of a point.

Lemma 5.2. *Let M be a reflexive $\mathcal{O}_X$-module. Let $\pi\colon \tilde{X} \to X$ be a resolution and p be a point in $\tilde{X}$. Denote by $\sigma\colon \tilde{X}' \to \tilde{X}$ the blow up of the point p, therefore we have the following diagram*

$$\begin{tikzcd} \tilde{X}' \arrow[r, "\sigma"] \arrow[dr, "\rho"] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

where $\rho := \pi\circ\sigma$.

Denote by $\mathcal{M} = (\pi^*M)^{\vee\vee}$ and $\mathcal{M}' = (\rho^*M)^{\vee\vee}$. Then the specialty defect of $\mathcal{M}$ is less or equal to the specialty defect of $\mathcal{M}'$.

Proof. Denote by

$$\mathcal{N}' = \mathcal{M}'^{\vee},$$
$$\mathcal{N} = \mathcal{M}^{\vee}.$$

Since $\rho = \pi\circ\sigma$, in order to compute $R^1\rho_*\mathcal{N}'$ we use the Leray spectral sequence. In this case the page $E_2^{(p,q)}$ of the spectral sequence is given by

$$
\begin{array}{c|ccc}
2 & 0 & 0 & 0 \\
1 & \pi_*\left(R^1\sigma_*\mathcal{N}'\right) & R^1\pi_*\left(R^1\sigma_*\mathcal{N}'\right) & 0 \\
0 & \pi_*\left(\sigma_*\mathcal{N}'\right) & R^1\pi_*\left(\sigma_*\mathcal{N}'\right) & 0 \\
\hline
 & 0 & 1 & 2
\end{array}
$$

The spectral sequence degenerates, therefore we obtain the following exact sequence

$$0 \to R^1\pi_*\left(\sigma_*\mathcal{N}'\right) \to R^1\rho_*\mathcal{N}' \to \pi_*\left(R^1\sigma_*\mathcal{N}'\right) \to 0. \tag{5.1.2}$$

Now by adjuction we have the following identification

$$\begin{aligned}
R^1\pi_*\left(\sigma_*\mathcal{N}'\right) &= R^1\pi_*\left(\sigma_*\mathscr{H}om_{\mathcal{O}_{\tilde{X}'}}\left(\sigma^*\pi^*M, \mathcal{O}_{\tilde{X}'}\right)\right) \\
&= R^1\pi_*\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}\left(\pi^*M, \sigma_*\mathcal{O}_{\tilde{X}'}\right) \\
&= R^1\pi_*\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}\left(\pi^*M, \mathcal{O}_{\tilde{X}}\right) \\
&= R^1\pi_*\mathcal{N}
\end{aligned} \tag{5.1.3}$$

By (5.1.2) and (5.1.3) we get

$$0 \to R^1\pi_*\mathcal{N} \to R^1\rho_*\mathcal{N}' \to \pi_*\left(R^1\sigma_*\mathcal{N}'\right) \to 0. \tag{5.1.4}$$

Therefore the specialty defect of $\mathcal{M}$ is less or equal to the specialty defect of $\mathcal{M}'$. □

The previous lemmas allow us to give a resolution where the full sheaf associated to M is generated by global sections. This is done in the following proposition.

Proposition 5.3. *If M is a reflexive $\mathcal{O}_X$-module, then there exists a unique minimal resolution $\rho\colon \tilde{X}' \to X$ such that $\mathcal{M}' := (\rho^*M)^{\vee\vee}$ is generated by global sections.*

Proof. Let M be a reflexive $\mathcal{O}_X$-module, $\pi\colon \tilde{X} \to X$ be the minimal resolution with exceptional divisor E and denote by $\mathcal{M} = (\pi^*M)^{\vee\vee}$. If $\mathcal{M}$ is generated by global sections, then we are done.

If $\mathcal{M}$ is not generated by global sections, then there exists a finite set of points $S = \{p_1, \dots, p_n\} \subset E$ where $\mathcal{M}$ fails to be generated by global sections.

Assume that the rank of $\mathcal{M}$ is r. Take r generic sections of $\mathcal{M}$ and consider the exact sequence given by the sections

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0 \tag{2.2.1}$$

where $S \subset \mathrm{Supp}(\mathcal{A}')$ (see the direct correspondence in Subsection 2.2.1).

Let $\mathfrak{E}$ be the union of the components of the exceptional divisor where the Gorensteim form has poles.

Denote by $\sigma_{S\cap\mathfrak{E}}\colon \tilde{X}' \to X$ the blow up at the points $S \cap \mathfrak{E}$. Therefore we have the following diagram

$$\begin{tikzcd} \tilde{X}' \arrow[r, "\sigma_{S\cap\mathfrak{E}}"] \arrow[dr, "\rho"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

where $\rho = \pi \circ \sigma_{S\cap\mathfrak{E}}$.

By Lemma 2.21 we know that the resolution $\rho\colon \tilde{X}' \to X$ is non-positive with respect to the canonical cycle. Denote by $\mathcal{M}_{S\cap\mathfrak{E}} = (\rho^* M)^{\vee\vee}$ the full sheaf associated to M. If $\mathcal{M}_{S\cap\mathfrak{E}}$ is generated by global sections, then we are done.

If $\mathcal{M}_{S\cap\mathfrak{E}}$ is not generated by global sections, then we repeat the process.

Since each blow up give us a non-positive resolution with respect to the canonical cycle, we can use Theorem 4.1.

Recall that the dimension of $R^1\pi_*\mathcal{M}$ as a $\mathbb{C}$-vector space always is less or equal to rp_g (see for example (2.2.12)). Now in each blow up the number $[\mathcal{A}'] \cdot [Z_K]$ decrease and by Lemma 5.2 the specialty defect does not decrease. Therefore at some moment we will get two cases:

1. The number $[\mathcal{A}'] \cdot [Z_K]$ is equal to d, then by Lemma 5.1 we can guarantee that this process will finish.

2. We have a point p in a component E_i of the exceptional divisor where the full sheaf fails to be globally generated and $\mathcal{O}_{Z_K}$ is zero in the component E_i. Let $\sigma_p\colon \tilde{X}' \to \tilde{X}$ be the blow up at the point p with exceptional divisor E'. Therefore we have the following diagram

$$\begin{tikzcd} \tilde{X}' \arrow[r, "\sigma_p"] \arrow[dr, "\rho"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

where $\rho = \pi \circ \sigma_p$.

Denote by $\mathcal{M}'$ the full sheaf associated to M in the resolution $\tilde{X}'$ and by r the rank of $\mathcal{M}'$. By our direct correspondence (Proposition 2.10) we know that taking r generic sections we get the exact sequence

$$0 \to \mathcal{O}^r_{\tilde{X}'} \to \mathcal{M}' \to \mathcal{A}'_0 \to 0, \tag{2.2.1}$$

and its dual is

$$0 \to \mathcal{N}' \to \mathcal{O}^r_{\tilde{X}'} \to \mathcal{A}_0 \to 0, \tag{2.2.8}$$

where $\mathcal{N}' := \mathcal{M}'^{\vee}$, $\mathcal{A}_0 := \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}'}}(\mathcal{A}'_0, \mathcal{O}_{\tilde{X}'})$ and $\mathcal{A}_0$ is a Cohen-Macaulay sheaf of dimension one. Denote by D the support of $\mathcal{A}_0$.

Applying the functor ρ_*- to the exact sequence (2.2.8) we get

$$0 \to N' \to \mathcal{O}^r_X \to \rho_*\mathcal{A}_0 \to R^1\rho_*\mathcal{N}' \to R^1\rho_*\mathcal{O}^r_{\tilde{X}'} \to 0. \tag{2.2.9}$$

Denote by $\mathcal{C}$ the image of $\mathcal{O}^r_X$.

Since we took the blow up in a point where the full sheaf failed to be globally generated we get that D intersects the exceptional divisor E'. Therefore we can use our inverse correspondence (Proposition 2.16) with the curve D, the module $\mathcal{C}$ and the same generators given by (2.2.9), this correspondence gives us the full sheaf $\mathcal{M}'$ but by Corollary 2.22 we know that $\mathcal{M}'$ is not a full sheaf, which is a contradiction.

Hence this case does not happen.

□

Remark 5.4. It is important to notice that the process given in the proof of the last proposition could finish and $[\mathcal{A}'] \cdot [Z_K] \neq d$.

The Proposition 5.3 allows us to give the following definition.

Definition 5.5. Let M be a reflexive $\mathcal{O}_X$-module. The resolution given by Proposition 5.3 is called *the minimal resolution adapted to M.*

Given M a reflexive $\mathcal{O}_X$-module, the minimal resolution adapted to M allows us to obtain new results. In the following sections we study the relation between the minimal resolution adapted to M and the properties of the full sheaf associated to $\mathcal{M}$.

5.2 Specialty Lemma

Let M be a reflexive $\mathcal{O}_X$-module and consider $\pi\colon (\tilde{X}, E) \to (X, x)$ the minimal resolution adapted to M. In this subsection we study how the specialty defect of the full sheaf associated to M behaves under the blow up of a point of the minimal adapted resolution. Later in this thesis this result will be important.

First we have a general result.

Proposition 5.6. *Let M be a reflexive $\mathcal{O}_X$-module and $\pi\colon \tilde{X} \to X$ be resolution with E the exceptional divisor. Let p be a point in E such that $\mathcal{M}$ is generated by global sections in the point p. Denote by $\sigma\colon \tilde{X}' \to \tilde{X}$ the blow up of the point p and E' the exceptional divisor of σ, therefore we have the following diagram*

$$\begin{tikzcd} \tilde{X}' \arrow[r, "\sigma"] \arrow[dr, "\rho"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

where $\rho := \pi \circ \sigma$.

Denote by $\mathcal{M} := (\pi^* M)^{\vee\vee}$ and $\mathcal{M}' := (\rho^* M)^{\vee\vee}$. Then the specialty defect of $\mathcal{M}$ and $\mathcal{M}'$ concide.

PROOF. Denote by

$$\mathcal{N}' = \mathcal{M}'^{\vee},$$
$$\mathcal{N} = \mathcal{M}^{\vee}.$$

By exactly the same arguments as in the proof of Lemma 5.2, we get the exact sequence

$$0 \to R^1\pi_*\mathcal{N} \to R^1\rho_*\mathcal{N}' \to \pi_*\left(R^1\sigma_*\mathcal{N}'\right) \to 0. \tag{5.1.4}$$

Therefore we need to prove that $\pi_*\left(R^1\sigma_*\mathcal{N}'\right) = 0$.

Now consider the exact sequence given by the natural map from π^*M to its double dual

$$0 \longrightarrow T \longrightarrow \pi^*M \longrightarrow \mathcal{M} \longrightarrow \mathcal{S} \longrightarrow 0,$$

where T is the kernel and $\mathcal{S}$ is the cokernel. Notice that the support of $\mathcal{S}$ is the set S.

The last exact sequence can be split as follows

$$0 \longrightarrow T \longrightarrow \pi^*M \longrightarrow \pi^*M/T \longrightarrow 0,$$
$$0 \longrightarrow \pi^*M/T \longrightarrow \mathcal{M} \longrightarrow \mathcal{S} \longrightarrow 0.$$

Applying the functor σ^*- to the last two exact sequences we obtain

$$0 \longrightarrow K_1 \longrightarrow \sigma^*T \longrightarrow \rho^*M \longrightarrow \sigma^*\pi^*M/T \longrightarrow 0,$$
$$0 \longrightarrow K_2 \longrightarrow \sigma^*\pi^*M/T \longrightarrow \sigma^*\mathcal{M} \longrightarrow \sigma^*\mathcal{S} \longrightarrow 0,$$

where K_1 and K_2 are the modules that make the last sequences exact. Remember that σ^*- is just a right exact functor.

Hence we split the previous exact sequences as follows

$$0 \longrightarrow K_1 \longrightarrow \sigma^*T \longrightarrow H_1 \longrightarrow 0,$$
$$0 \longrightarrow H_1 \longrightarrow \rho^*M \longrightarrow \sigma^*\pi^*M/T \longrightarrow 0,$$
$$0 \longrightarrow K_2 \longrightarrow \sigma^*\pi^*M/T \longrightarrow H_2 \longrightarrow 0,$$
$$0 \longrightarrow H_2 \longrightarrow \sigma^*\mathcal{M} \longrightarrow \sigma^*\mathcal{S} \longrightarrow 0.$$

Dualizing the first, second and third exact sequences we get

$$\begin{aligned} H_1^\vee &\cong 0, \quad \text{because } \sigma^*T \text{ is supported in the exceptional divisor,}\\ (\sigma^*\pi^*M/T)^\vee &\cong (\rho^*M)^\vee, \quad \text{by the previous identification,}\\ H_2^\vee &\cong (\sigma^*\pi^*M/T)^\vee, \quad \text{because } K_2 \text{ is supported in the exceptional divisor.} \end{aligned}$$

Hence as $\mathcal{N}' = \mathcal{M}'^\vee \cong (\rho^*M)^\vee$ we get $\mathcal{N}' \cong (\sigma^*\pi^*M/T)^\vee$.

Finally dualizing the fourth exact sequence and using the previous identifications we get

$$0 \longrightarrow (\sigma^*\mathcal{M})^\vee \longrightarrow \mathcal{N}' \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}'}}(\sigma^*\mathcal{S}, \mathcal{O}_{\tilde{X}'}) \longrightarrow 0.$$

Since the point p does not belong to the support of $\mathcal{S}$, we get that the support of $\sigma^*\mathcal{S}$ is zero dimensional, therefore $\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}'}}(\sigma^*\mathcal{S}, \mathcal{O}_{\tilde{X}'})$ is equal to zero. Hence we get

$$R^1\sigma_*\left(\sigma^*\mathcal{M}\right)^\vee \cong R^1\sigma_*\mathcal{N}'.$$

Since $\mathcal{M}$ is locally free and we obtain $\tilde{X}'$ taking the blow up in the point p we get

$$R^1\sigma_*\left(\left(\sigma^*\mathcal{M}\right)^\vee\right) = R^1\sigma_*\left(\sigma^*\left(\mathcal{O}^r_{\tilde{X}}\right)^\vee\right) = R^1\sigma_*\mathcal{O}^r_{\tilde{X}'} = 0.$$

Hence $R^1\sigma_*\mathcal{N}'$ is equal to zero. □

Now we understand better how the specialty defect behaves. The following two results give us more information on the specialty.

Corollary 5.7. *Let M be a reflexive $\mathcal{O}_X$-module. Denote by $\pi\colon \tilde{X} \to X$ the minimal resolution adapted to M and $\mathcal{M} = (\pi^*M)^{\vee\vee}$ the full associated to M. If $\mathcal{M}$ is special, then the full sheaf associated to M in the minimal resolution is special.*

Proof. Let $\pi_{\min}\colon \tilde{X}_{\min} \to X$ be the minimal resolution of X. Let d_0 be the specialty defect of the full sheaf associated to M in the minimal resolution and d be the specialty defect of $\mathcal{M}$. By Lemma 5.2 we have that $d_0 \leq d$ and by hypothesis d is equal to zero, therefore d_0 is equal to zero. □

Theorem 5.8. *Let M be a reflexive $\mathcal{O}_X$-module. Denote by $\pi\colon \tilde{X} \to X$ the minimal resolution adapted to M and $\mathcal{M} = (\pi^*M)^{\vee\vee}$ the full associated to M. If $\mathcal{M}$ is special, then M is a special reflexive module.*

Proof. We need to prove that for any resolution $\rho\colon \hat{X} \to X$, the full sheaf $\hat{\mathcal{M}} = (\rho^*M)^{\vee\vee}$ is special (Definition 1.14).

Let $\pi_{\min}\colon \tilde{X}_{\min} \to X$ be the minimal resolution of X. If the minimal resolution coincides with the minimal resolution adapted to M, then by Proposition 5.6 we are done.

Suppose that the minimal resolution and the minimal resolution adapted to M do not coincide. Let $\rho\colon \hat{X} \to X$ be a resolution. By taking a finite succession of blowing ups in different points we obtain a resolution $\breve{\rho}\colon \breve{X} \to X$ such that it satisfies the following diagram

$$\begin{tikzcd} \breve{X} \arrow[r, "o"] \arrow[d, "\nu"] & \tilde{X} \arrow[d, "\pi"] \\ \hat{X} \arrow[r, "\rho"] & \tilde{X}_{min} \end{tikzcd}$$

where ν and o are a composition of blowings up in points and $\breve{\rho} = \rho \circ \nu$.

By Proposition 5.6 the full sheaf associated to M in the resolution $\breve{X}$ is special. By Lemma 5.2 the specialty defect of the full sheaf associated to M in the resolution $\hat{X}$ is less or equal to zero, hence M is special. □

Corollary 5.9. *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Then there exist special reflexives modules.*

Proof. Let $\pi\colon (\tilde{X}, E) \to (X, x)$ be the minimal resolution of (X, x) with exceptional divisor $E = \bigcup_{i=1}^{n} E_i$

Suppose that there exists a component E_j such the canonical cycle is zero in this component. Let D be a smooth curve transverse to the exceptional divisor E_j. By Proposition 3.2 taking $\mathcal{C} = \pi_* \mathcal{O}_D$ we can construct a special full sheaf $\mathcal{M}$. Denote by $M = \pi_* \mathcal{M}$ the reflexive $\mathcal{O}_X$-module associated to the full sheaf $\mathcal{M}$. By construction the minimal resolution is the minimal resolution adapted to M, therefore by Proposition 5.6 the module M is a special reflexive module.

If the canonical cycle is different to zero in any component of the exceptional divisor, then by taking some blowing ups in different points we can obtain a resolution $\pi'\colon \tilde{X}' \to X$ where there exists a irreducible component of the exceptional divisor such that the canonical cycle is zero in this component. Now we use the same idea as in the previous case. □

Now let us say something about special modules. They will be very important in the following chapter, in this moment we just give one property that they satisfy.

Proposition 5.10. *Let M be a special reflexive $\mathcal{O}_X$-module and $\pi\colon \tilde{X} \to X$ be a non-positive resolution with respect to the canonical cycle. Denote by $\mathcal{M}$ the special full sheaf associated to M and assume that the rank of $\mathcal{M}$ is r. Take r generic sections and consider the exact sequence given by the sections*

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Dualizing the exact sequence (2.2.1) we get

$$0 \to \mathcal{N} \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{A} \to 0, \tag{2.2.11}$$

where $\mathcal{A} = \mathscr{Ext}^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}', \mathcal{O}_{\tilde{X}})$, D is the support of $\mathcal{A}$ and $n\colon \tilde{D} \to D$ its normalization.

Applying the functor π_*- to the exact sequence (2.2.11) we get the exact sequence

$$0 \to \pi_*\mathcal{N} \to \mathcal{O}_X^r \to \pi_*\mathcal{A} \to R^1\pi_*\mathcal{N} \to R^1\pi_*\mathcal{O}_{\tilde{X}}^r \to 0.$$

Denote by $\mathcal{C}$ as the image of $\mathcal{O}_X^r$.

Then the modules $\mathcal{C}$, $\pi_*\mathcal{A}$ and $\pi_* n_* \mathcal{O}_{\tilde{D}}$ are equal.

Proof. By the direct correspondence given in Subsection 2.2.1 we know that the support of $\mathcal{A}'$ is a curve D that intersects the exceptional divisor in a finite set.

By Proposition 2.8 and Proposition 2.9 we get that $\mathcal{A}$ is contained in $n_*\mathcal{O}_{\tilde{D}}$.

Applying the functor π_*- to the exact sequence (2.2.11) and taking in account that $\mathcal{M}$ is a special full sheaf, we get the exact sequence

$$0 \to N \to \mathcal{O}_X^r \to \pi_*\mathcal{A} \to 0, \tag{2.1.1}$$

this tells us that $\mathcal{C}$ is equal to $\pi_*\mathcal{A}$.

The exact sequence (2.1.1) give us r generators of the $\mathcal{O}_X$-module $\pi_*\mathcal{A}$. Since $\pi_*\mathcal{A}$ is contained in $\pi_* n_* \mathcal{O}_{\tilde{D}}$, then we can add to the previous set of generators a finite collection of sections of

$\pi_* n_* \mathcal{O}_{\tilde{D}}$ such that they generate it. Hence we obtain the following diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & \mathcal{O}_X^r \arrow[r] \arrow[d] & \pi_*\mathcal{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & \mathcal{O}_X^s \arrow[r] \arrow[d] & \pi_* n_* \mathcal{O}_{\tilde{D}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[d] & \mathcal{O}_X^{s-r} \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By Corollary 2.20 or by our direct construction at the singularity we get that the module N' is reflexive.

The support of K is zero dimensional, therefore dualizing the last row we get that $H^{\vee}$ is isomorphic to $\mathcal{O}_X^{s-r}$. Now dualizing the first column and using the previous identification we get

$$0 \to \mathcal{O}_X^{s-r} \to N'^{\vee} \to N^{\vee} \to 0. \tag{5.2.1}$$

The exact sequence (5.2.1) is an element of the group $\operatorname{Ext}^1_{\mathcal{O}_X}\left(N^{\vee}, \mathcal{O}_X^{s-r}\right)$ which is zero because $N^{\vee}$ is reflexive. This tells us that $N'^{\vee}$ is isomorphic to $N^{\vee} \oplus \mathcal{O}_X^{s-r}$.

Since N and N' are reflexive, dualizing the previous identification tell us that N' is isomorphic to $N \oplus \mathcal{O}_X^{s-r}$. Hence the previous diagram is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & \mathcal{O}_X^r \arrow[r] \arrow[d] & \pi_*\mathcal{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \oplus \mathcal{O}_X^{s-r} \arrow[r] \arrow[d] & \mathcal{O}_X^s \arrow[r] \arrow[d] & \pi_* n_* \mathcal{O}_{\tilde{D}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{s-r} \arrow[r] \arrow[d] & \mathcal{O}_X^{s-r} \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now the last row of this diagram is an element of the group $\operatorname{Ext}^1_{\mathcal{O}_X}\left(K, \mathcal{O}_X^{s-r}\right)$ which is zero because the support of K has dimension zero. This tells us that $\mathcal{O}_X^{s-r}$ is isomorphic to $\mathcal{O}_X^{s-r} \oplus K$.

Notice that

$$\operatorname{Ext}^2_{\mathcal{O}_X}\left(\mathcal{O}_X^{s-r}, \mathcal{O}_X\right) = 0,$$
$$\operatorname{Ext}^2_{\mathcal{O}_X}\left(\mathcal{O}_X^{s-r} \oplus K, \mathcal{O}_X\right) = \operatorname{Ext}^2_{\mathcal{O}_X}\left(K, \mathcal{O}_X\right).$$

Since the modules $\mathcal{O}_X^{s-r}$ and $\mathcal{O}_X^{s-r} \oplus K$ are isomorphic we get that $\operatorname{Ext}^2_{\mathcal{O}_X}(K, \mathcal{O}_X)$ must be equal to zero. By this identification and Theorem 1.4 we get that K must be equal to zero.

This tells us that $\pi_*\mathcal{A}$ is equal to $\pi_* n_* \mathcal{O}_{\tilde{D}}$. $\square$

5.3 Dimension of $R^1\pi_*\left(\mathcal{M} \otimes \mathcal{M}^\vee\right)$

Let M be a special $\mathcal{O}_X$-module and consider $\pi\colon (\tilde{X}, E) \to (X, x)$ the minimal resolution adapted to M. In this section we use the minimal resolution adapted to M to compute the dimension as $\mathbb{C}$-vector space of $R^1\pi_*\left(\mathcal{M} \otimes \mathcal{M}^\vee\right)$.

It is important to notice that the sheaf $\mathcal{M} \otimes \mathcal{M}^\vee$ is isomorphic to the sheaf $\mathscr{H}om_{\mathcal{O}_{\tilde{X}}}(\mathcal{M}, \mathcal{M})$, hence if we want to study the deformations of $\mathcal{M}$ as a locally free sheaf, then our formula is giving us the dimension of the tangent space of the deformation functor (see for example [19, Section 19]).

The main objective of this section is to prove to following theorem.

Theorem 5.11. *Let M be a special $\mathcal{O}_X$-module and consider $\pi\colon (\tilde{X}, E) \to (X, x)$ the minimal resolution adapted to M. Denote by $\mathcal{M}$ the full sheaf associated to M, $\mathcal{N} = \mathcal{M}^\vee$ and assume that $\mathcal{M}$ has rank r. Then*

$$\dim_{\mathbb{C}}\left(R^1\pi_*\left(\mathcal{M} \otimes \mathcal{N}\right)\right) = r \dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right) = r^2 p_g.$$

In order to prove the theorem we need some previous work. Take r generic sections of $\mathcal{M}$ and consider the exact sequence given by the sections

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0, \tag{2.2.1}$$

dualizing the previous exact sequence we get

$$0 \to \mathcal{N} \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{A} \to 0, \tag{2.2.11}$$

where $\mathcal{A} = \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{A}', \mathcal{O}_{\tilde{X}})$.

Applying the functor $\mathcal{M} \otimes -$ to the exact sequence 2.2.11, we get

$$0 \to \mathcal{M} \otimes \mathcal{N} \to \mathcal{M}^r \to \mathcal{M} \otimes \mathcal{A} \to 0. \tag{5.3.1}$$

Applying the functor π_* to the previous exact sequence we get

$$\begin{tikzcd}
0 \arrow[r] & \pi_*\left(\mathcal{M} \otimes \mathcal{N}\right) \arrow[r] & \pi_*\left(\mathcal{M}^r\right) \arrow[r] & \pi_*\left(\mathcal{M} \otimes \mathcal{A}\right) \arrow[dll, dotted] & \\
& R^1\pi_*\left(\mathcal{M} \otimes \mathcal{N}\right) \arrow[r] & R^1\pi_*\left(\mathcal{M}^r\right) \arrow[r] & R^1\pi_*\left(\mathcal{M} \otimes \mathcal{A}\right) \arrow[r] & 0
\end{tikzcd} \tag{5.3.2}$$

Since the intersection of the support of $\mathcal{A}$ and the exceptional divisor is a finite set we get that $R^1\pi_*\left(\mathcal{M} \otimes \mathcal{A}\right)$ is equal to zero.

Lemma 5.12. *The map from $\pi_*\left(\mathcal{M}^r\right)$ to $\pi_*\left(\mathcal{M}\otimes\mathcal{A}\right)$ is a surjection.*

Proof. Applying the functor π_* tho the exact sequence (2.2.11) and taking in account that $\mathcal{M}$ is special we get

$$0\to\pi_*\mathcal{N}\to\mathcal{O}_X^r\to\pi_*\mathcal{A}\to 0. \tag{5.3.3}$$

Applying the functor $(\pi_*\mathcal{M})\otimes -$ to the previous exact sequence we get the surjection

$$\phi\colon\ (\pi_*\mathcal{M})\otimes\mathcal{O}_X^r\to(\pi_*\mathcal{M})\otimes(\pi_*\mathcal{A})\to 0.$$

Also we know that $(\pi_*\mathcal{M})\otimes\mathcal{O}_X^r$ is isomorphic to $(\pi_*\mathcal{M})^r$.

Now consider the natural map

$$\alpha\colon\ (\pi_*\mathcal{M})\otimes\pi_*\left(\mathcal{A}\right)\to\pi_*\left(\mathcal{M}\otimes\mathcal{A}\right).$$

Since the support of $\mathcal{A}$ intersects the exceptional divisor in a finite set, we can identify $\pi_*\left(\mathcal{M}\otimes\mathcal{A}\right)$ with $\mathcal{M}\otimes\mathcal{A}$.

Let $m\otimes a$ be section of $\mathcal{M}\otimes\mathcal{A}$. Since $\mathcal{M}$ is generated by global sections there exist global sections $\psi_1,\ldots,\psi_n$ of $\mathcal{M}$ and sections $f_1,\ldots,f_n$ of $\mathcal{O}_{\tilde{X}}$ defined near of the support of $\mathcal{A}$ such that $m=\sum_i\psi_i f_i$.

Denote by $m'\otimes a'=\sum(\psi_i\otimes f_i\cdot a)$. By construction we get that $\alpha(m'\otimes a')$ is $m\otimes a$, therefore the map α is a surjection.

Now consider the following diagram.

$$\begin{array}{ccc}
(\pi_*\mathcal{M})\otimes\mathcal{O}_X^r & \xrightarrow{\phi} & (\pi_*\mathcal{M})\otimes(\pi_*\mathcal{A})\\
\downarrow\rho & & \downarrow\alpha\\
(\pi_*\mathcal{M})^r & \xrightarrow{\theta} & (\pi_*\mathcal{M}\otimes\mathcal{A})
\end{array}$$

By construction this diagram commutes and since ϕ, α and ρ are surjections, we get that θ is also a surjection. □

By the previous lemma and by the exact sequence (5.3.2) we get that $R^1\pi_*\left(\mathcal{N}\otimes\mathcal{M}\right)$ and $R^1\mathcal{M}^r$ have the same dimension.

Now by Theorem 4.1 we get that

$$\dim_{\mathbb{C}}\left(R^1\pi_*\left(\mathcal{N}\otimes\mathcal{M}\right)\right)=\dim_{\mathbb{C}}\left(R^1\mathcal{M}^r\right)=r^2p_g. \tag{5.3.4}$$

This prove Theorem 5.11.

Chapter 6

The classification of special modules

Let M be a reflexive $\mathcal{O}_X$-module. In the previous chapter we constructed a special type of resolution where the full sheaf associated to M is generated by global sections. In this resolution we were able to study some aspects of the full sheaf.

In this chapter we study in detail the case when the full sheaf is special in the minimal resolution adapted to M. Notice that by Theorem 5.8 we can assume from the beginning that M is a special module.

The first thing that we prove is that in the minimal resolution adapted to M, the full sheaf associated to M it is determined by its first Chern class in the Picard group of the resolution. We are going to use our previous work and some results given by Artin-Verdier [4] and Esnault [13].

At the end of the chapter we use the minimal adapted resolution to M in order to define the combinatorial type of M.

6.1 The first Chern class of a special sheaf

Now we want to use the the minimal adapted resolution in order to obtain new information. In particular in this section we want to study the case of a special full sheaf and its first Chern class in the Picard group.

From now on, we will denote by M a reflexive $\mathcal{O}_X$-module, $\pi\colon (\tilde{X}, E) \to (X, x)$ the minimal adapted resolution to M, $\mathcal{M}$ the full sheaf associated to M and r the rank of $\mathcal{M}$.

Take r generic sections of $\mathcal{M}$ and consider the exact sequence given by the sections

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

By the direct correspondence (Subsection 2.2.1) we know that $\mathcal{A}'$ is a Cohen-Macaulay sheaf of dimension one and its support it is not contained in the exceptional divisor.

Following Artin-Verdier [4] we have the following proposition.

Proposition 6.1. *The sheaf $\mathcal{A}'$ is equal to a direct sum of $\mathcal{O}_{D_i}$ where each D_i is a curve transverse to the exceptional divisor and each curve intersects the exceptional divisor in a different point.*

Proof. It follows by Lemma 1.2 of [4] and from the fact that $\mathcal{M}$ is generated by global sections. □

Recall that by Proposition 5.10 the $\mathcal{O}_X$-module $\pi_*\mathcal{O}_D$ is the normalization of the curve $\pi(D)$. Now following Esnault [13, Lemma 2.4] we have the following result.

Lemma 6.2. *If $\mathcal{M}$ is special and does not have $\mathcal{O}_{\tilde{X}}$ as a direct summand, then the $\mathcal{O}_X$-module $(\pi_*\mathcal{M})^{\vee}$ is the module of relations of a minimal set of generators of $\pi_*\mathcal{A}'$. In this case the module $\pi_*\mathcal{A}'$ determines $\mathcal{M}$ up to isomorphism.*

Proof. By Proposition 6.1 we can identify $\mathcal{A}'$ with $\mathcal{O}_D$, where $\mathcal{O}_D$ is the direct sum of $\mathcal{O}_{D_i}$, each D_i is a curve transverse to the exceptional divisor and each curve intersects the exceptional divisor in a different point.

Therefore the exact sequence (2.2.1) can be written as follows

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{O}_D \to 0. \tag{2.2.1}$$

Dualizing the previous exact sequence we get

$$0 \to \mathcal{N} \to \mathcal{O}_{\tilde{X}}^r \to \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{O}_{\tilde{X}}) \to 0. \tag{2.2.11}$$

Since $\mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{O}_{\tilde{X}}) \cong \mathcal{O}_D$, the previous exact sequence can be written as follows

$$0 \to \mathcal{N} \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{O}_D \to 0.$$

Apply the functor π_*- to the exact sequence (2.2.11) and by the hypothesis of specialty, we get

$$0 \to N \to \mathcal{O}_X^r \to \pi_*\mathcal{O}_D \to 0, \tag{6.1.1}$$

where $\pi_*\mathcal{O}_D = \bigoplus \pi_*\mathcal{O}_{D_i}$ and N is $\pi_*\mathcal{N}$.

By Lemma 1.11 the module $\pi_*\mathcal{N}$ is isomorphic to the module $(\pi_*\mathcal{M})^{\vee}$. Notice that the module M does not have free submodules if and only if the module N does not have free submodules and by (2.1.1) this is equivalent to say that the minimal number of generators of $\pi_*\mathcal{O}_D$ as $\mathcal{O}_X$-module is r. This proves the first part of the proposition.

The second part follows from Esnault [13, Lemma 2.4]. □

By the previous work a natural question is: when the condition of being globally generated implies any of the first two conditions of Lemma 5.1? The following proposition gives a partial answer: if the module is special then the condition of being globally generated implies any of the first two conditions of Lemma 5.1.

Proposition 6.3. *Let M be a special $\mathcal{O}_X$-module, $\pi\colon (\tilde{X}, E) \to (X, x)$ the minimal adapted resolution to M, $\mathcal{M}$ the full sheaf associated to M and r the rank of $\mathcal{M}$. Take r generic sections and consider the exact sequence given by the sections*

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Then the support of $\mathcal{A}'$ does not intersect the support of $\mathcal{O}_{Z_K}$.

Proof. By Proposition 6.1 we can identify $\mathcal{A}'$ with $\mathcal{O}_D$, where $\mathcal{O}_D$ is the direct sum of $\mathcal{O}_{D_i}$, each D_i is a curve transverse to the exceptional divisor and each curve intersects the exceptional divisor in a different point.

Therefore we have to prove that D does not intersect the support of $\mathcal{O}_{Z_K}$.

Denote by

$$\mathcal{N} = (\mathcal{M})^{\vee},$$
$$N = \pi_*\mathcal{N}.$$

By Lemma 6.2 the module N is the module of relations of a minimal set of generators of $\pi_*\mathcal{O}_D$.

Let $D = D_1 \coprod \cdots \coprod D_l$ be the irreducible decomposition of D. Denote by $C_i := \pi(D_i)$, we have that C_i is a curve and by Proposition 5.10 we get that $\pi_*\mathcal{O}_D$ is the normalization of $\mathcal{O}_{\pi_*D}$.

Assume that there exists $p \in D \cap \operatorname{Supp}(Z_K)$ and without loss of generality we can assume that $p \in D_1$.

Consider $\sigma\colon \tilde{X}' \to \tilde{X}$ the blowup in the point p with exceptional divisor E'. Therefore we have the following diagram

$$\begin{tikzcd} \tilde{X}' \arrow[r, "\sigma"] \arrow[dr, "\rho"] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

where $\rho = \pi \circ \sigma$.

Since $p \in D \cap \operatorname{Supp}(Z_K)$, we get that the resolution $\rho\colon \tilde{X}' \to X$ is non-positive with respect to the canonical cycle.

Denote by $\tilde{D}_1$ the strict transform of D_1 and denote by $\tilde{D} = \tilde{D}_1 \coprod D_2 \coprod \cdots \coprod D_l \subset \tilde{X}'$. Observe that $\rho(\tilde{D}) = \pi(D)$, therefore $\rho_*\mathcal{O}_{\tilde{D}}$ is the normalization of $\mathcal{O}_{\pi_*D}$.

Take $\{s_1, \ldots, s_r\}$ a minimal set of generators of $\rho_*\mathcal{O}_{\tilde{D}}$ as $\mathcal{O}_X$-module and consider the exact sequence given by the generators

$$0 \to \mathcal{N}' \to \mathcal{O}^r_{\tilde{X}'} \to \mathcal{O}_{\tilde{D}} \to 0. \tag{6.1.2}$$

Applying the functor ρ_*- to the last exact sequence we get the sequence

$$0 \to \rho_*\mathcal{N}' \to \mathcal{O}^r_X \to \rho_*\mathcal{O}_{\tilde{D}} \to 0. \tag{6.1.3}$$

This sequence is exact because we took generators of $\rho_*\mathcal{O}_{\tilde{D}}$. This tells us that the dimension as $\mathbb{C}$-vector space of $R^1\rho_*\mathcal{N}'$ is rp_g.

By Proposition 3.2 we have that $\mathcal{M}'$ is a full special sheaf and by the exact sequence (6.1.3) we conclude that $\rho_*\mathcal{N}'$ is the module of relations of $\rho_*\mathcal{O}_{\tilde{D}}$.

By Lemma 1.11 we have $\rho_*\mathcal{N}' = (\rho_*\mathcal{M}')^{\vee}$, therefore the $\mathcal{O}_X$-modules $(\rho_*\mathcal{M}')^{\vee}$ and N are modules of relations of $\rho_*\mathcal{O}_{\tilde{D}}$. By Lemma 6.2 we get $\rho_*\mathcal{M}' = M$.

Now we compute $c_1(\mathcal{M}')$ in two differents ways.

First consider the exact sequence (6.1.2) given by the election of a minimal set of generators of $\rho_*\mathcal{O}_{\tilde{D}}$ as $\mathcal{O}_X$-module,

$$0 \to \mathcal{N}' \to \mathcal{O}^r_{\tilde{X}'} \to \mathcal{O}_{\tilde{D}} \to 0. \tag{6.1.2}$$

Dualizing this exact sequence we get

$$0 \to \mathcal{O}^r_{\tilde{X}'} \to \mathcal{M}' \to \mathscr{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{X}}) \to 0. \tag{6.1.4}$$

This tells us that the class $[\tilde{D}]$ of $\tilde{D}$ in Pic$\tilde{X}$ is the Chern class $c_1(\mathcal{M}')$ of $\mathcal{M}'$.

Now notice that $M = \pi_*\mathcal{M} = \rho_*\mathcal{M}'$. Now choosing another sections of M such that their set of degeneracy does not contain the point p we conclude that the class of $c_1(\mathcal{M}')$ in the Picard group does not intersects E', which is a contradiction because we said that the class of $c_1(\mathcal{M}')$ is $[\tilde{D}]$.

Therefore D does not intersect the support of $\mathcal{O}_{Z_K}$. □

Corollary 6.4. *If $\mathcal{M}$ is special, then all the conditions of Lemma 5.1 are equivalent.*

Now we study when a full sheaf is determined by its first Chern class in Pic$(\tilde{X})$. Recall that M is a special reflexive $\mathcal{O}_X$-module, $\pi\colon (\tilde{X}, E) \to (X, x)$ is the minimal resolution adapted to M, $\mathcal{M}$ is the full sheaf associated to M and r is the rank of $\mathcal{M}$.

Take $s_1, \dots, s_r$ generic sections of $\mathcal{M}$ and consider the exact sequence given by the sections

$$0 \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{M} \to \mathcal{A}' \to 0.$$

By Proposition 6.1 we can identify $\mathcal{A}'$ with $\mathcal{O}_D$ where $\mathcal{O}_D$ is the direct sum of $\mathcal{O}_{D_i}$ where each D_i is a curve transverse to the exceptional divisor and each curve intersects the exceptional divisor in a different point.

Locally in a trivialization U of $\mathcal{M}$ we have that the sections can be written as follows

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1r} \\ \vdots & \vdots & \vdots & \vdots \\ q_{r1} & q_{12} & \cdots & q_{rr} \end{pmatrix}$$

where each q_{ij} is an element of $\mathcal{O}_{\tilde{X}}(U)$.

Therefore p belongs to D if and only if the determinant of $Q(p)$ is equal to zero.

Since D is smooth, the matrix Q must have at least $r-1$ columns linearly independent. Therefore we can choose $r-1$ sections linear independent everywhere. These sections give us the exact sequence

$$0 \to \mathcal{O}^{r-1}_{\tilde{X}} \to \mathcal{M} \to \mathcal{L} \to 0.$$

where $\mathcal{L}$ is the line bundle $\det(\mathcal{M})$.

Lemma 6.5. *The dimension as $\mathbb{C}$-vector space of $R^1\pi_*\mathcal{L}$ is p_g.*

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}} \to \mathcal{L} \to \mathcal{O}_D \to 0 \tag{6.1.5}$$

Since $\mathcal{O}_D$ and $\mathcal{O}_{Z_K}$ have disjoint support, we obtain

$$\begin{aligned} \mathrm{Tor}_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{O}_{Z_K}) &= 0, \\ \mathcal{O}_D \otimes \mathcal{O}_{Z_K} &= 0. \end{aligned}$$

By the previous equalities applying the functor $-\otimes\mathcal{O}_{Z_K}$ to the exact sequence (6.1.5) we get

$$\mathcal{O}_{Z_K}\cong\mathcal{L}\otimes\mathcal{O}_{Z_K}. \tag{6.1.6}$$

Now applying the functor π_*- to the exact sequence (6.1.5) and using the last isomorphism we obtain

$$\begin{tikzcd}
0 \arrow[r] & \pi_*\mathcal{O}_{\tilde{X}} \arrow[r] & \pi_*\mathcal{L} \arrow[r] & \pi_*\mathcal{O}_D \arrow[r] & R^1\pi_*\mathcal{O}_{\tilde{X}} \arrow[r] \arrow[d] & R^1\pi_*\mathcal{L} \arrow[r] \arrow[d] & 0 \\
 & & & 0 \arrow[r] & R^1\pi_*\mathcal{O}_{Z_K} \arrow[r] & R^1\pi_*\mathcal{L}\otimes\mathcal{O}_{Z_K} \arrow[r] & 0
\end{tikzcd}$$

Since the diagram commutes and $R^1\pi_*\mathcal{O}_{\tilde{X}}$ and $R^1\pi_*\mathcal{O}_{Z_K}$ are isomorphic we conclude that $R^1\pi_*\mathcal{O}_{\tilde{X}}$ and $R^1\pi_*\mathcal{L}$ have the same dimension. □

Proposition 6.6. *If $\mathcal{M}$ is a special full sheaf, then $\mathcal{M}$ is determined by its first Chern class in $Pic(\tilde{X})$.*

PROOF. Take $s_1,\dots,s_{r-1}$ generic sections of $\mathcal{M}$ and consider the exact sequence given by the sections

$$0\to\mathcal{O}_{\tilde{X}}^{r-1}\to\mathcal{M}\to\mathcal{L}\to 0, \tag{6.1.7}$$

where $\mathcal{L}$ is the line bundle $\det(\mathcal{M})$.

Applying the fuctor π_*- to the exact sequence (6.1.7) we get

$$0\longrightarrow\pi_*\mathcal{O}_{\tilde{X}}^{r-1}\longrightarrow\pi_*\mathcal{M}\longrightarrow\pi_*\mathcal{L}\longrightarrow R^1\pi_*\mathcal{O}_{\tilde{X}}^{r-1}\longrightarrow R^1\pi_*\mathcal{M}\longrightarrow R^1\pi_*\mathcal{L}\longrightarrow 0. \tag{6.1.8}$$

Since

$$\begin{aligned}\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right)&=rp_g\quad\text{by Corollary 6.4,}\\ \dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{L}\right)&=\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{O}_{\tilde{X}}\right)\quad\text{by Lemma 6.5,}\end{aligned}$$

we get that the exact sequence (6.1.8) split as follows

$$0\longrightarrow\pi_*\mathcal{O}_{\tilde{X}}^{r-1}\longrightarrow\pi_*\mathcal{M}\longrightarrow\pi_*\mathcal{L}\longrightarrow 0.$$

Therefore $\pi_*\mathcal{M}\in\operatorname{Ext}^1_{\mathcal{O}_X}\left(\pi_*\mathcal{L},\mathcal{O}_X^{r-1}\right)$ and $\pi_*\mathcal{M}$ is reflexive. We conclude the proof by Lemma 1.9.ii in [4]. □

Let M be a special $\mathcal{O}_X$-module, $\pi\colon(\tilde{X},E)\to(X,x)$ be the minimal adapted resolution to M, $\mathcal{M}$ be the full sheaf associated to M and r the rank of $\mathcal{M}$. Take r generic sections and consider the following exact sequence

$$0\to\mathcal{O}_{\tilde{X}}^{r}\to\mathcal{M}\to\mathcal{A}'\to 0. \tag{2.2.1}$$

By Proposition 6.1 we can identify $\mathcal{A}'$ with $\mathcal{O}_D$, where $\mathcal{O}_D$ is the direct sum of $\mathcal{O}_{D_i}$, each D_i is a curve transverse to the exceptional divisor and each curve intersects the exceptional divisor in a different point.

Dualizing the exact sequence (2.2.1) and using the previous identification, we get the exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{O}_{\tilde{X}}) \to 0. \tag{2.2.11}$$

Since $\mathcal{E}xt^1_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{O}_{\tilde{X}}) \cong \mathcal{O}_D$, the previous exact sequence can be written as follows

$$0 \to \mathcal{N} \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{O}_D \to 0. \tag{6.1.9}$$

Applying the functor π_*- to the exact sequence (6.1.9) and taking in account that M is a special module, we get the exact sequence

$$0 \to N \to \mathcal{O}^r_X \to \pi_*\mathcal{O}_D \to 0.$$

Applying the inverse correspondence (Section 2.2.2) to the curve D and the module $\mathcal{C} = \pi_*\mathcal{O}_D$, we recover the sheaf $\mathcal{N}$.

Now applying the inverse correspondence to the curve D_i and the module $\mathcal{C}_i = \pi_*\mathcal{O}_{D_i}$, we obtain a full sheaf $\mathcal{M}_i$ and its dual $\mathcal{N}_i$.

We know that $\mathcal{O}_D = \oplus_{i=1}^l \mathcal{O}_{D_i}$, hence $\pi_*\mathcal{O}_D = \oplus_{i=1}^l \pi_*\mathcal{O}_{D_i}$.

Since the inverse correspondence commutes with direct sums and $\pi_*\mathcal{O}_D = \oplus_{i=1}^l \pi_*\mathcal{O}_{D_i}$, we have that $\mathcal{N} = \oplus_{i=1}^l \mathcal{N}_i$.

We have proved.

Corollary 6.7. *Let M be a special $\mathcal{O}_X$-module, $\pi\colon (\tilde{X}, E) \to (X, x)$ the minimal adapted resolution to M, $\mathcal{M}$ the full sheaf associated to M and r the rank of $\mathcal{M}$. Take r generic sections and consider the exact sequence given by the sections*

$$0 \to \mathcal{O}^r_{\tilde{X}} \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Denote by D the support of $\mathcal{A}'$. Consider

1. *$M = \bigoplus_{i=1}^s M_i$, where each M_i is indecomposable,*
2. *$D = \coprod_{i=1}^r D_i$, where each D_i is a irreducible component of D.*

Then there exists a bijection between the indecomposable modules M_i and the irreducible components D_i.

In order to guarantee that a full special sheaf is determined by its first Chern class we need the following lemma.

Lemma 6.8. *Let $\pi\colon (\tilde{X}, E) \to (X, x)$ be a non-positive resolution with respect to the canonical cycle such that for some irreducible component E_i of the exceptional divisor we have $E_i \not\subseteq$ Supp(Z_K). If D_1 and D_2 are two curves, each one transverse to E_i, then $\mathcal{O}_{\tilde{X}}(-D_1) \cong \mathcal{O}_{\tilde{X}}(-D_2)$.*

Proof. We want to prove that $\mathcal{O}_{\tilde{X}}(-D_1 + D_2)$ is isomorphic to $\mathcal{O}_{\tilde{X}}$.

Consider the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{\exp} \mathcal{O}^*_{\tilde{X}} \longrightarrow 0.$$

Applying the functor π_* to the previous exact sequence we get

$$\dots \longrightarrow H^1(\tilde{X},\mathbb{Z}) \longrightarrow H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) \xrightarrow{\exp} H^1(\tilde{X},\mathcal{O}^*_{\tilde{X}}) \xrightarrow{\delta} H^2(\tilde{X},\mathbb{Z}) \longrightarrow 0. \tag{6.1.10}$$

We know that the Picard group of $\tilde{X}$ is $H^1(\tilde{X},\mathcal{O}^*_{\tilde{X}})$ and the morphism δ is given by taking the first Chern class.

By hypothesis we know that $\delta(\mathcal{O}_{\tilde{X}}(-D_1+D_2)) = 0$. By the exact sequence (6.1.10) we get that there exist an element f in $H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$ such that the line bundle given by $\exp(f)$ is isomorphic to $\mathcal{O}_{\tilde{X}}(-D_1+D_2)$. Let us denote by $\mathcal{L}$ the line bundle given by $\exp(f)$.

Since we are working in a non-positive resolution with respect to the canonical cycle, we have the exact sequence

$$0 \to \omega_{\tilde{X}} \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{Z_K} \to 0. \tag{3.0.1}$$

Applying the functor π_*- and by Grauert-Riemenschneider Vanishing Theorem we get that $H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$ is isomorphic to $H^1(\tilde{X},\mathcal{O}_{Z_K})$.

Now we use Čech cohomology. Since f is an element of $H^1(\tilde{X},\mathcal{O}_{\tilde{X}})$, we have that f is represented by a 1-cocycle.

Taking in account $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\tilde{X}}(-D_1+D_2)$, we construct an open covering $\{U_\lambda\}$ of $\tilde{X}$ where f is represented by the 1-cocycle f_{ij} defined over $\mathcal{O}_{\tilde{X}}(U_i \cap U_j)$ such that f_{ij} is equal to zero if $U_i \cap U_j \cap \operatorname{Supp}(Z_K)$ is non-empty.

This tells us that f is zero in $H^1(\tilde{X},\mathcal{O}_{Z_K})$, hence the line bundle $\mathcal{L}$ must be trivial. □

The previous lemma give us the following theorem.

Theorem 6.9. *Let M be a special $\mathcal{O}_X$-module. Denote by $\pi\colon \tilde{X} \to X$ be the minimal resolution adapted to M and $\mathcal{M}$ the full sheaf associated to M. Then the full sheaf $\mathcal{M}$ is determined by its first Chern class in the Picard group of $\tilde{X}$.*

6.2 The classification of special reflexive modules

In this section we use all the previous work in order to construct the combinatorial classification of special modules. This classification uses the information given by the minimal adapted resolution.

Let M be a special, indecomposable $\mathcal{O}_X$-module, $\pi\colon (\tilde{X},E) \to (X,x)$ be the minimal resolution adapted to M, $\mathcal{M}$ be the full sheaf associated to M.

We construct a graph as follows:

1. Let $\mathfrak{G}_M$ be the dual graph of $\tilde{X}$.
2. In each vertex v_i, add as many arrows as the first Chern class of $\mathcal{M}$ intersects the exceptional divisor E_i.

Definition 6.10. The graph $\mathfrak{G}_M$ is *the combinatorial type of M*.

Now we need a definition (see [22, Definition 2.24]) and a lemma.

Definition 6.11. Let $\pi\colon \tilde{X} \to X$ be a resolution with exceptional divisor $E = \bigcup_{i=1}^n E_i$. Any irreducible component E_i is called *a divisor over X*.

Lemma 6.12. *Let $E'_1, \dots, E'_n$ be a collection of divisors over X. Then there exists a unique minimal resolution $\pi \colon \tilde{X} \to X$ with exceptional divisor $E = \bigcup_{i=1}^{l} E_i$ such that for any $i \in \{1, \dots, n\}$ we have that $E'_i = E_j$ for some j.*

Proof. Starting from the minimal resolution we can construct the minimal resolution for the divisors over X by taking a finite collection of blows up in different points. □

Now we present the principal theorem of this part of the thesis.

Theorem 6.13. *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Then there exists a bijection between the following sets:*

1. *The set of special $\mathcal{O}_X$-modules up to isomorphism.*
2. *The set of finite pairs $(E'_1, n_1), \dots, (E'_l, n_l)$ where each E'_i is a divisor over X and n_i is a positive integer, such the minimal resolution given by Lemma 6.12 is a non-positive resolution with respect to the canonical cycle and the Gorenstein form does not have any pole in the components $E'_1, \dots, E'_l$.*

Proof. Let M be a special $\mathcal{O}_X$-module and $\pi \colon \tilde{X} \to X$ be the minimal resolution adapted to M with exceptional divisor $E = \bigcup_{i=1}^{l} E_i$. Denote by $\mathcal{M}$ the full sheaf associated to M and by $n_j = c_1(\mathcal{M}) \cdot E_j$ for $j = 1, \dots, l$. We associate to the module M the pairs $(E_1, n_1), \dots, (E_k, n_k)$ such that n_j is different form zero.

Now consider $(E'_1, n_1), \dots, (E'_l, n_l)$ where each E'_i is a divisor over X and n_i is a positive integer and denote by $\pi \colon \tilde{X} \to X$ be the resolution given by Lemma 6.12. By hypothesis we know that $\pi \colon \tilde{X} \to X$ is a non-positive resolution with respect to the canonical cycle. Now for each positive integer n_j take a smooth curve D_j with n_j irreducible components such that D_j intersects only the irreducible component E_j and the intersection is transverse. Denote by $D = D_1 \coprod \cdots \coprod D_l$. By Proposition 3.2, taking $\mathcal{C} = \pi_* \mathcal{O}_D$ we construct a special full sheaf $\mathcal{M}$, denote by $\pi_* \mathcal{M} = M$. Now since each curve D_j intersects only one component of the exceptional divisor where the canonical cycle is zero, by Lemma 5.1 the resolution $\pi \colon \tilde{X} \to X$ is the minimal resolution adapted to M. Finally by Corollary 5.8 we get that M is a special module.

The bijection between both sets follows from Theorem 6.9. □

Corollary 6.14. *Let (X, x) be a complex analytic germ of a normal two-dimensional Gorenstein singularity. Then there exists a bijection between the following sets:*

1. *The set of special, indecomposable $\mathcal{O}_X$-modules up to isomorphism.*
2. *The set of pairs $(E'_1, 1)$ where E'_1 is a divisor over X, such the minimal resolution given by Lemma 6.12 is a non-positive resolution with respect to the canonical cycle and the Gorenstein form does not have any pole in the component E'_1.*

Proof. It follows from the Theorem 6.13. For the inverse just take a smooth irreducible curve D_1 such that intersects only the component E_1 and the intersection is transverse. □

Notice that if (X, x) is a rational double point, then the Corollary 6.14 is the McKay correspondence given by Artin and Verdier [4].

Corollary 6.15 ([4])**.** *Let (X, x) be a rational double point and denote by $\pi \colon \tilde{X} \to X$ the minimal resolution with exceptional divisor $E = \bigcup_{i=1}^{l} E_i$. Then there exists a bijection between the following sets:*

1. *The set of reflexive, indecomposable $\mathcal{O}_X$-modules up to isomorphism.*
2. *The set of pairs $(E_i, 1)$ where E_i is an irreducible component of the exceptional divisor E.*

PROOF. Since the singularity is a rational double point we know that in the minimal resolution the canonical cycle is equal to 0. This tells us that there exists a bijection between:

1. The set of pairs $(E_i, 1)$ where E_i is an irreducible component of the exceptional divisor E.
2. The set of pairs $(E_1', 1)$ where E_1' is a divisor over X, such the minimal resolution given by Lemma 6.12 is a non-positive resolution with respect to the canonical cycle and the Gorenstein form does not have any pole in the component E_1'.

By Corollary 6.14 and the previous discussion we know that there exists a bijection between:

1. The set of pairs $(E_i, 1)$ where E_i is an irreducible component of the exceptional divisor E.
2. The set of special, indecomposable $\mathcal{O}_X$-modules up to isomorphism.

Now let M be a reflexive $\mathcal{O}_X$-module, $\mathcal{M}$ the full sheaf associated to M and r the rank of $\mathcal{M}$. Take r generic sections and consider the exact sequence given by the sections

$$0 \to \mathcal{O}_{\tilde{X}}^r \to \mathcal{M} \to \mathcal{A}' \to 0. \tag{2.2.1}$$

Applying the functor $\pi_* -$ to the previous exact sequence and since the singularity is a rational double point, we conclude that $R^1\pi_*\mathcal{M} = 0$. Now by Theorem 4.1 and taking in account that the singularity is a rational double point we have that $\dim_{\mathbb{C}}\left(R^1\pi_*\mathcal{M}\right) = d$, where d is the specialty defect. Hence $d = 0$ and this tells us that $\mathcal{M}$ is a special full sheaf.

Finally by Lemma 5.1 we know that the minimal resolution is the minimal resolution adapted to M and by Theorem 5.8 we conclude that M is a special module. Hence any reflexive module is special and this give us the bijection between

1. The set of pairs $(E_i, 1)$ where E_i is an irreducible component of the exceptional divisor E.
2. The set of reflexive, indecomposable $\mathcal{O}_X$-modules up to isomorphism.

$\square$

Part II

Classification and properties of some classes of real singularities

Chapter 7

Background

7.1 Mixed function

Consider $\mathbb{C}^n$ with coordinates $z_1, \dots, z_n$. Let $\bar{z}_j$ be the complex conjugate of z_j. We will write $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$. To simplify notation we shall write $\mathbf{z} = (z_1, \dots, z_n)$, $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$, $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$. We also denote by $\mathbf{0}$ the origin in $\mathbb{C}^n$, by $\mathbb{C}^*$ the non-zero complex numbers and by $\mathbb{R}^+$ the positive real numbers.

Definition 7.1. Let $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ with $\mu_j, \nu_j \in \mathbb{N} \cup \{0\}$, set $\mathbf{z}^\mu = z_1^{\mu_1} \dots z_n^{\mu_n}$ and $\bar{\mathbf{z}}^\nu = \bar{z}_1^{\nu_1} \dots \bar{z}_n^{\nu_n}$.

We call f a *mixed analytic function* if f is a complex valued function $f\colon \mathbb{C}^n \to \mathbb{C}$ such that f has a convergent power series of variables $\mathbf{z}$ and $\bar{\mathbf{z}}$,

$$f(\mathbf{z}) = \sum_{\mu,\nu} c_{\mu,\nu} \mathbf{z}^\mu \bar{\mathbf{z}}^\nu \ .$$

We call f *mixed polynomial*, if f is a complex valued function $f\colon \mathbb{C}^n \to \mathbb{C}$ such that f is a polynomial in the variables $\mathbf{z}$ and $\bar{\mathbf{z}}$.

We consider f as a function $f\colon \mathbb{R}^{2n} \to \mathbb{R}^2$ in the $2n$ real variables $(\mathbf{x}, \mathbf{y})$ writing $f(\mathbf{z}) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$, taking $z_j = x_j + iy_j$ where $g, h\colon \mathbb{C}^n \cong \mathbb{R}^{2n} \to \mathbb{R}$ are real analytic functions. Recall that for any real analytic function $k\colon \mathbb{R}^{2n} \to \mathbb{R}$ we have

$$\frac{\partial k}{\partial z_j} = \frac{1}{2}\left(\frac{\partial k}{\partial x_j} - i\frac{\partial k}{\partial y_j}\right) , \qquad \frac{\partial k}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial k}{\partial x_j} + i\frac{\partial k}{\partial y_j}\right) .$$

So we have

$$\frac{\partial f}{\partial z_j} = \frac{\partial g}{\partial z_j} + i\frac{\partial h}{\partial z_j} , \qquad \frac{\partial f}{\partial \bar{z}_j} = \frac{\partial g}{\partial \bar{z}_j} + i\frac{\partial g}{\partial \bar{z}_j} .$$

As usual, we define the real gradients of g and h by

$$\mathrm{d}_{\mathbb{R}} g(\mathbf{x}, \mathbf{y}) = \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}, \frac{\partial g}{\partial y_1}, \dots, \frac{\partial g}{\partial y_n}\right) ,$$
$$\mathrm{d}_{\mathbb{R}} h(\mathbf{x}, \mathbf{y}) = \left(\frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n}, \frac{\partial h}{\partial y_1}, \dots, \frac{\partial h}{\partial y_n}\right) .$$

Following Oka [27] set

$$\mathrm{d}f(\mathbf{z}) = \left(\frac{\partial f(\mathbf{z})}{\partial z_1}, \dots, \frac{\partial f(\mathbf{z})}{\partial z_n}\right), \qquad \bar{\mathrm{d}}f(\mathbf{z}) = \left(\frac{\partial f(\mathbf{z})}{\partial \bar{z}_1}, \dots, \frac{\partial f(\mathbf{z})}{\partial \bar{z}_n}\right).$$

The following proposition is an useful criterium to determine whether a point $\mathbf{z} \in \mathbb{C}^n$ is a critical point of a mixed function f.

Proposition 7.2 (Oka's Criterium [27, Proposition 1])**.** *Let $\mathbf{z} \in \mathbb{C}^n$. The following two conditions are equivalent,*

1. *The vectors $\mathrm{d}_{\mathbb{R}}g(\mathbf{z})$ and $\mathrm{d}_{\mathbb{R}}h(\mathbf{z})$ are linearly dependent over $\mathbb{R}$.*
2. *There exists a complex number $\alpha \in \mathbb{S}^1$ such that $\overline{\mathrm{d}f(\mathbf{z})} = \alpha \bar{\mathrm{d}}f(\mathbf{z})$.*

We need the following condition which will be automatically satisfied by the family of polar weighted homogeneous polynomials that we will consider later.

Condition 7.3. *If the monomial z_j appears in f, then the monomial $\bar{z}_j$ does not appear in f.*

The following lemma is a generalization of [3, Proposition 11.1] by Arnold.

Lemma 7.4. *Fix $i \in \{1, \dots, n\}$. If f is a mixed polynomial with isolated singularity at the origin of $\mathbb{C}^n$ satisfying Condition 7.3, then there exist $a, b \in \mathbb{N} \cup \{0\}$ with $a + b \neq 0$, such that the monomial $z_i^a \bar{z}_i^b x$ appears in f, with $x \in \{z_1, \bar{z}_1, \dots, z_n, \bar{z}_n\}$.*

Proof. Assume that for all $a \geq 0$, $b \geq 0$ there are no monomial $z_i^a \bar{z}_i^b x$. By Condition 7.3, f does not have a linear term, if so, $f = 0$ has no singularity at the origin. Consider $\mathrm{d}f$ and $\bar{\mathrm{d}}f$ on the axis $z_1 = \cdots = z_{i-1} = z_{i+1} = \cdots = z_n = 0$. This axis is a subset of $f^{-1}(0)$ and we have that both gradient vectors vanish simultaneously. This means that the axis is included in the singular locus, which contradicts the fact that f has an isolated singularity at the origin. □

Following Oka [27, §2.3] we have the following definition.

Definition 7.5. Let $\mu_j = (\mu_{j,1}, \dots, \mu_{j,n})$ and $\nu_j = (\nu_{j,1}, \dots, \nu_{j,n})$ be multi-indices and let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a mixed polynomial written as

$$f(\mathbf{z}) = \sum_{j=1}^{m} c_j \mathbf{z}^{\mu_j} \bar{\mathbf{z}}^{\nu_j},$$

where $c_1, \dots, c_m$ are non-zero. Consider the following matrices

$$P = \begin{pmatrix} \mu_{1,1}+\nu_{1,1} & \cdots & \mu_{1,n}+\nu_{1,n} \\ \vdots & \vdots & \vdots \\ \mu_{m,1}+\nu_{m,1} & \cdots & \mu_{m,n}+\nu_{m,n} \end{pmatrix}, \qquad Q = \begin{pmatrix} \mu_{1,1}-\nu_{1,1} & \cdots & \mu_{1,n}-\nu_{1,n} \\ \vdots & \vdots & \vdots \\ \mu_{m,1}-\nu_{m,1} & \cdots & \mu_{m,n}-\nu_{m,n} \end{pmatrix}.$$

We say that f is *simplicial* if $m \leq n$ and the rank of the matrices P and Q are equal to m. We say that f is *radial full* (respectivelly *angular full*) if $n = m$ and P (respectively, Q) has rank n. If f is radial and angular full, then we say that f is *full*. We call the matrix P the *radial matrix* and Q the *angular matrix* of f.

Define the *associated Laurent polynomial* $\hat{f}\colon \mathbb{C}^{*n} \to \mathbb{C}$ by

$$\hat{f}(\mathbf{w}) = \sum_{j=1}^{m} c_j \mathbf{w}^{\mu_j - \nu_j} \ .$$

Theorem 7.6 ([27, Theorem 10])**.** *Let $f(\mathbf{z})$ be a full mixed polynomial and let $\hat{f}(\mathbf{w})$ be its associated Laurent polynomial. Then there exists a diffeomorphism $\phi\colon \mathbb{C}^{*n} \to \mathbb{C}^{*n}$ such that $\hat{f} \circ \phi = f|_{\mathbb{C}^{*n}}$.*

Corollary 7.7. *The associated Laurant polynomial $\hat{f}\colon \mathbb{C}^{*n} \to \mathbb{C}$ has no critical points.*

PROOF. Let Q be the angular matrix. As in [25, page 68] define the map $\psi_Q\colon \mathbb{C}^{*n} \to \mathbb{C}^{*n}$ by

$$\psi_Q(\mathbf{w}) = (w_1^{\mu_{1,1}-\nu_{1,1}} \dots w_n^{\mu_{1,n}-\nu_{1,n}}, \dots, w_1^{\mu_{m,1}-\nu_{m,1}} \dots w_n^{\mu_{m,n}-\nu_{m,n}}),$$

and define $h\colon \mathbb{C}^{*n} \to \mathbb{C}$ by $h(\mathbf{w}) = c_1 w_1 + \dots + c_m w_m$. Then we have that $\hat{f}(\mathbf{w}) = h(\psi_Q(\mathbf{w}))$. By [25, Assertion (1.3.2), page 109] ψ_Q is a $\det(Q)$-fold covering and clearly h has no critical points. □

An useful property of a radial full or angular full polynomial is that we can have more control on the coefficients c_j.

Lemma 7.8. *Let f be a mixed polynomial and suppose that k rows of the radial matrix P are linearly independent. Then under a change of coordinates we can assume that k coefficients are on $\mathbb{S}^1$.*

Lemma 7.9. *Let f be a mixed polynomial and suppose that k rows of the angular matrix Q are linearly independent. Then under a change of coordinates we can assume that k coefficients are on $\mathbb{R}^+$.*

Corollary 7.10 ([29, Lemma 8])**.** *If f is full, then under a change of coordinates we can assume that all the coefficients are* 1*.*

We are just going to prove Lemma 7.8 (actually it is just an adaptation of the proof of [29, Lemma 8]).

PROOF. [Proof of Lemma 7.8] We have $f(\mathbf{z}) = \sum_{j=1}^{m} c_j \mathbf{z}^{\mu_j} \bar{\mathbf{z}}^{\nu_j}$. We can apply a change of coordinates $z_j \to z_{\sigma(j)}$ with σ a permutation of $\{1, \dots, n\}$ so that the matrix

$$P' = \begin{pmatrix} \mu_{1,1}+\nu_{1,1} & \dots & \mu_{1,k}+\nu_{1,k} \\ \vdots & \vdots & \vdots \\ \mu_{k,1}+\nu_{k,1} & \dots & \mu_{k,k}+\nu_{k,k} \end{pmatrix}$$

is invertible.

We are going to construct a change of coordinates of the form $z_j \to e^{t_j} z_j$ where $t_j \in \mathbb{R}$ with $j = 1, \dots, k$. Write $c_j = e^{a_j}\theta_j$ and notice that we want some numbers $e^{t_j} \in \mathbb{R}^+$ such that

$$(\mu_{j,1}+\nu_{j,1})t_1 + \dots + (\mu_{j,k}+\nu_{j,k})t_k = -a_j$$

then we have the system

$$P'(t_1, \dots, t_k)^\top = (-a_1, \dots, -a_k)^\top \ .$$

Since P' is invertible we can solve this system. □

7.2 Polar weighted homogeneous polynomials

Let $p_1, \dots, p_n$ and $q_1, \dots, q_n$ be non-zero integers such that $\gcd(p_1, \dots, p_n) = 1$ and $\gcd(q_1, \dots, q_n) = 1$. Let $w \in \mathbb{C}^*$ written in its polar form $w = t\tau$, with $t \in \mathbb{R}^+$ and $\tau \in \mathbb{S}^1$. A *polar* $\mathbb{C}^*$*-action* on $\mathbb{C}^n$ with *radial weights* $(p_1, \dots, p_n)$ and *angular weights* $(q_1, \dots, q_n)$ is given by:

$$t\tau \bullet \mathbf{z} = (t^{p_1}\tau^{q_1} z_1, \dots, t^{p_n}\tau^{q_n} z_n) . \tag{7.2.1}$$

Definition 7.11. A mixed function $f \colon \mathbb{C}^n \to \mathbb{C}$ is *polar weighted homogeneous* if there exists $p_1, \dots, p_n$ positive integers, $q_1, \dots, q_n$ non-zero integers, a, c positive integers, and a polar $\mathbb{C}^*$-action given by (7.2.1) such that f satisfies the following functional equation

$$f(t\tau \bullet \mathbf{z}) = t^a \tau^c f(\mathbf{z}) . \tag{7.2.2}$$

We say that the polar weighted homogeneous function f has *radial weight type* $(p_1, \dots, p_n; a)$ and *angular weight type* $(q_1, \dots, q_n; c)$.

Sometimes it is more convenient to consider the *normalized radial weights* $(p_1', \dots, p_n')$ given by $p_i' = \frac{p_i}{a}$ and the *normalized angular weights* $(q_1, \dots, q_n)$ given by $q_i' = \frac{q_i}{c}$.

We will say that f is *generalized polar weighted homogeneous* if it satisfies (7.2.2) with $p_1, \dots, p_n$ and $q_1, \dots, q_n$ integers, i.e. some p_j or q_j can be zero or negative.

Remark 7.12. The definition of polar weighted homogeneous functions follows the original definition given in [10] but allowing the q_i's to be negative. Other authors (for instance [27, 8]) call polar weighted homogeneous functions to more general notions allowing the p_i's or q_i's to be zero; we call this more general definition generalized polar weighted homogeneous functions to emphasize the difference. Originally, the *angular weights* were called *polar weights* and this has caused some confusion in the literature because some authors (for instance [28, 8]) call *polar weighted homogeneous* to mixed functions which are weighted homogeneous with respect to the *angular weights* and not to both radial and angular weights. To avoid this ambiguity in [5] the authors propose to use the term mixed weighted homogeneous instead of what we call polar weighted homogeneous. We think it is better to keep the term polar weighted homogeneous for the original definition given in [10] and use the term *angular weights* instead of *polar weights* and respectively *angular weighted homogeneous*; the reason is that the polar coordinate system on the plane consists of two coordinates: the radial coordinate and the angular coordinate, and *polar* $\mathbb{C}^*$*-actions* are defined writing the acting element $w \in \mathbb{C}^*$ in its *polar form.*

Example. As examples of polar weighted homogeneous polynomials we have:

1. Complex weighted homogeneous polynomials are a particular case of polar weighted homogeneous polynomials with no $\bar{z}_j$ for $j = 1, \dots, n$ and with $p_j = q_j$ and $a = c$.

2. A mixed polynomial in $\mathbb{C}^n$ of the form

$$f(\mathbf{z}) = c_1 z_1^{a_1} \bar{z}_{\sigma(1)} + \dots + c_n z_n^{a_n} \bar{z}_{\sigma(n)}, \tag{7.2.3}$$

is called a *twisted Brieskorn-Pham polynomial* of class $\{a_1, \dots, a_n; \sigma\}$, where each $a_j \geq 2$, $j = 1, \dots, n$, the c_j are non-zero complex numbers and σ is a permutation of the set $\{1, \dots, n\}$ called the *twisting.*

Twisted Brieskorn-Pham polynomials were the first examples of polar weighted homogeneous polynomials and they appeared implicity in the work of Seade [35] and later were defined and studied by Ruas, Seade and Verjovsky in [34].

Notice that given a polar $\mathbb{C}^*$-action on $\mathbb{C}^n$, we get a radial $\mathbb{R}^+$-action on $\mathbb{C}^n$ given by

$$t * \mathbf{z} := (t^{p_1} z_1, \ldots, t^{p_n} z_n).$$

Sometimes we will be interested in the general case of real analytic maps $f \colon \mathbb{R}^n \to \mathbb{R}^k$, so we also consider the following definition.

Definition 7.13. Let $p_1, \ldots, p_n$ be integers with $\gcd(p_1, \ldots, p_n) = 1$. Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be an analytic map and consider an $\mathbb{R}^+$-action on $\mathbb{R}^n$ given by

$$t * \mathbf{x} := (t^{p_1} x_1, \ldots, t^{p_n} x_n) \ .$$

Let a be a positive integer. We call f a *radial weighted homogeneous map of type* $(p_1, \ldots, p_n; a)$ if

$$f(t * \mathbf{x}) = t^a f(\mathbf{x}) \ ,$$

where p_j is a positive integer for $j = 1, \ldots, n$. We say that f is a *generalized radial weighted homogeneous* if $p_1, \ldots, p_n$ are arbitrary integers.

Proposition 7.14 ([10, §3],[27, §2])**.** *Let $f(\mathbf{z})$ be a generalized polar weighted homogeneous function with radial weight type $(p_1, \ldots, p_n; a)$ and angular weight type $(q_1, \ldots, q_n; c)$. Then it satisfies the following properties:*

1. *Euler identities:*

$$af(\mathbf{z}) = \sum_{j=1}^{n} p_j z_j \frac{\partial f}{\partial z_j}(\mathbf{z}) + \sum_{j=1}^{n} p_j \bar{z}_j \frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}) \ ,$$
$$bf(\mathbf{z}) = \sum_{j=1}^{n} q_j z_j \frac{\partial f}{\partial z_j}(\mathbf{z}) - \sum_{j=1}^{n} q_j \bar{z}_j \frac{\partial f}{\partial \bar{z}_j}(\mathbf{z}) \ .$$

2. *The maps $\frac{\partial f}{\partial z_j}$ and $\frac{\partial f}{\partial \bar{z}_j}$ are also generalized weighted homogeneous.*

3. *The only critical value of f is 0.*

4. *Let $\alpha \neq 0$. The fiber $F_\alpha := f^{-1}(\alpha)$ is a manifold of real dimension $2(n-1)$ and it is canonical diffeomorphic to $F_1 = f^{-1}(1)$.*

5. *If the weights $p_1, \ldots, p_n$ are positive, then*
   (a) *The function f is indeed a polynomial.*
   (b) *The zero-set $V = f^{-1}(0)$ is contractible to the origin.*
   (c) *The restriction $f \colon (\mathbb{C}^n \setminus V) \to \mathbb{C}^*$ is a locally trivial fibration.*
   (d) *The map*

$$\phi = \frac{f}{|f|} \colon \mathbb{S}^{2n-1}_\epsilon \setminus K_\epsilon \to \mathbb{S}_1 \ , \tag{7.2.4}$$

   is a fiber bundle, for any $\epsilon > 0$.

(e) *The fibration* $f_{\mathbb{S}^1} := f|_{f^{-1}(\mathbb{S}^1)} \colon f^{-1}(\mathbb{S}^1) \to \mathbb{S}^1$ *is equivalent to the fibration* (7.2.4).

Furthermore, if the origin is an isolated singularity of V

(f) $V \setminus \{\mathbf{0}\}$ *is smooth.*

(g) *The sphere* $\mathbb{S}^{2n-1}_\epsilon$ *of radius* ϵ *around* $\mathbf{0}$ *is transverse to* V *for any* $\epsilon > 0$.

(h) *Let* $K_\epsilon := V \cap \mathbb{S}^{2n-1}_\epsilon$. *Then for any* $\epsilon', \epsilon > 0$, $K_{\epsilon'}$ *and* K_ϵ *are* $\mathbb{S}^1$*-equivariantly diffeomorphic (Compare with [31, Proposition 3.1.3]).*

Remark 7.15. If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a radial weighted homogeneous map analogous to item 2 of Proposition 7.14 we have that $\frac{\partial f}{\partial x_i}$ is also radial weighted homogeneous.

In this thesis we restrict to the case when the radial weights are positive and non-zero angular weights, that is, we are only interested in polar weighted homogeneous polynomials.

Lemma 7.16. *If* $f\colon \mathbb{C}^n \to \mathbb{C}$ *is polar weighted homogeneous and* $\mathbf{z}$ *is a critical point of* f*, then* $t\lambda \bullet \mathbf{z}$ *is a critical point for all* $t\lambda \in \mathbb{C}^*$.

The analogous statement is true for a radial weighted map.

Proof. We will prove it for polar weighted homogeneous polynomials, the other case is analogous.

Since $\mathbf{z}$ is a critical point of f, by Lemma 7.2 there exists $\alpha \in \mathbb{S}^1$ such that

$$\overline{\frac{\partial f(\mathbf{z})}{\partial z_j}} = \alpha \frac{\partial f(\mathbf{z})}{\partial \bar{z}_j}, \qquad j \in \{1, \dots, n\}\ .$$

Since $\frac{\partial f}{\partial z_j}$ and $\frac{\partial f}{\partial \bar{z}_j}$ are also polar weighted homogeneous, for any $t \in \mathbb{R}^+$ and $\lambda \in \mathbb{S}^1$

$$\overline{\frac{\partial f(t\lambda \bullet \mathbf{z})}{\partial z_j}} = t^{a-p_j} \lambda^{-b+q_j} \overline{\frac{\partial f(\mathbf{z})}{\partial z_j}} = t^{a-p_j} \lambda^{b+q_j} \frac{\alpha}{\lambda^{2b}} \frac{\partial f(\mathbf{z})}{\partial \bar{z}_j} = \frac{\alpha}{\lambda^{2b}} \frac{\partial f(t\lambda \bullet \mathbf{z})}{\partial \bar{z}_j}\ ,$$

therefore $t\lambda \bullet \mathbf{z}$ is a critical point of f. □

Finally let us say something about the Condition 7.3 and the property of being polar weighted homogeneous polynomial.

Lemma 7.17. *Let* $f\colon \mathbb{C}^n \to \mathbb{C}$ *be a mixed polynomial. If* f *does not satisfies Condition 7.3, then it cannot be weighted homogeneous with respect to the* $\mathbb{S}^1$*-action.*

Proof. Since f does not satisfies Condition 7.3, then for some j we have that the monomials z_j and $\overline{z}_j$ appear in f. This tells us that we can write f as follows

$$f(\mathbf{z}) = z_j + \overline{z}_j + g(\mathbf{z}),$$

where g is a mixed function.

Suppose that f is weighted homogeneous with respect to the $\mathbb{S}^1$-action, therefore there exist $q_1, \dots, q_n$ non-zero integers and c a positive integer such that $\gcd(q_1, \dots, q_n) = 1$ and f satisfies the following functional equation

$$f(\tau \bullet \mathbf{z}) = \tau^c f(\mathbf{z})\ . \tag{7.2.5}$$

where $\tau \in \mathbb{S}^1$ and

$$\tau \bullet \mathbf{z} = (\tau^{q_1} z_1, \dots, \tau^{q_n} z_n)\ .$$

Now we have that

$$f(\tau \bullet \mathbf{z}) = \tau^{q_j} z_j + \tau^{-q_j} \overline{z}_j + g(\tau \bullet \mathbf{z}) \ .$$

The previous equiality and (7.2.5) tell us that $q_j = c = 0$ but this is impossible by definition. Therefore f cannot be weighted homogeneous with respect to the $\mathbb{S}^1$-action. $\square$

Chapter 8

The classification of polar weighted homogeneous polynomials

Let $f\colon \mathbb{C}^3 \to \mathbb{C}$ be a complex weighted homogeneous polynomial with isolated critical point. Let $V = f^{-1}(0)$ be its zero-set and consider its link given by $K = V \cap \mathbb{S}^5$. It is now a classical result by Orlik and Wagreich [31, §3.1] that the link of such polynomial is equivariantly diffeomorphic to the link of a polynomial in one of six classes given explicitly in the aforementioned paper.

In this chapter we generalize Orlik and Wagreich classes for polar weighted homogeneous polynomials with isolated critical point. The organization of this section is as follows. In Section 8.1 we prove that polar weighted homogeneous polynomials with isolated critical point at the origin under small perturbation of their coefficients remain with isolated critical point (Corollary 8.7). In Section 8.2 we give the classes of mixed polinomials which generalize Orlik and Wagreich classes. In contrast to the complex case, these classes of mixed polynomials are not automatically polar weighted homogeneous, so we compute the explicit conditions for these families to be polar weighted homogeneous with isolated singularity at the origin (Theorem 8.12 and Theorem 8.17). As a result of these computations we list the classes which are full polar weighted homogeneous polynomials (Corollary 8.14). In Section 8.3 we prove that the diffeomorphism type of the link of a polar weighted homogeneous polynomial with isolated singularity at the origin does not change under small perturbation of the coefficients of the polynomial (Theorem 8.20).

8.1 Isolated critical point under perturbation of coefficients

The aim of this section is to prove that given a polar weighted homogeneous polynomial with isolated critical point, with a small perturbation of its coefficients it still has isolated critical point.

Definition 8.1. Let $f = (f_1, \dots, f_m)\colon (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^m, \mathbf{0})$ be a map where $m \le n$ and $f_j\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial for $j \in \{1, \dots, m\}$. Suppose that

$$f_j(\mathbf{x}) = \sum_{l=1}^{k_j} c_{j,l} P_{j,l}(\mathbf{x}) ,$$

where $c_{j,l} \in \mathbb{R}^*$ and $P_{j,l}$ are monomials with coefficient 1.

We can identify the set of coefficients $c_{j,l}$ of f (up to a permutation) with a point in $\mathbb{R}^{k_1+\cdots+k_m}$. Let $\epsilon > 0$ and $\mathbb{B}(\mathbf{0}, \epsilon)$ be the open ball in $\mathbb{R}^{k_1+\cdots+k_m}$ centered at the origin with radius ϵ and let $p \in \mathbb{B}(\mathbf{0}, \epsilon)$ with coordinates $p = (p_{j,l})$, $j = 1, \ldots, m$ and $l = 1, \ldots, k_j$.

We can consider the polynomials

$$f_{j,p}(\mathbf{x}) = \sum_{l=1}^{k_j} (c_{j,l} + p_{j,l}) P_{j,l}(\mathbf{x}) \ ,$$

and the map

$$f_p = (f_{1,p}, \ldots, f_{k,p}) \colon \mathbb{R}^n \to \mathbb{R}^m \ . \tag{8.1.1}$$

Suppose that f has an isolated critical point at the origin. We say that f is *stable under a small perturbation of its coefficients*, if there exist $\epsilon > 0$ small enough such that f_p has an isolated critical point at the origin for all $p \in \mathbb{B}(\mathbf{0}, \epsilon)$.

Remark 8.2. Suppose that f has an isolated critical point at the origin, let $\mathbf{x} \in \mathbb{R}^n$ be a regular point of f and let $M_1, \ldots, M_k$ be all the minors of size $m \times m$ of the Jacobian matrix of f.

Each minor M_j is a polynomial on the variables $x_1, \ldots, x_n$ and if we fix the variables and allow to change the coefficients of f, we have that M_j is also a polynomial on the coefficients $c_{1,1}, \ldots, c_{m,k_m}$.

Therefore we think M_j as a polynomial

$$M_j \colon \mathbb{R}^n \times \mathbb{R}^{k_1+\cdots+k_m} \to \mathbb{R} \ . \tag{8.1.2}$$

That f is stable under a small perturbation of its coefficient is equivalent to say that there exist $\epsilon > 0$ such that for any $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we have $M_j(\mathbf{x}, p) \neq 0$ for some $j \in \{1, \ldots, k\}$ and every $p \in \mathbb{B}(\mathbf{0}, \epsilon) \subset \mathbb{R}^{k_1+\cdots+k_m}$.

The following lemma is a direct consequence of Lemma 7.16.

Lemma 8.3. *Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a radial weighted homogeneous map with $m \leq n$, $(x_1', \ldots, x_n') \in \mathbb{R}^n$ a regular point of f and $M(x_1, \ldots, x_n)$ be a $m \times m$ minor of the Jacobian matrix of f such that $M(x_1', \ldots, x_n') \neq 0$, then $M(t \bullet (x_1', \ldots, x_n')) \neq 0$ for all $t \in \mathbb{R}^+$.*

Proof. Suppose that

$$M(x_1, \ldots, x_n) = \det \begin{pmatrix} \frac{\partial f_{j_1}}{\partial x_{k_1}} & \cdots & \frac{\partial f_{j_1}}{\partial x_{k_m}} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_{j_m}}{\partial x_{k_1}} & \cdots & \frac{\partial f_{j_m}}{\partial x_{k_m}} \end{pmatrix} . \tag{8.1.3}$$

Let $(p_1, \ldots, p_n; a)$ be the radial weights of f. The partials derivatives of f are radial weighted homogeneous and they satisfy

$$\frac{\partial f_i(t \bullet (x_1, \ldots, x_n))}{\partial x_j} = t^{a-p_j} \frac{\partial f_i(x_1, \ldots, x_n)}{\partial x_j}. \tag{8.1.4}$$

Therefore by (8.1.3) and (8.1.4) we have

$$M(t \bullet (x_1, \ldots, x_n)) = t^{ma - p_{k_1} - \ldots p_{k_m}} M(x_1, \ldots, x_n) \ .$$

Therefore $M(t \bullet (x_1', \ldots, x_n')) = t^{ma - p_{k_1} - \ldots p_{k_m}} M(x_1', \ldots, x_n') \neq 0$ for all $t \in \mathbb{R}^+$. □

Corollary 8.4. *Let f be a polar weighted homogeneous polynomial and $\mathbf{z}_0 \in \mathbb{C}$ a regular point of f. Let $M(\mathbf{z})$ be a 2×2 minor of the Jacobian matrix of f, seen as a real analytic map, such that $M(\mathbf{z}_0) \neq 0$, then $M(t \bullet \mathbf{z}_0) \neq 0$ for all $t \in \mathbb{R}^+$.*

Proof. Since any polar polar weighted homogeneous polynomial is a radial weighted homogeneous map, the proof follows from Lemma 8.3. □

Proposition 8.5. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a radial weighted homogeneous map. If f has an isolated critical point at the origin, then f is stable under a small perturbation of its coefficients.*

Proof. Let $\mathbf{x} \in \mathbb{S}_1^{n-1}$. By Lemma 7.16 the origin is the only critical point of f, therefore $\mathbf{x}$ is a regular point of f and there exist a minor $M_{\mathbf{x}}$ of size $m \times m$ with $M_{\mathbf{x}}(\mathbf{x}) \neq 0$ and an open set $U_{\mathbf{x}} \subset \mathbb{R}^n$ such that $\mathbf{x} \in U_{\mathbf{x}}$ and

$$|M_{\mathbf{x}}(\mathbf{x}) - M_{\mathbf{x}}(\mathbf{y})| < \frac{|M_{\mathbf{x}}(\mathbf{x})|}{2}\ , \quad \text{for every } \mathbf{y} \in U_{\mathbf{x}}.$$

Therefore we have a cover of $\mathbb{S}_1^{n-1}$ consisting of $\{U_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{S}_1^{n-1}}$ and since it is compact, we have a finite subcover $\{U_j\}$. Denote by M_j the minor corresponding to the open set U_j and consider M_j as in (8.1.2). Now consider the following function $D_j\colon \overline{U}_j \to \mathbb{R}^+$

$$D_j(\mathbf{y}) = \sup\{\epsilon > 0 \,|\, M_j(\mathbf{y}, p) \neq 0 \text{ for every } p \in \mathbb{B}(\mathbf{0}, \epsilon) \subset \mathbb{R}^{k_1+\cdots+k_m}\}\ ,$$

where $\overline{U}_j$ is the closure of U_j and take $\epsilon_j = \min\{\, D_j(\mathbf{y}) \;|\; \mathbf{y} \in \overline{U}_j \,\}$. Consider $\epsilon = \min\{\epsilon_1, \dots, \epsilon_j\}$, therefore for every $0 < \epsilon' \leq \epsilon$, $\mathbf{y} \in \mathbb{S}_1^{n-1}$ and $p \in \mathbb{B}(\mathbf{0}, \epsilon')$ we have $M_j(\mathbf{y}, p) \neq 0$, for some minor M_j.

Now using Lemma 8.3 we have that the same holds for any $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and any $p \in \mathbb{B}(\mathbf{0}, \epsilon')$ for every $0 < \epsilon' \leq \epsilon$. □

Corollary 8.6. *Let $f_p = (f_{1,p}, \dots, f_{k,p})\colon \mathbb{R}^n \to \mathbb{R}^m$ be a family of radial weighted homogeneous maps as in* (8.1.1)*. Then the subspace U of $\mathbb{R}^{k_1+\cdots+k_m}$ of parameters p for which f_p has an isolated singularity is an open set.*

Corollary 8.7. *If f is a polar weighted homogeneous polynomial with isolated singularity, then f is stable under a small perturbation of its coefficients.*

8.2 Classification of polar weighted homogeneous polynomials in $\mathbb{C}^3$

In this section we want to study polar weighted homogeneous polynomials $f\colon \mathbb{C}^3 \to \mathbb{C}$ with isolated singularity at the origin. We do it in four steps:

First step We define families of mixed polynomials which contain terms which are necessary in order to have isolated singularity.

Second step We give conditions on the exponents of the elements of these families to be polar weighted homogeneous polynomials.

Third step Under a suitable change of coordinates we simplify the coefficient of these families taking them to a *special form.*

Fourth step In the *special form* we give conditions to have isolated singularity.

Following Orlik and Wagreich [31, §3.1] we have the following definition.

Definition 8.8. A mixed function $f(\mathbf{z})$ is said to be of class **I** (respectively **II**,..., **V**) if there is a permutation σ of the set $\{1, 2, 3\}$ and non-zero complex numbers $\alpha_1, \alpha_2, \alpha_3$ such that $f(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})$ is equal to

I. $\alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3}$,

II. $\alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} g_2$,

III. $\alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} g_3 + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} g_2$,

IV. $\alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} g_1 + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} g_2$,

V. $\alpha_1 z_1^{a_1} \bar{z}_1^{b_1} g_2 + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} g_3 + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} g_1$,

where $g_j \in \{z_j, \bar{z}_j\}$ and, a_j and b_j are non-negative integers.

Remark 8.9. By Lemma 7.17 if a mixed polynomial does not satisfies Condition 7.3 it cannot be weighted homogeneous with respect to the $\mathbb{S}^1$-action, therefore polar weighted homogeneous polynomials satisfy Condition 7.3. Definition 8.8 lists all possible polynomials that one can get applying Lemma 7.4 to a mixed function with isolated singularity. Notice that taking $b_i = 0$ and $g_i = z_i$ for $i = 1, 2, 3$ in classes **I** to **V** we recover Orlik and Wagreich classes **I** to **V** of irreducible complex weighted homogeneous polynomials, their class **VI** corresponds to class **III** taking $a_2 = a_3 = 1$.

In contrast with Orlik and Wagreich the classes in Definition 8.8 are not necessarily polar weighted homogeneous, for example,

$$f(\mathbf{z}) = |z_1|^2 + |z_2|^2 + |z_3|^2$$

is a mixed function of class **I** but it is not polar weighted homogeneous. For this reason one has to find the conditions that the a_i and b_i should satisfy in order to get a polar weighted homogeneous polynomial.

Remark 8.10. Using the change of coordinates $z_i \mapsto \bar{z}_i$ we can always assume that $g_i = z_i$ but in this case $a_i - b_i$ can be positive, negative or zero. Hereafter we assume that $g_j = z_j$.

We will frequently use the following basic lemma.

Lemma 8.11. *Let $z \in \mathbb{C}$ and $t \in \mathbb{R}$. If $z + t\bar{z}$ belongs to $\mathbb{R}$, then $t = 1$ or $z \in \mathbb{R}$.*

Theorem 8.12. *Let f be a mixed function of one of the classes of Definition 8.8. Then the following conditions must be satisfied in order to f be polar weighted homogeneous:*

Class I *$a_j - b_j \neq 0$ with $j = 1, 2, 3$.*

Class II

a) *$a_j - b_j \neq 0$ with $j = 1, 2, 3$ and $a_2 \pm b_2 \neq 1$.*

b) *$a_1 - b_1 \neq 0$, $a_2 - b_2 = 1$, $b_2 \neq 0$ and $a_3 = b_3$.*

Class III *$a_1 - b_1 \neq 0$ and $a_2 - b_2$, $a_3 - b_3$ are not both -1. Also:*

a) *$a_2 \pm b_2$ and $a_3 \pm b_3$ are not 1.*

b) *$(a_2 + b_2)(a_3 + b_3) > 1$, $a_2 - b_2 = 1$ and $a_3 - b_3 = 1$.*

c) *$a_2 = a_3 = 1$ and $b_2 = b_3 = 0$.*

Class IV

a) *$a_i - b_i \neq 0$ for $i = 1, 2, 3$, $a_1 \pm b_1 \neq 1$ and $(a_1, a_2) \neq (b_1 - 1, b_2 + 2)$.*

b) *$a_2 = b_2$, $a_1 - b_1 = 1$ and $b_1 \neq 0$.*

c) *$a_3 = b_3$, $a_1 - b_1 \neq 0$, $a_1 + b_1 > 1$ and $(a_1, a_2) = (b_1 - 1, b_2 + 2)$.*

Class V *$(a_1 - b_1)(a_2 - b_2)(a_3 - b_3) \neq -1$ and*

$$\begin{cases} (a_{i-1}, a_{i+1}) \neq (b_{i-1} + 1, b_{i+1}), \\ (a_{i-1}, a_{i+1}) \neq (b_{i-1} - 1, b_{i+1} + 2), \end{cases} \qquad i = 1, 2, 3.$$

Proof.

Class I Suppose that
$$f(\mathbf{z}) = \alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} \ .$$

In this case the radial and angular matrices are

$$P = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 0 & a_2+b_2 & 0 \\ 0 & 0 & a_3+b_3 \end{pmatrix} \ , \quad Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 0 & a_2-b_2 & 0 \\ 0 & 0 & a_3-b_3 \end{pmatrix} \ .$$

In order to f be polar weighted homogeneous we want to find solutions to the system

$$P(p_1, p_2, p_3)^\top = (1, 1, 1)^\top \ , \qquad Q(q_1, q_2, q_3)^\top = (1, 1, 1)^\top \ ,$$

with $p_j \in \mathbb{Q}^+$ and $q_j \in \mathbb{Q} \setminus \{0\}$.

If some $a_j - b_j = 0$, then we can not solve the system, therefore $a_j - b_j \neq 0$ for $j = 1, 2, 3$.

The solution of the system give us the normalized radial and angular weights so we need to take m, m', M and M' to get the weights.

Class II Suppose that
$$f(\mathbf{z}) = \alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} z_2 \ .$$

In this case the radial and angular matrices are

$$P = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 0 & a_2+b_2 & 0 \\ 0 & 1 & a_3+b_3 \end{pmatrix} \ , \quad Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 0 & a_2-b_2 & 0 \\ 0 & 1 & a_3-b_3 \end{pmatrix} \ .$$

We have two cases:

a) The easiest case is that P and Q are invertible. In this case we have that

$$a_j - b_j \neq 0 \ , \qquad j = 1, 2, 3 \ .$$

The weights are just the solution of the system given by P and Q as in the previous case.

b) By construction, P is always invertible, so suppose that Q is not invertible but the system has solution. In this case we have that $a_1 - b_1$, $a_2 - b_2 \neq 0$ and $a_3 - b_3 = 0$. Therefore we have

$$Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 0 & a_2-b_2 & 0 \\ 0 & 1 & 0 \end{pmatrix} ,$$

and since we want $Q(q_1, q_2, q_3)^\top = (1,1,1)^\top$, therefore we have $q_2 = 1$ and $a_2 - b_2 = 1$. Under this assumptions we have

$$P = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 0 & 2a_2-1 & 0 \\ 0 & 1 & 2a_3 \end{pmatrix} ,$$

although the matrix is invertible, if $b_2 = 0$ (i.e. $a_2 = 1$), then $p_2 = 1$, so p_3 must be 0 but we do not allow this kind of solutions. Therefore $b_2 \neq 0$. If f satisfies the aforementioned conditions, then it is polar weighted homogeneous and the weights are just the solutions to the systems given by P and Q.

Class III Suppose that

$$f(\mathbf{z}) = \alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} z_3 + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} z_2 .$$

In this case the radial and angular matrices are

$$P = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 0 & a_2+b_2 & 1 \\ 0 & 1 & a_3+b_3 \end{pmatrix} , \quad Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 0 & a_2-b_2 & 1 \\ 0 & 1 & a_3-b_3 \end{pmatrix} .$$

We have basically two cases:

a) If P and Q are invertible, then we have

$$\begin{aligned} \det P &= (a_1 + b_1)\left((a_2 + b_2)(a_3 + b_3) - 1\right) \neq 0 , \\ \det Q &= (a_1 - b_1)\left((a_2 - b_2)(a_3 - b_3) - 1\right) \neq 0 \end{aligned}$$

and since we always have $a_1 + b_1 \neq 0$, then the conditions are

$$\begin{aligned} (a_2 + b_2)(a_3 + b_3) &\neq 1 , \\ (a_1 - b_1)\left((a_2 - b_2)(a_3 - b_3) - 1\right) &\neq 0 . \end{aligned}$$

b) Suppose Q is not invertible but P it is invertible. Since $a_1 - b_1$ must be different from 0, then suppose that $(a_2 - b_2)(a_3 - b_3) = 1$.
We have two cases, the first one is $a_2 - b_2 = a_3 - b_3 = -1$, then

$$Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} ,$$

but in this case we can not find q_2 and q_3 such that $q_2 - q_3 = 1$ and $q_3 - q_2 = 1$, so we do not have to consider this case.
The second case is $a_2 - b_2 = a_3 - b_3 = 1$. Under this assumption, we have

$$Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} .$$

The angular weights are deduced from this matrix and the additional condition comes from the requirement to have non-zero integers.

c) Suppose P is not invertible but Q it is invertible, therefore $a_2 + b_2 = a_3 + b_3 = 1$ and the only solutions to this equation are:

$$\begin{aligned} &a_2 = a_3 = 1 \text{ and } b_2 = b_3 = 0, \\ &a_2 = b_3 = 1 \text{ and } b_2 = a_3 = 0, \\ &a_3 = b_2 = 1 \text{ and } a_2 = b_3 = 0, \\ &a_2 = a_3 = 0 \text{ and } b_2 = b_3 = 1. \end{aligned}$$

and since $(a_2 - b_2)(a_3 - b_3) \neq 1$, then the solutions are

$$\begin{aligned} &a_2 = b_3 = 1 \text{ and } b_2 = a_3 = 0, \\ &a_3 = b_2 = 1 \text{ and } a_2 = b_3 = 0, \end{aligned}$$

consider the equations

$$a_2 = b_3 = 1 \text{ and } b_2 = a_3 = 0,$$

therefore

$$P = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

but in this case $q_2 = 1$ and $q_3 = 0$ but all the angular weights must be non-zero rational numbers, therefore this case does not happen. The case $a_3 = b_2 = 1$ and $a_2 = b_3 = 0$ does not happen by an analogous argument.

d) Suppose P and Q are not invertible. Using the last ideas we have that the only solution (up to a change of coordinates) is

$$a_2 = a_3 = 1 \text{ and } b_2 = b_3 = 0,$$

therefore

$$P = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

therefore f is polar weighted homogeneous.

Class IV Suppose that

$$f(\mathbf{z}) = \alpha_1 z_1^{a_1} \bar{z}_1^{b_1} + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} z_1 + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} z_2 .$$

In this case the radial and angular matrices are

$$P = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 1 & a_2+b_2 & 0 \\ 0 & 1 & a_3+b_3 \end{pmatrix}, \quad Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 1 & a_2-b_2 & 0 \\ 0 & 1 & a_3-b_3 \end{pmatrix}.$$

a) Suppose P and Q are invertible, then $a_j - b_j \neq 0$.
Now if $a_1 + b_1 = 1$, then $p_1 = 1$ but this gives us that $p_2 = 0$ and this can not happen. Using the same idea we can check that $a_1 - b_1 \neq 1$.
Consider the system $Q(q_1, q_2, q_3)^\top = (1, 1, 1)^\top$, solving this system we get that

$$q_3 = \frac{(a_1 - b_1)(a_2 - b_2 - 1) + 1}{(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)},$$

and since q_3 can not be 0, then $(a_1 - b_1)(a_2 - b_2 - 1) \neq -1$. Since P is always invertible, hence we just have to consider the following cases:

b) Suppose $a_2 = b_2$, then the angular matrix is

$$Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & a_3-b_3 \end{pmatrix} ,$$

hence $a_1 - b_1$ must be 1. Notice that if $b_1 = 0$, then the radial matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2a_2 & 0 \\ 0 & 1 & a_3+b_3 \end{pmatrix} ,$$

but this implies that $p_2 = 0$ but this can not happen.

c) Suppose $a_2 \neq b_2$ and $a_3 = b_3$. Then the angular matrix is

$$Q = \begin{pmatrix} a_1-b_1 & 0 & 0 \\ 1 & a_2-b_2 & 0 \\ 0 & 1 & 0 \end{pmatrix} ,$$

therefore $q_2 = 1$ and since $q_1 = \frac{1}{a_1-b_1}$, then we have $1 + (a_2 - b_2)(a_1 - b_1) = (a_1 - b_1)$. The radial matrix is

$$P = \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 1 & a_2+b_2 & 0 \\ 0 & 1 & 2a_3 \end{pmatrix} ,$$

notice that if $a_1 + b_1 = 1$, then again $p_2 = 0$ but this can not happen.

Class V Suppose that

$$f(\mathbf{z}) = \alpha_1 z_1^{a_1} \bar{z}_1^{b_1} z_2 + \alpha_2 z_2^{a_2} \bar{z}_2^{b_2} z_3 + \alpha_3 z_3^{a_3} \bar{z}_3^{b_3} z_1 .$$

In this case the radial and angular matrices are

$$P = \begin{pmatrix} a_1+b_1 & 1 & 0 \\ 0 & a_2+b_2 & 1 \\ 1 & 0 & a_3+b_3 \end{pmatrix} , \quad Q = \begin{pmatrix} a_1-b_1 & 1 & 0 \\ 0 & a_2-b_2 & 1 \\ 1 & 0 & a_3-b_3 \end{pmatrix} .$$

We have that

$$\det P = (a_1 + b_1)(a_2 + b_2)(a_3 + b_3) + 1 ,$$
$$\det Q = (a_1 - b_1)(a_2 - b_2)(a_3 - b_3) + 1 ,$$

hence P is always invertible. Suppose that Q is invertible, therefore

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} (a_2+b_2)(a_3+b_3) & -(a_3+b_3) & 1 \\ 1 & (a_1+b_1)(a_3+b_3) & -(a_1+b_1) \\ -(a_2+b_2) & 1 & (a_1+b_1)(a_2+b_2) \end{pmatrix} ,$$
$$Q^{-1} = \frac{1}{\det Q} \begin{pmatrix} (a_2-b_2)(a_3-b_3) & -(a_3-b_3) & 1 \\ 1 & (a_1-b_1)(a_3-b_3) & -(a_1-b_1) \\ -(a_2-b_2) & 1 & (a_1-b_1)(a_2-b_2) \end{pmatrix} ,$$

and since the normalized weights satisfy

$$(p_1, p_2, p_3)^\top = P^{-1}(1, 1, 1)^\top$$
$$(q_1, q_2, q_3)^\top = Q^{-1}(1, 1, 1)^\top,$$

hence

$$p_j = \frac{(a_{j-1} + b_{j-1})(a_{j+1} + b_{j+1} - 1) + 1}{\det P} ,$$
$$q_j = \frac{(a_{j-1} - b_{j-1})(a_{j+1} - b_{j+1} - 1) + 1}{\det Q} ,$$

for $j = 1, 2, 3 \mod 3$.

Since q_j must be non zero, then

$$(a_{j-1} - b_{j-1})(a_{j+1} - b_{j+1} - 1) \neq -1 ,$$

for $j = 1, 2, 3 \mod 3$.

If $\det Q = 0$, then the only solutions (up to a change of coordinates) are

$$a_1 - b_1 = -1 \text{ and } a_2 - b_2 = a_3 - b_3 = 1 ,$$
$$a_1 - b_1 = a_2 - b_2 = a_3 - b_3 = -1 .$$

In both cases the system given by Q has no solutions.

$\square$

Corollary 8.13. *Let $f : \mathbb{C}^3 \to \mathbb{C}$ be a polar weighted homogeneous polynomial belonging to some class of Theorem 8.12 and $n \in \mathbb{Q} \setminus \{0\}$. Then its normalized radial and angular weights are given by:*

Class I $p'_i := \frac{1}{a_i+b_i}$, $\quad q'_i := \frac{1}{a_i-b_i}$.

Class II.a

$$p'_1 := \frac{1}{a_1+b_1} , \qquad p'_2 := \frac{1}{a_2+b_2} , \qquad p'_3 := \frac{a_2+b_2-1}{(a_2+b_2)(a_3+b_3)} ,$$
$$q'_1 := \frac{1}{a_1-b_1} , \qquad q'_2 := \frac{1}{a_2-b_2} , \qquad q'_3 := \frac{a_2-b_2-1}{(a_2-b_2)(a_3-b_3)} .$$

Class II.b

$$p'_1 := \frac{1}{a_1+b_1} , \qquad p'_2 := \frac{1}{1+2b_2} , \qquad p'_3 := \frac{b_2}{(1+2b_2)b_3} ,$$
$$q'_1 := \frac{1}{a_1-b_1} , \qquad q'_2 := 1 , \qquad q'_3 := \frac{n}{a_1-b_1}.$$

Class III.a

$$p'_1 := \frac{1}{a_1+b_1} , \qquad p'_2 := \frac{1-(a_3+b_3)}{(1-(a_2+b_2)(a_3+b_3))} , \qquad p'_3 := \frac{1-(a_2+b_2)}{(1-(a_2+b_2)(a_3+b_3))} ,$$
$$q'_1 := \frac{1}{a_1-b_1} , \qquad p'_2 := \frac{1-(a_3-b_3)}{(1-(a_2-b_2)(a_3-b_3))} , \qquad p'_3 := \frac{1-(a_2-b_2)}{(1-(a_2-b_2)(a_3-b_3))} .$$

Class III.b *The radial weights are the same as in **Class III.a** but*

$$q'_1 := \frac{1}{a_1-b_1}, \quad q'_2 := \frac{a_1-b_1-n}{a_1-b_1}, \quad q'_3 := \frac{n}{a_1-b_1}, \quad n \neq a_1 - b_1.$$

Class III.c *The angular weights are the same as in **Class III.a** but*

$$p'_1 := \frac{1}{a_1+b_1}, \quad p'_2 := \frac{a_1+b_1-n}{a_1+b_1}, \quad p'_3 := \frac{n}{a_1+b_1}, \quad 1 \leq n \leq a_1+b_1-1.$$

Class IV.a

$$p'_1 := \tfrac{1}{a_1+b_1}\ , \qquad p'_2 := \tfrac{a_1+b_1-1}{(a_1+b_1)(a_2+b_2)}\ , \qquad p'_3 := \tfrac{(a_1+b_1)(a_2+b_2-1)+1}{(a_1+b_1)(a_2+b_2)(a_3+b_3)}\ ,$$
$$q'_1 := \tfrac{1}{a_1-b_1}\ , \qquad q'_2 := \tfrac{a_1-b_1-1}{(a_1-b_1)(a_2-b_2)}\ , \qquad q'_3 := \tfrac{(a_1-b_1)(a_2-b_2-1)+1}{(a_1-b_1)(a_2-b_2)(a_3-b_3)}\ .$$

Class IV.b

$$p'_1 := \tfrac{1}{1+2b_1}\ , \quad p'_2 := \tfrac{b_1}{(1+2b_1)a_2}\ , \quad p'_3 := \tfrac{a_2(1+2b_1)-b_1}{(1+2b_1)a_2(a_3+b_3)}\ ,$$
$$q'_1 := 1\ , \quad q'_2 := 1 - q'_3(a_3-b_3)\ , \quad q'_3 \in \mathbb{Q} \text{ with } q'_3 \neq 0 \text{ and } q'_3(a_3-b_3) \neq 1.$$

Class IV.c

$$p'_1 := \tfrac{1}{a_1+b_1}\ , \qquad p'_2 := \tfrac{a_1+b_1-1}{(a_1+b_1)(a_2+b_2)}\ , \qquad p'_3 := \tfrac{(a_1+b_1)(a_2+b_2-1)+1}{(a_1+b_1)(a_2+b_2)2b_3}\ ,$$
$$q'_1 := \frac{1}{a_1-b_1}\ , \qquad q'_2 := 1\ , \qquad q'_3 := \frac{n}{a_1-b_1}.$$

Class V *Define*

$$\begin{aligned} r_i &:= (1+(a_i+b_i)(a_{i+1}+b_{i+1}) - (a_{i-1}+b_{i-1})) \\ s_i &:= (1+(a_i-b_i)(a_{i+1}-b_{i+1}) - (a_{i-1}-b_{i-1})), \end{aligned}$$

for $i = 1, 2, 3$. *Then*

$$p'_i := \tfrac{r_i}{1+(a_1+b_1)(a_2+b_2)(a_3+b_3)} \qquad q'_i := \tfrac{s_i}{1+(a_1-b_1)(a_2-b_2)(a_3-b_3)}\ .$$

Proof. Solving the systems appearing in the proof of Theorem 8.12 for each of the classes. □

Corollary 8.14. *Let $f\colon \mathbb{C}^3 \to \mathbb{C}$ be a polar weighted homogeneous polynomial belonging to some class of Theorem 8.12. Then*

1. *f is full if it is of one of the classes:* ***I***, ***II.a***, ***III.a***, ***IV.a***, ***V***.
2. *f is radial full if it is of one of the classes:* ***II.b***, ***III.b***, ***IV.b***, ***IV.c***.

Corollary 8.15. *Let $f\colon \mathbb{C}^3 \to \mathbb{C}$ be a polar weighted homogeneous polynomial belonging to some class of Theorem 8.12. Then there exists a change of coordinates such that we get:*

Class I

$$z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2} + z_3^{a_3}\bar{z}_3^{b_3}.$$

Class II.a

$$z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2} + z_3^{a_3}\bar{z}_3^{b_3}z_2.$$

Class II.b

$$z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2} + \tau z_3^{a_3}\bar{z}_3^{b_3}z_2, \quad \tau \in \mathbb{S}^1.$$

Class III.a

$$z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}z_2.$$

Class III.b

$$z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2}z_3 + \tau z_3^{a_3}\bar{z}_3^{b_3}z_2, \quad \tau \in \mathbb{S}^1.$$

Class III.c

$$z_1^{a_1}\bar{z}_1^{b_1} + z_2 z_3.$$

Class IV.a

$$z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2}z_1 + z_3^{a_3}\bar{z}_3^{b_3}z_2.$$

Class IV.b

$$z_1^{a_1}\bar{z}_1^{b_1} + \tau z_2^{a_2}\bar{z}_2^{b_2}z_1 + z_3^{a_3}\bar{z}_3^{b_3}z_2, \quad \tau \in \mathbb{S}^1.$$

Class IV.c

$$z_1^{a_1}\bar{z}_1^{b_1} + \tau z_2^{a_2}\bar{z}_2^{b_2}z_1 + z_3^{a_3}\bar{z}_3^{b_3}z_2, \quad \tau \in \mathbb{S}^1$$

Class V

$$z_1^{a_1}\bar{z}_1^{b_1}z_2 + z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}z_1.$$

PROOF. By Corollary 8.14 the polynomials in classes **I**, **II.a**, **III.a**, **IV.a**, **V** are full, so just apply Lemma 7.8 and Lemma 7.9. Also by Corollary 8.14 the polynomials in classes **II.b**, **III.b**, **IV.b**, **IV.c** are only radial full. Applying Lemma 7.8 we can assume that all the coefficients are in $\mathbb{S}^1$. Now, consider for instance Class **IV.c**. Its angular matrix has rows pairwise linearly independent, hence, applying Lemma 7.9 we can make any two coefficients equal to 1. Suppose $\alpha_1 = \alpha_2 = 1$, taking the change of coordinates $(z_1, z_2, z_3) \mapsto (z_1, \bar{\alpha}_3 z_2, z_3)$ we get

$$z_1^{a_1}\bar{z}_1^{b_1} + \alpha_3^{b_2-a_2}z_2^{a_2}\bar{z}_2^{b_2}z_1 + z_3^{a_3}\bar{z}_3^{b_3}z_2.$$

For $\alpha_2 = \alpha_3 = 1$, let $\alpha_1 = e^{i\theta}$ and let $\tau = e^{i\frac{\theta}{a_1-b_1}}$. Taking the change of coordinates $(z_1, z_2, z_3) \mapsto (\tau z_1, z_2, z_3)$ we get the expresion we want.

The other classes which are radial full are analogous. $\square$

Definition 8.16. Each of the polynomials given in Corollary 8.15 are called the *special representative* of its corresponding subclass.

Theorem 8.17. *Let $f\colon \mathbb{C}^3 \to \mathbb{C}$ be the special representative of some subclass. Then*

1. *If f is of one of the classes **I**, **II.a**, **III.a**, **III.c**, **IV.a** or **V**, then f has an unique singularity at the origin.*
2. *If f is of one of the classes **II.b**, **III.b** or **IV.b**, then f has an unique singularity at the origin if and only if $\tau \neq -1$.*
3. *If f is of the classe **IV.c** then f has an unique singularity if and only if $\tau \neq 1$.*

PROOF.

Class I We have

$$\begin{aligned} \overline{df(\mathbf{z})} &= (a_1 \bar{z}_1^{a_1-1} z_1^{b_1}, a_2 \bar{z}_2^{a_2-1} z_2^{b_2}, a_3 \bar{z}_3^{a_3-1} z_3^{b_3})\ , \\ \bar{d}f(\mathbf{z}) &= (b_1 z_1^{a_1} \bar{z}_1^{b_1-1}, b_2 z_2^{a_2} \bar{z}_2^{b_2-1}, b_3 z_3^{a_3} \bar{z}_3^{b_3-1})\ . \end{aligned} \tag{8.2.1}$$

Suppose that (z_1, z_2, z_3) is a critical point of f, then by Proposition 7.2 there exist $\alpha \in \mathbb{S}^1$ such that

$$\overline{df(\mathbf{z})} = \alpha \bar{d}f(\mathbf{z})\ .$$

The previous equality and (8.2.1) give us the following system

$$\begin{aligned} a_1 \bar{z}_1^{a_1-1} z_1^{b_1} &= \alpha b_1 z_1^{a_1} \bar{z}_1^{b_1-1}\ , \\ a_2 \bar{z}_2^{a_2-1} z_2^{b_2} &= \alpha b_2 z_2^{a_2} \bar{z}_2^{b_2-1}\ , \\ a_3 \bar{z}_3^{a_3-1} z_3^{b_3} &= \alpha b_3 z_3^{a_3} \bar{z}_3^{b_3-1}\ . \end{aligned} \tag{8.2.2}$$

Suppose $z_j \neq 0$ for some $j \in \{1, 2, 3\}$, then by (8.2.2) we have

$$a_j \bar{z}_j^{a_j-1} z_j^{b_j} = \alpha b_j z_j^{a_j} \bar{z}_j^{b_j-1}\ ,$$

computing the norm we have

$$a_j |z_j|^{a_j+b_j-1} = b_j |z_j|^{a_j+b_j-1}\ ,$$

so $a_j = b_j$ which can not occur. Then $\mathbf{0}$ is the only critical point.

Class II.a We have that $a_j - b_j \neq 0$ with $j = 1, 2, 3$ and $a_2 \pm b_2 \neq 1$. Again, computing $\overline{df(\mathbf{z})}$, $\bar{d}f(\mathbf{z})$ and using Proposition 7.2, there exist $\alpha \in \mathbb{S}^1$ which gives the following system

$$a_1 \bar{z}_1^{a_1-1} z_1^{b_1} = \alpha b_1 z_1^{a_1} \bar{z}_1^{b_1-1}\ , \tag{8.2.3}$$

$$a_2 \bar{z}_2^{a_2-1} z_2^{b_2} + \bar{z}_3^{a_3} z_3^{b_3} = \alpha b_2 z_2^{a_2} \bar{z}_2^{b_2-1}\ , \tag{8.2.4}$$

$$a_3 \bar{z}_3^{a_3-1} z_3^{b_3} \bar{z}_2 = \alpha b_3 z_3^{a_3} \bar{z}_3^{b_3-1} z_2\ . \tag{8.2.5}$$

Using the ideas of the previous case, it is clear that $z_1 = 0$. Now suppose that $z_2 \neq 0$, we have two cases:

1. If $z_3 \neq 0$, considering equation (8.2.5) and using the norm we get that $a_3 = b_3$ which can not occur.
2. If $z_3 = 0$, considering equation (8.2.4) and again using the norm we get $a_2 = b_2$.

We conclude that $z_2 = 0$ and using equation (8.2.4) we get that $z_3 = 0$. Therefore the only critical point is $\mathbf{0}$.

Class II.b We have that $a_1 \neq b_1$, $a_2 - b_2 = 1$, $a_3 = b_3$ and $b_2 \neq 0$. By Proposition 7.2, there exist $\alpha \in \mathbb{S}^1$ such that

$$a_1 \bar{z}_1^{a_1-1} z_1^{b_1} = \alpha b_1 z_1^{a_1} \bar{z}_1^{b_1-1}\ , \tag{8.2.6}$$

$$a_2 \bar{z}_2^{a_2-1} z_2^{b_2} + \bar{\tau} \bar{z}_3^{a_3} z_3^{b_3} = \alpha b_2 z_2^{a_2} \bar{z}_2^{b_2-1}\ , \tag{8.2.7}$$

$$\bar{\tau} a_3 \bar{z}_3^{a_3-1} z_3^{b_3} \bar{z}_2 = \alpha \tau b_3 z_3^{a_3} \bar{z}_3^{b_3-1} z_2\ . \tag{8.2.8}$$

Again, we have that $z_1 = 0$ and we can simplify the equations to get

$$\begin{aligned} a_2|z_2|^{2(a_2-1)} + \bar{\tau}|z_3|^{2a_3} &= \alpha(a_2-1)z_2^{a_2}\bar{z}_2^{a_2-2} \,, \\ \bar{\tau}a_3|z_3|^{2(a_3-1)}z_3\bar{z}_2 &= \alpha\tau a_3 z_3|z_3|^{2(a_3-1)}z_2 \,. \end{aligned} \tag{8.2.9}$$

If $z_3 = 0$, then

$$a_2|z_2|^{2(a_2-1)} = \alpha(a_2-1)z_2^{a_2}\bar{z}_2^{a_2-2} \,,$$

therefore z_2 must be 0.

If $z_2 = 0$, then

$$\bar{\tau}|z_3|^{2a_3} = 0 \,,$$

therefore z_3 must be 0.

Now suppose $z_2, z_3 \neq 0$. We can simplify equations (8.2.9) to the following equations

$$\begin{aligned} a_2|z_2|^{2(a_2-1)} + \bar{\tau}|z_3|^{2a_3} &= \alpha(a_2-1)z_2^{a_2}\bar{z}_2^{a_2-2} \,, \\ \bar{\tau}\bar{z}_2 &= \alpha\tau z_2 \,, \end{aligned}$$

so we get

$$\tau a_2|z_2|^{2(a_2-1)} + \tau\bar{\tau}|z_3|^{2a_3} = \bar{\tau}(a_2-1)|z_2|^{2(a_2-1)} \,,$$

therefore

$$|z_2|^{2(a_2-1)}(\tau a_2 - \bar{\tau}(a_2-1)) + |z_3|^{2a_3} = 0 \,, \tag{8.2.10}$$

in particular $\tau a_2 - \bar{\tau}(a_2-1)$ must be a real number and by Lemma 8.11 we have that $\tau \in \mathbb{R}$, hence $\tau = \pm 1$.

We can simplify equation (8.2.10) to

$$|z_2|^{2(a_2-1)}\tau + |z_3|^{2a_3} = 0 \,, \tag{8.2.11}$$

so we have that:

1. If $\tau \neq -1$, then the only critical point is the origin.
2. If $\tau = -1$, then f does not have isolated singularity, for instance the point $(0,1,1)$ satisfies (8.2.11) and by Lemma 7.16 all the points of the form $(0, t^{p_2}, t^{p_3})$ for $t \in \mathbb{R}^+$ are singular.

Class III.a Suppose that (z_1, z_2, z_3) is a critical point of f. We have that $a_1 - b_1 \neq 0$ and $a_2 - b_2$, $a_3 - b_3$ are not both -1, also $a_2 \pm b_2$ and $a_3 \pm b_3$ are not 1. By Proposition 7.2, there exist $\alpha \in \mathbb{S}^1$ such that

$$\begin{aligned} a_1\bar{z}_1^{a_1-1}z_1^{b_1} &= \alpha b_1 z_1^{a_1}\bar{z}_1^{b_1-1} \,, \\ a_2\bar{z}_2^{a_2-1}z_2^{b_2}\bar{z}_3 + \bar{z}_3^{a_3}z_3^{b_3} &= \alpha b_2 z_2^{a_2}\bar{z}_2^{b_2-1}z_3 \,, \\ \bar{z}_2^{a_2}z_2^{b_2} + a_3\bar{z}_3^{a_3-1}z_3^{b_3}\bar{z}_2 &= \alpha b_3 z_3^{a_3}\bar{z}_3^{b_3-1}z_2 \,. \end{aligned} \tag{8.2.12}$$

As before, we have that $z_1 = 0$. Since $a_2 + b_2$ and $a_3 + b_3$ are not 1, then $z_2 = 0$ if and only if $z_3 = 0$.

For $z_1 = 0$ and $z_2 z_3 \neq 0$ by Theorem 7.6 and Corollary 7.7 the point $(0, z_2, z_3)$ is not a critical point of $z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}z_2$.

Class III.b We have $a_1 - b_1 \neq 0$ and $a_2 - b_2$, $a_3 - b_3$ are not both -1, also $a_2 + b_2$, $a_3 + b_3$ are not 1 and $a_2 - b_2 = a_3 - b_3 = 1$. The set of equations given by Proposition 7.2 is also given by (8.2.12). As before, the first equation implies that $z_1 = 0$.

Suppose that $(0, z_2, z_3)$ is a critical point of f, we can simply equations (8.2.12) to get

$$a_2 \bar{z}_2^{a_2-1} z_2^{a_2-1} \bar{z}_3 + \bar{z}_3^{a_3} z_3^{a_3-1} = \alpha(a_2 - 1) z_2^{a_2} \bar{z}_2^{a_2-2} z_3 \; , \tag{8.2.13}$$

$$\bar{z}_2^{a_2} z_2^{a_2-1} + a_3 \bar{z}_3^{a_3-1} z_3^{a_3-1} \bar{z}_2 = \alpha(a_3 - 1) z_3^{a_3} \bar{z}_3^{a_3-2} z_2 \; . \tag{8.2.14}$$

It is clear that $z_2 = 0$ if and only if $z_3 = 0$.

Suppose $z_2, z_3 \neq 0$. Now we have that

$$\begin{aligned} f(0, z_2, z_3) &= z_2 z_3 (z_2^{a_2-1} \bar{z}_2^{a_2-1} + \tau z_3^{a_3-1} \bar{z}_3^{a_3-1}) \; , \\ &= z_2 z_3 (|z_2|^{2(a_2-1)} + \alpha |z_3|^{2(a_3-1)}) = 0 \; . \end{aligned}$$

Just as for the class **II.b** we have that if $\alpha = -1$, then $(0, t^{p_2}, t^{p_3})$ are singular points of f. Therefore f has an isolated singularity if and only if $\tau \neq -1$.

Class III.c It is immediate.

Class IV.a We have that $a_j - b_j \neq 0$, $a_1 \pm b_1 \neq 1$ and $(a_1 - b_1)(a_2 - b_2 - 1) \neq -1$. We have the following equations

$$a_1 \bar{z}_1^{a_1-1} z_1^{b_1} + \bar{z}_2^{a_2} z_2^{b_2} = \alpha b_1 z_1^{a_1} \bar{z}_1^{b_1-1} \; , \tag{8.2.15}$$

$$a_2 \bar{z}_2^{a_2-1} z_2^{b_2} \bar{z}_1 + \bar{z}_3^{a_3} z_3^{b_3} = \alpha b_2 z_2^{a_2} \bar{z}_2^{b_2-1} z_1 \; , \tag{8.2.16}$$

$$a_3 \bar{z}_3^{a_3-1} z_3^{b_3} \bar{z}_2 = \alpha b_3 z_3^{a_3} \bar{z}_3^{b_3-1} z_2 \; . \tag{8.2.17}$$

If $z_1 = 0$, then by (8.2.15) and (8.2.16)

$$\begin{aligned} \bar{z}_2^{a_2} z_2^{b_2} &= 0 \; , \\ \bar{z}_3^{a_3} z_3^{b_3} &= 0 \; , \end{aligned}$$

therefore $z_2 = z_3 = 0$.

If $z_2 = 0$, then by (8.2.15)

$$a_1 \bar{z}_1^{a_1-1} z_1^{b_1} = \alpha b_1 z_1^{a_1} \bar{z}_1^{b_1-1} \; ,$$

which implies

$$a_1 |z_1|^{a_1+b_1-1} = b_1 |z_1|^{a_1+b_1-1} \; ,$$

and this only happens if $z_1 = 0$.

If $z_3 = 0$, then by (8.2.16)

$$a_2 \bar{z}_2^{a_2-1} z_2^{b_2} \bar{z}_1 = \alpha b_2 z_2^{a_2} \bar{z}_2^{b_2-1} z_1 \; ,$$

therefore

$$a_2 |z_2|^{a_2+b_2-1} |z_1| = b_2 |z_2|^{a_2+b_2-1} |z_1| \; ,$$

so $z_1 = 0$ or $z_2 = 0$.

Finally suppose z_1, z_2, z_3 are not 0, then by (8.2.17)

$$a_3|z_3|^{a_3+b_3-1}|z_2| = b_3|z_3|^{a_3+b_3-1}|z_2| \ ,$$

but this implies $a_3 = b_3$.

Therefore f has an isolated singularity at the origin.

Class IV.b We have that $a_2 = b_2$, $a_3 - b_3 \neq 0$, $a_1 - b_1 = 1$ and $b_1 \neq 0$. Using the ideas of the previous cases, we have

$$a_1\bar{z}_1^{a_1-1}z_1^{b_1} + \bar{\tau}\bar{z}_2^{a_2}z_2^{b_2} = \alpha b_1 z_1^{a_1}\bar{z}_1^{b_1-1} \ , \tag{8.2.18}$$
$$\bar{\tau}a_2\bar{z}_2^{a_2-1}z_2^{b_2}\bar{z}_1 + \bar{z}_3^{a_3}z_3^{b_3} = \alpha\tau b_2 z_2^{a_2}\bar{z}_2^{b_2-1}z_1 \ , \tag{8.2.19}$$
$$a_3\bar{z}_3^{a_3-1}z_3^{b_3}\bar{z}_2 = \alpha b_3 z_3^{a_3}\bar{z}_3^{b_3-1}z_2 \ . \tag{8.2.20}$$

Let (z_1, z_2, z_3) be a critical point of f.

If z_1, z_2, z_3 are not 0, then by (8.2.20)

$$a_3|z_3|^{a_3+b_3-1}|z_2| = b_3|z_3|^{a_3+b_3-1}|z_2| \ ,$$

so $a_3 = b_3$ but this can not happen.

Suppose $z_1 = 0$, then by (8.2.18)

$$\bar{\tau}\bar{z}_2^{a_2}z_2^{b_2} = 0 \ ,$$

so $z_2 = 0$ and by (8.2.19)

$$\bar{z}_3^{a_3}z_3^{b_3} = 0 \ ,$$

therefore $z_3 = 0$.

If $z_2 = 0$, then by (8.2.18)

$$a_1\bar{z}_1^{a_1-1}z_1^{b_1} = \alpha b_1 z_1^{a_1}\bar{z}_1^{b_1-1} \ ,$$

hence

$$a_1|z_1|^{a_1+b_1-1} = b_1|z_1|^{a_1+b_1-1} \ ,$$

but this only happen if $z_1 = 0$ and therefore $z_3 = 0$.

If $z_3 = 0$ and $z_1, z_2 \neq 0$, then by (8.2.18) and (8.2.19)

$$a_1|z_1|^{2(a_1-1)} + \bar{\tau}|z_2|^{a_2+b_2} = \alpha b_1 z_1^2|z_1|^{2(a_1-2)} \ , \tag{8.2.21}$$
$$\bar{\tau}z_2|z_2|^{2(a_2-1)}\bar{z}_1 = \alpha\tau z_2|z_2|^{2(a_2-1)}z_1 \ . \tag{8.2.22}$$

We have using (8.2.22)

$$\bar{\tau}\bar{z}_1 = \alpha\tau z_1 \ ,$$

therefore by (8.2.21) and since $a_1 - b_1 = 1$,

$$|z_1|^{2(a_1-1)}(a_1\tau - a_1\bar{\tau} + \bar{\tau}) + |z_2|^{2a_2} = 0$$

and by Lemma 8.11 this only happen if $\tau = \pm 1$.

If $\tau = 1$, then
$$|z_1|^{2(a_1-1)} + |z_2|^{2a_2} = 0$$
but this can not happen.

If $\tau = -1$, then all the points of the form $(t^{p_1}, t^{p_2}, 0)$ are singular points.

Therefore f has an isolated singularity if and only if $\tau \neq -1$.

Class IV.c We have that $a_3 = b_3$, $1 + (a_2 - b_2)(a_1 - b_1) = a_1 - b_1$, $a_1 + b_1 \neq 1$ and $a_1 - b_1, a_2 - b_2$ are not 0. The system is

$$a_1 \bar{z}_1^{a_1-1} z_1^{b_1} + \bar{\tau} \bar{z}_2^{a_2} z_2^{b_2} = \alpha b_1 z_1^{a_1} \bar{z}_1^{b_1-1} \ , \tag{8.2.23}$$
$$\bar{\tau} a_2 \bar{z}_2^{a_2-1} z_2^{b_2} \bar{z}_1 + \bar{z}_3^{a_3} z_3^{b_3} = \alpha \tau b_2 z_2^{a_2} \bar{z}_2^{b_2-1} z_1 \ , \tag{8.2.24}$$
$$a_3 \bar{z}_3^{a_3-1} z_3^{b_3} \bar{z}_2 = \alpha b_3 z_3^{a_3} \bar{z}_3^{b_3-1} z_2 \ . \tag{8.2.25}$$

If $z_1 = 0$, then by (8.2.23)
$$\bar{\tau} \bar{z}_2^{a_2} z_2^{b_2} = 0 \ ,$$
therefore $z_2 = 0$ and by (8.2.24)
$$\bar{z}_3^{a_3} z_3^{b_3} = 0 \ ,$$
so $z_3 = 0$.

If $z_2 = 0$, then by (8.2.23)
$$a_1 \bar{z}_1^{a_1-1} z_1^{b_1} = \alpha b_1 z_1^{a_1} \bar{z}_1^{b_1-1} \ ,$$
computing the norm
$$a_1 |z_1|^{a_1+b_1-1} = b_1 |z_1|^{a_1+b_1-1} \ ,$$
this only happen if $z_1 = 0$.

If $z_3 = 0$, then by (8.2.24)
$$\bar{\tau} a_2 \bar{z}_2^{a_2-1} z_2^{b_2} \bar{z}_1 = \alpha \tau b_2 z_2^{a_2} \bar{z}_2^{b_2-1} z_1 \ ,$$
using the norm
$$a_2 |z_2|^{a_2+b_2-1} |z_1| = b_2 |z_2|^{a_2+b_2-1} |z_1| \ ,$$
hence if z_2, z_3 are not 0, then $a_2 = b_2$ but this can not happen, therefore $z_1 = 0$ or $z_2 = 0$.

If z_1, z_2, z_3 are not 0, then the equation (8.2.25)
$$a_3 \bar{z}_3^{a_3-1} z_3^{b_3} \bar{z}_2 = \alpha b_3 z_3^{a_3} \bar{z}_3^{b_3-1} z_2 \ ,$$
give us
$$\bar{z}_2 = \alpha z_2 \ .$$

Using the polar action we can assume that $z_2 \in \mathbb{R}^+$, therefore the last equality give us $\alpha = 1$ and we can simplify equation (8.2.24) to
$$a_2 w + r = b_2 \bar{w} \ , \tag{8.2.26}$$
where $w = \bar{\tau} z_2^{a_2+b_2-1} \bar{z}_1$ and $r = |z_3|^{2a_3}$.

Therefore $a_2 w - b_2 \bar{w}$ must be a real and by Lemma 8.11 this only happen if $w \in \mathbb{R}$, so $\tau z_1 \in \mathbb{R}$. We can simplify the equation (8.2.26) to

$$w(a_2 - b_2) + r = 0 \; . \tag{8.2.27}$$

Notice that $1 + (a_2 - b_2)(a_1 - b_1) = a_1 - b_1$ only has the solutions

$$\begin{aligned} a_2 = b_2 \; , \qquad & a_1 - b_1 = 1 \; , \quad \text{or} \\ a_2 - b_2 = 2 \; , \qquad & a_1 - b_1 = -1 \; , \end{aligned}$$

but $a_2 \neq b_2$, therefore $a_2 - b_2 = 2$ and $a_1 - b_1 = -1$.

So we can simplify (8.2.27) to

$$2w + r = 0 \; ,$$

this only happen if $\tau z_1 \in \mathbb{R}^-$.

Multiplying equation (8.2.23) by $\bar{z}_1$ we get

$$a_1 \bar{z}_1^{a_1} z_1^{b_1} + \bar{\tau} \bar{z}_2^{a_2} z_2^{b_2} \bar{z}_1 = b_1 z_1^{a_1} \bar{z}_1^{b_1} \; ,$$

since $w = \bar{\tau} z_2^{a_2+b_2-1} \bar{z}_1$,

$$a_1 \bar{z}_1^{a_1} z_1^{b_1} + z_2 w = b_1 z_1^{a_1} \bar{z}_1^{b_1} \; ,$$

and $b_1 = a_1 + 1$, therefore

$$a_1 \bar{z}_1^{a_1} z_1^{a_1+1} + z_2 w = (a_1 + 1) z_1^{a_1} \bar{z}_1^{a_1+1} \; ,$$

so

$$|z_1|^{2a_1} (a_1 z_1 - (a_1 + 1) \bar{z}_1) + z_2 w = 0 \; , \tag{8.2.28}$$

but $|z_1|, z_2 w \in \mathbb{R}$, so by Lemma 8.11 the only solution is $z_1 \in \mathbb{R} \setminus \{0\}$ and since $w = \bar{\tau} z_2^{a_2+b_2-1} \bar{z}_1 \in \mathbb{R}^-$, then τ must be ± 1.

If $\tau = -1$, equation (8.2.28) becomes

$$-|z_1|^{2a_1} z_1 - z_2^{a_2+b_2} z_1 = 0 \; ,$$

hence

$$z_1^{2a_1} + z_2^{a_2+b_2} = 0 \; ,$$

but this can not happen since z_1 and z_2 are real numbers different from 0.

If $\tau = 1$, equation (8.2.28) becomes

$$-z_1^{2a_1} + z_2^{a_2+b_2} = 0 \; ,$$

therefore $(-1, 1, 2^{\frac{1}{2a_3}})$ is a singular point.

Then f has an isolated singularity if and only if $\tau \neq 1$.

Class V We have $1+(a_1-b_1)(a_2-b_2)(a_3-b_3)\neq 0$. The system is

$$a_1\bar{z}_1^{a_1-1}z_1^{b_1}\bar{z}_2+\bar{z}_3^{a_3}z_3^{b_3}=\alpha b_1 z_1^{a_1}\bar{z}_1^{b_1-1}z_2\ , \tag{8.2.29}$$
$$\bar{z}_1^{a_1}z_1^{b_1}+a_2\bar{z}_2^{a_2-1}z_2^{b_2}\bar{z}_3=\alpha b_2 z_2^{a_2}\bar{z}_2^{b_2-1}z_3\ , \tag{8.2.30}$$
$$\bar{z}_2^{a_2}z_2^{b_2}+a_3\bar{z}_3^{a_3-1}z_3^{b_3}\bar{z}_1=\alpha b_3 z_3^{a_3}\bar{z}_3^{b_3-1}z_1\ . \tag{8.2.31}$$

If $z_1=0$, then by (8.2.31)

$$\bar{z}_2^{a_2}z_2^{b_2}=0\ ,$$

so $z_2=0$ and by (8.2.29)

$$\bar{z}_3^{a_3}z_3^{b_3}=0\ ,$$

therefore $z_3=0$.

If $z_2=0$, then $z_3=0$ and by (8.2.30)

$$\bar{z}_1^{a_1}z_1^{b_1}=0,$$

so $z_1=0$.

For $z_1z_2z_3\neq 0$ by Theorem 7.6 and Corollary 7.7 the point (z_1,z_2,z_3) is not a critical point of f.

Therefore the origin is the only singularity. □

8.3 Diffeomorphism type of the link under perturbation of the coefficients

In Section 8.1 we proved that given a polar weighted homogeneous polynomial with isolated critical point, with a small perturbation of its coefficients it still has isolated critical point. In this section we prove that under such perturbation the diffeomorphism type of the link does not change. We follow the proof of [31, Theorem 3.1.4] by Orlik and Wagreich.

Proposition 8.18. *Let $f\colon \mathbb{C}^3\to\mathbb{C}$ be a polar weighted homogeneous polynomial of radial weight type $(p_1,p_2,p_3;a)$ and angular weight type $(q_1,q_2,q_3;b)$ with isolated singularity at the origin. Then f can be written as*

$$f(\mathbf{z})=h(\mathbf{z})+g(\mathbf{z}),$$

where h belongs to one of the classes of Definition 8.8, h and g have no monomials in common and both are also polar weighted homogeneous polynomials of radial weight type $(p_1,p_2,p_3;a)$ and angular weight type $(q_1,q_2,q_3;b)$.

Proof. Applying Lemma 7.4 several times we obtain that f must contain a polynomial h in one of the classes. □

Definition 8.19. Let $f\colon \mathbb{C}^3 \to \mathbb{C}$ be a polar weighted homogeneous polynomial. By Proposition 8.18 it can be written in the form $f(\mathbf{z}) = h(\mathbf{z}) + g(\mathbf{z})$ where h belongs to one of the classes of Definition 8.8, h and g have no monomials in common and both are also polar weighted homogeneous polynomials. We say that f *corresponds* to that class.

Let $f\colon \mathbb{C}^3 \to \mathbb{C}$ be a polar weighted homogeneous polynomial with isolated singularity at the origin. Let $V = f^{-1}(0)$ and $K = V \cap \mathbb{S}^5$. By Proposition 8.18 we can write $f(\mathbf{z}) = h(\mathbf{z}) + g(\mathbf{z})$. Let

$$f = \sum_{j=1}^{r} \alpha_j M_j \ ,$$

where M_j is a monomial on the variables $z_i, \bar{z}_i$ for $i = 1, 2, 3$ and

$$h = \sum_{j=1}^{3} \alpha_j M_j \ , \qquad g = \sum_{j=4}^{r} \alpha_j M_j \ .$$

Given $\mathbf{w} = (w_1, \ldots, w_r) \in \mathbb{C}^r$ consider the mixed function

$$f_{\mathbf{w}}(\mathbf{z}) = \sum_{j=1}^{r} w_j M_j(\mathbf{z})$$

and let $V_{\mathbf{w}} = f_{\mathbf{w}}^{-1}(0) \subset \mathbb{C}^3$ be its zero-locus and $K_{\mathbf{w}} = V_{\mathbf{w}} \cap \mathbb{S}^5$ its link. Notice that for $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$ we have $f_\alpha = f$, $V_\alpha = V$ and therefore $K = K_\alpha$. Hence we have a family of polar weighted homogeneous polynomials $f_{\mathbf{w}}\colon \mathbb{C}^n \to \mathbb{C}$ where the parameter $\mathbf{w}$ belongs to the parameter space $\mathbb{C}^r$.

We want to construct a manifold M with a $\mathbb{S}^1$-action, an open set $U \subset \mathbb{C}^r$ and a map $\phi\colon M \to U$, such that the action leaves $\phi^{-1}(\mathbf{w})$ invariant for all $\mathbf{w} \in U$, $\phi^{-1}(\mathbf{w}) \cong K_{\mathbf{w}}$ equivariantly and ϕ is a locally trivial fibration.

Consider the function $k\colon \mathbb{C}^{r+3} \to \mathbb{C}$ given by

$$k(\mathbf{z}, w_1, \ldots, w_r) = \sum_{j=1}^{r} w_j M_j(\mathbf{z}) \ ,$$

let

$$\begin{aligned} N &:= k^{-1}(0) = \bigsqcup_{\mathbf{w} \in \mathbb{C}^r} V_{\mathbf{w}} \times \{\, \mathbf{w} \,\} \subset \mathbb{C}^{r+3} \ , \\ C &:= \mathbb{S}^5 \times \mathbb{C}^r, \end{aligned}$$

let $\phi_0\colon \mathbb{C}^{r+3} \to \mathbb{C}^r$ be the projection onto the last r coordinates and set $\phi_1 := \phi_0|_N$.

Define

$$\begin{aligned} U &:= \{\mathbf{w} \in \mathbb{C}^r \mid \phi_1^{-1}(\mathbf{w}) = V_{\mathbf{w}} \times \{\, \mathbf{w} \,\} \text{ has an isolated singularity at } 0\} \ , \\ M &:= C \cap \phi_1^{-1}(U) = \bigsqcup_{\mathbf{w} \in U} K_{\mathbf{w}} \times \{\, \mathbf{w} \,\} \subset \mathbb{C}^{r+3} \ , \\ \phi &:= \phi_1|_M \colon M \to U \ . \end{aligned}$$

Notice that for any $\mathbf{w} \in U$ we have that $\phi^{-1}(\mathbf{w}) = K_{\mathbf{w}} \times \{\, \mathbf{w} \,\}$ and by Corollary 8.6 and Corollary 8.7 U is an open set. If $f_{\mathbf{w}}$ is a family of polar weighted homogeneous polynomials, another way to see that U is open (pointed out to us by the referee) is the following: the singular locus

$$W = \left\{ (\mathbf{z}, \mathbf{w}) \in \mathbb{S}^{2n+1} \times \mathbb{C}^r \;\Big|\; \begin{array}{l} f_{\mathbf{w}}(\mathbf{z}) = 0 \text{ is singular at } (\mathbf{z}, \mathbf{w}) \\ \text{as a mixed variety in } \{p\} \times \mathbb{C}^n \end{array} \right\}$$

is a real algebraic set as it is defined by the vanishing of 2×2 minors of the Jacobian matrices of the real and imaginary part of $f_p(\mathbf{z})$. In particular it is a closed set. So the projection of the complement of W onto $\mathbb{C}^r$ is open and it is precisely U. We have that

$$\tau \circ (z_1, z_2, z_3, w_1, \dots, w_r) = (\tau^{q_1} z_1, \tau^{q_2} z_2, \tau^{q_3} z_3, w_1, \dots, w_r)\,, \qquad \tau \in \mathbb{S}^1$$

is the required $\mathbb{S}^1$-action on M.

Theorem 8.20. *The map $\phi\colon M \to U$ is a locally trivial fibration.*

Proof. The proof is a generalization of the proof of [31, Theorem 3.1.4] by Orlik and Wagreich.

Step 1: The map $\phi\colon M \to U$ has no critical points.

Let $m = (\mathbf{z}, w_1, \dots, w_r) \in M$ and $\mathbf{w} = (w_1, \dots, w_r)$. Let $k_1, k_2\colon \mathbb{C}^{r+3} \cong \mathbb{R}^{2(r+3)} \to \mathbb{R}$ be the real and imaginary parts of k respectively. Consider the matrix of partial derivatives at m

$$A = \begin{pmatrix} \frac{\partial k_1}{\partial x_1}(m) & \frac{\partial k_1}{\partial y_1}(m) & \dots & \frac{\partial k_1}{\partial y_{r+3}}(m) \\ \frac{\partial k_2}{\partial x_1}(m) & \frac{\partial k_2}{\partial y_1}(m) & \dots & \frac{\partial k_2}{\partial y_{r+3}}(m) \end{pmatrix}\,,$$

we are taking coordinates $z_j = x_j + iy_j$ for $j = 1, 2, 3$ and $w_{j-3} = x_j + iy_j$ for $j = 4, \dots, r$.

Since $m \in \phi_1^{-1}(U)$, the point $\mathbf{z} \in \mathbb{C}^3$ is a regular point of $f_{\mathbf{w}}$ and the six first columns of A are precisely the Jacobian of $f_{\mathbf{w}}$ at $\mathbf{z} \in \mathbb{C}^3$ therefore the rank of A is 2 and m is a regular point of k.

Let T_mN and T_mC denote the tangent spaces at m to N and C respectively. We know that T_mC is the real hyperplane orthogonal to $(\mathbf{z}, 0, \dots, 0)$.

Using the radial action given by f, we have an action on $\mathbb{C}^{r+3}$ given by

$$t * (\mathbf{z}, w_1, \dots, w_r) := (t \bullet \mathbf{z}, w_1, \dots, w_r)$$

for any $t \in \mathbb{R}^+$.

With this, we have that $k(t * (\mathbf{z}, w_1, \dots, w_r)) = t^a k(\mathbf{z}, w_1, \dots, w_r)$, therefore if we denote by

$$v := \frac{d}{dt}(t * m)|_{t=1} = (p_1 z_1, p_2 z_2, p_3 z_3, 0, \dots, 0)$$

then $v \in T_mN$ and $v \notin T_mC$, therefore T_mN and T_mC intersect transversely at m.

So we can denote by T_mM the tangent space at m to M and we have that $T_mM = T_mN \cap T_mC$.

We need to prove that

$$\ker \phi_0 + T_mM = \mathbb{C}^{r+3}\,.$$

Since $T_mC = \{(v_1, v_2, v_3) \in \mathbb{C}^3 \mid \Re\langle \mathbf{z}, (v_1, v_2, v_3)\rangle = 0\} \times \mathbb{C}^r$ and $T_mM = T_mN \cap T_mC$, then it is enough to prove that

$$\ker \phi_0 + T_mN = \mathbb{C}^{r+3}.$$

Denote by $\{e_1, \dots, e_{2(r+3)}\}$ the canonical basis of $\mathbb{R}^{2(r+3)} \cong \mathbb{C}^{r+3}$, so we have that $e_j \in \ker \phi_0$ for $j = 1, \dots, 6$.

Notice that

$$Ae_{2j-1}^\top = \left(\frac{\partial k_1}{\partial x_j}(m), \frac{\partial k_2}{\partial x_j}(m)\right)^\top, \quad Ae_{2j}^\top = \left(\frac{\partial k_1}{\partial y_j}(m), \frac{\partial k_2}{\partial y_j}(m)\right)^\top$$

for $j = 1, 2, 3$.

Since $m \in \phi_1^{-1}(U)$, then there exists two vectors e_{j_1}, e_{j_2} such that $Ae_{j_1}^\top \neq 0$, $Ae_{j_2}^\top \neq 0$ and $Ae_{j_1}^\top + tAe_{j_2}^\top \neq 0$ for all $t \in \mathbb{R}$. Therefore we have two vectors $e_{j_1}, e_{j_2} \in \ker \phi_0$ such that $e_{j_1}, e_{j_2} \notin T_m N$ and

$$\text{span } \{e_{j_1}, e_{j_2}\} \cap T_m N = \mathbf{0} \ ,$$

therefore the intersection is transversal.

Step 2: The map $\phi\colon M \to U$ is proper.

Let L be a compact subset of $U \subset \mathbb{C}^r$. We have that

$$M = (\mathbb{S}^5 \times \mathbb{C}^r) \cap N \cap (\mathbb{C}^3 \times U) \ ,$$

and

$$\phi^{-1}(L) = (\mathbb{S}^5 \times \mathbb{C}^r) \cap N \cap (\mathbb{C}^3 \times L) = (\mathbb{S}^5 \times L) \cap N \subset \mathbb{C}^{3+r} \ .$$

Since $\mathbb{S}^5 \times L$ is close and bounded and N is closed in $\mathbb{C}^{3+r}$, hence $\phi^{-1}(L)$ is compact in $\mathbb{C}^{3+r}$, therefore $\phi^{-1}(L)$ it is also compact in M with the subspace topology. This proves that ϕ is proper.

Since $\phi\colon M \to U$ is a proper submersion, by Ehresmann Fibration Theorem it is a smooth fibre bundle over it image. □

Corollary 8.21. *Let $\phi\colon M \to U$ be as in Theorem 8.20. Let $\tilde{\mathbf{w}} \in U$ and consider the polar weighted homogeneous polynomial with isolated critical point $f_{\tilde{\mathbf{w}}}$. Then there exist a ball $\mathbb{B}(\tilde{\mathbf{w}}, \epsilon)$ centred at $\tilde{\mathbf{w}}$ such that for any $\mathbf{w} \in \mathbb{B}(\tilde{\mathbf{w}}, \epsilon)$ the link $K_{\mathbf{w}}$ of $f_{\mathbf{w}}$ is diffeomorphic to the link $K_{\tilde{\mathbf{w}}}$ of $f_{\tilde{\mathbf{w}}}$.*

Remark 8.22. Recall that we considered $f\colon \mathbb{C}^3 \to \mathbb{C}$ being a polar weighted homogeneous polynomial with isolated singularity at the origin written as $f(\mathbf{z}) = h(\mathbf{z}) + g(\mathbf{z})$ by Proposition 8.18 and such that

$$f = \sum_{j=1}^{r} \alpha_j M_j \ ,$$

where M_j is a monomial on the variables $z_i, \bar{z}_i$ for $i = 1, 2, 3$ and

$$h = \sum_{j=1}^{3} \alpha_j M_j \ , \qquad g = \sum_{j=4}^{r} \alpha_j M_j \ .$$

By Theorem 8.17 the vector $\mathbf{w}_0 = (1, 1, 1, 0, \dots, 0) \in U$ (except for the class **IV.c** for which we take $\mathbf{w}_0 = (1, -1, 1, 0, \dots, 0) \in U$), and also $\alpha \in U$. It may happen that the set $U \subset \mathbb{C}^r$ is not connected, in this case we cannot conclude that $K_{\mathbf{w}_0} = \phi^{-1}(\mathbf{w}_0)$ is diffeomorphic to $K_\alpha = \phi^{-1}(\alpha)$ as in the complex case. If this is the case, one has to study the connected components of U.

This phenomenon is shown in two variables in an example given by Oka in [28, Example 59] or [5, §3.2].

Corollary 8.23. *Let f be a polar weighted homogeneous polynomial with isolated singularity at the origin and let K be its link.*

- *In the classes **II.b**, **III.b** **IV.b** or **IV.d**, the diffeomorphism type of the link K of f is the same for any $\tau \neq -1$. In particular, we can take $\tau = 1$.*
- *In the classe **IV.c**, the diffeomorphism type of the link K of f is the same for any $\tau \neq 1$. In particular we can take $\tau = -1$.*

Proof. Let f be a special representative of each of the aforementioned classes. If we vary τ in $\mathbb{C}$ by Theorem 8.17 for classes **II.b**, **III.b** **IV.b** or **IV.d** the locus where f has non-isolated critical point is the non-positive real ray. Also by Theorem 8.17 for class **IV.c** the locus where f has non-isolated critical point is the non-negative real ray. Therefore in these cases U is connected. □

The word classification in the title is meant in a coarse sense: by Proposition 8.18 every polar weighted homogeneous polynomial with isolated singularity corresponds to one of the subclasses of Corollary 8.15. By Remark 8.22 the parameter space can have several connected components, so the natural step to follow is to study the topology of the Milnor fibre (Milnor number, characteristic polynomial, etc.) for the special representatives of each subclass and then study how this topology change when we change connected component. This will appear in a future work.

Chapter 9

The embedding method

Given a mixed function a natural question is: Is it possible to find a holomorphic function such that both zero set are topological equivalent?

The first example of a mixed function which is topological equivalent to a holomorphic function was given by Ruas, Seade and Verjosky in [34], they proved that the zero set of the polynomial

$$f(\mathbf{z}) = z_1^{a_1}\bar{z}_1^{b_1} + \cdots + z_{n-1}^{a_{n-1}}\bar{z}_{n-1}^{b_{n-1}} + z_n^{a_n}\bar{z}_n^{b_n}, \quad \text{with } a_i > b_i \geq 1$$

is homeomorphic to the Brieskorn variety defined by the polynomial $z_1^{a_1-b_1} + \cdots + z_n^{a_n-b_n}$, hence the corresponding links are also homeomorphic, and that the corresponding Milnor fibrations are topologically equivalent [34, Theorem 4.1]. Later Oka generalized their result to the family of simplicial polynomials [29, 20] and proved that the Milnor fibrations are diffeomorphic.

In this chapter we will introduce "the embedding method" which will allow us to give a different proof of the result given by Oka and later will be used in order to define the mixed GSV index.

9.1 Oka's isotopy theorem

9.1.1 The real embedding

Consider the map given by

$$\begin{aligned} c_n\colon \mathbb{C}^n &\to \mathbb{C}^n, \\ \mathbf{z} &\mapsto \bar{\mathbf{z}}. \end{aligned} \tag{9.1.1}$$

Let $\mathrm{id}_n\colon \mathbb{C}^n \to \mathbb{C}^n$ be the identity map and consider the following embedding

$$\begin{aligned} e_n = (\mathrm{id}_n, c_n)\colon \mathbb{C}^n &\to \mathbb{C}^n \oplus \mathbb{C}^n, \\ \mathbf{z} &\mapsto (\mathbf{z}, \bar{\mathbf{z}}). \end{aligned} \tag{9.1.2}$$

Remark 9.1. Notice that the image $H = \bar{e}_n(\mathbb{C}^n)$ is a $2n$-dimensional real subspace of $\mathbb{C}^n \oplus \mathbb{C}^n$.

We are going to generalize the map e_n. Set $\underline{n} = \{1, 2, \ldots, n\}$, so the cardinality $|\underline{n}|$ of $\underline{n}$ is n. Let K be a subset of $\underline{n}$, such that $|K| = k$. Define the projection

$$\begin{aligned} p_K\colon \mathbb{C}^n &\to \mathbb{C}^k, \\ (z_1, \ldots, z_n) &\mapsto (z_{i_1}, z_{i_2}, \ldots, z_{i_k}), \qquad \text{where } i_r \in K, \text{ for } r = 1, \ldots, k. \end{aligned} \tag{9.1.3}$$

Let $K, L \subset \underline{n}$ such that $|K| = k$ and $|L| = l$. Define the map

$$\begin{aligned} e_{K,L} = (p_K, c_l \circ p_L) \colon \mathbb{C}^n &\to \mathbb{C}^k \oplus \mathbb{C}^l \\ (z_1, \dots, z_n) &\mapsto \big((z_{i_1}, z_{i_2}, \dots, z_{i_k}), (\bar{z}_{j_1}, \bar{z}_{j_2}, \dots, \bar{z}_{j_l})\big). \end{aligned}$$

So we have that $e_n = e_{\underline{n},\underline{n}}$.

Let $f \colon \mathbb{C}^n \to \mathbb{C}$ be a mixed function. We define the following two subsets of $\underline{n}$

$$\begin{aligned} K &= \{\, i \in \underline{n} \mid \text{the variable } z_i \text{ appears in } f \,\}, \\ L &= \{\, j \in \underline{n} \mid \text{the variable } \bar{z}_j \text{ appears in } f \,\}. \end{aligned}$$

We also ask that $K \cup L = \underline{n}$, otherwise, let J be the complement of $K \cup L$ in $\underline{n}$ and suppose $|J| = j$, hence we can consider the function f defined on $\mathbb{C}^{n-j}$ instead of $\mathbb{C}^n$ using the coordinates with indices in $K \cup L$. Suppose that $|K| = k$, $|L| = l$ and consider the complex space $\mathbb{C}^k \oplus \mathbb{C}^l$ with variables $z_1, \dots, z_{i_k}, w_1, \dots, w_{j_l}$. Define a function $F \colon \mathbb{C}^k \oplus \mathbb{C}^l \to \mathbb{C}$ by replacing in f each variable $\bar{z}_{j_s}$ with $j_s \in L$ by the corresponding complex variable w_{j_s}. The function F is called the *holomorphic function associated to the mixed function f*. It is easy to see that the following diagram is commutative

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "e_{K,L}"] \arrow[dr, "f"'] & \mathbb{C}^k \oplus \mathbb{C}^l \arrow[d, "F"] \\ & \mathbb{C} \end{tikzcd}$$

so we have that

$$f = F \circ e_{K,L}. \tag{9.1.4}$$

Set

$$\begin{aligned} V_f &= f^{-1}(0), & V_f^* &= V_f - \{0\}, \\ V_F &= F^{-1}(0), & V_F^* &= V_F - \mathrm{Sing} V_F. \end{aligned}$$

Let $\mathbf{z} \in V_f^*$. As usual denote by $T_{\mathbf{z}} V_f^*$ the tangent space of V_f^* at the point $\mathbf{z}$. We use the analogous notation for the points in V_F^*.

As before, denote by H the image $e_{K,L}(\mathbb{C}^n)$. Notice that

$$V_f \cong V_F \cap H. \tag{9.1.5}$$

Lemma 9.2. *Let $\mathbf{z} \in V_f^*$. Then $e_{K,L}(\mathbf{z}) \in V_F^*$.*

PROOF. Since $\mathbf{z} \in V_f^*$ we have that $\mathrm{rank}_{\mathbb{R}} Df_{\mathbf{z}} = 2$. Suppose $e_{K,L}(\mathbf{z}) \in \mathrm{Sing} V_F$, then $\mathrm{rank}_{\mathbb{R}} DF_{e_{K,L}(\mathbf{z})} < 2$. By (9.1.4) and the chain rule we have that $Df_{\mathbf{z}} = DF_{e_{K,L}(\mathbf{z})} \circ e_{K,L}$. Hence

$$2 = \mathrm{rank}_{\mathbb{R}} Df_{\mathbf{z}} = \mathrm{rank}_{\mathbb{R}}(DF_{e_{K,L}(\mathbf{z})} \circ e_{K,L}) \le \mathrm{rank}_{\mathbb{R}} DF_{e_{K,L}(\mathbf{z})} < 2,$$

which clearly is a contradiction. $\square$

Lemma 9.3. *Let $\mathbf{z} \in V_f$ such that $e_{K,L}(\mathbf{z}) \in V_F^*$. Then, $\mathbf{z} \in V_f^*$ if and only if $T_{e_{K,L}(\mathbf{z})} V_F^* \pitchfork H$.*

PROOF. Suppose $\mathbf{z} \in V_f^*$, then $\dim_{\mathbb{R}} T_{\mathbf{z}} V_f^* = 2n-2$. Since $e_{K,L}(\mathbf{z}) \in V_F^*$ we have that $\dim_{\mathbb{R}} T_{e_{K,L}(\mathbf{z})} V_F^* = 2(k+l)-2$. We also have that $\dim_{\mathbb{R}} H = 2n$. On the other hand we have that

$$T_{\mathbf{z}} V_f^* \cong T_{e_{K,L}(\mathbf{z})} V_F^* \cap H. \tag{9.1.6}$$

Hence, $\operatorname{codim}_{\mathbb{R}} T_{e_{K,L}(\mathbf{z})} V_F^* \cap H = \operatorname{codim}_{\mathbb{R}} T_{e_{K,L}(\mathbf{z})} V_F^* + \operatorname{codim}_{\mathbb{R}} H$, and this is the case only if $T_{e_{K,L}(\mathbf{z})} V_F^* \pitchfork H$.

Conversely, suppose that $T_{e_{K,L}(\mathbf{z})} V_F^* \pitchfork H$. We have that $f = F \circ e_{K,L}$ and by the chain rule $Df_{\mathbf{z}} = DF_{e_{K,L}(\mathbf{z})} \circ e_{K,L}$. The rank of the map $e_{K,L}$ is $2n$, since $e_{K,L}(\mathbf{z}) \in V_F^*$, we have that $\dim_{\mathbb{R}} T_{e_{K,L}(\mathbf{z})} V_F^* = 2(k+l)-2$. Since $T_{e_{K,L}(\mathbf{z})} V_F^* \pitchfork H$ we have that $\dim_{\mathbb{R}} T_{e_{K,L}(\mathbf{z})} V_F^* \cap H = 2n-2$. Hence, by (9.1.6)

$$\dim_{\mathbb{R}} T_{\mathbf{z}} V_f^* = \ker Df_{\mathbf{z}} = 2n-2.$$

Therefore the rank of $Df_{\mathbf{z}}$ is 2 and $\mathbf{z} \in V_f^*$. $\square$

Consider the standard Hermitian product $\langle\ ,\ \rangle$ on $\mathbb{C}^n$; the Hermitian inner product on the direct sum $\mathbb{C}^k \oplus \mathbb{C}^l$ gives also the standard Hermitian product

$$\begin{gathered} \langle\ ,\ \rangle \colon \mathbb{C}^k \oplus \mathbb{C}^l \to \mathbb{C}, \\ \langle (\mathbf{z}, \mathbf{w}), (\mathbf{z}', \mathbf{w}') \rangle = \sum_{j=1}^{k} z_j \bar{z}'_j + \sum_{j=1}^{l} w_j \bar{w}'_j. \end{gathered}$$

Recall that the Euclidean inner product on $\mathbb{C}^n$ is given by the real part of the Hermitian inner product.

Denote by $\mathbb{S}_{\varepsilon}^{2n-1}$ the $(2n-1)$-sphere in $\mathbb{C}^n$ of radius $\varepsilon > 0$ with centre at the origin. Let $\mathbf{z} \in \mathbb{C}^n$ and consider the following tangent spaces:

$$\begin{aligned} T_{\mathbf{z}} \mathbb{S}_{\|\mathbf{z}\|}^{2n-1} &= \{ v \in \mathbb{C}^n \mid \Re \langle \mathbf{z}, v \rangle = 0 \}, \\ T_{e_n(\mathbf{z})} \mathbb{S}_{\|e_n(\mathbf{z})\|}^{4n-1} &= \{ w \in \mathbb{C}^n \oplus \mathbb{C}^n \mid \Re \langle e_n(\mathbf{z}), w \rangle = 0 \}. \end{aligned}$$

We have the following proposition:

Proposition 9.4.

$$e_n \colon T_{\mathbf{z}} \mathbb{S}_{\|\mathbf{z}\|}^{2n-1} \to T_{e_n(\mathbf{z})} \mathbb{S}_{\|e_n(\mathbf{z})\|}^{4n-1}. \tag{9.1.7}$$

PROOF. For $v \in T_{\mathbf{z}} \mathbb{S}_{\|\mathbf{z}\|}^{2n-1}$ we have $\Re \langle \mathbf{z}, v \rangle = \sum_{j=1}^{n} z_j \bar{v}_j = 0$, then

$$\Re \langle e(\mathbf{z}), e(v) \rangle = \Re \langle (\mathbf{z}, \bar{\mathbf{z}}), (v, \bar{v}) \rangle = \Re\Big(\sum_{j=1}^{n} z_j \bar{v}_j + \sum_{j=1}^{n} \bar{z}_j v_j \Big) = 2\Re\Big(\sum_{j=1}^{n} z_j \bar{v}_j \Big) = 0.$$

$\square$

9.1.2 Oka's isotopy theorem

Following [29, 20] consider the following families of mixed polynomials

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, with $a_i, b_i \in \mathbb{N}$ and $a_i > b_i \geq 0$, $i = 1, \dots, n$..

$$f_A(\mathbf{z}) = z_1^{a_1}\bar{z}_1^{b_1} + z_2^{a_2}\bar{z}_2^{b_2} + z_3^{a_3}\bar{z}_3^{b_3}, \tag{9.1.8}$$

$$f_B(\mathbf{z}) = z_1^{a_1}\bar{z}_1^{b_1}z_2 + z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}, \qquad (a_2, a_3) \neq (b_2 + 2, b_3 - 1), \tag{9.1.9}$$

$$f_C(\mathbf{z}) = z_1^{a_1}\bar{z}_1^{b_1}z_2 + z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}z_1, \qquad \begin{cases} (a_1 - b_1)(a_2 - b_2)(a_3 - b_3) \neq -1, \\ (a_{i-1}, a_{i+1}) \neq (b_{i-1} + 1, b_{i+1}), \\ (a_{i-1}, a_{i+1}) \neq (b_{i-1} - 1, b_{i+1} + 2), \\ i = 1, 2, 3. \end{cases} \tag{9.1.10}$$

With the given conditions on the exponents, these correspond respectively (up to a change of coordinates) to classes **I**, **IV.a** and **V** of polar weighted homogeneous polynomials given in [11, Theorem 4.5], which generalize Orlik and Wagreich classes of complex weighted homogeneous polynomials with isolated critical point, given in [31, §3.1]. These mixed polynomials are simplicial full [11, Corollary 4.7] and have isolated critical point [11, Theorem 4.10].

Now define

$$h_j(z_j) = \begin{cases} z_j & \text{if} \quad a_j - b_j > 0 \\ \bar{z}_j & \text{if} \quad a_j - b_j < 0. \end{cases}$$

for $j = 1, 2, 3$ and consider the maps

$$\begin{aligned} g_A(\mathbf{z}) &= h_1(z_1)^{|a_1 - b_1|} + h_2(z_2)^{|a_2 - b_2|} + h_3(z)_3^{|a_3 - b_3|}, \\ g_B(\mathbf{z}) &= h_1(z_1)^{|a_1 - b_1|}z_2 + h_2(z_2)^{|a_2 - b_2|}z_3 + h_3(z_3)^{|a_3 - b_3|}, \\ g_C(\mathbf{z}) &= h_1(z_1)^{|a_1 - b_1|}z_2 + h_2(z_2)^{|a_2 - b_2|}z_3 + h_3(z_3)^{|a_3 - b_3|}z_1, \end{aligned}$$

Consider the map $f_{l,t}\colon \mathbb{C}^3 \to \mathbb{C}$ given by $f_{l,t}(\mathbf{z}) := (1-t)f_l(\mathbf{z}) + tg_l(\mathbf{z})$ for $0 \leq t \leq 1$ and consider the family $V_{l,t} = f_{l,t}^{-1}(0)$ where $l = A, B, C$.

Lemma 9.5. *The function $f_{l,t}(\mathbf{z})\colon \mathbb{C}^3 \to \mathbb{C}$ has an isolated critical point for any $0 \leq t \leq 1$ and $l = A, B, C$.*

Proof.

Case $l = A$ Under a change of coordinates we can assume that

$$g_A(\mathbf{z}) = z_1^{a_1 - b_1} + z_2^{a_2 - b_2} + z_3^{a_3 - b_3},$$

and this case was proved by Oka [29, Lemma 1].

Case $l = B$ Under a change of coordinates we can assume that

$$g_B(\mathbf{z}) = z_1^{a_1 - b_1}z_2 + h_2(z_2)^{|a_2 - b_2|}z_3 + h_3(z_3)^{|a_3 - b_3|}.$$

The case $a_2 > b_2$ and $a_3 > b_3$ was proved by Oka [29, Lemma 9].

Suppose that $a_2 < b_2$ and $a_3 > b_3$, then

$$g_B(\mathbf{z}) = z_1^{a_1 - b_1}z_2 + \bar{z}_2^{b_2 - a_2}z_3 + z_3^{a_3 - b_3}.$$

For $t=0,1$ the assertion is true by [11, Theorem 4.10], therefore suppose $t\in(0,1)$. We have that

$$f_{B,t}(\mathbf{z})=(1-t)(z_1^{a_1}\bar{z}_1^{b_1}z_2+z_2^{a_2}\bar{z}_2^{b_2}z_3+z_3^{a_3}\bar{z}_3^{b_3})+t(z_1^{a_1-b_1}z_2+\bar{z}_2^{b_2-a_2}z_3+z_3^{a_3-b_3}).$$

Suppose that (z_1,z_2,z_3) is a critical point of $f_{B,t}$, then by Proposition 7.2 exist $\alpha\in\mathbb{S}^1$ such that

$$\overline{df_{B,t}(\mathbf{z})}=\alpha\bar{d}f_{B,t}(\mathbf{z}).$$

This equality gives us the following system

$$\bar{z}_1^{a_1-b_1-1}\bar{z}_2((1-t)a_1|z_1|^{2b_1}+t(a_1-b_1))=\alpha(1-t)b_1z_1^{a_1}\bar{z}_1^{b_1-1}z_2 \tag{9.1.11}$$

$$\bar{z}_1^{a_1-b_1}((1-t)|z_1|^{2b_1}+t)+(1-t)a_2\bar{z}_2^{a_2-1}z_2^{b_2}\bar{z}_3=\alpha\bar{z}_2^{b_2-a_2-1}z_3((1-t)b_2|z_2|^{2a_2}+t(b_2-a_2)) \tag{9.1.12}$$

$$\bar{z}_3^{a_3-b_3-1}((1-t)a_3|z_3|^{2b_3}+t(a_3-b_3))+z_2^{b_2-a_2}((1-t)|z_2|^{2a_2}+t)=\alpha(1-t)(b_3z_3^{a_3}\bar{z}_3^{b_3-1}). \tag{9.1.13}$$

Suppose $z_3\neq 0$ and let $s=\min\{j\mid z_k\neq 0,\ k\geq j\}$.

If $s=1$, multiplying (9.1.11) by $\bar{z}_1$ we get

$$\bar{z}_1^{a_1-b_1}\bar{z}_2((1-t)a_1|z_1|^{2b_1}+t(a_1-b_1))=z_1^{a_1-b_1}z_2(\alpha(1-t)b_1|z_1|^{2b_1})\ ,$$

computing the norm and since $|z_1|,|z_2|\neq 0$ we have

$$(1-t)a_1|z_1|^{2b_1}+t(a_1-b_1)=(1-t)b_1|z_1|^{2b_1}\ ,$$

therefore

$$(a_1-b_1)((1-t)|z_1|^{2b_1}+t)=0$$

and this can not happen since $(a_1-b_1)\neq 0$, $t\in(0,1)$ and $|z_1|>0$.

If $s=2$, using (9.1.12) we get a contradiction in the same way as in the previous case.

If $s=3$, using (9.1.13) again we get a contradiction using the same idea.

Finally, suppose that $z_3=0$ and define $k=\min\{j\mid z_k=0,\ k\geq j\}$.

If $k=3$, by (9.1.13) and the fact that $a_3-b_3\neq 1$ (see conditions of family **IV.a**)

$$(1-t)|z_2|^{2a_2}+t=0\ ,$$

and this can not happen.

If $k=2$, by (9.1.12)

$$(1-t)|z_1|^{2b_1}+t=0\ ,$$

but again, this can not happen.

Therefore $k=1$, so the origin is the only critical point.

The cases $a_2>b_2$, $a_3<b_3$ and $a_2<b_2$, $a_3<b_3$ are analogous.

Case $l = C$ For $t = 0, 1$ the assertion is true by [11, Theorem 4.10], therefore suppose $t \in (0, 1)$. If $a_j > b_j$ for $j = 1, 2, 3$ then,

$$f_{C,t}(\mathbf{z}) = (1-t)(z_1^{a_1}\bar{z}_1^{b_1}z_2 + z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}z_1) + t(z_1^{a_1-b_1}z_2 + z_2^{a_2-b_2}z_3 + z_3^{a_3-b_3}z_1).$$

Suppose that (z_1, z_2, z_3) is a critical point of $f_{C,t}$, then by Proposition 7.2 exist $\alpha \in \mathbb{S}^1$ such that

$$\overline{df_{C,t}(\mathbf{z})} = \alpha \bar{d} f_{C,t}(\mathbf{z}).$$

This equality give us the following system

$$\bar{z}_1^{a_1-b_1-1}\bar{z}_2((1-t)a_1|z_1|^{2b_1} + t(a_1-b_1)) + \bar{z}_3^{a_3-b_3}((1-t)|z_3|^{2b_3} + t) = \alpha(1-t)b_1 z_1^{a_1}\bar{z}_1^{b_1-1}z_2 \tag{9.1.14}$$

$$\bar{z}_2^{a_2-b_2-1}\bar{z}_3((1-t)a_2|z_2|^{2b_2} + t(a_2-b_2)) + \bar{z}_1^{a_1-b_1}((1-t)|z_1|^{2b_1} + t) = \alpha(1-t)b_2 z_2^{a_2}\bar{z}_2^{b_2-1}z_3 \tag{9.1.15}$$

$$\bar{z}_3^{a_3-b_3-1}\bar{z}_1((1-t)a_3|z_3|^{2b_3} + t(a_3-b_3)) + \bar{z}_2^{a_2-b_2}((1-t)|z_2|^{2b_2} + t) = \alpha(1-t)b_3 z_3^{a_3}\bar{z}_3^{b_3-1}z_1 \tag{9.1.16}$$

If $z_1 = 0$, then by (9.1.16)

$$\bar{z}_2^{a_2-b_2}((1-t)|z_2|^{2b_2} + t) = 0,$$

therefore $z_2 = 0$ and then by (9.1.14)

$$\bar{z}_3^{a_3-b_3}((1-t)|z_3|^{2b_3} + t) = 0,$$

hence $z_3 = 0$.

Suppose that $z_3 = 0$, then by (9.1.15)

$$z_1^{b_1-a_1}((1-t)|z_1|^{2a_1} + t) = 0,$$

therefore $z_1 = 0$.

Finally suppose that $z_j \neq 0$ for $j = 1, 2, 3$. Here is were we use the embedding method.

Consider the function $H_{C,t} \colon \mathbb{C}^6 \to \mathbb{C}$ given by

$$H_{C,t}(\mathbf{w}) = w_1^{a_1-b_1}w_2((1-t)w_1^{b_1}w_4^{b_1}+t) + w_2^{a_2-b_2}w_3((1-t)w_2^{b_2}w_5^{b_2}+t) + w_3^{a_3-b_3}w_1((1-t)w_3^{b_3}w_6^{b_3}+t)$$

and the map

$$\begin{aligned} e \colon \mathbb{C}^3 &\to \mathbb{C}^6 \\ \mathbf{z} &\mapsto (\mathbf{z}, \bar{\mathbf{z}}), \end{aligned}$$

therefore

$$f_{C,t}(\mathbf{z}) = H_{C,t}(e(\mathbf{z})) .$$

Since e is a $\mathbb{R}$-linear embedding, its differential is e itself. Let $\mathbf{z} \in \mathbb{C}^3$ and consider $v \in \mathbb{C}^3$ as a *real* tangent vector of $\mathbb{C}^3$ at $\mathbf{z}$, hence we have that

$$d(f_{C,t})_{\mathbf{z}}(v) = \boldsymbol{\nabla} H_{C,t}(e(\mathbf{z})) \cdot e(v) .$$

Notice that even though $\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))$ is $\mathbb{C}$-linear, the map $d(f_{C,t})_{\mathbf{z}}$ is only $\mathbb{R}$-linear since e is only $\mathbb{R}$-linear. Now we have that

$$\begin{aligned}\boldsymbol{\nabla} H_{C,t}(\mathbf{w}) = \Big(&(1-t)(a_1 w_1^{a_1-1} w_4^{b_1} w_2 + w_3^{a_3} w_6^{b_3}) + t((a_1-b_1) w_1^{a_1-b_1-1} w_2 + w_3^{a_3-b_3}),\\ &(1-t)(w_1^{a_1} w_4^{b_1} + a_2 w_2^{a_2-1} w_5^{b_2} w_3) + t(w_1^{a_1-b_1} + (a_2-b_2) w_2^{a_2-b_2-1} w_3),\\ &(1-t)(w_2^{a_2} w_5^{b_2} + a_3 w_3^{a_3-1} w_6^{b_3} w_1) + t(w_2^{a_2-b_2} + (a_3-b_3) w_3^{a_3-b_3-1} w_1),\\ &(1-t) b_1 w_1^{a_1} w_4^{b_1-1} w_2,\\ &(1-t) b_2 w_2^{a_2} w_5^{b_2-1} w_3,\\ &(1-t) b_3 w_3^{a_3} w_6^{b_3-1} w_1\Big).\end{aligned}$$

Therefore

$$\begin{aligned}\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z})) = \Big(&z_1^{a_1-b_1-1} z_2((1-t)a_1|z_1|^{2b_1} + t(a_1-b_1)) + z_3^{a_3-b_3}((1-t)|z_3|^{2b_3} + t),\\ &z_2^{a_2-b_2-1} z_3((1-t)a_2|z_2|^{2b_2} + t(a_2-b_2)) + z_1^{a_1-b_1}((1-t)|z_1|^{2b_1} + t),\\ &z_3^{a_3-b_3-1} z_1((1-t)a_3|z_3|^{2b_3} + t(a_3-b_3)) + z_2^{a_2-b_2}((1-t)|z_2|^{2b_2} + t),\\ &(1-t) b_1 z_1^{a_1} \bar{z}_1^{b_1-1} z_2,\\ &(1-t) b_2 z_2^{a_2} \bar{z}_2^{b_2-1} z_3,\\ &(1-t) b_3 z_3^{a_3} \bar{z}_3^{b_3-1} z_1\Big).\end{aligned}$$

Since $z_j \neq 0$ for $j = 1, 2, 3$, then $e(\mathbf{z})$ is a regular point of the holomorphic map $H_{C,t}$ and the kernel of $\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))$ is a complex subspace of complex codimension 1, that is, of real codimension 2. In order to prove that $\mathbf{z}$ is a regular point of $f_{C,t}$ we need to find a 2-dimensional *real* subspace in the image $e(\mathbb{C}^3)$ which is complementary to the kernel of $\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))$.

Consider the following *real* basis of $\mathbb{C}^3$:

$$\begin{aligned} v_1 &= (z_1, 0, 0)\ , & iv_1 &= (iz_1, 0, 0)\ ,\\ v_2 &= (0, z_2, 0)\ , & iv_2 &= (0, iz_2, 0)\ ,\\ v_3 &= (0, 0, z_3)\ , & iv_3 &= (0, 0, iz_3)\ ,, \end{aligned}$$

and define

$$\begin{aligned} P_j &= (1-t)(a_j+b_j)|z_j|^{2b_j} + t(a_j-b_j)\ ,\\ Q_j &= (1-t)(a_j-b_j)|z_j|^{2b_j} + t(a_j-b_j)\ .\\ R_j &= (1-t)|z_{j-1}|^{2b_{j-1}} + t. \end{aligned} \tag{9.1.17}$$

We have that

$$\begin{aligned} \boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(v_1)^\top &= z_1^{a_1-b_1} z_2 P_1 + z_3^{a_3-b_3} z_1 R_1, & \boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(iv_1)^\top &= i(z_1^{a_1-b_1} z_2 Q_1 + z_3^{a_3-b_3} z_1 R_1)\ ,\\ \boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(v_2)^\top &= z_2^{a_2-b_2} z_3 P_2 + z_1^{a_1-b_1} z_2 R_2, & \boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(iv_2)^\top &= i(z_2^{a_2-b_2} z_3 Q_2 + z_1^{a_1-b_1} z_2 R_2)\ ,\\ \boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(v_3)^\top &= z_3^{a_3-b_3} z_1 P_3 + z_2^{a_2-b_2} z_3 R_3, & \boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(iv_3)^\top &= i(z_3^{a_3-b_3} z_1 Q_3 + z_2^{a_2-b_2} z_3 R_3)\ . \end{aligned} \tag{9.1.18}$$

We want to check that $e(v_j)$ and $e(iv_j)$, $j = 1,2,3$, are not in the kernel of $\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))$. Suppose that

$$\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z})) \cdot e(v_1)^\top = z_1^{a_1-b_1} z_2 P_1 + z_3^{a_3-b_3} z_1 R_1 = 0. \tag{9.1.19}$$

Multiplying equations (9.1.14), (9.1.15) and (9.1.16) by $\bar{z}_1$, $\bar{z}_2$ and $\bar{z}_3$ respectively we get

$$\bar{z}_1^{a_1-b_1} \bar{z}_2((1-t)a_1|z_1|^{2b_1} + t(a_1-b_1)) + \bar{z}_3^{a_3-b_3} \bar{z}_1 R_1 = \alpha(1-t)b_1 z_1^{a_1-b_1} z_2 |z_1|^{2b_1} \tag{9.1.20}$$

$$\bar{z}_2^{a_2-b_2} \bar{z}_3((1-t)a_2|z_2|^{2b_2} + t(a_2-b_2)) + \bar{z}_1^{a_1-b_1} \bar{z}_2 R_2 = \alpha(1-t)b_2 z_2^{a_2-b_2} z_3 |z_2|^{2b_2} \tag{9.1.21}$$

$$\bar{z}_3^{a_3-b_3} \bar{z}_1((1-t)a_3|z_3|^{2b_3} + t(a_3-b_3)) + \bar{z}_2^{a_2-b_2} \bar{z}_3 R_3 = \alpha(1-t)b_3 z_3^{a_3-b_3} z_1 |z_3|^{2b_3} \tag{9.1.22}$$

using equation (9.1.20) and the conjugate of (9.1.19) we have

$$\begin{aligned} \alpha(1-t)b_1 z_1^{a_1-b_1} |z_1|^{2b_1} z_2 &= \bar{z}_1^{a_1-b_1} \bar{z}_2((1-t)a_1|z_1|^{2b_1} + t(a_1-b_1) - P_1) \\ &= -\bar{z}_1^{a_1-b_1} \bar{z}_2(1-t)b_1|z_1|^{2b_1}, \end{aligned}$$

therefore

$$\alpha z_1^{a_1-b_1} z_2 = -\bar{z}_1^{a_1-b_1} \bar{z}_2.$$

Now multiplying (9.1.21) by $z_1^{a_1-b_1} z_2$ we get

$$\begin{aligned} &\alpha z_1^{a_1-b_1} z_2 (1-t) b_2 z_2^{a_2-b_2} |z_2|^{2b_2} z_3 = -z_2^{a_2-b_2} z_3 \bar{z}_1^{a_1-b_1} \bar{z}_2 (1-t) b_2 |z_2|^{2b_2} \\ &= \bar{z}_2^{a_2-b_2} \bar{z}_3 z_1^{a_1-b_1} z_2((1-t)a_2|z_2|^{2b_2} + t(a_2-b_2)) + |z_1|^{2(a_1-b_1)} |z_2|^2 R_2 \end{aligned}$$

Hence we get

$$\begin{aligned} &-\Re(z_2^{a_2-b_2} z_3 \bar{z}_1^{a_1-b_1} \bar{z}_2)(1-t)b_2|z_2|^{2b_2} \\ &= \Re(\bar{z}_2^{a_2-b_2} \bar{z}_3 z_1^{b_1-a_1} z_2)((1-t)a_2|z_2|^{2b_2} + t(a_2-b_2)) + |z_1|^{2(a_1-b_1)}|z_2| R_2 \\ &= \Re(z_2^{a_2-b_2} z_3 \bar{z}_1^{a_1-b_1} \bar{z}_2)((1-t)a_2|z_2|^{2b_2} + t(a_2-b_2)) + |z_1|^{2(a_1-b_1)}|z_2| R_2 \end{aligned}$$

so

$$\Re(z_2^{a_2-b_2} z_3 \bar{z}_1^{a_1-b_1} \bar{z}_2) P_2 + |z_1|^{2(a_1-b_1)} |z_2| R_2 = 0. \tag{9.1.23}$$

Also

$$\begin{aligned} &-\Im(z_2^{a_2-b_2} z_3 \bar{z}_1^{a_1-b_1} \bar{z}_2)(1-t)b_2|z_2|^{2b_2} \\ &= \Im\left(\bar{z}_2^{a_2-b_2} \bar{z}_3 z_1^{b_1-a_1} z_2((1-t)a_2|z_2|^{2b_2} + t(a_2-b_2)) + |z_1|^{2(a_1-b_1)}|z_2| R_2\right) \\ &= -\Im(z_2^{a_2-b_2} z_3 \bar{z}_1^{a_1-b_1} \bar{z}_2)((1-t)a_2|z_2|^{2b_2} + t(a_2-b_2)) \end{aligned}$$

so

$$\Im(z_2^{a_2-b_2} z_3 \bar{z}_1^{a_1-b_1} \bar{z}_2) Q_2 = 0$$

and since $Q_2 \neq 0$, then $z_2^{a_2-b_2} z_3 \bar{z}_1^{a_1-b_1} \bar{z}_2 \in \mathbb{R}$ and using (9.1.23) we get

$$z_2^{a_2-b_2} z_3 P_2 + z_1^{a_1-b_1} z_2 R_2 = 0. \tag{9.1.24}$$

Using equation (9.1.21) we get

$$\bar{z}_2^{a_2-b_2}\bar{z}_3((1-t)a_2|z_2|^{2b_2}+t(a_2-b_2))-\bar{z}_2^{a_2-b_2}\bar{z}_3P_2=\alpha(1-t)b_2z_2^{a_2-b_2}z_3|z_2|^{2b_2}$$

hence

$$-\bar{z}_2^{a_2-b_2}\bar{z}_3=\alpha z_2^{a_2-b_2}z_3.$$

Therefore equation (9.1.21) is just

$$\begin{aligned}0&=\bar{z}_2^{a_2-b_2}\bar{z}_3((1-t)a_2|z_2|^{2b_2}+t(a_2-b_2)+(1-t)b_2|z_2|^{2b_2})+\bar{z}_1^{a_1-b_1}\bar{z}_2R_2\\&=\bar{z}_2^{a_2-b_2}\bar{z}_3P_2+\bar{z}_1^{a_1-b_1}\bar{z}_2R_2\\&=z_2^{a_2-b_2}z_3P_2+z_1^{a_1-b_1}z_2R_2\\&=\boldsymbol{\nabla}H_{C,t}(e(\mathbf{z}))\cdot e(v_2)^\top.\end{aligned}$$

In the same way we get

$$\boldsymbol{\nabla}H_{C,t}(e(\mathbf{z}))\cdot e(v_3)^\top=0.$$

This give us the system

$$\begin{aligned}z_1^{a_1-b_1}z_2P_1+z_3^{a_3-b_3}z_1R_1&=0\\z_2^{a_2-b_2}z_3P_2+z_1^{a_1-b_1}z_2R_2&=0\\z_3^{a_3-b_3}z_1P_3+z_2^{a_2-b_2}z_3R_3&=0\end{aligned}$$

therefore

$$z_3^{a_3-b_3}z_1(R_1R_2R_3+P_1P_2P_3)=0$$

but this can not happen since $z_3^{a_3-b_3}z_1\neq 0$ and $R_j,P_j>0$ for all $j=1,2,3$.
Using a similiar argument we can deduce that

$$\boldsymbol{\nabla}H_{C,t}(e(\mathbf{z}))\cdot e(v_j)^\top\neq 0\qquad \boldsymbol{\nabla}H_{C,t}(e(\mathbf{z}))\cdot e(iv_j)^\top\neq 0 \tag{9.1.25}$$

for all $j=1,2,3$.
Finally suppose that for all $j=1,2,3$, there exists $s_j\in\mathbb{R}$ such that

$$\boldsymbol{\nabla}H_{C,t}(e(\mathbf{z}))\cdot e(v_j+s_jiv_j)=0.$$

this gives us the system

$$\begin{aligned}z_1^{a_1-b_1}z_2(P_1+is_1Q_1)+z_3^{a_3-b_3}z_1R_1(1+is_1)&=0\\z_2^{a_2-b_2}z_3(P_2+is_2Q_2)+z_1^{a_1-b_1}z_2R_2(1+is_2)&=0\\z_3^{a_3-b_3}z_1(P_3+is_3Q_3)+z_2^{a_2-b_2}z_3R_3(1+is_3)&=0\end{aligned}$$

therefore

$$z_3^{a_3-b_3}z_1\left(R_1(1+is_1)R_2(1+is_2)R_3(1+is_3)+(P_1+is_1Q_1)(P_2+is_2Q_2)(P_3+is_3Q_3)\right)=0,$$

and since $z_j \neq 0$ for all $j = 1, 2, 3$, then

$$R_1(1 + is_1)R_2(1 + is_2)R_3(1 + is_3) + (P_1 + is_1Q_1)(P_2 + is_2Q_2)(P_3 + is_3Q_3) = 0,$$

and using the norm we get

$$|R_1(1 + is_1)R_2(1 + is_2)R_3(1 + is_3)| = |(P_1 + is_1Q_1)(P_2 + is_2Q_2)(P_3 + is_3Q_3)|. \tag{9.1.26}$$

We have that $0 < R_{j+1} < Q_j < P_j$ therefore

$$|(1 + is_j)R_{j+1}|^2 = R_{j+1}^2 + s_j^2 R_{j+1}^2 < P_j^2 + s_j^2 Q_j^2 = |P_j + is_jQ_j|^2$$

hence

$$|R_1(1 + is_1)R_2(1 + is_2)R_3(1 + is_3)|^2 < |(P_1 + is_1Q_1)(P_2 + is_2Q_2)(P_3 + is_3Q_3)|^2$$

and this can not happen by (9.1.26).

Therefore for some $j \in \{1, 2, 3\}$ we have that $\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(v_j + siv_j) \neq 0$ for all $s \in \mathbb{R}$. Thus we have found a 2-dimensional *real* subspace in the image $e(\mathbb{C}^3)$ which is complementary to the kernel of $\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))$ and therefore $\mathbf{z}$ is a regular point of $f_{C,t}$. So the origin is the only critical point.

Using the same method we can prove the assertion for the rest of the cases.

□

Again consider the map $f_{l,t}\colon \mathbb{C}^3 \to \mathbb{C}$ given by $f_{l,t}(\mathbf{z}) := (1 - t)f_l(\mathbf{z}) + tg_l(\mathbf{z})$ for $0 \leq t \leq 1$ and the family $V_{l,t} = f_{l,t}^{-1}(0)$ where $l = A, B, C$.

Proposition 9.6. *For any $0 \leq t \leq 1$ and $r > 0$, the sphere $\mathbb{S}_r^5$ intersects transversely $V_{l,t}$ for $l = A, B, C$.*

PROOF.

1. The case $l = A$ was proved by Oka [29, Lemma 2].

2. Let $l = B$. Under a change of coordinates we can suppose that $a_1 > b_1$.

   The case $a_2 > b_2$, $a_3 > b_3$ was proved by Oka [29, Lemma 10].

   Suppose that $a_2 < b_2$, $a_3 > b_3$, then

   $$f_{B,t}(\mathbf{z}) = (1 - t)(z_1^{a_1}\bar{z}_1^{b_1}z_2 + z_2^{a_2}\bar{z}_2^{b_2}z_3 + z_3^{a_3}\bar{z}_3^{b_3}) + t(z_1^{a_1-b_1}z_2 + \bar{z}_2^{b_2-a_2}z_3 + z_3^{a_3-b_3}).$$

   For $t = 0, 1$ the assertion is true since f_B and g_B are polar weighted homogeneous with isolated singularity. Therefore suppose $t \in (0, 1)$.

   Consider the function $H_t\colon \mathbb{C}^6 \to \mathbb{C}$ given by

   $$H_{B,t}(\mathbf{w}) = w_1^{a_1-b_1}w_2((1-t)w_1^{b_1}w_4^{b_1}+t)+w_5^{b_2-a_2}w_3((1-t)w_2^{b_2}w_5^{b_2}+t)+w_3^{a_3-b_3}((1-t)w_3^{b_3}w_6^{b_3}+t)$$

and the map

$$\begin{aligned} e\colon \mathbb{C}^3 &\to \mathbb{C}^6 \\ \mathbf{z} &\mapsto (\mathbf{z}, \bar{\mathbf{z}}), \end{aligned}$$

therefore

$$f_{B,t}(\mathbf{z}) = H_{B,t}(e(\mathbf{z})).$$

Now we have that

$$\begin{aligned} \boldsymbol{\nabla} H_{B,t}(\mathbf{w}) = \Big(&(1-t)a_1 w_1^{a_1-1} w_4^{b_1} w_2 + t(a_1-b_1) w_1^{a_1-b_1-1} w_2, \\ &(1-t)(w_1^{a_1} w_4^{b_1} + a_2 w_2^{a_2-1} w_5^{b_2} w_3) + t w_1^{a_1-b_1}, \\ &(1-t)(w_2^{a_2} w_5^{b_2} + a_3 w_3^{a_3-1} w_6^{b_3}) + t(w_5^{b_2-a_2} + (a_3-b_3) w_3^{a_3-b_3-1}), \\ &(1-t) b_1 w_1^{a_1} w_4^{b_1-1} w_2, \\ &(1-t)(b_2 w_2^{a_2} w_5^{b_2-1} w_3) + t(b_2-a_2) w_5^{b_2-a_2-1} w_3, \\ &(1-t) b_3 w_3^{a_3} w_6^{b_3-1} \Big). \end{aligned}$$

Therefore

$$\begin{aligned} \boldsymbol{\nabla} H_{B,t}(e(\mathbf{z})) = \Big(&z_1^{a_1-b_1-1} z_2((1-t)a_1|z_1|^{2b_1} + t), \\ &(1-t)(a_2 z_2^{a_2-1} \bar{z}_2^{b_2} z_3) + z_1^{a_1-b_1}((1-t)|z_1|^{2b_1} + t), \\ &z_3^{a_3-b_3-1}((1-t)a_3|z_3|^{2b_3} + t(a_3-b_3)) + \bar{z}_2^{b_2-a_2}((1-t)|z_2|^{2a_2} + t), \\ &(1-t) b_1 z_1^{a_1} \bar{z}_1^{b_1-1} z_2, \\ &\bar{z}_2^{b_2-a_2-1} z_3((1-t) b_2 |z_2|^{2a_2} + t(b_2-a_2)), \\ &(1-t) b_3 z_3^{a_3} \bar{z}_3^{b_3-1} \Big). \end{aligned}$$

By Proposition 9.4 we have that

$$e\colon T_{\mathbf{z}} \mathbb{S}^5_{\|\mathbf{z}\|} \to T_{e(\mathbf{z})} \mathbb{S}^{11}_{\|e(\mathbf{z})\|}. \tag{9.1.27}$$

Let $\mathbf{z} = (z_1, z_2, z_3) \in V_{B,t}$ with $t \neq 0, 1$. We want to find a tangent vector to $e(V_{B,t})$ at $e(\mathbf{z})$ which is not in $T_{e(\mathbf{z})} \mathbb{S}^{11}_{\|e(\mathbf{z})\|}$.

Consider the following vectors in $\mathbb{C}^3$

$$v_1 = (z_1, 0, 0) \quad v_2 = (0, z_2, 0) \quad v_3 = (0, 0, z_3),$$

and define

$$\begin{aligned} P_j &= (1-t)(a_j+b_j)|z_j|^{2b_j} + t(a_j-b_j) & \tilde{P}_2 &= (1-t)(a_2+b_2)|z_2|^{2a_2} + t(b_2-a_2) \\ R_{j+1} &= (1-t)|z_j|^{2b_j} + t & \tilde{R}_3 &= (1-t)|z_2|^{2a_2} + t \end{aligned}$$

for $j = 1, 3$.

We have that

$$\begin{aligned}
\boldsymbol{\nabla} H_{B,t}(e(\mathbf{z}))\cdot e(v_1)^\top &= z_1^{a_1-b_1} z_2 P_1\\
\boldsymbol{\nabla} H_{B,t}(e(\mathbf{z}))\cdot e(v_2)^\top &= \bar{z}_2^{b_2-a_2} z_3 P_2 + z_1^{a_1-b_1} z_2 R_2\\
\boldsymbol{\nabla} H_{B,t}(e(\mathbf{z}))\cdot e(v_3)^\top &= z_3^{a_3-b_3} P_3 + \bar{z}_2^{b_2-a_2} z_3 R_3
\end{aligned}$$

Let $r_1, r_2, r_3 \in \mathbb{R}$, then

$$\begin{aligned}
&\boldsymbol{\nabla} H_{B,t}(e(\mathbf{z}))\cdot e(r_1v_1+r_2v_2+r_3v_3)^\top\\
&= z_1^{a_1-b_1} z_2 P_1 r_1 + r_2(\bar{z}_2^{b_2-a_2} z_3 P_2 + z_1^{a_1-b_1} z_2 R_2) + r_3(z_3^{a_3-b_3} P_3 + \bar{z}_2^{b_2-a_2} z_3 R_3)\\
&= z_1^{a_1-b_1} z_2(P_1 r_1 + r_2 R_2) + \bar{z}_2^{b_2-a_2} z_3(r_2 P_2 + r_3 R_3) + z_3^{a_3-b_3} r_3 P_3,
\end{aligned}$$

also we have

$$f_{B,t}(\mathbf{z}) = z_1^{a_1-b_1} z_2 R_2 + \bar{z}_2^{b_2-a_2} z_3 R_3 + z_3^{a_3-b_3} R_1 = 0.$$

Consider the system

$$\begin{aligned}
P_1 r_1 + r_2 R_2 &= R_2\\
r_2 P_2 + r_3 R_3 &= R_3\\
r_3 P_3 &= R_1
\end{aligned}$$

whose solution is given by

$$r_1 = \frac{R_2(1-r_2)}{P_1} \quad r_2 = \frac{R_3(1-r_3)}{P_2} \quad r_3 = \frac{R_1}{P_3}. \tag{9.1.28}$$

Since $0 < R_{j+1} < P_j$, we have that r_1, r_2 and r_3 are all positive real numbers.

Therefore using (9.1.28) we get that

$$\boldsymbol{\nabla} H_{B,t}(e(\mathbf{z}))\cdot e(r_1v_1+r_2v_2+r_3v_3)^\top = 0,$$

and also

$$\Re\langle \mathbf{z}, (r_1z_1, r_2z_2, r_3z_3)\rangle = r_1|z_1|^2 + r_2|z_2|^2 + r_3|z_3|^2 \neq 0$$

therefore we have that $r_1v_1+r_2v_2+r_3v_3 \notin T_{\mathbf{z}}\mathbb{S}^5_{\|\mathbf{z}\|}$, so the intersection with $V_{B,t}$ is transversal.

The other cases are just analogous.

3. Suppose $l = C$. For $t = 0, 1$ the assertion is true since f_C and g_C are polar weighted homogeneous with isolated singularity. Suppose $t \in (0,1)$.

If $a_j > b_j$ for $j = 1, 2, 3$ then,

$$f_{C,t}(\mathbf{z}) = (1-t)(z_1^{a_1}\bar{z}_1^{b_1} z_2 + z_2^{a_2}\bar{z}_2^{b_2} z_3 + z_3^{a_3}\bar{z}_3^{b_3} z_1) + t(z_1^{a_1-b_1} z_2 + z_2^{a_2-b_2} z_3 + z_3^{a_3-b_3} z_1)\ .$$

Using again the definitions given in (9.1.17) we obtain the first three equations in (9.1.18)

$$\begin{aligned}
\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(v_1)^\top &= z_1^{a_1-b_1} z_2 P_1 + z_3^{a_3-b_3} z_1 R_1\\
\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(v_2)^\top &= z_2^{a_2-b_2} z_3 P_2 + z_1^{a_1-b_1} z_2 R_2\\
\boldsymbol{\nabla} H_{C,t}(e(\mathbf{z}))\cdot e(v_3)^\top &= z_3^{a_3-b_3} z_1 P_3 + z_2^{a_2-b_2} z_3 R_3.
\end{aligned}$$

Since
$$f_{C,t}(\mathbf{z}) = z_1^{a_1-b_1} z_2 R_2 + z_2^{a_2-b_2} z_3 R_3 + z_3^{a_3-b_3} z_1 R_1 = 0,$$
then usign the same idea as the previous case we can prove that there exist $r_1, r_2, r_3 \in \mathbb{R}^+$ such that
$$\begin{aligned} \boldsymbol{\nabla} H_{C,t}(e(\mathbf{z})) \cdot e(r_1 v_1 + r_2 v_2 + r_3 v_3)^\top &= 0, \\ \Re\langle \mathbf{z}, r_1 v_1 + r_2 v_2 + r_3 v_3 \rangle &\neq 0, \end{aligned}$$
therefore the intersection is transversal.

The other cases are just analogous. □

Remark 9.7. The last case was conjectured by Oka in [29] and we gave this demonstration in the congress "Brazil-Mexico 2nd meeting on Singularities" held in Salvador, Bahía, Brazil in 2015 using the embedding method. Later this result was proved in [20] using different ideas.

Fix $\varepsilon > 0$ and let $l \in \{A, B, C\}$. Since $f_{l,t}$ has isolated singularity at the origin, there exist $\delta > 0$ such that for any $\eta \in \mathbb{C}$ such that $|\eta| \leq \delta$ the fibers $f_{l,t}^{-1}(\eta)$ and the sphere $\mathbb{S}^5_\varepsilon$ intersect transversely. Let
$$\begin{aligned} \mathcal{N}(\varepsilon, \delta) &:= \{(\mathbf{z}, t) \in \mathbb{C}^3 \times I \,|\, |f_{l,t}(\mathbf{z})| = \delta, \|\mathbf{z}\| \leq r\} \\ \mathcal{N}_t(\varepsilon, \delta) &:= \{\mathbf{z} \in \mathbb{C}^3 \,|\, |f_{l,t}(\mathbf{z})| = \delta, \|\mathbf{z}\| \leq r\}. \end{aligned}$$
Note that $f_{l,t} \colon \mathcal{N}_t(\varepsilon, \delta) \to \mathbb{S}_\delta$ is the Milnor fibration of $f_{l,t}$ on the Milnor tube. Applying Ehresmann's fibration theorem to the projections
$$\pi \colon \mathbb{S}^5_r \times I \to I \qquad \pi' \colon \mathcal{N}(\varepsilon, \delta) \to I\ ,$$
we obtain:

Theorem 9.8 (Isotopy Theorem)**.** *Let $\varepsilon > 0$ small enough. Choose $\delta > 0$ sufficiently small such that for any $\eta \in \mathbb{C}$ with $|\eta| \leq \delta$, the fibre $f_{l,t}^{-1}(\eta)$ and $\mathbb{S}^5_r$ intersect transversely. Then,*

1. *There exists an isotopy $h_t \colon (\mathbb{S}^5_\varepsilon, K_{0,\varepsilon}) \to (\mathbb{S}^5_\varepsilon, K_{t,\varepsilon})$ such that $f_{l,t}(h_t(\mathbf{z})) = f_{l,0}(\mathbf{z})$ for any $\mathbf{z}$ with $|f_{l,0}(\mathbf{z})| \leq \delta$.*
2. *The Milnor fibrations*
$$\begin{aligned} f_{l,0} \colon \mathcal{N}_0(\varepsilon, \delta) &\to \mathbb{S}^1_\delta\ , \\ f_{l,t} \colon \mathcal{N}_t(\varepsilon, \delta) &\to \mathbb{S}^1_\delta\ , \end{aligned}$$
are C^∞ equivalent for any $t \in [0, 1]$ and with $l = A, B, C$.

Corollary 9.9. *The Milnor fibrations of $f_l(\mathbf{z})$ and $g_l(\mathbf{z})$ are C^∞ equivalent.*

9.2 The mixed GSV-index

The GSV index was defined by Gómez-Mont, Seade and Verjovsky for vector fields on an isolated complex hypersurface [15].

The idea of the construction is the following: Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function with an isolated critical point at the origin and denote by $V := f^{-1}(0)$. Consider also v a continuous vector field on V with an isolated singularity at the origin.

Since V has an isolated singularity, the vector field $\overline{\boldsymbol{\nabla} f}$ is normal to $V^* := V \setminus \{\mathbf{0}\}$ for the usual hermitian inner product in $\mathbb{C}^n$. Therefore the set $\{\overline{\boldsymbol{\nabla} f}, v\}$ is a 2-frame at each point in V^*, and up to homotopy, it can be assumed to be orthonormal.

Let K be the link of V, then we have a continuous map

$$\phi_v = (\overline{\boldsymbol{\nabla} f}, v)\colon K \to W(2,n) \tag{9.2.1}$$

where $W(2,n)$ is the Stiefel manifold of complex orthonormal 2-frames in $\mathbb{C}^n$

Definition 9.10 ([6, Definition 3.2.1])**.** The GSV index of v at $\mathbf{0} \in V_f$, $\mathrm{Ind}_{\mathrm{GSV}}(v,\mathbf{0})$ is the degree of the map ϕ_v.

In this section we will generalize the GSV index for some mixed function under a hypothesis.

9.2.1 The complex embedding

Given a complex vector space V we can define its *conjugate* $\bar{V}$, which is again a *complex* vector space, which as an abelian group, $\bar{V}$ is the same as V, but $\mathbb{C}$ acts by scalar multiplication on $\bar{V}$ in a new way: $\lambda \in \mathbb{C}$ acts on $\bar{V}$ as $\bar{\lambda}$ used to act on V; so if we denote by $\cdot$ the scalar multiplication on $\bar{V}$ we have

$$\lambda \cdot v := \bar{\lambda} v.$$

Let $\bar{\mathbb{C}}^n$ be the *conjugate* of $\mathbb{C}^n$. Consider the map given by

$$\begin{aligned} \bar{c}_n\colon \mathbb{C}^n &\to \bar{\mathbb{C}}^n, \\ \mathbf{z} &\mapsto \bar{\mathbf{z}}. \end{aligned} \tag{9.2.2}$$

it is an isomorphism of *complex* vector spaces, since it is $\mathbb{C}$-linear:

$$\bar{c}_n(\lambda \mathbf{z}) = \overline{\lambda \mathbf{z}} = \bar{\lambda}\bar{\mathbf{z}} = \lambda \cdot \bar{\mathbf{z}} = \lambda \cdot \bar{c}_n(\mathbf{z}). \tag{9.2.3}$$

Consider the following embedding

$$\begin{aligned} \bar{e}_n = (\mathrm{id}_n, c_n)\colon \mathbb{C}^n &\to \mathbb{C}^n \oplus \bar{\mathbb{C}}^n, \\ \mathbf{z} &\mapsto (\mathbf{z}, \bar{\mathbf{z}}). \end{aligned} \tag{9.2.4}$$

Remark 9.11. Notice that the embedding $\bar{e}_n$ is $\mathbb{C}$-linear, since both coordinate maps are $\mathbb{C}$-linear. Hence, the image $H = \bar{e}_n(\mathbb{C}^n)$ is an n-dimensional complex subspace of $\mathbb{C}^n \oplus \bar{\mathbb{C}}^n$.

As before, let $K, L \subset \underline{n}$ such that $|K| = k$ and $|L| = l$. Define the $\mathbb{C}$-linear map

$$\begin{aligned} \bar{e}_{K,L} = (p_K, \bar{c}_l \circ p_L)\colon \mathbb{C}^n &\to \mathbb{C}^k \oplus \bar{\mathbb{C}}^l \\ (z_1, \ldots, z_n) &\mapsto \big((z_{i_1}, z_{i_2}, \ldots, z_{i_k}), (\bar{z}_{j_1}, \bar{z}_{j_2}, \ldots, \bar{z}_{j_l})\big). \end{aligned}$$

So we have that $\bar{e}_n = \bar{e}_{\underline{n},\underline{n}}$.

Consider the conjugate Hermitian product $\langle\ ,\ \rangle_{\bar{\mathbb{C}}^n} \colon \bar{\mathbb{C}}^n \to \mathbb{C}$ on $\bar{\mathbb{C}}^n$:

$$\langle \mathbf{z}, \mathbf{z}' \rangle_{\bar{\mathbb{C}}^n} = \sum_{j=1}^{n} \bar{z}_j z'_j.$$

The Hermitian inner product on the direc sum $\mathbb{C}^k \oplus \bar{\mathbb{C}}^l$ gives the Hermitian product

$$\begin{gathered} \langle\ ,\ \rangle_{k,l} \colon \mathbb{C}^k \oplus \bar{\mathbb{C}}^l \to \mathbb{C}, \\ \langle (\mathbf{z}, \mathbf{w}), (\mathbf{z}', \mathbf{w}') \rangle_{k,l} = \langle \mathbf{z}, \mathbf{z}' \rangle + \langle \mathbf{w}, \mathbf{w}' \rangle_{\bar{\mathbb{C}}^n} = \sum_{j=1}^{k} z_j \bar{z}'_j + \sum_{j=1}^{l} \bar{w}_j w'_j. \end{gathered} \tag{9.2.5}$$

9.2.2 All the variables

As in Subsection 9.1.1, consider the complex space $\mathbb{C}^n \oplus \bar{\mathbb{C}}^n$ with variables

$$(\mathbf{z}, \mathbf{w}) = (z_1, z_2, \dots, z_n, w_1, w_2, \dots, w_n),$$

and the complex embedding

$$\begin{aligned} \bar{e}_n = (\mathrm{id}_n, c_n) \colon \mathbb{C}^n &\to \mathbb{C}^n \oplus \bar{\mathbb{C}}^n, \\ \mathbf{z} &\mapsto (\mathbf{z}, \bar{\mathbf{z}}); \end{aligned} \tag{9.2.6}$$

For $i = 1, 2, \dots, n$, define the vectors $\mathsf{n}_i \in \mathbb{C}^n \oplus \bar{\mathbb{C}}^n$ with coordinates

$$z_j = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \qquad w_j = \begin{cases} -1 & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

that is, we have

$$\begin{aligned} \mathsf{n}_1 &= (1, \dots, 0, \underbrace{-1}_{n+1}, \dots, 0), \\ \mathsf{n}_2 &= (0, 1 \dots, 0, 0, \underbrace{-1}_{n+2}, \dots, 0), \\ &\ \vdots \\ \mathsf{n}_i &= (0, \dots, \underbrace{1}_{i}, \dots, 0, 0 \dots, \underbrace{-1}_{n+i}, \dots, 0), \\ &\ \vdots \\ \mathsf{n}_n &= (0, \dots, \underbrace{1}_{n}, 0, \dots, \underbrace{-1}_{2n}). \end{aligned}$$

Also define its corresponding orthogonal complex subspaces H_i with respect to the Hermitian inner product on $\mathbb{C}^n \oplus \bar{\mathbb{C}}^n$ given in (9.2.5)

$$H_i = \{\, (\mathbf{z}, \mathbf{w}) \in \mathbb{C}^n \oplus \bar{\mathbb{C}}^n \mid z_i = \bar{w}_i \,\} = \{\, (\mathbf{z}, \mathbf{w}) \in \mathbb{C}^n \oplus \bar{\mathbb{C}}^n \mid \langle (\mathbf{z}, \mathbf{w}), \mathsf{n}_i \rangle_{n,n} = 0 \,\}.$$

Remark 9.12. Notice that the image H of the embedding $\bar{e}_n$ is precisely the intersection of the subspaces H_i, that is,

$$\bar{e}_n(\mathbb{C}^n) = H = \bigcap_{i=1}^{n} H_i.$$

Moreover, H is the orthogonal complement with respect to the Hermitian inner product $\langle\ ,\ \rangle_{n,n}$ of the complex subspace $\langle \mathsf{n}_1, \dots, \mathsf{n}_n \rangle$ of $\mathbb{C}^n \oplus \bar{\mathbb{C}}^n$ generated by the vectors $\{\mathsf{n}_1, \dots, \mathsf{n}_n\}$.

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a mixed function and let $F\colon \mathbb{C}^n \oplus \bar{\mathbb{C}}^n \to \mathbb{C}$ be its associated holomorphic function. We have that the following commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\bar{e}_n"] \arrow[dr, "f"'] & \mathbb{C}^n \oplus \bar{\mathbb{C}}^n \arrow[d, "F"] \\ & \mathbb{C}. \end{tikzcd} \tag{9.2.7}$$

Set

$$V_f = f^{-1}(0), \qquad V_f^* = V_f - \{0\},$$
$$V_F = F^{-1}(0), \qquad V_F^* = V_F - \mathrm{Sing} V_F.$$

Notice that

$$V_f \cong V_F \cap H. \tag{9.2.8}$$

Let ∇F be the gradient of F at $(\mathbf{z}, \mathbf{w}) \in \mathbb{C}^n \oplus \bar{\mathbb{C}}^n$ given by

$$\nabla F(\mathbf{z}, \mathbf{w}) = \Big(\frac{\partial F}{\partial z_1}(\mathbf{z}, \mathbf{w}), \dots, \frac{\partial F}{\partial z_n}(\mathbf{z}, \mathbf{w}), \frac{\partial F}{\partial w_1}(\mathbf{z}, \mathbf{w}), \dots, \frac{\partial F}{\partial w_n}(\mathbf{z}, \mathbf{w}) \Big).$$

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a mixed function such that

1. All the variables z_i and $\bar{z}_i$, for $i = 1, \dots, n$ appear in f.

2. It has an isolated critical point at the origin.

3. For every $\mathbf{z} \in V_f^*$ we have that $\bar{e}_n(\mathbf{z}) \in V_F^*$.

Let $\mathbb{S}_\varepsilon^{2n-1} \subset \mathbb{C}^n$ be a sphere of suficiently small radius $\varepsilon > 0$ with centre at the origin and let $L_f = V_f \cap \mathbb{S}_\varepsilon^{2n-1}$ be the link of f at 0. We have that L_f is a compact oriented smooth manifold of dimension $2n-3$.

Let v be a continuous vector field on V_f with isolated singularity at 0, or goal is to define an integer-valued index for v.

Proposition 9.13. *Let $v = (v_1, \dots, v_n)$ be a continuous vector field on $V_f \subset \mathbb{C}^n$ with isolated singularity at* 0*. Then, the vectors*

$$\{\mathsf{n}_1, \dots, \mathsf{n}_n, \nabla F(\bar{e}_{n,n}(\mathbf{z}))\} \tag{9.2.9}$$

are $\mathbb{C}$-linearly independent for every $\mathbf{z} \in V_f^*$.

PROOF. By condition 3, the vector $\nabla F(\bar{e}_{n,n}(\mathbf{z}))$ is not zero. Suppose that

$$\nabla F(\bar{e}_{n,n}(\mathbf{z})) = \alpha_1 \mathsf{n}_1 + \cdots + \alpha_n \mathsf{n}_n, \qquad \text{with } \alpha_i \in \mathbb{C}.$$

Then we have that (remember the scalar product on $\mathbb{C}^n \oplus \bar{\mathbb{C}}^n$)

$$\frac{\partial F}{\partial z_i}(\bar{e}_{n,n}(\mathbf{z})) = \alpha_i, \quad \frac{\partial F}{\partial w_i}(\bar{e}_{n,n}(\mathbf{z})) = -\bar{\alpha}_i, \quad \text{for } i = 1, \ldots, n.$$

By the definition of F or the commutative diagram (9.2.7) we have that

$$\frac{\partial f}{\partial z_i}(\mathbf{z}) = \frac{\partial F}{\partial z_i}(\bar{e}_{n,n}(\mathbf{z})), \quad \frac{\partial f}{\partial \bar{z}_i}(\mathbf{z}) = \frac{\partial F}{\partial w_i}(\bar{e}_{n,n}(\mathbf{z})), \quad \text{for } i = 1, \ldots, n. \tag{9.2.10}$$

Hence

$$\overline{\frac{\partial f}{\partial z_i}}(\mathbf{z}) = -\frac{\partial f}{\partial \bar{z}_i}(\mathbf{z}), \quad \text{for } i = 1, \ldots, n,$$

but by Proposition 7.2, this implies that $\mathbf{z}$ is a critical point of f, which is a contradiction. □

Now it is important to assume the following hypothesis

Hypothesis 9.14. The vectors $\{\mathsf{n}_1, \ldots, \mathsf{n}_n, \nabla F(\bar{e}_{n,n}(\mathbf{z})), \bar{e}_{n,n}(v(\mathbf{z}))\}$ are $\mathbb{C}$-linearly independent for every $\mathbf{z} \in V_f^*$.

By Hypothesis 9.14, the set $\{\mathsf{n}_1, \ldots, \mathsf{n}_n, \nabla F(\bar{e}_{n,n}(\mathbf{z})), \bar{e}_{n,n}(v(\mathbf{z}))\}$ is a complex $(n+2)$-frame in $\mathbb{C}^n \oplus \bar{\mathbb{C}}^n$ at each point $\mathbf{z}$ in V_f^*, and up to homotopy, it can be assumed to be orthonormal. Hence we have a continuos map from V_f^* to the Stiefel manifold $W(2n, n+2)$ of complex $(n+2)$-frames in $\mathbb{C}^{2n}$, which restricted to the link L_f gives a continuos map

$$\phi_v \colon L_f \to W(2n, n+2).$$

The Stiefel manifold $W(2n, n+2)$ is $2n-4$-connected (see [36]), that is,

$$\pi_i\big(W(2n, n+2)\big) = \begin{cases} 0 & \text{for } i \leq 2n-4, \\ \mathbb{Z} & \text{for } i = 2n-3. \end{cases}$$

By Hurewicz Theorem we have that

$$H_{2n-3}\big(W(2n, n+2); \mathbb{Z}\big) \cong \mathbb{Z}. \tag{9.2.11}$$

Hence the map ϕ_v has a well defined degree $\deg(\phi_v) \in \mathbb{Z}$ defined by considering the homomorphism induced in homology

$$(\phi_v)_* \colon H_{2n-3}(L_f; \mathbb{Z}) \cong \mathbb{Z} \to H_{2n-3}(W(2n, n+2); \mathbb{Z}) \cong \mathbb{Z},$$

and taking the image of the fundamental class $[L_f] \in H_{2n-3}(L_f; \mathbb{Z})$ of L_f, that is, $\deg(\phi_v) = (\phi_v)_*([L_f])$.

Definition 9.15. Given a continuous vector field v on V_f with isolated singularity at 0, we define the *mixed GSV-index* of v by

$$\mathrm{Ind}_{mGSV}(v, 0) = \deg(\phi_v).$$

9.2.3 Not all the variables

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a mixed function.

As in Subsection 9.1.1, set $\underline{n} = \{1, 2, \ldots, n\}$ and define the subsets K and L of $\underline{n}$

$$K = \{\, i \in \underline{n} \mid \text{the variable } z_i \text{ appears in } f \,\},$$
$$L = \{\, j \in \underline{n} \mid \text{the variable } \bar{z}_j \text{ appears in } f \,\}.$$

with $K \cup L = \underline{n}$. Suppose that $|K| = k$, $|L| = l$ and $|K \cap L| = m$ (it can be zero), then we have that

$$n = k + l - m. \tag{9.2.12}$$

Consider the complex space $\mathbb{C}^k \oplus \bar{\mathbb{C}}^l$ with variables

$$(\mathbf{z}, \mathbf{w}) = (z_1, \ldots, z_{i_k}, w_1, \ldots, w_{j_l}),$$

and the complex embedding

$$\begin{aligned} \bar{e}_{K,L} = (p_K, \bar{c}_l \circ p_L)\colon \mathbb{C}^n &\to \mathbb{C}^k \oplus \bar{\mathbb{C}}^l \\ (z_1, \ldots, z_n) &\mapsto \big((z_{i_1}, z_{i_2}, \ldots, z_{i_k}), (\bar{z}_{j_1}, \bar{z}_{j_2}, \ldots, \bar{z}_{j_l})\big). \end{aligned}$$

For $i \in K \cap L$, define the m vectors $\mathsf{n}_i \in \mathbb{C}^k \oplus \bar{\mathbb{C}}^l$ with coordinates

$$z_j = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \qquad w_j = \begin{cases} -1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

that is, we have

$$\begin{aligned} \mathsf{n}_1 &= (1, \ldots, 0, \underbrace{-1}_{n+1}, \ldots, 0), \\ \mathsf{n}_2 &= (0, 1 \ldots, 0, 0, \underbrace{-1}_{n+2}, \ldots, 0), \\ &\ \vdots \\ \mathsf{n}_i &= (0, \ldots, \underbrace{1}_{i}, \ldots, 0, 0 \ldots, \underbrace{-1}_{n+i}, \ldots, 0), \\ \mathsf{n}_i &= (0, \ldots, \underbrace{1}_{i}, \ldots, 0, 0 \ldots, \underbrace{-1}_{n+i}, \ldots, 0), \\ &\ \vdots \\ \mathsf{n}_n &= (0, \ldots, \underbrace{1}_{n}, 0, \ldots, \underbrace{-1}_{2n}). \end{aligned}$$

Also define its corresponding orthogonal complex subspaces H_i with respect to the Hermitian inner product on $\mathbb{C}^n \oplus \bar{\mathbb{C}}^n$ given in (9.2.5)

$$H_i = \{\, (\mathbf{z}, \mathbf{w}) \in \mathbb{C}^k \oplus \bar{\mathbb{C}}^l \mid z_i = \bar{w}_i \,\} = \{\, (\mathbf{z}, \mathbf{w}) \in \mathbb{C}^k \oplus \bar{\mathbb{C}}^l \mid \langle (\mathbf{z}, \mathbf{w}), \mathsf{n}_i \rangle_{k,l} = 0 \,\}.$$

Remark 9.16. Notice that the image H of the embedding $\bar{e}_n$ is precisely the intersection of the subspaces H_i, that is,

$$\bar{e}_{K,L}(\mathbb{C}^n) = H = \bigcap_{i=1}^{m} H_i.$$

Moreover, H is the orthogonal complement with respect to the Hermitian inner product $\langle\ ,\ \rangle_{k,l}$ of the complex subspace $\langle \mathsf{n}_1, \dots, \mathsf{n}_m \rangle$ of $\mathbb{C}^k \oplus \bar{\mathbb{C}}^l$ generated by the vectors $\{\mathsf{n}_1, \dots, \mathsf{n}_m\}$.

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a mixed function and let $F\colon \mathbb{C}^k \oplus \bar{\mathbb{C}}^l \to \mathbb{C}$ be its associated holomorphic function. We have that the following commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\bar{e}_{K,L}"] \arrow[dr, "f"'] & \mathbb{C}^k \oplus \bar{\mathbb{C}}^k \arrow[d, "F"] \\ & \mathbb{C}. \end{tikzcd} \tag{9.2.13}$$

Set

$$V_f = f^{-1}(0), \qquad V_f^* = V_f - \{0\},$$
$$V_F = F^{-1}(0), \qquad V_F^* = V_F - \mathrm{Sing}V_F.$$

Notice that

$$V_f \cong V_F \cap H. \tag{9.2.14}$$

Let ∇F be the gradient of F at $(\mathbf{z}, \mathbf{w}) \in \mathbb{C}^n \oplus \bar{\mathbb{C}}^n$ given by

$$\nabla F(\mathbf{z}, \mathbf{w}) = \Big(\frac{\partial F}{\partial z_{i_1}}(\mathbf{z}, \mathbf{w}), \dots, \frac{\partial F}{\partial z_{i_k}}(\mathbf{z}, \mathbf{w}), \frac{\partial F}{\partial w_{j_1}}(\mathbf{z}, \mathbf{w}), \dots, \frac{\partial F}{\partial w_{j_l}}(\mathbf{z}, \mathbf{w}) \Big).$$

We also ask that the mixed function $f\colon \mathbb{C}^n \to \mathbb{C}$ satisfies

1. It has an isolated critical point at the origin.

2. For every $\mathbf{z} \in V_f^*$ we have that $\bar{e}_{K,L}(\mathbf{z}) \in V_F^*$.

Let $\mathbb{S}_\varepsilon^{2n-1} \subset \mathbb{C}^n$ be a sphere of suficiently small radius $\varepsilon > 0$ with centre at the origin and let $L_f = V_f \cap \mathbb{S}_\varepsilon^{2n-1}$ be the link of f at 0. We have that L_f is a compact oriented smooth manifold of dimension $2n-3$.

Let v be a continuous vector field on V_f with isolated singularity at 0, or goal is to define an integer-valued index for v.

Proposition 9.17. *Let $v = (v_1, \dots, v_n)$ be a continuous vector field on $V_f \subset \mathbb{C}^n$ with isolated singularity at* 0*. Then, the vectors*

$$\{\mathsf{n}_1, \dots, \mathsf{n}_m, \nabla F(\bar{e}_{K,L}(\mathbf{z}))\} \tag{9.2.15}$$

are linearly independent for every $\mathbf{z} \in V_f^*$.

Proof. By condition 2, the vector $\nabla F(\bar{e}_{K,L}(\mathbf{z}))$ is not zero. Suppose that

$$\nabla F(\bar{e}_{K,L}(\mathbf{z})) = \alpha_1 \mathsf{n}_1 + \cdots + \alpha_m \mathsf{n}_m, \qquad \text{with } \alpha_i \in \mathbb{C}.$$

Then we have that (remember the scalar product on $\mathbb{C}^n \oplus \bar{\mathbb{C}}^n$)

$$\begin{aligned} &\frac{\partial F}{\partial z_i}(\bar{e}_{K,L}(\mathbf{z})) = \alpha_i, \quad \text{for } i \in K \cap L, && \frac{\partial F}{\partial z_i}(\bar{e}_{K,L}(\mathbf{z})) = 0, \quad \text{for } i \in K - L,\\ &\frac{\partial F}{\partial w_i}(\bar{e}_{K,L}(\mathbf{z})) = -\bar{\alpha}_i, \quad \text{for } i \in K \cap L, && \frac{\partial F}{\partial w_i}(\bar{e}_{K,L}(\mathbf{z})) = 0, \quad \text{for } i \in L - K. \end{aligned} \tag{9.2.16}$$

By the definition of F or the commutative diagram (9.2.13) we have that

$$\begin{aligned} &\frac{\partial f}{\partial z_i}(\mathbf{z}) = \frac{\partial F}{\partial z_i}(\bar{e}_{K,L}(\mathbf{z})), \quad \text{for } i \in K, && \frac{\partial f}{\partial z_i}(\mathbf{z}) = 0, \quad \text{for } i \notin K,\\ &\frac{\partial f}{\partial \bar{z}_i}(\mathbf{z}) = \frac{\partial F}{\partial w_i}(\bar{e}_{K,L}(\mathbf{z})), \quad \text{for } i \in L, && \frac{\partial f}{\partial \bar{z}_i}(\mathbf{z}) = 0, \quad \text{for } i \notin L. \end{aligned} \tag{9.2.17}$$

Hence combining (9.2.16) and (9.2.17) we get

$$\begin{aligned} \overline{\frac{\partial f}{\partial z_i}}(\mathbf{z}) = \bar{\alpha}_i = -\frac{\partial f}{\partial \bar{z}_i}(\mathbf{z}), &\quad \text{for } i \in K \cap L,\\ \overline{\frac{\partial f}{\partial z_i}}(\mathbf{z}) = \overline{\frac{\partial F}{\partial z_i}}(\mathbf{z}) = 0 = \frac{\partial f}{\partial \bar{z}_i}(\mathbf{z}), &\quad \text{for } i \in K - L,\\ \overline{\frac{\partial f}{\partial z_i}}(\mathbf{z}) = 0 = \frac{\partial F}{\partial w_i}(\bar{e}_{K,L}(\mathbf{z})) = \frac{\partial f}{\partial \bar{z}_i}(\mathbf{z}) &\quad \text{for } i \in L - K. \end{aligned}$$

but by Proposition 7.2, this implies that $\mathbf{z}$ is a critical point of f, which is a contradiction. □

Now as before we need an hypothesis.

Hypothesis 9.18. The vectors $\{\mathsf{n}_1, \dots, \mathsf{n}_n, \nabla F(\bar{e}_{n,n}(\mathbf{z})), \bar{e}_{n,n}(v(\mathbf{z})), \bar{e}_{K,L}(v(\mathbf{z}))\}$ are $\mathbb{C}$-linearly independent for every $\mathbf{z} \in V_f^*$.

By Hypothesis 9.18, the set $\{\mathsf{n}_1, \dots, \mathsf{n}_m, \nabla F(\bar{e}_{K,L}(\mathbf{z})), \bar{e}_{K,L}(v(\mathbf{z}))\}$ is a complex $(m+2)$-frame in $\mathbb{C}^k \oplus \bar{\mathbb{C}}^l$ at each point $\mathbf{z}$ in V_f^*, and up to homotopy, it can be assumed to be orthonormal. Hence we have a continuous map from V_f^* to the Stiefel manifold $W(k+l, m+2)$ of complex $(m+2)$-frames in $\mathbb{C}^{k+l}$, which restricted to the link L_f gives a continuous map

$$\phi_v \colon L_f \to W(k+l, m+2).$$

The Stiefel manifold $W(k+l, m+2)$ is $2(k+l-m-2)$-connected (see [36]), thus, by (9.2.12), it is $2(n-2)$-connected, that is,

$$\pi_i\big(W(2n, n+2)\big) = \begin{cases} 0 & \text{for } i \leq 2(n-2),\\ \mathbb{Z} & \text{for } i = 2n-3. \end{cases}$$

By Hurewicz Theorem we have that

$$H_{2n-3}\big(W(2n, n+2); \mathbb{Z}\big) \cong \mathbb{Z}. \tag{9.2.18}$$

Hence the map ϕ_v has a well defined degree $\deg(\phi_v) \in \mathbb{Z}$ defined by considering the homomorphism induced in homology

$$(\phi_v)_* \colon H_{2n-3}(L_f;\mathbb{Z}) \cong \mathbb{Z} \to H_{2n-3}(W(m,k);\mathbb{Z}) \cong \mathbb{Z},$$

and taking the image of the fundamental class $[L_f] \in H_{2n-3}(L_f;\mathbb{Z})$ of L_f, that is, $\deg(\phi_v) = (\phi_v)_*([L_f])$.

Definition 9.19. Given a continuous vector field v on V_f with isolated singularity at 0, we define the *mixed GSV-index* of v by

$$\mathrm{Ind}_{mGSV}(v,0) = \deg(\phi_v).$$

Remark 9.20. We are working in order to eliminate the Hypothesis 9.18 and 9.18. This will appear in a future paper.

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