UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
POSGRADO EN CIENCIAS FÍSICAS
INSTITUTO DE FÍSICA, INSTITUTO DE CIENCIAS NUCLEARES

CURVATURE FOR A LIE-ALGEBRAIC SPACE-TIME

TESIS

QUE PARA OPTAR POR EL GRADO DE
MAESTRO EN FÍSICA

P R E S E N T A:
D I E G O V I D A L C R U Z P R I E T O

Director de Tesis:
Dr. José David Vergara Oliver
Instituto de Ciencias Nucleares

Ciudad Universitaria, Cd. Mx. Septiembre 2017

Diego Vidal-Cruzprieto

Curvature for a Lie-algebraic Space-time

Master's Thesis

– Monograph –

September 13, 2017

To my parents, Elsa and Paco. You laid the foundations of my life and enrich them enduringly through your acts of love. Please consider this work as a token from my deepest gratefulness.

To Alberto, the other recessive gene amongst the family. Inadvertently you have taught me the sound of perseverance.

To the ever-loving and supportive Vidal-Correa family. Your heartwarming embrace means a lot to me. It gives me a sense of belonging.

To Anas, Bárbara and Carlos. For being the living proof that there is more to family than just blood. You are my kin for life.

Anas, there will never be enough words to describe what you have done for me. You were a true friend during dark times, you signaled a path towards finding myself.

Bárbara, the embodiment of the living force of nature. Your determination serves as a source of inspiration on a daily basis. The days that passed upon us have made you both the beacon and compass of my soul.

Carlos, I've shared a home with you. Words written here will never weigh the same as those we exchange in our deep conversations. Thank you, for every single day. In one year you have distilled the best things that lurk within me.

Prof. Vergara. I will always admire your erudition, brightness and sensitivity when it comes to solve a problem. Thanks for all the time you have deposited on me. I owe who I am as a scientist to you.

Prof. Rosenbaum. I appraise all those midday discussions, they gave me an stout and oriented physical insight. When you talk about physics, elegance is summoned into the room.

To my comrades in NoFM. Sebastián, Benjamín, Gabriela, Óscar. You ignited the spark that transformed into flare and soon enough will become the blaze that shall melt the walls. Mejía, Eveillard, Vera, Muciño, Julián, Claudio, Emiliana, thanks for all the joyful moments.

And last but not least, to John. Long live Bolzano-Weierstrass.

The author acknowledges partial support from DGAPA-UNAM grant IN 103716

Preface

Here come the golden words

place(s),
month year

First name Surname
First name Surname

Part I

Resumen

¿Qué es la *geometría no-conmutativa*? Resolver esta pregunta no será una tarea sencilla y una prueba de ello es que tomará la gran mayoría de este texto proveer un bosquejo de la respuesta deseada.

Podemos pensar que es una generalización de la geometría diferencial usual. Con el fin de reforzar esta idea pensemos en una variedad suave equipada con coordenadas reales $\mathcal{M}$; además, permitámonos considerar que las funciones que actúan sobre el son de clase $C^{\infty}(\mathcal{M})$, las funciones infinito-diferenciables. En algunas ocasiones elegimos ignorar el hecho de que éstas funciones constituyen una álgebra conmutativa sobre los números reales $\mathcal{A}$, sus características de álgebra se pueden ver un modo inmediato: la adición y multiplicación de funciones está bien definida punto a punto.

La alusión a dicha álgebra no es gratuita, contiene un vínculo conceptual profundo entre álgebra y geometría. Consideremos por un instante que sólo conocemos el álgebra $\mathcal{A}$ sin saber como la obtuvimos, toda referencia a la variedad, es decir a la geometría se ha desvanecido. Ésta situación puede parecer desventajosa a primera instancia, sin embargo es posible reconstruir la variedad $\mathcal{M}$ desde la información contenida en el álgebra. De este modo se puede plantear una equivalencia entre ambas estructuras. Ésto es la esencia del teorema de Gel'fand-Neimark.

La geometría no-conmutativa generaliza ésta notción reemplazando el álgebra conmutativa con una no-conmutativa, por ende, usando la verrsión no-conmutativa del teorema de Gel'fand-Neimark, podemos encontrar una geometría no-conmutativa asociada al álgebra que estamos considerando. Es importante mencionar que a este nivel la noción de variedad desaparece, aún así, es conveniente ver esto como una ganancia, ya que permite generalizar conceptos geométricos sin hacer referencia a una variedad.

Como apunte final a éste tema, permitámonos detallar en que sentido se pierde este concepto: dado que cualquier álgebra C^* conmutativa puede ser vista como el álgebra C^* de algunas funciones aactuando sobre un espacio de Hausdorff localmente compacto, un álgebra C^* no-conmutativa será considerada como el álgebra de funciones continúas sobre un *espacio no-conmutativo*. Esto implica que ahora podremos centrar nuestra atención sobre el álgebra de las funciones en lugar de hacerlo sobre los espacios, y es por ello que decimos que el concepto de variedad desaparece, pues además de ya no ser necesario, la localidad que implica un punto puede llevar a varios problemas conceptuales.
¿Por qué queremos estudiar la geometría no-conmutativa? Una respuesta inmediata es: en mecańica cuántica usualmente encontramos álgebras no-conmutativas estudian y describen la naturaleza microscópica del espacio-tiempo. Permitámonos elaborar más éste punto, pensemos en el álgebra de Heisenberg $[x, p] = i\hbar$, en ésta base para dicha álgebra es posible encontrar una geometría asociada, y tomar conceptos de ella para investigar la dinámica cuántica de un sistema. Además, consideremos el álgebra de momento angular $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$, la cual bajo la interpretación usual es la álgebra de rotaciones de la esfera S_2. Desde el punto de vista de la geometría

no-conmutativa, éstos operadores deben concebirse como coordenadas que definen una nueva esfera: la esfera no-conmutativa, que exhibe un nuevo conjunto de características que emergen de la no-conmutatividad. Es una esfera en la que el radio se identifica con el *operador Casimir*[Mad99] L^2 y por ende el espacio se cuantiza, en el sentido de que si queremos mover en la dirección radial, no podemos hacerlo de un modo continúo; en lugar de esto, nos vemos obligados a hacerlo en pasos discretos definidos por $\ell(\ell+1)$.

Ésta *cuantización radial* revela que la geometría no-conmutativa puede dar un modo natural y sencillo para cuantizar el espacio-tiempo, lo cual a su vez es la segunda respuesta a la pregunta planteada anteriormente. No debemos ver esto como una coincidencia afortunada, ya que hay evidencia que indica que el espacio-tiempo no debe ser continúo en escalas pequeñas, y una forma de implementar esto es mediante una descripción del mismo con geoemtría no-conmutativa.

Dado que estamos lidiando con un campo que se encuentra en investigación activa, hay diferentes versiones de lo que es la geometría no-conmutativa, todas están basas en las bases estipuladas anteriormente, pero se desarrollan a través de conceptos y formalismos distintos.

La primera y más célebre es la aproximación funciones de Connes [Kha13, Con95], en la cual se desarrolla un nuevo cálculo diferencial por medio de la introducción de una *tripleta espectral* $(\mathcal{A}, \mathcal{H}, D, J)$, donde $\mathcal{A}$ es una álgebra no-conmutativa, $\mathcal{H}$ es un espacio de Hilbert donde la realización de $\mathcal{A}$ está dada por la álgebra de operadores acotados $\mathcal{B}(\mathcal{A})$, finalmente D es el operador el cual codificará la gran mayoría de la información geométrica. Finalmente, la isometría antilineal J induce la estructura real en la tripleta, no es fundamental para el formalismo pero se introduce para teer contacto con aplicaciones en la física.

Con esto, Connes fue capaz de definir una cálculo diferencial que generaliza al usual y sirve para la geometría no-conmutativa, además, le permite construir el análogo no-conmutatico de la integral, la *traza de Dixmier* que puede ser calculada usando el *residuo de Wodzicki*; con éstas herramientas Connes pudo proponer una acción geométrica o espectral, la cual bajo ciertas condiciones, es capaz de generar el Lagrangiano del modelo estánda acoplado a la interacción gravitacional de la teoría de Einstein. Éste resultado es importante en varios niveles pero nos gustaría señalar uno de ellos: la materia ahora se puede pensar como una consecuencia de la no-conmutatividad del espacio-tiempo.

El enfoque de Majid a la geometría no-conmutativa tiene mucho en común con el de Connes, sin embargo es un camino un poco más directo. Es más específico ya que se considera que la no-conmutatividad del espacio-tiempo es del tipo álgebra de Lie $[x^\mu, x^\nu] = C^{\mu\nu}{}_\lambda x^\lambda$. Para definir un cálculo diferencual es necesario introducir uno-formas álgebra-valuadas $\Omega^1(\mathcal{A})$, que obedecen una relación heredada del conmutador del álgebra de Lie $[dx^\mu, x^\nu] = D^{\mu\nu}{}_\lambda dx^\lambda$, donde $D^{\mu\nu}{}_\lambda$ son términos constantes proporcionales a las constantes de estructura con un término simétrico adicional que se elige bajo consideraciones físicas.

Además, se pide que el elemento del línea g sea un elemento central del álgebra, esto garantiza la existencia de un inverso cuya acción en un elemento del álgebra siempre dé el mismo resultado. Esto ayuda a construir entidades que se parecen a los tensores en éste marco de trabajo y por ende poder establecer una relación directa con la teoría de Einstein.

Finalmente, haciendo uso de la teoría algebraica de las conexiones desarrollada por Koszul [Kos86], se encuentra una fórmula para la conexión con términos adicionales relacionados a correcciones cuánticas provenientes de la no-conmutatividad del espacio-tiempo; en nuestro trabajo se eligió truncar la expresión para sólo considerar términos de primer orden en el parámetro cuántico (i.e. la longitud de Planck).

Con todas éstas consideraciones, Majid puede analizar espacios-tiempos en particular [MR94], entre ellos el *bicrossproduct model* [BM14] y algunos otros. Nuestra motivación principal emerge de aquí, pues buscamos analizar cualquier espacio-tiemopo del tipo álgebra de Lie. Logramos esto emulando el procedimiento de Majid con dos consideraciones más: primero se debe comenzar desde un espacio-tiempo clásico cuyas simetrías deben ser promovidas al álgebra del espacio-tiempo. En otras palabras, la álgebra de los vectores de Killing para un espacio-tiempo en particular se promueve al álgebra no conmutativa que describe al espacio-tiempo considerada junto con la métrica. Segundo, dado que ya tenemos un elemento de línea, la condición de centralidad fija el término simétrico del conmutador entre un elemento de la base de unno-formas y un generador del álgebra. Esto proviene del hecho de que la ambigüedad en dicha elección debe ser eliminada por argumentos que provengan de consideraciones físicas. La diferencia en nuestro método es que buscamos cuantizar un espacio-tiempo que posea simetrías en un nivel clásico.

En este texto presentamos un procedimiento general para estudiar un espacio-tiempo del tipo álgebra de Lie usando geometría no-conmmutativa. Para lograr esto, empezamos con una breve revisión de geometría diferencial clásica con miras a cubrir conceptos de haces fibrados. Elegimos hacer esto porque un paso intermedio hacia el teorema de Gel'fand-Neimark es el teorema de Serre-swan que hace uso de varias nociones de haces fibrados, en particular, haces vectoriales; después los relaciones con módulos proyectivos sobre álgebras.

Posteriormente introducimos nociones básicas de geometría no-conmutativa, donde presentamos aspectos clave tales como: álgebras C^*, el teorema de Gel'fand-Neimark, una revisión de módulos, particularmente los proyectivos.

Después explicamos como definir un cálculo diferencial en ésta geometría siguiendo el formalismo de Majid. Con todos éstos elementos podemos presentar de un modo general una característica fundamental de nuestra investigación: la condición de centralidad. Haciendo uso de ésta condición junto con el marco de trabajo usual de la geometría no-conmutativa nos lleva a uno de los principales alcances de nuestra investigación.

Luego nos dedicamos a definir las bases y alcances de nuestro trabajo, calculamos los análogos no-conmutativos de los tensores de Riemann y de Ricci junto con el escalar de Ricci; todas éstas entidades se usan para construir el análogo al tensor de Einstein, el cual adquiere correciones cuánticas de primer orden que emergen de la no-conmutatividad. Todo esto es hecho haciendo uso de la aproximación de Majid, donde el cálculo diferencual universal asociado a un álgebra es usado para construir las análogías mencionados anteriormente.

Para concluir, presentamos algunas aplicaiones. La primera es el *bicrossproduct model*, la cual era una de las principales motivaciones de este trabajo. También consideramos un álgebra de Lie bi-dimensional en el contexto de un espacio-tiempo conformemente plano bi y cuatri-dimensional. Nuestro último ejemplo es el modelo Friedmann-Robertson-Walker, que después de ser analizado con nuestro tratamiento da origen a una anomalía en el tensor de Einstein, lo cual es un hallazgo relevante, ya que proviene de la no-conmutatividad del espacio tiempo en lugar del lugar usual: la teoría cuántica de la materia.

Contents

Part II

Preliminaries

1

Introduction

It seems proper to pose the question from the very beginning: What is *non-commutative geometry*? We would like to point out the fact that finding the answer it is indeed a bold and a stark task, and it is going to take the vast majority of the following work to provide an sketch of such objective.

We may think of it as a generalization of ordinary differential geometry. In order to endorse this idea, let us think of $\mathcal{M}$ a smooth manifold with real coordinates; moreover, let us consider $C^{\infty}(\mathcal{M})$, which is the family of smooth real-valued functions acting upon the manifold. Sometimes we ignore the fact that this functions constitute a commutative algebra over the real numbers $\mathcal{A}$ in the following sense: we can add and multiply them.

The allusion of this algebra is not just a trivial statement, it endows a deep link between algebra and geometry. Consider that we only know the algebra $\mathcal{A}$; we do not know how we obtained it, so every trace of the geometry has vanished. This scenario might look perilous; however it is possible to reconstruct the manifold $\mathcal{M}$ from the algebra $\mathcal{A}$. Hence, this structures are to be deemed equivalent. This is the essence of the commutative Gel'fand-Neimark theorem.

Non-commutative geometry generalizes this notion by replacing the commutative algebra with a non-commutative one; thus, by using the non-commutative version of the Gel'fand-Neimark theorem, we are able to find a non-commutative geometry for the algebra under consideration. It is important to mention that the notion of a manifold is lost, nevertheless, this is a gain rather than a loss, because it allows us to generalize geometrical concepts.

As a final remark on this subject, we will detail the sense in which it is lost: since any commutative C^*-algebra can be understood as the C^*-algebra of some functions over a locally compact Hausdorff space, a non-commutative C^*-algebra will be considered as the algebra of continuous functions over some *non-commutative space*. This implies that now we shall fix our attention upon the algebras of functions rather than the spaces, and this is why the concept

of manifold disappears.
Why do we want to study non-commutative geometry? A preemptive reply would be: in quantum mechanics we encounter non-commutative algebras study the microscopic nature of space-time. Let us elaborate more on the first answer: think of the Heisenberg algebra $[x, p] = i\hbar$, on this basis for th algebra it is possible to find a geometry, and use concepts borrowed from it to study the quantum dynamics of a system. Moreover, consider the angular momentum algebra $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$, which, according to the usual interpretation, is the algebra of infinitesimal rotations of the sphere S_2. From the non-commutative geometry viewpoint, these operators are conceived as coordinates and they define a new sphere: the non-commutative sphere, which exhibits a new set of features that arise from non-commutativity. It is a sphere in which the radius is identified with the *Casimir operator*[Mad99] L^2 and hence the space is quantized in the sense that, if we want to move in radial direction, we can not do it anymore in a continuous fashion; instead, we are obliged to do it in discrete steps defined by $\ell(\ell + 1)$.

This *radial quantization* reveals that non-commutative geometry might provide a natural and straightforward path to quantize space-time, which is our second reply to the posed question. Nonetheless this is not just a fortunate coincidence, but it is to be expected, as there is evidence that indicates that space-time must not be continuous at small scales, and a form to remove this is to consider that it is described by non-commutative geometry.

Since we are dealing with a field which is being actively researched, there are different versions of what is non-commutative geometry, all of them based on the grounds above stated, but developing different formalisms and concepts.

The first and most renowned is Connes' functional analytic approach [Kha13, Con95], in which he develops a new differential calculus by introducing an *spectral triple* $(\mathcal{A}, \mathcal{H}, D, J)$, where $\mathcal{A}$ is a non-commutative algebra, $\mathcal{H}$ is a Hilbert space where $\mathcal{A}$ is realized as an algebra of bounded operators $\mathcal{B}(\mathcal{A})$, and D is the Dirac operator which shall encompass all of the geometrical information. Finally, the antilinear isometry J introduces the real structure on the triple, it is not fundamental in the formalism but it is introduced for physical applications.

With this, Connes was able to define a differential calculus for the non-commutative geometry, as well as to construct the non-commutative analogue of the integral, the so-called *Dixmier trace*, which can be calculated by making use of the *Wodzicki residue*; with these tools Connes was able to propose a geometrical action, which, applied in an specific model, is able to yield the Lagrangian of the Standard model coupled to the usual Einstein gravity; this is remarkable in many layers, but we would like to highlight one: matter may be thought as a consequence of a non-commutative space-time.

Majid's approach to non-commutative geometry shares many features with Connes' version; however it is a more straightforward path to non-commutative geometry. This formalism is more specific, given that one considers the non-commutativity of the space-time to be of the Lie-algebraic type $[x^\mu, x^\nu] = C^{\mu\nu}{}_\lambda x^\lambda$. In order to define a differential calculus we need to introduce the one-forms over the algebra $\Omega^1(\mathcal{A})$, which follow a relation inherited from the Lie algebra commutator $[dx^\mu, x^\nu] = D^{\mu\nu}{}_\lambda dx^\lambda$, where $D^{\mu\nu}{}_\lambda$ are some constants that are proportional to the structure constants plus a symmetric term which it is chosen on physical grounds.

Additionally, one demands that the line-element g is a central element of the algebra; this guarantees the existence of an inverse and that, regardless of how it acts on an element of the algebra it will give the same result. This aids to build tensor-like entities on this framework and thus mimic Einstein's theory.

Finally, making use of the algebraic theory of connections developed by Koszul [Kos86], a formula is found for the connection plus quantum corrections originated from non-commutativity; then it is chosen to truncate the expression so that only terms that are of first order in the quantum parameter (i.e. Plank's length) are considered.

With all of these tools, Majid [MR94] is able to analyze particular space-times such as the *bicrossproduct model* [BM14] and some other examples. Our main motivation emerges from here, for we seek to study any Lie-algebraic space-time. This is achieved by following Majid with two additional considerations: we need to start from a classical space-time, and if it possesses symmetries, then such symmetries must be promoted to be the non-commutative algebra. In other words, the Killing vectors algebra for a particular space-time is taken as the non-commutative algebra to be considered along with the metric. Second, since we already have a line-element, then the centrality condition fixes the symmetric term in the commutator between a one-form basis and an algebra generator, this is inspired in Majid's claim that it must come from physical grounds, but now we are trying to introduce a way to promptly quantize any classical space-time that has symmetries.

In this work, we present a general framework to study a quantum space-time of the Lie-algebraic kind via non-commutative geometry. To achieve this task, we start with a brief review of classical differential geometry aiming towards the concept of fiber bundles. We do this because a middle step to the Gel'fand-Neimark theorem is the Serre-Swan theorem which relies on various notions of fiber bundles, in particular, vector bundles; and then it links them to projective modules over algebras.

So, we introduce some basic notions of non-commutative geometry where we present key aspects of it such as: C^*-algebras, the Gel'fand-Naimark theorem, and an overview on modules, mainly on those of projective kind for the reasons afore-mentioned; then we take an overview on how to define a differential calculus in this geometry following Majid's formalism. With all these elements we are ready to present in a general way a key feature of our research: that is the centrality condition. Using it along the usual non-commutative geometry framework is one of the main insights of this research.

After we define the outset of our framework, we compute the non-commutative analogues of the Riemann and Ricci tensors along with the Ricci scalar, all of these elements are put together to build an Einstein tensor that gets first order quantum corrections from its non-commutativity. All of this is done on Majid's non-commutative approach where the universal differential calculus associated to an algebra is used to build analogues of classical geometric entities.

Finally, we present some applications. The first one is the bicrossproduct model, which was one of the main motivations of this work, we consider a two dimensional Lie algebra in two and four dimensional conformally flat spacetimes. Our last example is the Friedmann-Robertson-Walker model, which, after being analyzed through our treatment, gives origin to an anomaly, which is a remarkable finding, since it arises from non-commutative grounds instead of the usual way, in which it comes from a quantum theory of matter.

2

Classical differential geometry

Throughout this section most of the content may be found in [Ble05, Nak05, CBDDB82]. We shall try to keep our exposition brief, for the main purpose of this chapter is to provide a succinct review of the necessary concepts essential to formulate non-commutative geometry.

2.1 Manifolds

Definition 2.1 (Topology). *Consider a set $\mathcal{M}$, furthermore let us regard it as the arbitrary union of some subsets U_i (i.e., $\mathcal{M} = \cup_{i \in I} U_i$ for some index set I). In addition, let*

$$\phi_i : U_i \to \mathbb{R}^n$$

to be an injective function such that $\phi_i(U_i)$ is an open set. We assume that for all $i, j \in I$ the mapping

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is a C^∞ mapping (i.e., every partial derivative is continuous regardless its order)- We say that a subset $V \subset \mathcal{M}$ is open if $\phi_i(U_i \cap V)$ is open for all $i \in I$. Then the collection of such open sets is called the topology of $\mathcal{M}$ relative to $\{\phi_i | i \in I\}$.

Definition 2.2 (Atlas). *Assume that the topology of $\mathcal{M}$ is Hausdorff (i.e., it is always possible to find disjoint neighborhoods for different elements). Provided that $\mathcal{M}$ is the finite union of the U_i subsets, then $\{\phi_i | i \in I\}$ is called an atlas of $\mathcal{M}$ and each mapping ϕ_i is a chart.*

Definition 2.3 (Differential structure). *If the union of two atlantes is an atlas again, they are deemed as equivalent. An equivalence class of atlantes is called a differential structure over $\mathcal{M}$.*

Definition 2.4 (Manifold). *$\mathcal{M}$ with a differentiable structure is called a C^∞-manifold.*

Definition 2.5 (Diffeomorphism). *Let f be a smooth map between manifolds*

$$f : \mathcal{M} \to \mathcal{N}$$

such that for all charts in the atlas,

$$\phi_i : U_i \to \mathbb{R}^n \text{ in } \mathcal{M} \quad \text{and} \quad \psi_i : V_i \to \mathbb{R}^n \text{ in } \mathcal{N}$$

we have that

$$\psi_j \circ \phi_i^{-1} : \phi_i(U_i \cap f^{-1}(V_j)) \to \mathbb{R}^n \in C^\infty.$$

In other words, the smoothness of the function is independent of the atlantes considered in both manifolds. If f has an inverse then it is called a diffeomorphism.

With this now we have set some terrain to study functions and their global properties on manifolds. The vast majority of our analysis uses familiar concepts from calculus, and at most generalizes them. If we want to study the local behavior of functions then we must introduce a generalization of a directional derivative, the *tangent vector*; before that it is necessary to introduce an special kind of function, a *curve.*

Definition 2.6 (Curve). *A curve γ through a point $x \in \mathcal{M}$ is a map $\gamma : (a,b) \to \mathcal{M}$ with $a < 0 < b$ such that maps the origin to x (i.e., $\gamma(0) = x$).*

Definition 2.7 (Tangent vector). *Two curves γ_1 and γ_2 are said to be equivalent if for any chart ϕ*

$$\frac{d}{dx}(\phi \circ \gamma_1)\bigg|_0 = \frac{d}{dx}(\phi \circ \gamma_1)\bigg|_0.$$

An equivalence class of curves through x is called a tangent vector, which is associated to the base point, in this case x. The vector is denoted either by $\gamma'(0)$ or

$$\frac{d}{dt}\gamma(t)\bigg|_0.$$

Definition 2.8 (Tangent space). *The set of all tangent vectors at x is denoted as $T_x\mathcal{M}$ and is referred to as the tangent space at x, and it possesses a vector space structure.*

We were able to create vectors as objects that stand on points whose direction is given by an equivalence class of curves. In an heuristic sense, the vector joins the point where it is standing to a point infinitesimally close in the direction of the curve, that is why is a proper generalization for a directional derivative on manifolds.

Definition 2.9 (Derivative of a function). *Let $f \in C^\infty$ and $\gamma'(0) \in T_x\mathcal{M}$, then the derivative of f along the curve γ is*

$$(f \circ \gamma)'(0) =: \gamma'(0).$$

Definition 2.10 (Vector field). *Let $TM := \cup_{x\in\mathcal{M}} T_x\mathcal{M}$. A vector field on $\mathcal{M}$ is a function*

$$V : \mathcal{M} \to T\mathcal{M}$$

such that $V_x \in T_x\mathcal{M}$ and the function $x \mapsto V_x(f)$ is in $C^\infty(\mathcal{M})$ we shall denote this function as $V(f)$. The set of all vector fields on $\mathcal{M}$ is denoted by $\Gamma(T\mathcal{M})$.

Definition 2.11 (Coordinate vector fields). *Consider a chart $\phi : U \to \mathbb{R}^n$ then its coordinated vectors fields are given by*

$$\partial_i|_x := \frac{d}{dt}\phi^{-1}(\phi(x) + te_i)\Big|_{t=0}$$

with $e_i \in \mathbb{R}^n$ is the canonical basis. With the coordinate vector fields we can express any vector field V locally as

$$V = v^i\partial_i,$$

If we take another chart $\bar{\phi} : \bar{U} \to \mathbb{R}^n$ then for a point in the overlap $x \in U \cap \bar{U}$ we have

$$\bar{v}^j = v^i\partial_i(\bar{\phi} \circ \phi^{-1})|_{\phi(x)}. \tag{2.1}$$

In this sense we can think of vector fields as an array of functions v^i that lurk on coordinated domains and transform obeying 2.1 as shown above, finally there is a distinguished vector field related to any two we want to study, this is the commutator and it is helpful to study symmetries of fields.

Definition 2.12 (Commutator). *Consider two vector fields $Y, Z \in \Gamma(T\mathcal{M})$, then its commutator $[Y, Z]$ defines the following vector field*

$$[Y, Z]_x(f) := Y_x(Z(f)) - Z_x(Y(f)),$$

where we shall drop the subscript for the sake of simplicity. Additionally it satisfies $[Y, Z] = -[Z, Y]$ and the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

A generalization of vectors are tensors, which are entities that are valued on both vector and dual vector spaces.

Definition 2.13 (Tensors). *Let E and F be vector spaces over the field $\mathbb{R}$, we denote the dual space of E as E^*. The space of multilinear functions (F-valued tensors) is $T^{p,q}(E,F)$ each of its members is of the form*

$$T^{p,q}(E,F) \ni f : E^* \times \underbrace{\cdots}_{p} \times E^* \times E \times \underbrace{\cdots}_{q} \times E.$$

If $u_1, \ldots, u_p \in E^*$ and $v^1, \ldots, v^q \in E$ then

$$u_1 \otimes \cdots \otimes u_p \otimes v^1 \otimes \cdots \otimes v^q \in T^{p,q}(E,\mathbb{R}) := T^{p,q}(E),$$

if a basis is provided for both the space and the dual then we may write any element as

$$f = f^{i_1 \cdots i_p}_{j_1 \cdots j_q} e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_q}$$

where $f^{i_1 \cdots i_p}_{j_1 \cdots j_q} \in \mathbb{R}$, $e_{i_1}, \ldots, e_{i_p}$ is a basis for E and $\omega^{j_1}, \ldots, \omega^{j_q}$ is a basis for E^*

Differential forms are a core-concept needed to define calculus in manifolds, integration relies on their use and they open the gate towards the study of global properties through the means of the de Rham cohomology.

Definition 2.14 (One-form). *Let $T^{p,q}(\mathcal{M}) = \cup_{x \in \mathcal{M}} T^{p,q}(T_x\mathcal{M})$. A one-form is a function $\alpha : \mathcal{M} \to T^{0,1}(\mathcal{M})$ with $\alpha_x \in T^{0,1}(T_x\mathcal{M})$ and for any $Y \in \Gamma(T\mathcal{M})$ there is a function $\alpha(Y)$ that*

$$\alpha(Y)(x) = \alpha_x(Y_x) \in C^\infty,$$

if we proceed in an analogue fashion to the vectors, we may also provide a chart ϕ, then we have a coordinated dual basis $dx^1, \ldots, dx^n$ and are linear functionals that act on the vector basis as $dx^i(\partial_j) = \delta^i_j$ where δ^i_j is the Kronecker delta.

Definition 2.15 (Wedge product). *Let us consider q one-forms, its wedge product is a totally antisymmetric tensor product, the simplest case is*

$$dx^i \wedge dx^j := dx^i \otimes dx^j - dx^j \otimes dx^i,$$

in general we shall denote it by

$$dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

Definition 2.16 (q-form). *A differential form of order q is a totally antisymmetric tensor $T^{0,q}(\mathcal{M})$.It may be locally expressed upon an open set U with a chart ϕ as*

$$\eta = \frac{1}{q!} \eta_{i_1 \cdots i_q} dx^{i_q} \wedge \cdots \wedge dx^{i_q}$$

, with $\eta_{i_1 \cdots i_q} = \eta(\partial_{i_1}, \cdots \partial_{i_q}) \in C^\infty$. This set is denoted as $\Omega^q(\mathcal{M})$

Definition 2.17 (Exterior derivative). *Let* $f \in C^\infty(\mathcal{M})$, $X \in \Gamma(T\mathcal{M})$ *and* $\eta \in \Omega^q(\mathcal{M})$. *The exterior derivative* $d\eta$ *of* η *is a* $q+1$*-form such that for any* $X_1, \ldots, X_{q+1} \in \Gamma(T\mathcal{M})$ *we obtain*

$$d\eta(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-)^{i+1} X_i(\eta(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{q+1})) + \sum_{i<j<n} (-)^{i+j} \eta([X_i, X_j], X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{q+1}).$$

And it complies to the following properties

$$\begin{aligned} d :&\Omega^q(\mathcal{M}) \to \Omega^{q+1}(\mathcal{M}) \\ d(\alpha \wedge \beta) =& d\alpha \wedge \beta + (-)^p \alpha \wedge d\beta \\ d^2 =& d \circ d = 0, \end{aligned}$$

where $\alpha \in \Omega^p(\mathcal{M})$. *All of the above expressed with respect to a chart* $\phi : U \to \mathbb{R}^n$ *is*

$$\begin{aligned} d\eta =& \frac{1}{q!} d(\eta_{i_1 \ldots i_q}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} \\ =& \frac{1}{q!} \partial_i (\eta_{i_1 \ldots i_q}) dx^{i_1} \wedge \cdots \wedge dx^{i_q}. \end{aligned}$$

Definition 2.18 (Hodge star). *The Hodge star operation which we shall denote by* $*$, *is a linear map that in a manifold of dimension* n *is defined by*

$$* : \Omega^q(\mathcal{M}) \to \Omega^{n-q}(\mathcal{M})$$

$$*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}) = \frac{\sqrt{|g|}}{(m-r)!} \varepsilon^{\mu_1 \cdots \mu_q}{}_{\nu_{q+1} \cdots \nu_n} dx^{\nu_{q+1}} \wedge \cdots \wedge dx^{\nu_n}.$$

2.2 Fibre Bundles

Fibre bundles have been extensively used throughout physics, namely in gauge theory where they provide an elegant formalism where interactions are seen as consequences of an astonishing geometrical framework endowed at each point of space-time. However, they also provide a good starting point if we are ought to codify the geometry in an algebra, this is due to the fact that a middle point before the Gel'fand-Neimark theorem is the Serre-Swan theorem, and it establishes a link between a sections in particular kind of fibre bundles, the vector bundles and a morphisms of projective module over an algebra.

Given that the Gel'fand-Neimark theorem is one of the key elements of the non-commutative geometry program, we are going to study them from their most abstract notion to their applications.

Definition 2.19 (Fibre bundle). *Let us denote a fibre bundle by* (E, Π, X, F, G), *where*

1. *A topological space* E, *which we are going to as the total space.*
2. *Another topological space* X, *the base space.*
3. *A surjection* $\Pi : E \to X$, *the projection.*
4. *The fibre* F *which is also a topological space. It may be thought to be standing on a point of the base space. This becomes clearer if we make the following association:* $F_x \simeq \Pi^{-1}(X)$.
5. *A group* G *known as the structure group.*
6. *A covering* $\{U_\alpha\}$ *of* X *equipped with homeomorphisms* $\phi_\alpha(x, f)$ *such that* $\forall\, U_\alpha\ \exists\, \phi_\alpha : U_\alpha \times F \to \Pi^{-1}(U_\alpha)$ *and they are well defined in the following sense:* $\Pi \circ \phi_\alpha(x, f) = x$ *for* $x \in U_\alpha$ *and* $f \in F$.

The main idea behind this definition is allowing the decomposition of the total space into a base whose each point possesses a fibre. Such association is performed via a projection which needs to be properly defined, to achieve this we introduce a covering of the base space, and additionally some functions ϕ_α which map the union of fibres above the neighborhood U_α of some point in the base space, to the product space $U_\alpha \times F$; this is also known as a *local trivialization* and it might be regarded as the **untwisting of the fibre space**.

As it should be expected, the definition of the fibre bundle is independent of the the covering, i.e.: the coordinates. Nevertheless, when discussing physics it is not rare to introduce coordinates to relate these concepts to observations.

Definition 2.20 (Transition functions). *Allow us to consider two coverings* (U_α, ϕ_α), (U_β, ϕ_β) *with* $U_\alpha \cap U_\beta \neq \emptyset$, *then the transition functions may be defined as follows:*

$$
\begin{aligned}
t_{\alpha\beta} :=& \phi_\alpha^{-1} \circ \phi_\beta : F \to F, \\
f_\alpha =& t_{\alpha\beta} f_\beta .
\end{aligned}
$$

They must comply the following consistency conditions:

- $t_{\alpha\alpha}(x) = id \quad (x \in U_\alpha)$.
- $t_{\alpha\beta}(x) = t_{\beta\alpha}(x)^{-1} \quad (x \in U_\alpha \cap U_\beta)$.
- $t_{\alpha\beta}(x) \cdot t_{\beta\gamma}(x) = t_{\alpha\gamma}(x) \quad (x \in U_\alpha \cap U_\beta \cap U_\gamma)$.

We have used the $\cdot$ *product which we have not defined yet, but this shall be clarified promptly in definition 2.21, which is to be stated just ahead.*

These functions tell us how fibres must be *glued* together in the overlap of two neighborhoods and thus allow to explore the global structure of the bundle.

Definition 2.21 (Structure group). *The set of all transition functions generates the structure group* $G = \{t_{\alpha\beta}\}$; *this gives us a more refined idea on the domain and the image of this functions, which is given by:*

$$
t_{\alpha\beta} : U_\alpha \cap U_\beta \to G.
$$

Remark 2.22 (Transition functions and local trivializations). The transition functions naturally relate fibres, thus this can be extended to local trivializations in the following fashion

$$\phi_\alpha(x, f) = \phi_\beta(x, t_{\alpha\beta} f).$$

Example 2.23 (Trivial bundle). If we consider the case where all the transition functions are the identity map, i.e.: $\{t_{\alpha\beta} = id\}$; the fibre bundle is known as a *trivial bundle.*

Example 2.24 (Möbius strip and cylinder). Consider the following scenario: the circle S^1 is the base space, the fibre is $F = [-1, 1]$ and the transition group $G = \mathbb{Z}_2 = id, g$ where g is a *twist*; we may choose as coverings U_1 and U_2 such that $U_1 \cap U_2 = A \cup B$. The transition functions in this case are

$$t_{11}(x) = id_{S^1} = t_{22}(x), \quad t_{12}(x) = \begin{cases} id \text{ if } x \in A \\ g \text{ if } x \in B \end{cases} .$$

Therefore, $f_2 = t_{21}(x) f_1$; so if $x \in A$ there is no twist at all, however for $x \in B$ we need to twist and thus we obtain the Möbius strip. If we had chosen $G = \{id\}$ then the outcome would have been the cylinder, which is the trivial bundle $S^1 \times [-1, 1]$.

It is quite evident that the possible set of transition functions for a bundle is not unique at all, and actually this is the essence of gauge theory as we shall appreciate immediately.

Definition 2.25 (Gauge degrees of freedom). *Consider a covering* $\{U_\alpha\}$ *with two sets of local trivializations* $\phi_\alpha(x)$ *and* $\tilde{\phi}_\alpha$*, thus the transition functions for such trivializations are:*

$$t_{\alpha\beta}(x) = \phi_\alpha^{-1} \circ \phi_\beta,$$
$$\tilde{t}_{\alpha\beta}(x) = \tilde{\phi}_\alpha^{-1} \circ \tilde{\phi}_\beta,$$

we define a map $g_\alpha(x)$ *at each point of the base space* X *such that*

$$g_\alpha(x) := \phi_\alpha^{-1} \circ \tilde{\phi}_\alpha.$$

*Furthermore, we require that this map is both an **homeomorphism** and an element of the **structure group** G.*

Remark 2.26 (Transition functions and gauge degrees of freedom). If we take the definition given above, we reach the following result:

$$\tilde{t}_{\alpha\beta}(x) = g_\alpha^{-1} \circ t_{\alpha\beta} \circ g_\beta(x).$$

In physics we usually refer to $t_{\alpha\beta}$ as *gauge transformations* and they carry the information required to merge local charts; the *gauge degrees of freedom* are always to be taken within a local chart U_α.

Definition 2.27 (Section). *Allow us to consider a fibre bundle and an smooth map s which abides to:*

$$s : X \to E,$$
$$\Pi \circ s = id_X.$$

Evidently, $s(x) \in F_x = \Pi^{-1}(x)$*. Let us denote the set of sections in the base space as* $\Gamma(X, F)$*; if* $U \subset X$ *then this is the set of local sections* $\Gamma(U, F)$*.*

Definition 2.28 (Tangent bundle). *A manifold* X *contains an additional structure, its tangent bundle:* TX *which is defined as*

$$TM := \bigcup_{p \in X} T_pX,$$

where T_pX *is the tangent space at* p*.*

Example 2.29 (Vector fields). Let X be a manifold and consider its fibre set to be the tangent bundle TX. Now if we regard a vector field $V \in \mathcal{X}(X)$ as a map that at each point $p \in X$ it assigns a vector $V|_p \in T_pX$, then we unveil the use of a section; if set of all sections is denoted by $\Gamma(X, TX)$ we can do a natural identification with the set of all vector fields on the manifold $\mathcal{X}(X)$.

2.3 Reconstruction of bundles

So far we have accomplished the following: we started with a manifold, associated additional structures to it and then introduced a series of mappings to establish a relation between them.

This leads us to the following question: **what is the minimal information needed to reconstruct the fibre bundle?** The answer to this is rather simple, provided that we know the *base space, the fibre, the structure group, the set of coverings and the transition functions* $(X, F, G, \{U_\alpha\}, t_{\alpha\beta})$ there is enough data to find the *projection, total space and the local trivialisations* (Π, E, ϕ_α), as it may be seen below.

Definition 2.30 (Union of trivialisations). *Given the set of coverings* $\{U_\alpha\}$ *and the fibre manifold* F *we introduce the union of trivialisations:*

$$\Xi = \bigcup_\alpha U_\alpha \times F$$

Definition 2.31 (Fibre equivalence). *Allow us to consider* $(p, f) \in U_\alpha \times F$ *and* $(q, f') \in U_\beta \times F$*, they are said to be equivalent* $(p, f) \sim (q, f')$ *iff* $p = q$ *and the elements of the fiber are related by a transition function, i.e.:* $f' = t_{\alpha\beta}(x)f$*.*

With these tools we are set to recover a fibre bundle, thus now we are able to study them without redundant data.

Definition 2.32 (Fibre bundle (revisited)). *The fibre bundle now is the quotient set given by:*

$$E = \Xi / \sim,$$

its elements shall be denoted as $[(p, f)]$.

Definition 2.33 (Projection (revisited)). *Let us take the before-mentioned element and map it in this way*

$$\Pi : [(p, f)] \mapsto p.$$

Definition 2.34 (Local trivialisation (Revisited)). *We continue with the modus operandi that we have adopted lately,*

$$\phi_\alpha : (p, f) \mapsto [(p, f)].$$

2.4 Vector bundles

There is a case of special interest for us: what if the fibre is a vector space? This is of relevance to our purposes because, as we stressed before, it will allow us to relate algebraic entities to geometrical ones, in particular: sections of vector bundles to morphisms of projective modules over an algebra.

Definition 2.35 (Real fibre). *Let us consider a base manifold* X *with* $\dim(X) = d$ *and a fibre* $F = \mathbb{R}^k$ *with fibre dimension* $\dim(F) = k$*; hence the dimension of the bundle is* $\dim(E) = d + k$*. In this case the transition functions are elements of* $GL(k, \mathbb{R})$

Example 2.36 (Tangent bundle (revisited)). If we consider the tangent bundle TX associated to a manifold X we can think of it as a vector bundle with fibre $F = \mathbb{R}^d$. Here $\dim(X) = d = \dim(F)$ and consequently $dim(E) = 2d$. Furthermore consider a point $u \in TX$ whose projection lies in the intersection of at least two covers, i.e.: $\Pi(u) = p \in U_\alpha \cap U_\beta$. Next, we introduce coordinate coordinate systems for U_α and U_β respectively: $x^a = \phi_\alpha(p)$ and $y^a = \phi_\beta(p)$. Let us recall that in this bundle a point u gives us both a point p in the base space and a vector V standing on said point, this vector is coordinate-independent and we make this explicit with:

$$V = V^a \frac{\partial}{\partial x^a} = \bar{V}^a \frac{\partial}{\partial y^a},$$

we can do a coordinate transformation following the usual procedure

$$V^a = \left. \frac{\partial x^a}{\partial y^b} \right|_p \bar{V}^b := G^a_b(p) \bar{V}^b,$$

where $\{G^a_b(p)\} \in GL(d, \mathbb{R})$. With the sum of all of the above we can conclude that a tangent bundle is $(E = TM, \Pi, X, \mathbb{R}^d, GL(d, \mathbb{R}))$ and its sections correspond to vector fields on X, i.e.: $\mathcal{X} = \Gamma(X, TX)$.

Example 2.37 (The tangent bundle of a 2-sphere). In this case it is customary to use the south (s) and north (n) pole as references for the coverings,

$$U_S := S^2 - s \quad U_N := S^2 - n,$$

in this case we define their respective coordinates as the pairs: (U, V) and (X, Y), which are nothing more than the stereographic mappings of the sphere; they are related by

$$U = \frac{X}{X^2 + Y^2} \quad V = -\frac{Y}{X^2 + Y^2}.$$

Following the last example, $u \in TS^2$ with $\Pi(u) = p \in U_S \cap U_N$, the trivialisations are:

$$\phi_S^{-1}(u) = (p, V_S^a) \quad \phi_N^{-1}(u) = (p, V_N^a),$$

we know that the transition function $t_{SN}(p) \in GL(2, \mathbb{R})$ and that is in fact the jacobian, so without further ado:

$$t_{SN}(p) = \frac{\partial(U, V)}{\partial(X, Y)} = \frac{1}{r^2} \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

If we consider the vectors as elemetns of the fibre F, we see that non-trivial elements of the structure group act on them by a rotation of an angle 2θ followed by a rescaling.

Definition 2.38 (Line bundle). *A line bundle is a vector bundle whose fibre is one-dimensional ($F = \mathbb{R}$ or $\mathbb{C}$), the structure group $GL(1, \mathbb{C}) = \mathbb{C} - 0$ is Abelian.*

Example 2.39 (Quantum mechanics). Let us consider 3-dimensional quantum mechanics, in this case the base space is $X = \mathbb{R}^3$ with fibre $F = \mathbb{C}$ and the line bundle has the form $E = \mathbb{R}^3 \times \mathbb{C}$. The wavefunction corresponds to a section in this bundle.

2.4.1 Sections of vector bundles

Now it is clear that in the case of a vector bundle, a section associates a point in the base space with itself plus a vector. Thus a section can be regarded as vector standing on a point and therefore if we are determined to do algebra with it, the natural path is to employ vector algebra.

Definition 2.40 (Section algebra). *Let s, s' be sections of a vector bundle; their addition and multiplication emulate the ones from vector algebra but in this case they are point-wise*

$$(s + s')(p) = s(p) + s'(p),$$
$$(fs)(p) = f(p)s(p),$$

where $p \in X$ and $f \in \mathcal{F}(X)$ belongs to the space of functions over the base space.

Remark 2.41 (Triviality). A bundle is trivial iff its principal bundle has a global section.

2.4.2 Frames

Allow ourselves to consider an a tangent bundle TX, we know that if we choose a chart U_α on the base space X parametrized by coordinates x^μ there shall be a natural basis $\{\partial/\partial x^\mu\}$ on the fibre. If we set $\dim(X) = d$ we know that it is always possible to choose d linearly independent vector fields over our open covering U_α, but not necessarily this can be extended for the whole manifold X.

Definition 2.42 (Local basis). *Let us define the components for both the natural and orthonormal basis vectors*

$$\frac{\partial}{\partial x^\mu} = (0, \ldots, 0, \underset{\mu}{1}, 0, \ldots, 0)$$

$$\hat{e}_\alpha = (0, \ldots, 0, \underset{\alpha}{1}, 0, \ldots, 0),$$

as it will be seen later, these vectors define a local frame over U_α.

Definition 2.43 (Frame). *Consider a vector bundle ($F = \mathbb{R} or \mathbb{C}$). If we choose a chart U_α we know that the mapping $\Pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{R}^k$ is trivial, and furthermore, we may choose k independent sections that are said to define a frame over the covering U_α.*

We can conceive extracting the components of a vector as the mapping

$$V = V^\alpha e_\alpha(p) \mapsto \{V^\alpha\} \in F,$$

with the following trivialisation

$$\phi_\alpha^{-1}(V) = (p, \{V^\alpha(p)\}),$$

which by definition is

$$\phi_\alpha(p, \{0, \ldots, 0, \underset{\alpha}{1}, 0, \ldots, 0\}) = e_\alpha(p)$$

2.5 Principal Bundles

As we have stated several times throughout this text, there is a wide variety of particular cases within the fibre bundle structure. For instance, we can ask ourselves what happens if the fibre F turns out to be the structure group G, this is defined as a *principal bundle.*

Definition 2.44 (Principal Bundle). *A principal bundle has a fibre identical to the structure group $F = G$. This is often denoted as $P(X, G)$ and it is called a $G-$budle over X. The transition function acts on the fibre on the left as always, however now we can define a right action.*

Definition 2.45 (Right action). *Let $\phi_\alpha : U_\alpha \times G \to \Pi^{-1}(U_\alpha)$ be the trivialization given by $\phi_\alpha^{-1}(u) = (p, g_\alpha)$ with $u \in \Pi^{-1}(U_\alpha), p = \Pi(u)$. With these elements, we can say the the right action of G on $\Pi^{-1}(U_\alpha)$ is defined by: $\phi_\alpha^{-1}(ua) = (p, g_\alpha a)$ or equivalently*

$$ua = \phi_\alpha(p, g_\alpha a),$$

for any $a \in G$. Most of the time this is denoted by $P \times G \to P$ or $(u, a) \mapsto ua$; it is important to note the $\Pi(ua) = \Pi(u)$.

Remark 2.46 (Fibre construction). Let us take $u_1, u_2 \in \Pi^{-1}(p)$, there exists $a \in G$ such that $u_1 = u_2 a$. Thus, if $\Pi(u) = p$ we can construct the whole fibre as $\Pi^{-1}(p) = \{ua | a \in G\}$.

Definition 2.47 (Canonical local trivialization). *Consider a section s_α over U_α; there is a preferred local trivialization $\phi_\alpha : U_\alpha \times G \to \Pi^{-1}(U_\alpha)$ and it is defined by: for $u \in \Pi^{-1}(p), p \in U_\alpha$ there is a unique $g_u \in G$ such that $u = s_\alpha(p) g_u$, then:*

$$\phi_\alpha^{-1}(u) = (p, g_u)$$

w.r.t. this trivialization, the section now can be expressed as

$$s_\alpha(p) = \phi_\alpha(p, e)$$

Definition 2.48 (Frame bundle). *Consider the set of frames over a manifold $L_p X$, the union throughout all the points is defined as $LX := \bigcup_{p \in X} L_p X$.*

Remark 2.49 (Unveiling the bundle structure). Take a chart U_α with coordinates x^μ, then $T_p X$ has $\{\partial / \partial x^\mu\}$ for basis. A frame $u = \{X_1, \ldots, X_d\}$ at p is expressed as

$$X_\alpha = X^\mu{}_\alpha \frac{\partial}{\partial x^\mu}\bigg|_p \qquad 1 \leq \alpha \leq d,$$

where as it has been seen before $\{X^\mu{}_\alpha\} \in GL(d, \mathbb{R})$, the trivialization is:

$$\begin{aligned} &\phi_\alpha : U_\alpha \times GL(d, \mathbb{R}) \to \Pi^{-1}(U_\alpha), \\ &\phi_\alpha^{-1}(u) = (p, (X^\mu{}_\alpha)), \end{aligned}$$

the bundle structure becomes apparent if we consider the following points

1. Taking a frame u at p we define $\Pi_L : LX \to M$ by $\Pi_L(u) = p$.

2. The action of $g \in GL(d, \mathbb{R})$ upon the frame $U = \{X_1, \ldots, X_d\}$ is simply $(u, a) \mapsto ua$, where ua now defines a new frame at p, given by $Y_\beta = X_\alpha a^\alpha{}_\beta$. This also can be stated that given any two frames X and Y $\exists a \in GL(d, \mathbb{R})$ that can relate them. Therefore $GL(d, \mathbb{R})$ acts on LX **transitively**.
3. Consider an overlap with two associated frames X and Y, the transition function is:

$$t^L_{\alpha\beta}(p) = \left.\frac{\partial x^\mu}{\partial y^\nu}\right|_p \in GL(d, \mathbb{R})$$

Remark 2.50 (Frame bundle from the tangent bundle). Accordingly with the past definition and remark, providing a tangent bundle TX we can construct the frame bundle LX which will have the same transition functions.

2.6 Connections

Here we aim to generalize the usual notion that are already familiar from the usual Riemannian manifolds, here we will note that the connection needs a more abstract description with geometrical meaning and the importance of algebraic methods will start to play a role in some extent. The connection contains a lot of physics, two of its applications that have defined major milestones are in the context of gauge fields and Berry's phases.

Definition 2.51 (Vertical subspace). *Consider u to be an element of a principal bundle $P(X, G)$ and let G_p be the fibre at $\Pi(u) = p$. We shall denote the vertical subspace as V_uP and understand it as a subspace of T_uP* ***which is tangent to the fibre*** *G_p at u.*

Remark 2.52 ($T_pX \neq T_uP$). The tangent space T_pX of a base manifold X does not coincides with the vertical subspace T_uP of a principal bundle P.

Lemma 2.53 (Construction of the vertical subspace). *Consider the group G that acts both as the transition group and a fibre and take its tangent space at the identity, i.e., the algebra $\mathfrak{g} = T_eG$. If $A \in \mathfrak{g}$ its right action is defined by*

$$R_{\exp(tA)}u := u \exp(tA) := \sigma(t, u),$$

which conversely defines a curve parametrized by t that passes through u. Such an action does not implies a translation on the base space, and therefore the curve lurks entirely within G_p; another way of stating this is by saying that

$$p = \Pi(u) = \Pi(u \exp(tA)).$$

Define a vector $A^\# \in T_uP$ by:

$$A^{\#}f(u) = \frac{d}{dt}f(u\exp(tA))|_{t=0},$$

which by construction is tangent to P at u and therefore $A^{\#} \in V_uP$. If we iterate this procedure at each point of P we may generate a vector field known as the ***fundamental vector field****.*

Definition 2.54 (Fundamental vector field). *Let $A \in \mathfrak{g} = T_eG$, there is a natural isomorphisms between vector spaces that defines the vector $A^{\#}$, and it is given by*

$$\begin{aligned} \# : \mathfrak{g} &\to V_uP, \\ A &\mapsto A^{\#}. \end{aligned}$$

Definition 2.55 (Horizontal subspace). *The horizontal subspace which shall be denoted by H_uP is defined as the complement of V_uP in T_uP.*

Definition 2.56 (Connection (1:The Separator)). *Allow us to consider a principal bundle $P(X, G)$, a connection on such bundle is a unequivocal separation of the tangent space at T_uP into its vertical V_uP and horizontal subspaces H_uP. It complies with the following:*

- $T_uP = V_uP \oplus H_uP$.
- *Consequently, a vector field X becomes uniquely separated as $X = X^H + X^V$ with $X^H \in H_uP$ $X^V \in V_uP$.*[1]
- *$\forall\ u \in P$ $g \in G$ we have $H_{ug}P = R_{g^*}H_uP$; i.e.: behaves well under flow.*

Definition 2.57 (Connection (2: As a one-form)). *Consider a one-form $\omega \in \mathfrak{g} \otimes T^*P$ which obeys the following conditions*[2]

- $\omega(A^{\#}) = A$,
- $R_g^*\omega = Ad_{g^{-1}}\omega$,

speaking in other terms, given $X \in T_uP$,

$$R_g^*\omega_{ug}(R_g^*X) = g^{-1}\omega_u(X)g,$$

with this we can think of the horizontal subspace as the set that contains the vectors that are not projected by the connection.

$$H_uP := \{X \in T_uP|\omega(X) = 0\}.$$

Remark 2.58 (The one-form as a projection). The one-form projects T_uP onto the vertical component $V_uP \simeq \mathfrak{g}$

[1] R_{g^*} Is the induced map $R_{g^*} : T_hG \to T_hgG$.
[2] R_g^* stands for the pullback of the right action

Definition 2.59 (Connection (3:Local trivializations)). *A connection assigns to each local trivialization (choice of gauge) a Lie-algebra-valued one-form ω. In this sense we need to summon both a covering U_α and a local section σ_α. The connection one-form is*

$$\mathcal{A}_\alpha := \sigma_\alpha^* \omega \in \mathfrak{g} \otimes \Omega^1(U_\alpha).$$

Or we may think of this in the converse direction: if we have $\mathcal{A}_\alpha$ (which is Lie-algebra valued) defined on U_α we can rebuild an ω such that under a pullback by σ_α^* we get $\mathcal{A}_\alpha$.

It is necessary to address the issue of unicity of such form on P; in that sense we must ask ourselves what is ought to be when we consider two open coverings $U_\alpha \cap U_\beta \neq \emptyset$ and identify that uniqueness means that in such overlap $\omega_\alpha = \omega_\beta$. Locally this leads to the next *compatibility condition*:

Definition 2.60 (Compatibility condition for the one-form). *Consider two coverings U_α, U_β, a transitions function $t_{\alpha\beta}$ defined among them and a connection one-form, the compatibility condition is stated as follows,*

$$\mathcal{A}_\beta(p) = t_{\alpha\beta}^{-1}(p)\mathcal{A}_\alpha t_{\alpha\beta}(p) + t_{\alpha\beta}^{-1}(p) dt_{\alpha\beta}(p).$$

Now if the principal bundle is non-trivial, then there might not be a globally defined section; so the pullback of $\sigma^*\omega_\alpha = \mathcal{A}_\alpha$ only exists locally.

Example 2.61 (U(1) as a fibre). Envisage $P(X, U(1))$, consider overlapping charts $U_\alpha, U\beta$ with their respective local connections $\mathcal{A}_\alpha, \mathcal{A}_\beta$. Let the transition function that shall communicate this charts is

$$t_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1),$$
$$t_{\alpha\beta} = e^{i\Lambda(p)} \quad \Lambda(p) \in \mathbb{R},$$

therefore, the local connections are related in the following fashion:

$$\begin{aligned} \mathcal{A}_\beta(p) &= t_{\alpha\beta}^{-1}\mathcal{A}_\alpha t_{\alpha\beta} + t_{\alpha\beta}^{-1} dt_{\alpha\beta}, \\ &= e^{-i\Lambda(p)}\mathcal{A}_\alpha(p) e^{i\Lambda(p)} + e^{-i\Lambda(p)} de^{i\Lambda(p)}, \\ &= \mathcal{A}_\alpha(p) + id\Lambda(p) \end{aligned}$$

Remark 2.62 (About ω and $\mathcal{A}_\alpha$). ω is defined globally over the principal bundle; there might many connection one-forms but they share the same information about this geometric entity. Nevertheless, a local connection such as $\mathcal{A}_\alpha$ is only defined on the trivialization of the bundle and therefore they cannot have any global information.

Proposition 2.63 (Equivalence of definitions). *The definitions 2.56 and 2.57 are equivalent, to prove this we need to show that ω separates T_uP into V_uP and H_uP.Then it suffices to show that the horizontal subspace is carried along under the right action, this is, emulating what we knew for the vertical subspace:*

$$R_{g^*} H_u P = H_{ug} P$$

Proof. Take $X \in H_uP$, construct $R_{g^*}X \in T_{ug}P$ and apply the connection upon it

$$\begin{aligned}\omega(R_{g^*}X) = R_g^*\omega(X) &= g^{-1} \underbrace{\omega(X)}_{=0, X\in H_uP} g \\ &= 0.\end{aligned}$$

Therefore $R_{g^*}X \in H_{ug}P$. Which can be thought as: any $Y \in H_{ug}P$ may be written as $Y = R_{g^*}X$ for some $X \in H_uP$.

We shall find out that definitions 2.57 and 2.59 are equivalent as well.

Proof. Let us define an algebra valued one-form as:

$$\omega_\alpha := g_\alpha^{-1} \Pi^* \mathcal{A}_\alpha g_\alpha + g_\alpha^{-1} d_P g_\alpha,$$

where g_i is the canonical local trivialization given by $\phi_\alpha^{-1}(u) = (p, g_\alpha)$ for $u = \sigma_\alpha(p)g_\alpha$. Let $A^\# \in V_uP, A \in \mathfrak{g}$, i.e.: $\Pi_* A^\# = 0$, hence

$$\begin{aligned}\omega_\alpha(A^\#) = g_\alpha^{-1} d_P g_\alpha(A^\#) &= g_\alpha^{-1}(u) \frac{dg_\alpha(u \exp(tA))}{dt}\bigg|_{t=0} \\ &= g_\alpha^{-1}(u) g_\alpha(u) \frac{d \exp(tA)}{dt}\bigg|_{t=0} = A.\end{aligned}$$

Furthermore, consider $X \in T_uP$, $h \in G$, we develop:

$$R_h^*\omega_\alpha(X) = \omega_\alpha(R_{h^*}X) = g_{\alpha uh}^{-1} \mathcal{A}_\alpha(\Pi_* R_{h^*}X) g_{\alpha uh} + g_{\alpha uh}^{-1} d_P g_{\alpha uh}(R_{h^*}X),$$

where we have $g_{\alpha uh} = g_{\alpha u}h$ and $\Pi R_h = \Pi$, thus $\Pi_* R_{h^*}X = \Pi_* X$. Consequently we have

$$\begin{aligned}R_h^*\omega_\alpha(X) &= h^{-1} g_{\alpha u}^{-1} \mathcal{A}_\alpha(\Pi_* X) g_{\alpha u} h + h^{-1} g_{\alpha u}^{-1} d_P g_{\alpha u}(X) h \\ &= h^{-1}\omega_\alpha(X) h.\end{aligned}$$

Which readily demonstrates that this local connection satisfies the axioms of the connection one-form.

2.6.1 Horizontal lift

The definition of vertical subspace instantly leads us to think that we are basically dealing with information that after being projection shall be vanished. The horizontal lift actually seeks to avoid this situation: we want to keep some information after projecting.

Definition 2.64 (Horizontal lift). *Consider the base curve $c : [0, 1] \to X$ and its horizontal lift $\tilde{c} : [0, 1] \to P$. they obey these requirements:*

- *$\Pi\tilde{c} = c$.*
- *The tangent vector of $\tilde{c}$ is horizontal.*

2.6.2 Holonomy

In some situations we can have a closed curve in the base space $c(0) = c(1)$ with a non-closed horizontal lift $\tilde{c}(0 \neq \tilde{c}(1))$, the holonomy measures this non-closure of the lift under the action of the transition group between fibres.

Definition 2.65 (Holonomy group). *Let $u \in P$, $\Pi(u) = p \in X$; additionally consider the set of loops at p; $C_p(X) = \{c : [0,1] \to X | c(0) = p = c(1)\}$, then the holonomy group is defined as:*

$$\Phi_u := \{g \in G | \tau_c(u) = ug, \;\; c \in C_p(X)\}$$

where $\tau_c : \Pi^{-1} \to \Pi^{-1}$ is a transformation compatible with the group action, i.e.: $\tau_c(ug) = \tau_c(u)g$

Example 2.66 (The helix). Let the base manifold be $U(1) \simeq S^1$ and consider a projection given by

$$\begin{aligned} \Pi :& \mathbb{R} \to U(1), \\ & u \mapsto e^{i2\pi u}. \end{aligned}$$

By construction, the connection shall assign an horizontal subspace to each u we choose, in this case it is the following tangent space $T_u\mathbb{R}$. Now, the base loop is obeys $c(0) = c(1)$, but the lift is under the action of the holonomy group, which in this case is $\Phi = \mathbb{Z}$, so the lift shall be

$$\tilde{c}(0) = 0 \quad \tilde{c}(1) = e \cdot g$$

2.7 Curvature

The curvature has a direct physical interpretation in the sense that its local expression is the force tensor of the corresponding theory, e.g.: in electromagnetism it is the Faraday tensor.

Given that the fibres have a group structure, we need to generalize the concept of differential form so we can differentiate a Lie algebra-valued form.

Definition 2.67 (Vector-valued r-form). *Consider a vector space V of dimension k and a principal bundle P a vector valued $r-$form is the following multilinear map:*

$$\phi : \underbrace{TP \wedge \cdots \wedge TP}_{r-times} \to V,$$

which belongs to $\phi \in \Omega^r(P) \otimes V$. If we introduce a basis for the vector space $\{e_\alpha\}$ and consider $\phi^\alpha \in \Omega^r(P)$, ϕ may be written as:

$$\phi = \phi^\alpha \otimes e_\alpha$$

Definition 2.68 (Covariant derivative). *Let $X_1, \ldots, X_{r+1} \in T_uP$, ω the connection one-form of the bundle and $\phi \in \Omega^r(P) \otimes V$, the covariant derivative of ϕ is:*

$$D\phi(X_1, \ldots, X_{r+1}) := d_p(X_1^H, \ldots, X_{r+1}^H)$$

Definition 2.69 (Curvature). *The curvature two-form $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$ is defined as the covariant derivative of the connection one-form, thus:*

$$\Omega := D\omega,$$

which satisfies for $a \in G$,

$$R_a^* \Omega = a^{-1} \Omega a.$$

Definition 2.70 (Commutator). *Let $\xi = \xi^\alpha \otimes T_\alpha \in \Omega^p(P) \otimes \mathfrak{g}$, $\quad \eta = \eta^\alpha \otimes T_\alpha \in \Omega^q(P) \otimes \mathfrak{g}$ be $\mathfrak{g}-$valued forms where $\{T_\alpha\}$ is a basis for $\mathfrak{g}$. Their commutator is defined as*

$$\begin{aligned}[]
[\xi, \eta] &: = \xi \wedge \eta - (-1)^{pq} \eta \wedge \xi \\
&= [T_\alpha, T_\beta] \otimes \xi^\alpha \wedge \eta^\beta = f_{\alpha\beta}{}^\gamma T_\gamma \otimes \xi^\alpha \wedge \eta^\beta
\end{aligned}$$

Theorem 2.71 (Curvature and Cartan's structure equation). *Consider ω, Ω and $X, Y \in T_uP$, then we have*

$$\Omega(X, Y) = d_P \omega(X, Y) + [\omega(X), \omega(Y)].$$

Theorem 2.72 (Ambrose-Singer). *Consider a principal bundle over a connected manifold. The holonomy group of a point u_0 is the same as the subalgebra of $\mathfrak{g}$ spanned by*

$$\Omega_u(X, Y) \quad X, Y \in H_uP,$$

where $u \in P$ has the same horizontal lift as u_0.

Remark 2.73 (Physical interpretation). As the reader may recall, in a principal bundle we may face the situation where the horizontal lift of a loop does not closes. The holonomy is the quantity that measures such discrepancy, but so does the curvature. That is why the Ambrose-Singer theorem shows us that we can express the holonomy group in terms of the curvature.

Definition 2.74 (Curvature's local form). *Denote the local form of the curvature Ω by $\mathcal{F}$ and consider a section σ defined on a chart U; it is defined as follows:*

$$\mathcal{F} := \sigma^* \Omega,$$

Remark 2.75 (Curvature in terms of gauge potential). Written using the gauge potential we get

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.$$

Remark 2.76 (Component expression). If we parametrize our chart with coordinates, we arrive to

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_\mu dx^\mu \\ \mathcal{F} &= \frac{1}{2}\mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \\ \mathcal{F}_{\mu\nu} &= \mathcal{A}_{\mu,\nu} - \mathcal{A}_{\nu,\mu} + [\mathcal{A}_\mu, \mathcal{A}_\nu]\end{aligned}$$

Remark 2.77 (Compatibility conditions). Let U_α and U_β be overlapping charts with their respective field strengths $\mathcal{F}_\alpha, \mathcal{F}_\beta$, the compatibility condition is stated as:

$$\mathcal{F}_\alpha = Ad_{t_{\alpha\beta}^{-1}} \mathcal{F}_\beta = t_{\alpha\beta}^{-1} \mathcal{F}_\beta t_{\alpha\beta}$$

Definition 2.78 (Bianchi identity). *Let Ω be the curvature and $\mathcal{F}$ its local form, the Bianchi identity is*

$$D\Omega = 0,$$

or, locally

$$d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0.$$

2.8 Applications

Throughout this section we will review some basic applications, particularly in gauge theory. but also give some notions of the connection's role in the Berry phase.

2.8.1 U(1) gauge theory

Since $U(1)$ is one dimensional and Abelian, thus its structure constants vanish. Furthermore, we shall consider this theory in a base space that resembles the usual vacuum space-time, i.e.: 4-dimensional Minkowski, given that the base space is contractible to a point, the bundle is trivial $P = \mathbb{M}^4 \times U(1)$ and then we need to provide a single local trivialization, consequently a single gauge potential and strength:

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_\mu dx^\mu \\ \mathcal{F} &= d\mathcal{A} \\ d\mathcal{F} &= d^2\mathcal{A} = 0,\end{aligned}$$

in components this is

$$\mathcal{F}_{\mu\nu} = \partial_\nu \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\nu$$
$$\partial_\lambda \mathcal{F}_{\mu\nu} + \partial_\nu \mathcal{F}_{\lambda\mu} + \partial_\mu \mathcal{F}_{\nu\lambda} = 0.$$

The action for the theory is

$$S_{\text{Maxwell}}[\mathcal{A}] = \frac{1}{4}\int_{\mathbb{M}^4} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}.$$

2.8.2 Dirac magnetic monopole

Allow us to consider $U(1)$ again but in this case the base space is to be topologically non-trivial: $X = \mathbb{R}^3 - \{0\} \simeq S^2$. This changes drastically the scenario because now the bundle is not trivial and we are going to need at least two charts, which are defined by

$$U_S = \{(\theta,\phi)|\frac{\pi}{2} - \varepsilon \leq \theta \leq \pi\} \quad U_N = \{(\theta,\phi)|\frac{\pi}{2} \leq \theta \leq \pi + \varepsilon\},$$

we know that there is a global connection ω, given that we have provided two charts we shall obtain two local gauge potentials

$$\mathcal{A}_S = \sigma_S^* \omega \quad \mathcal{A}_N = \sigma_N^* \omega.$$

A monopole in the origin would be expressed by:

$$\nabla \cdot \mathbf{B} = 4\pi g \delta^3(0),$$

a solution is

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3},$$

whose vector potentials may be written as

$$\mathbf{A}_S = -g\frac{1+\cos\theta}{r\sin\theta}\hat{e}_\phi \quad \mathbf{A}_N = g\frac{1-\cos\theta}{r\sin\theta}\hat{e}_\phi,$$

in terms of one-forms

$$\mathcal{A}_S = -ig(1+\cos\theta)d\phi \quad \mathcal{A}_N = ig(1-\cos\theta)d\phi.$$

The transition function is

$$t_{SN}(\phi) = e^{i\varphi(\phi)},$$

the gauge potentials are related by a gauge transformation

$$\mathcal{A}_N = t_{NS}^{-1}(\mathcal{A}_S + d)t_{NS} = \mathcal{A}_S + id\varphi,$$

hence

$$d\varphi = \frac{1}{i}(\mathcal{A}_N - \mathcal{A}_S) = 2gd\phi,$$

where we want to avoid a multi-valued transition function for it must be unique, thus

$$\frac{\Delta\varphi}{2\pi} = 2g \in \mathbb{Z},$$

therefore the magnetic charge is quantized.

2.8.3 Yang-Mills theory

Let us consider the simplest gauge group $SU(2)$ over the Minkowski spacetime $\mathbb{M}^4$, similarly to what we encountered in electromagnetism, since the base space is contractible the bundle is trivial and a single gauge potential will provide a full description

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu = \mathcal{A}^a_\mu T_a dx^\mu,$$

where T_a are the generators of the algebra

$$[T_a, T_b] = i\varepsilon_{abc} T_c,$$

hence, the field strength is

$$\begin{aligned} \mathcal{F} &= d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \frac{1}{2}\mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \\ \mathcal{F}_{\mu\nu} &= \partial_\nu \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\nu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ \mathcal{F}^a_{\mu\nu} &= \partial_\nu \mathcal{A}^a_\mu - \partial_\mu \mathcal{A}^a_\nu + \varepsilon_{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu, \end{aligned}$$

the Bianchi identity is

$$D\mathcal{F} = d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0,$$

and the action is

$$S_{YM}[\mathcal{A}] = \frac{1}{2} \int_{\mathbb{M}^4} tr(\mathcal{F} \wedge *\mathcal{F}).$$

2.8.4 Instantons

Usually, when one encounters the task of calculating a functional integral, it is necessary to find its minimum and then calculate the quantum fluctuations around it. Additionally, said integral is only well defined in euclidean space. The local minima are called *instantons* and we shall consider them in euclidean space for an $SU(2)$ Yang-Mills theory. The euclidean action for such a theory is

$$S^E_{YM}[\mathcal{A}] = -\frac{1}{2} \int_{\mathbb{E}^4} tr(\mathcal{F} \wedge *\mathcal{F}),$$

where the Hodge dual is w.r.t euclidean space. The equations of motion for the theory are

$$D\mathcal{F} = 0 \quad D * \mathcal{F} = 0,$$

the first equation is geometrical, it could be thought as an identity while the second is a differential equation for the field strength, which may be solved by

$$\mathcal{F} = \pm\mathcal{F},$$

when we substitute this solution to the equations of motion into the action we get

$$S^E_{YM}[\mathcal{A}] = \mp\frac{1}{2}\int_{\mathbb{E}^4} tr(\mathcal{F}\wedge\mathcal{F}).$$

The field strength is given by

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A},$$

the field configurations that render finite actions are the only ones that shall contribute to the functional integral, this means that at infinity we need the field strength to vanish, this is obtained via a *pure gauge*

$$\begin{aligned}
&\mathcal{A}(x) \to g^{-1}(x)dg(x) \qquad \text{as } |x|\to\infty \\
&\Rightarrow \mathcal{F} \to d(g^{-1}dg) + g^{-1}dg\wedge g^{-1}dg \\
&\qquad = -g^{-1}dgg^{-1}\wedge dg + g^{-1}dgg^{-1}\wedge dg = 0.
\end{aligned}$$

To study this in terms of fibre bundles, let us compactify $\mathbb{E}^4$ by adding the infinity and then mapping to S^4; allow ourselves to call this infinity south pole and the origin north pole; this effectively separates $\mathbb{E}^4$ into two pieces, two hemispheres

$$U_S = \{x\in\mathbb{E}^4 ||x| \le R-\varepsilon\} \quad U_N = \{x\in\mathbb{E}^4 ||x| \ge R+\varepsilon\},$$

the pure gauge requirement implies that a gauge potential needs to vanish, let us choose $\mathcal{A}_S = 0$, then

$$\mathcal{A}_N = t^{-1}_{NS}dt_{NS},$$

with this we identify the transition function with g and regard it as a map $g: S^3 \to SU(2)$, this is because $S^3 = U_N\cap U_S$ is the equator; before continuing, let us note that the natural identification between $SU(2)$ and S^3 is provided by

$$t^2\mathbb{I}_2 + t^i\sigma_i \in SU(2) \leftrightarrow \mathbf{t}^2 + (t^4)^2 = 1.$$

This map may be classified according to $\pi_3(SU(2)) = \mathbb{Z}$; the classification is given by the winding number

- Constant map: class 0.

$$g_0 : x\in S^3 \mapsto e\in SU(2).$$

- Identity map: class 1.

$$g_1 : x\in SU(2) \mapsto \frac{x^4\mathbb{I}_2 + x^i\sigma_i}{\mathbf{x}^2 + (x^4)^2}.$$

- $n-$map: class n.

$$g_n : x \in SU(2) \mapsto \left(\frac{x^4 \mathbb{I}_2 + x^i \sigma_i}{\mathbf{x}^2 + (x^4)^2} \right)^n .$$

If we recall electromagnetism, integrating the field strength, a two-form, over the sphere yielded the magnetic charge of a monopole. It would be quite appropriate to expect that integrating a four-form over S^4 leads to a physically meaningful quantity. The action defines a natural four-form since

$$S^E_{YM}[\mathcal{A}] = \mp \frac{1}{2} \int_M tr(\mathcal{F} \wedge \mathcal{F}) = \mp \frac{1}{2} \int_{\partial M} dtr(\mathcal{F} \wedge \mathcal{F}),$$

$tr\mathcal{F}^2$ is closed on the four-sphere since it is boundless.

$$dtr(\mathcal{F} \wedge \mathcal{F}) = tr(\underbrace{d\mathcal{F}}_{-[\mathcal{A},\mathcal{F}]} \wedge \mathcal{F} + \mathcal{F} \wedge \underbrace{d\mathcal{F}}_{-[\mathcal{A},\mathcal{F}]}) = -tr([\mathcal{A}, \mathcal{F}]\mathcal{F} + \mathcal{F}[\mathcal{A}, \mathcal{F}]) = 0,$$

which means that the closed form y locally exact and $\exists K$ such that $tr\mathcal{F}^2 = dK$, such form is

$$K = tr(\mathcal{A} d\mathcal{A} + \frac{2}{3}\mathcal{A}^3),$$

furthermore upon integration we obtain

$$\int_{S^4} tr\mathcal{F}^2 = \int_{U_N} tr\mathcal{F}^2 = \int_{U_N} dK = \int_{\partial U_N} K = \int_{S^3} K$$

since $F = 0$ in S^3 and $d\mathcal{A} = \mathcal{F} - \mathcal{A}^2 = -\mathcal{A}^2$, the form becomes

$$K = tr(-\mathcal{A}^3 + \frac{2}{3}\mathcal{A}^3) = -\frac{1}{3}\mathcal{A}^3,$$

consequently we have

$$\int_{S^4} tr\mathcal{F}^2 = -\frac{1}{3} \int_{S^3} \mathcal{A}^3,$$

thus the entity $tr\mathcal{F}^2$ or conversely $tr\mathcal{A}^3$ contain topological information about the bundle, and in fact this is encoded in the winding number defined by

$$n := \frac{1}{2} \int_{S^4} tr \left(\frac{i\mathcal{F}^2}{2\pi} \right) = \frac{1}{24\pi^2} \int_{S^3} tr\mathcal{A}^3.$$

3

Non-commutative geometry

In this section we will outline how to encode geometrical data into a commutative algebra, or more properly: how to think of fibre bundles as projective modules over a commutative algebra. This leads us to an speculation; if we choose a non-commutative algebra it will encode the information of what we shall call a non-commutative geometry, where several notions of geometry remain unchanged, others are generalized and some can not be used anymore.

Most of the content of this chapter was taken from [DB86, Dix82].

3.1 C^*-algebras

The C^*-algebras arise naturally in the algebraic formulation of Quantum Mechanics and Quantum Field Theory. In addition they are an essential tool to prove one of the main results in this section: the Gel'fand-Neimark theorem.

Definition 3.1 (Associative linear algebra). *Let $\mathcal{A}$ be a linear space over a field F, we shall refer to it as an associative linear algebra if for $x, y \in \mathcal{A}$ we have the product map $\cdot$ such that $\forall x, y, z \in \mathcal{A}, \lambda \in F$ we have*

$$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

1.

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z;$$

2.

$$x \cdot (y + z) = x \cdot y + x \cdot z; \quad (y + z) \cdot x = y \cdot x + z \cdot x$$

3.

$$\lambda(x \cdot y) = (\lambda x) \cdot y = x \cdot (\lambda y).$$

If $F = \mathbb{C}(= \mathbb{R})$ it is said that we have a complex (real) associative linear algebra.

For the sake of simplicity we shall omit the $\cdot$ from now on, also, we are going to refer to an associative linear algebra just as an algebra unless some confusion can arise.

Definition 3.2 (Commutative algebra). *An algebra is said to be commutative if $\forall\, x, y \in \mathcal{A}$*

$$xy = yx.$$

Definition 3.3 (Algebra with identity). *If $\exists\, e \in \mathcal{A}$ such that $\forall x \in \mathcal{A}$ we have*

$$ex = x = xe,$$

then it is said that the algebra possesses an identity.

Definition 3.4 (Ideal). *Consider a subspace I of an algebra $\mathcal{A}$, such space is said to be a left ideal if $j \in I, x \in \mathcal{A}$*

$$jx \in I,$$

following a complementary path, the right ideal is defined as

$$xj \in I.$$

Furthermore, a two-sided ideal is both a left and right. A proper ideal is an ideal that does not matches the algebra, i.e.: $I \neq \mathcal{A}$. If an algebra has no two-sided ideals besides $\{\emptyset\}$ is called a simple algebra.

Remark 3.5 (ideals on commutative algebras). If an algebra $\mathcal{A}$ is commutative, then there is no distinction bewteen left, right and two-sided ideals.

Definition 3.6 ($*-$algebra). *An algebra over $\mathbb{C}$ equipped with an involution which is the following mapping:*

$$\begin{aligned} \mathcal{A} &\to \mathcal{A} \\ x &\mapsto x^*, \end{aligned}$$

is a $*-$algebra if $\forall x, y \in \mathcal{A}, \lambda \in \mathbb{C}$ we have

- $$(x+y)^* = x^* + y^*;$$
- $$(\lambda x)^* = \bar{\lambda} x^*;$$

- $$(xy)^* = y^* x^*;$$

- $$x^{**} = x.$$

Definition 3.7 ($*-$subalgebra). *If we have a subalgebra B of $\mathcal{A}$, then it is a $*-$subalgebra if*

$$x \in B \Rightarrow x^* \in B.$$

Definition 3.8 ($*-$ideal). *An ideal I of an algebra $\mathcal{A}$ is a $*-$ideal if*

$$x \in I \Rightarrow x^* \in I.$$

Definition 3.9 ($*-$homomorphism). *Consider a mapping between $*-$algebras $\mathcal{A}$ and B,*

$$\begin{aligned} \phi : \mathcal{A} \to B \\ \phi(xy) = \phi(x)\phi(y) \\ \phi(x^*) = \phi(x)^*, \end{aligned}$$

if the mapping is bijective then it is a $*-$isomorphism of $\mathcal{A}$ onto B.

Definition 3.10 (Normed algebra). *An algebra $\mathcal{A}$ is normed if it has mapping $|| \ ||$ called norm that satisfies*

$$||xy|| \leq ||x||||y||. \tag{3.1}$$

Definition 3.11 (Banach algebra). *If said normed algebra is complete, it is a $*-$Banach algebra.*

Definition 3.12 (Isometries in $*-$algebras). *Let an algebra $\mathcal{A}$ have an involution $*$ such that*

$$||x^*|| = ||x||,$$

then the involution is isometric. Additionally, if the $*$ morphism ($* : \mathcal{A} \to \mathcal{B}$) complies with

$$||x^*|| = ||x||, \tag{3.2}$$

then the algebras $\mathcal{A}, \mathcal{B}$ are said to be isometrically $*-$isomorphic. (3.2) *is known as the C^* condition.*

Definition 3.13 (C^*-algebra). *Let $\mathcal{A}$ be a $*-$Banach algebra, if it satisfies the C^* condtion then it is a C^*-algebra, and it follows that*

$$||x^* x|| = ||x||^2.$$

Example 3.14 (Algebra of bounded linear operators $\mathcal{B}(\mathcal{H})$). In this case, the involution is the adjoint map, the multiplication is is the compostion of operators and the norm of $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ is given by

$$||\mathcal{O}|| = \sup\{||\mathcal{O}\xi|| : ||\xi|| \leq 1, \xi \in \mathcal{H}\}.$$

Example 3.15 (Continuous functions on a Hausdorff space). Let $\mathcal{C}(\mathcal{M})$ be the algebra of continuous functions on a Hausdorff space M, the product is pointwise multiplication and the involution being the complex conjugation with norm

$$||f||_\infty = \sup_{x \in M} |f(x)|,$$

which defines a commutative C^*-algebra.

Example 3.16 ($\mathbb{M}_(\mathbb{C})$). Consider the set of $n \times n$ matrices with complex arguments, the product is matrix multiplication and the involution is the hermitian conjugate, the norm is given by

$$||T|| = \sup T_{ij}$$

Definition 3.17 (Representation). *Let $\mathcal{A}$ be a C^*-algebra, the pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and π is a $*-$morphism is said to be a representation if π maps as follows*

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}).$$

Furthermore, the representation is faithful if $\ker(\pi) = \{\emptyset\}$. In case of being faithful, it can be proved that this happens iff $||\pi(x)|| = ||x||$ or that $\pi(x) > 0$ if $x > 0$.

Definition 3.18 (Irreducible representation). *The representation is irreducible if all of $x \in \mathcal{B}(\mathcal{H})$ which commute with all $y \in \pi(\mathcal{A})$, are multiples of the identity operator.*

Definition 3.19 (Unitary equivalence). *Consider a unitary operator U and two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$, they are unitary equivalent if*

$$\pi'(x) = U^* \pi(x) U$$

Definition 3.20 ($*$-automorphism). *Consider two $*-$algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, an invertible map $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ is said to be an automorphism if for all $a, b \in \mathcal{A}_1$*

- $\sigma(e) = e$.
- $\sigma(a)^* = \sigma(a^*)$.
- $\sigma(ab) = \sigma(a)\sigma(b)$.

It is important to stress the fact that the $*$-automorphisms from a C^*-algebra to itself (i.e. $\sigma : \mathcal{A} \to \mathcal{A}$) are themselves a group, which we shall denote by $\mathrm{Aut}(\mathcal{A})$*.*

Since a $*$-automorphism preserves the algebraic structure, it is natural two grasp two algebras as the same if they are to be related by an automorphism.

Definition 3.21 (Unitary group over an algebra). *Let $\mathcal{A}$ be a C^*-algebra with $u \in \mathcal{A}$ such that complies with $uu^* = e = u^*u$. These u form the unitary group $U(\mathcal{A})$.*

These unitary elements provide a natural way to construct an special kind of $*$-automorphisms.

Definition 3.22 (Inner automorphisms). *An automorphism $\sigma \in \mathrm{Aut}(\mathcal{A})$ is said to be an inner automorphism if $\sigma(a) = uau^*$ for all $a \in \mathcal{A}$ and $u \in U(\mathcal{A})$, and it may be written as $\sigma = \mathrm{Ad}_u$. Furthermore, the inner automorphisms $\mathrm{Inn}(\mathcal{A}) \subset \mathrm{Aut}(\mathcal{A})$ constitute a subgroup.*

From the definition above it is clear that an inner automorphisms is not uniquely determined by an element $u \in U(\mathcal{A})$. Actually one may consider $\mathrm{Ad}_u = \mathrm{Ad}_u$, consequently $uau^* = vav^*$ for all $a \in \mathcal{A}$ and therefore $v^*uau^*v = a$ given that $v \in U(\mathcal{A})$. this implies that

$$v^*uau^*v - a = 0 \tag{3.3}$$
$$v^*uau^*v - v^*uu^*va = 0 \tag{3.4}$$
$$v^*u(au^*v - u^*va) = 0 \tag{3.5}$$
$$v^*u[a, u^*v] = 0, \tag{3.6}$$

which leads us to conclude that $u^*v \in \mathcal{Z}(\mathcal{A})$, this allows us to set $v = uz$ with $z \in \mathcal{Z}(U(\mathcal{A}))$, this result is stated in the following lemma.

Lemma 3.23 (Link between the inner automorphism and the unitary group of an algebra). *As it was stated above, the inner automorphisms comply to*

$$\mathrm{Inn}(\mathcal{A}) \simeq U(\mathcal{A})/U(\mathcal{Z}(\mathcal{A})). \tag{3.7}$$

The definition for the other automorphisms that are not inner is immediate.

Definition 3.24 (Outer automorphism). *The outer automorphisms are elements of the following equivalence class*

$$\mathrm{Out}(\mathcal{A}) \simeq \mathrm{Aut}(\mathcal{A})/\mathrm{Aut}(\mathcal{A}). \tag{3.8}$$

If the algebra $\mathcal{A}$ under consideration is commutative, then the automorphism group $\mathrm{Aut}(\mathcal{A})$ is the symmetry group of the topological space that naturally arises from $\mathcal{A}$. Additionally if $\mathcal{A} = C^{\infty}(\mathcal{M})$ for some manifold $\mathcal{M}$ then all of the $\sigma \in \mathrm{Aut}(\mathcal{A})$ are diffeomorfisms; considering a commutative algebra renders the group of inner transformations $\mathrm{Inn}(\mathcal{A})$ into a trivial group, which implies that all of the automorphisms are outer $\mathrm{Out}(\mathcal{A}) \simeq \mathrm{Aut}(\mathcal{A})$ which tells us that there is no commutative analogue of an inner automorphisms.

In the non-commutative realm $\mathrm{Out}(\mathcal{A})$ is the symmetry group of the space-time under consideration, on the other hand given that there was no commutative equivalent of the inner automorphisms, the natural identification that can be established is to interpret them as the group of gauge transformations.

3.2 Gel'fand-Naimark theorem

3.2.1 Commutative case

The Gel'fand-Naimark theorem may be stated in an heuristic level in the following fashion: given any commutative C^*-algebra$\mathcal{C}$, there is a way to reconstruct a Hausdorff space M such that $\mathcal{C}$ is isometrically $*$-isomorphic to the algebra of complex valued continuous functions $\mathcal{C}(M)$.

Definition 3.25 (Non-zero linear functionals). *Consider $\mathcal{A}$ to be a commutative Banach algebra (a generalisation of a C^*-algebra) and definitionne $\hat{\mathcal{A}}$ as the set of non-zero multiplicative linear functionals, i.e.:*

$$\hat{\mathcal{A}} = \{\phi : \mathcal{A} \to \mathbb{C} | \phi(xy) = \phi(x)\phi(y), \text{ with } \phi(x) \neq= 0, \forall x\},$$

it can be proved that $\phi \in \hat{\mathcal{A}}$ is continuous and that $||\phi|| \leq 1$.

Definition 3.26 (Gel'fand transform). *For all $x \in \mathcal{A}$, we definitionne a map that shall be the abstract analogue of a Fourier transform,*

$$\hat{x} : \mathcal{A} \to \mathbb{C}$$
$$\hat{x}(\phi) = \phi(x),$$

such operation is called the Gel'fand transform

Definition 3.27 (Gel'fand topology). *A topology on $\hat{\mathcal{A}}$ is a Gel'fand topology if we take the weakest topology where all functions $\hat{x}$ are continuous.*

Definition 3.28 (Structure space). *The structure space is $\hat{\mathcal{A}}$ equipped with a Gel'fand topology, there is a bijection between the maximal ideals in $\mathcal{A}$ and elements of $\hat{\mathcal{A}}$, because of this, the structure space is often called the maximal ideal space of $\mathcal{A}$.*

Definition 3.29 (Gel'fand representation). *Consider the algebra homomorphism*

$$\begin{aligned} \hat{} : \mathcal{A} &\to \mathcal{C}_0(\mathcal{A}) \\ x &\mapsto \hat{x}, \end{aligned}$$

this is the Gel'fand representation and it is neither injective, surjective nor norm-preserving. However, if we are dealing with C^*-algebras it is an isometric $*-$isomorphism of $\mathcal{A}$ onto $\mathcal{C}_0(\hat{\mathcal{A}})$. Additionaly, we know that $\mathcal{C}_0(\hat{\mathcal{A}})$ has the $||\ \ ||_\infty$ norm, and this holds

$$||\hat{x}||_\infty \leq ||x||$$

which guarantees the continuity of the isomorphism.

If we have two representations denoted by ρ_1 and ρ_2 they are said to be *unitary equivalent* if there is an $u \in \mathcal{B}(\mathcal{H})$ such that $\rho_1 = u^* \rho_2 u$. This defines an equivalence relation, the structure space $\hat{\mathcal{A}}$ is the set of these unitary equivalence classes.

Definition 3.30 (Spectrum). *Let $x \in \mathcal{A}$, its spectrum is*

$$\sigma_{\mathcal{A}} = \{\lambda \in \mathbb{C} | x - \lambda e \text{ is not invertible in } \mathcal{A}\}$$

Definition 3.31 (Spectral radius). *Let x be an element of a Banach algebra, its spectrum is compact subset of the complex plane and this holds*

$$|x|_\sigma = \lim_{n \to \infty} ||x^n||^{1/n} \leq ||x|| \tag{3.9}$$

where

$$|x|_\sigma = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(x)\}$$

is known as the spectral radius.

Remark 3.32. the folowing result can be proven if one considers that the multiplicative linear functionals on a commutative Banach algebra are relate to the points in the spectra associated with the elements of $\mathcal{A}$.

$$||\hat{x}||_\infty = |x|_\sigma \leq ||x||$$

Theorem 3.33 (Gel'fand-Naimark). *Let $\mathcal{A}$ be a commutative C^*-algebra. Then the Gel'fand representation is an isometric $*-$isomorphism of $\mathcal{A}$ onto $\mathcal{C}_0(\hat{\mathcal{A}})$. In particular, $(x^*)\hat{} = \bar{\hat{x}}$ for every $x \in \mathcal{A}$.*

Proof. *Let us recall a central idea, we know that $x \to \hat{x}$ is a homorphism that maps $\mathcal{A}$ onto $\mathcal{C}(\hat{\mathcal{A}})$, to prove the isometry of the involution we follow the original argumentation by Gel'fand and Neimark.*

Consider an algebra $\mathcal{A}$, for every element h of such algebra that complies with $h = h^*$ the C^*-condition renders the following chain of consequences

$$||h^2|| = ||h||^2$$
$$||h^{2^n}|| = ||h||^{2^n},$$

which, may be stated as

$$||h^{2^n}||^{1/2^n} = ||h||$$

which, if we fix our attention on (3.9) *we arrive to the following:*

$$||h|| = |h|_\sigma.$$

In particular, we may choose h to be $h = x^*x$, and thus $||x^*x|| = |x^*x|_\sigma$; since, we have the following

$$\sigma(x^*) = \overline{\sigma(x)},$$

we observe that

$$|x^*|_\sigma = |x|_\sigma,$$

and by summoning the submultiplicavity of the normed algebra (3.1) *we get two results that prove the equality of $||x^*|| = ||x||$*

$$||x^*|| \cdot ||x|| = ||x^*x|| = |x^*x|_\sigma \leq |x^*|_\sigma |x|_\sigma = |x|_\sigma^2 \leq ||x||^2$$
$$||x|| \cdot ||x^*|| = ||xx^*|| = |xx^*|_\sigma \leq |x|_\sigma |x^*|_\sigma = |x^*|_\sigma^2 \leq ||x^*||^2,$$

the first implies that $||x^*|| \leq ||x||$ while the second conversely renders $||x|| \leq ||x^*||$, which leads us to $||x|| = ||x^*||$. Now we need to prove that if $h = h^*$, then the mapping $\phi(h) \in \hat{\mathcal{A}}$ is real. Consider $z = h + ite$, definitionne $\phi(h) := \alpha + i\beta$ with $t, \alpha, \beta \in \mathbb{R}$, thus

$$\phi(z) = \phi(h + ite) = \phi(h) + it\underbrace{\phi(e)}_{1} = \alpha + i(\beta + t),$$

additionally, we know that

$$z^*z = h^2 + t^2 e,$$

so that

$$\alpha^2 + (\beta + t)^2 = |\phi(z)|^2 \leq ||z||^2 = ||z^*z|| \leq ||h^2|| + t^2$$
$$\alpha^2 + \beta^2 + 2t\beta \leq ||h^2||,$$

given that the last equation must hold $\forall t \in \mathbb{R}$, we conclude that $\beta = 0$ and then it is real. With this we are ready and set to prove that ϕ is a $*-$map.

Consider a generic $x \in \mathcal{A}$, it may be written as $x = h + ik$, the "real" and "imaginary" parts are definitionned as

$$\Re(x) := \frac{x + x^*}{2} \quad \Im(x) := \frac{x - x^*}{2i},$$

let us stress the fact that $\Re^*(x) = \Re(x)$, $\Im^*(x) = \Im(x)$, $x^* = \Re(x) - i\Im(x)$, we have $\forall \phi \in \hat{\mathcal{A}}$

$$(x^*)\hat{}(\phi) = \phi(x^*) = \phi(\Re(x) - i\Im(x)) = \overline{\phi(\Re(x) + i\Im(x))} = \overline{\phi(x)} = \overline{\hat{x}(\phi)}.$$

This promptly proves that the Gel'fand representation is a $*-$map.

Now that we have shown that the Gel'fand representation is a $*-$isomorphism of $\mathcal{A}$ into $\mathcal{C}(\hat{\mathcal{A}})$; let us study the B the range of $x \to \hat{x}$; we know that it is a normed subalgebra of $\mathcal{C}(\hat{\mathcal{A}})$, separates the points of $\hat{\mathcal{A}}$, never vanishes (in $\hat{\mathcal{A}}$) and it is closed under complex conjugation; with this elements and the Stone-Weierstrass theorem we conclude that $B = \mathcal{C}(\hat{\mathcal{A}})$, ahd thus B is uniformly dense in said set and therefore the mapping $x \to \hat{x}$ is onto.

3.3 Non-commutative case

Now we are ready to take all of the above into the non-commutative realm, having an algebra $\mathcal{A}$ we need to reconstruct a Hilbert space $\mathcal{H}$ that represents such algebra as a norm-closed $*-$subalgebra. First of all, we need to demonstrate that the involution in a general C^*-algebra is continuous and then introduce an equivalent C^*-norm with isometric involution.

3.3.1 Continuity of the involution

Typically, one definitionnes a C^*-algebra with the C^*-norm condition, i.e.:$||x^*x|| = ||x||^2$ which promptly implies that $||x^*|| = ||x||$ and therefore $x \mapsto x^*$ is continuous. However this last statement might not be obvious, in that sense we must proceed to prove such continuity.

Proposition 3.34 (The involution of a C^*-algebra is continuous). *To prove this we need closure and continuity*

*Proof. Let $\{h_n\}$ be a convergent sequence in $H(\mathcal{A}) = \{h \in \mathcal{A} | h^* = h\}$ with limit $h + ik$ where $h, k \in H(\mathcal{A})$; given that $h_n - h \to ik$ then it converges and also $||h_n - h|| \leq 1$. We summon the spectral mapping for polynomials [**?**] and arrive to*

$$\sigma_{\mathcal{A}}([h_n - h]^2 - [h_n - h]^4) = \{\lambda^2 - \lambda^4 | \lambda \in \sigma_{\mathcal{A}}(h_n - h)\},$$

furthermore, given that $||x|| = |x|_\sigma$ and that $\sigma_{\mathcal{A}} \in \mathbb{R}$ we have

$$||[h_n - h]^2 - [h_n - h]^4|| = \sup\left\{\lambda^2 - \lambda^4 | \lambda \in \sigma_{\mathcal{A}}([h_n - h])\right\}$$
$$\leq \sup\{\lambda^2 | \lambda \in \sigma_{\mathcal{A}}([h_n - h])\} = ||[h_n - h]^2||,$$

taking the limit to infinity we get $|| - k^2 - k^4|| \leq ||k^2||$ *and therefore*

$$\sup\{\lambda^2 + \lambda^4 | \lambda \in \sigma_{\mathcal{A}}(k)\} \leq \sup\{\lambda^2 | \lambda \in \sigma_{\mathcal{A}}(k)\}.$$

Let us choose μ *such that* $\mu^2 = \sup\{\lambda^2 | \lambda \in \sigma_{\mathcal{A}}(k)\}$*. Hence, given that* $\mu^2 + \mu^4 \leq \mu^2$ *we conclude that* $\mu = 0$*; this readily implies that* $||k|| = |k|_\sigma = 0$*, consequently* $k = 0$ *which leads us to conclude that* $H(\mathcal{A})$ *is closed. Suppose that* $x_n \to x$ *and* $x_n^* \to y$*, then* $x_n + x_n^* \to x + y$ *and* $()x_n - x_n^*)/i \to (x - y)/i$ *are explicitly hermitian expressions. Since we have proven that* $H(\mathcal{A})$ *is closed, we are allowed to equate and we have*

$$x + y = x^* + y^* \quad x - y = y^* - x^*$$
$$\Rightarrow x = y^*,$$

we have shown that the graph of the $*$ *mapping is closed, then by means of the closed graph theorem we conclude that* $*$*(the involution) is continuous.*

Now we are about to show that there is an equivalent isometric C^*-norm that is equivalent to the original.

Proposition 3.35. *Consider* $\mathcal{A}$ *to be a* C^*-*algebra, then the following norm*

$$||x||_o = ||x^* x||^{1/2}$$

is and equivalent C^*-*norm on* $\mathcal{A}$ *such that* $||x^*||_o = ||x||_o$ *and therefore if* h *is hermitian we have* $||h||_o = ||h||$.

Proof. Given the closure and the continuity we know that $\exists M \geq 1$ *such that* $||x^*|| \leq M||x|| \quad \forall x \in A$*, thus*

$$M^{-1/2}||x|| \leq ||x^*||^{1/2}||x||^{1/2} = ||x||_o \leq M^{1/2}||x||,$$

which implies that $|| \, ||_o$ *and* $|| \, ||$ *are equivalent, and moreover, the o norm is homogeneous and submultiplicative; let us prove the triangle inequality*

$$||x + y||_o^2 = ||(x + y)^*(x + y)|| \leq ||x^* x|| + ||y^+ y|| + ||x^* y + y^* x||,$$

then we only need to show that $||x^* y + y^* x|| \leq 2||x||_o||y||_o$*; we can show this if we begin with the following statement which is valid for any positive integer* n

$$||(x^*y)^{2^{n-1}} + (y^+x)^{2^{n-1}}||^2 \tag{3.10}$$
$$= ||(x^*y)^{2^n} + (y^*x)^{2^n} + (x^*y)^{2^{n-1}}(y^*x)^{2^{n-1}} + (y^*x)^{2^{n-1}}(x^*y)^{2^{n-1}}|| \tag{3.11}$$
$$\leq ||(x^*y)^{2^n} + (y^*x)^{2^n}|| + 2(||x^*x|| \cdot ||y^*y||)^{2^{n-1}}. \tag{3.12}$$

We know that for every $\varepsilon > 0, \quad \exists n \in \mathbb{Z}$ *such that*

$$||(x^*y)^{2^n}|| \leq (|x^*y|_\sigma^2 + \varepsilon)^{2^{n-1}}$$
$$||(y^*x)^{2^n}|| \leq (|y^*x)|_\sigma^2 + \varepsilon)^{2^{n-1}},$$

the following can be proven

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\},$$

from this, it follows that

$$||(x^*y)^{2^n}|| \leq (|x^*y|_\sigma |y^*x|_\sigma + \varepsilon)^{2^{n-1}} \leq (||x^*y|| \cdot ||y^*x|| + \varepsilon)^{2^{n-1}}$$
$$\leq (||x^*x|| \cdot ||y^*y|| + \varepsilon)^{2^{n-1}}$$
$$\text{and} \quad ||(y^*x)^{2^n}|| \leq (||x^*x|| \cdot ||y^*y|| + \varepsilon)^{2^{n-1}},$$

so that

$$||(x^*y)^{2^n} + (y^*x)^{2^n}|| \leq 2(||x^*x|| \cdot ||y^*y|| + \varepsilon)^{2^{n-1}}, \tag{3.13}$$

taking (3.13) *into a recursion with* (3.12) *arrive to*

$$||(x^*y)^{2^{k-1}} + (y^*x)^{2^{k-1}}||^2 \leq 4(||x^*x|| \cdot ||y^*y|| + \varepsilon)^{2^{k-1}},$$

this expression is valid for any finite k*, in particular we may set it to be* $k = 2$ *and obtain*

$$||x^*y + y^*x||^2 \leq 4(||x^*x|| \cdot ||y^*y|| + \varepsilon).$$

Hence $||x^*y+y^*x|| \leq 2||x||_o||y||_o$ *which implies that* $||\ ||_o$ *is a* C^**-algebra equivalent norm on* $\mathcal{A}$*. Additionally we have that for all hermitian* $h \in \mathcal{A}, \quad ||h||_o = ||h^*h||^{1/2} = ||h||$*, concluding that* $||x^*||_o = ||x||_o$ *for all* $x \in \mathcal{A}$*.*

3.3.2 The Gel'fand-Naimark-Segal construction

We aim to represent an algebra $\mathcal{A}$ as a norm-closed $*-$subalgebra of bounded linear operatores on a Hilbert space, this is achieved by means of the GNS construction.

Definition 3.36 (Positive linear functional (State)). *Let* ϕ *be a linear functional such that*

$$\phi(x^*x) \geq 0 \quad \forall x \in \mathcal{A}$$
$$(x|y) := p(y^*x) \quad x, y \in \mathcal{A}$$

This inner product is linear in x*, conjugate-linear in* y *and as it would be expected:*

$$(x|x) \geq 0 \quad (x|y) = \overline{(y|x)} \quad |(x|y)|^2 \leq (x|x)(y|y),$$

in particular if we set $y = e$ and adopt the ϕ notation we obtain

$$|\phi(x)|^2 \leq \phi(x^*x) \quad \phi(x^*) = \overline{\phi(x)},$$

where we have used that $\phi(e) = 1$. This inner product might be degenerate, i.e.: $(x|x) = 0$ for $x \neq 0$.

These states form a set $\mathcal{S}(\mathcal{A})$ that happens to be a convex space, this can be seen from the fact that for any $\phi_\alpha, \phi_\beta \in \mathcal{S}(\mathcal{A})$ we can construct a convex hull and such will be in the set of states, i.e.: $\lambda\phi_\alpha + (1-\lambda)\phi_\beta \in \mathcal{S}(\mathcal{A})$.

Definition 3.37 (Pure states). *Let $\mathcal{S}(\mathcal{A})$ be the set of states of an algebra $\mathcal{A}$, an state ψ is said to be pure if $\psi \in \partial\mathcal{S}(\mathcal{A})$.*

Proposition 3.38. *Now, consider we have an state ϕ and definitionne the following set*

$$\mathcal{N}_\phi := \{x \in \mathcal{A} | \phi(x^*x) = 0\},$$

then this set must be an ideal of $\mathcal{A}$.

Proof. *Consider $x, y \in \mathcal{A}$, then*

$$\phi(x^*y^*yx) \leq ||y||^2\phi(x^*x) = 0.$$

This ideal induces a pre-Hilbert space given by the equivalence class $\mathcal{A}/\mathcal{N}_\phi$ with the following inner product

$$\begin{gathered}\mathcal{A}/\mathcal{N}_\phi \times \mathcal{A}/\mathcal{N}_\phi \to \mathbb{C} \\ (x + \mathcal{N}_\phi | y + \mathcal{N}_\phi) \mapsto \phi(y^*x),\end{gathered}$$

the completion of this equivalence class is the space we shall be dealing with in the representation.

Definition 3.39 (Representation in the Hilbert space of bounded operators). *Suppose you are given an element $a \in \mathcal{A}$, we can associate an operator $\pi(a) \in \mathcal{B}(\mathcal{H})$ such that*

$$\pi(a)(b + \mathcal{N}_\phi) =: ab + \mathcal{N}_\phi,$$

furthermore, it has the following properties

$$\begin{gathered}\pi(a+b) = \pi(a) + \pi(b) \quad \pi(\lambda a) = \lambda\pi(a) \quad \pi(ab) = \pi(a)\pi(b) \\ (\pi(a)(x + \mathcal{N}_\phi)|y + \mathcal{N}_\phi) = (x + \mathcal{N}_\phi|\pi(a)(y + \mathcal{N}_\phi)),\end{gathered}$$

it can be shown that $||\pi(a)|| \leq ||a||$ and therefore $\pi(a) \in \mathcal{B}(\mathcal{A}/\mathcal{N}_\phi)$. Additionally, there is a unique extension of $\pi(a)$ into $\pi_\phi(a) \in \mathcal{B}(\mathcal{H}_\phi)$ that obeys

$$\pi_\phi(a_1a_2) = \pi_\phi(a_1)\pi_\phi(a_2)$$
$$\pi_\phi(a^*) = (\pi_\phi(a))^*,$$

which is a $*$-morphism, or a representation for the algebra

$$\pi_\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$$
$$a \mapsto \pi_\phi(a).$$

Definition 3.40 (Cyclic vector). *The state ϕ is called a vector state, this implies that $\forall a \in \mathcal{A}\ \exists\, \xi_\phi \in \mathcal{H}_\phi$ such that*

$$(\xi_\phi|\pi_\phi(a)\xi_\phi) = \phi(a),$$

the vector is defined as

$$\xi_\phi =_e + \mathcal{N}_\phi.$$

The cyclic vector generates the Hilbert space in the sense that the set $\{\pi_\phi(a)\xi_\phi | a \in \mathcal{A}\}$ is dense $\mathcal{A}/\mathcal{N}_\phi$.

Theorem 3.41 (Gel'Fand-Neimark). *Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is isometrically $*$-isomorphic to a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$.*

Proof. *First of all we need to show that the representation is bounded, which is easily demonstrated if we consider an arbitrary element $b + \mathcal{N}_\phi \in \mathcal{A}/\mathcal{N}_\phi$ and calculates its norm*

$$||\pi(a)(b+\mathcal{N}_\phi)||^2 = \phi(b^*a^*ab) \leq ||a||^2\phi(b^*b) = ||a||^2||b+\mathcal{N}_\phi||, \tag{3.14}$$

since this was done for an arbitrary element we conclude that

$$||\pi(a)|| \leq ||a|| \tag{3.15}$$

3.4 Modules

Definition 3.42 (Module). *Let $\mathcal{A}$ be an algebra over $\mathbb{C}$; moreover consider a vector space $\mathcal{E}$ over $\mathbb{C}$, it is said to be a tight module if it carries a right representation of $\mathcal{A}$.*

$$\mathcal{E} \times \mathcal{A} \ni (\eta, a) \mapsto \eta a \in \mathcal{E}$$
$$\eta(ab) = (\eta a)b$$
$$\eta(a+b) = \eta a + \eta b$$
$$(\eta + \xi)a = \eta a + \xi a,$$

for all $a, b \in \mathcal{A}\ \ \eta, \xi \in \mathcal{E}$.

Definition 3.43 (Opposite algebra). *Let $\mathcal{A}$ be an algebra, its opposite $\mathcal{A}^o$ possess elements a^o with a natural bijection towards $a \in \mathcal{A}$; multiplication is given by*

$$a^o b^o = (ba)^o.$$

If we embed the module structure within the opposition a new structure arises

Definition 3.44 (Enveloping algebra). *If we consider a right (left) module $\mathcal{E}_r$ ($\mathcal{E}_l$) over an algebra $\mathcal{A}$ it may be turned into a left (right) module of $\mathcal{A}^o$ by this following means:*

$$a^o \eta = \eta a \quad (a\eta = \eta a^o),$$

for η belonging to either of both modules and $a \in \mathcal{A}$. With this, we have the necessary elements to definitionne the enveloping algebra as

$$\mathcal{A}^e = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^o.$$

With this, we may regard any $\mathcal{A}$-bimodule $\mathcal{E}$ structure as a right (left) $\mathcal{A}^e$-module by setting

$$\eta(a \otimes b^o) = b\eta a \quad ((a \otimes b^o)\eta = a\eta b).$$

We may also analyse the role of the center of the algebra $\mathcal{Z}(\mathcal{A})$ of the algebra; it is possible to definitionne modules by considering the center as a commutative algebra. In this sense if we have an $\mathcal{A}$-module it is possible to produce an analogous structure over the center $\mathcal{Z}(\mathcal{A})$ while the converse may not always hold. Additionally, we should stress the fact that regardless of the commutativity of the center $\mathcal{Z}(\mathcal{A})$ a right or left module structure over it should be considered to be distinct.

Definition 3.45 (Generating family). *Consider Λ to be any directed set, then its generating family $(e_\lambda)_{\lambda \in \Lambda}$ for the right module $\mathcal{E}$ renders any $\eta \in \mathcal{E}$ as*

$$\sum_{\lambda \in \Lambda} e_\lambda a_\lambda$$

which is not unique and $a_\lambda \in \mathcal{A}$; moreover, only a finite amount of terms in such sum may differ from zero.

A module is said to be free if it admits a basis. If a module is finitely generated is labeled as a finite type module.

Example 3.46 (Complex module). Allow ourselves to consider the following module: $\mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A} =: \mathcal{A}^N$, whose elements η can be written as

$$\eta = \sum_{j=1}^{N} e_j a_j,$$

and may be thought as an N-dimensional vector basis with entires in $\mathcal{A}$ as long e_j is the canonical basis of $\mathbb{C}^N$. This module is *both free and finite.*

The module basis is deeply connected with the triviality of a bundle. For instance, consider the algebra of smooth functions over the sphere $C^\infty(S^2)$ and the Lie- algebra of smooth vector fields over it $\mathcal{X}(S^2)$. In general it is not possible to solve the constraints among the basis elements to get a free basis, in this example this situation happens in the following way: we know that $\mathcal{X}(S^2)$ is *finite module* over $C^\infty(S^2)$, we may provide the three-element basis

$$\{Y_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} x_k \partial_k | \sum_{i=1}^{3} (x_i)^2 = 1\},$$

please note that the basis is not free since we have a constraint

$$\sum_{i=1}^{3} x_i Y_i = 0,$$

this tells us something we already knew, there are not two globally definitionned vector fields over the sphere that may serve as a basis for it. In other words, this tells us that the tangent bundle TS^2 over S^2 is non-trivial.

3.4.1 Projective Modules

Definition 3.47 (Projective module). *Let us definitionne a projective right $\mathcal{A}$-module as an entity that complies to any of the following*

1. *The lifting property; given surjective homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ of right $\mathcal{A}$-modules, any homomorphism $\lambda : \mathcal{E} \to \mathcal{N}$ may be lifted to another homomorphism $\tilde{\lambda} : \mathcal{E} \to \mathcal{M}$, and the following holds*

$$\rho \circ \tilde{\lambda} = \lambda$$

$$\begin{array}{rcl} id : \mathcal{M} & \longleftrightarrow & \mathcal{M} \\ \tilde{\lambda} \uparrow & & \downarrow \rho \\ \lambda : \mathcal{E} & \longrightarrow & \mathcal{N} \\ & & \downarrow \\ & & 0 \end{array}$$

2. *Every surjective module morphism $\rho : \mathcal{M} \to \mathcal{E}$ splits, in the sense that there shall always be a module morphism $s : \mathcal{E} \to \mathcal{M}$ such that $\rho \circ s = id_{\mathcal{E}}$.*
3. *The module $\mathcal{E}$ may be regarded as a part of direct sum of a free module, i.e.: if there exists a free module $\mathcal{F}$ and a module $\mathcal{E}'$ we have*

$$\mathcal{F} = \mathcal{E} \oplus \mathcal{E}'$$

Definition 3.48 (Lift of a projective module). *Consider a projective module of finite type $\mathcal{E}$ with surjection $\rho : \mathcal{A}^N \to \mathcal{E}$, then a lift is a mapping $\tilde{\lambda} : \mathcal{E} \to \mathcal{A}^N$*

$$\begin{array}{rcl} & \rho \circ \tilde{\lambda} = id_{\mathcal{E}} & \\ id : \mathcal{A}^N & \longleftrightarrow & \mathcal{A}^N \\ \tilde{\lambda} \uparrow & & \downarrow \rho \\ \lambda : \mathcal{E} & \longrightarrow & \mathcal{E} \\ & & \downarrow \\ & & 0 \end{array}$$

Definition 3.49 (Idempotent decomposition). *With this, we are able to construct an idempotent endomorphism $p \in End_{\mathcal{A}} A^N \simeq \mathbb{M}_N(\mathcal{A})$ rendered by*

$$p = \tilde{\lambda} \circ p,$$

this allows us to decompose any free module as the following direct sum of submodules

$$\mathcal{A}^N = p\mathcal{A}^N \oplus (1-p)\mathcal{A}^N,$$

which promptly states that ρ and $\tilde{\lambda}$ are inverse isomorphism, moreover they map $\mathcal{E}$ isomorphically to $p\mathcal{A}^N$.

The module $\mathcal{E}$ is projective iff there exists an idempotent $p \in \mathbb{M}_N(\mathcal{A})$ such that $\mathcal{E} = p\mathcal{A}^N$; in this sense any element of the **module** may be thought as the N-dimensional vectors ξ with entries in $\mathcal{A}$ which are invariant under p, such that $p\xi = \xi$.

Lemma 3.50 (Isomorphism between modules). *Consider $\mathcal{M}$ to be an compact manifold of finite dimension. Then The $C^\infty(\mathcal{M})$-module $\mathcal{E}$ is isomorphic to a module $\Gamma(E, \mathcal{M})$ of smooth sections of a bundle $E \to \mathcal{M}$ iff it is finite projective.*

3.5 Differential calculus

Derivations are the non-commutative generalisations of vectors, which in the commutative case are differential operators that shall act upon functions defined throughout the manifold.

Definition 3.51 (Derivation). *A derivation $\mathcal{X}$ is an operator $\mathcal{X} : \mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is a C^*-algebra*

$$\mathcal{X}(ab) = \mathcal{X}(a)b + a\mathcal{X}(b), \tag{3.16}$$

it forms a vector space $\mathrm{Der}(\mathcal{A})$ *and it also forms a Lie algebra, where the Lie bracket is defined*

$$[\mathcal{X}, \mathcal{Y}](a) = \mathcal{X}(\mathcal{Y}(a)) - \mathcal{Y}(\mathcal{X}(a)), \tag{3.17}$$

the involution acts as $\mathcal{X}^*(a) = (\mathcal{X}(a^*))^*$. A derivation is said to be hermitian if $\mathcal{X}^* = \mathcal{X}$.

In the case where we consider $\mathcal{A} = C^\infty(\mathcal{M})$ for some manifold $\mathcal{M}$ the derivations become the sections of the tangent bundle.

Definition 3.52 (Inner derivation). *Let $a, b \in \mathcal{A}$, we define the following operator*

$$\mathrm{ad}_a(b) := [a, b], \tag{3.18}$$

which is a derivation that from this point on we shall call inner derivations $\mathrm{Int}(\mathcal{A})$ *which is a vectorial subspace of* $\mathrm{Der}(\mathcal{A})$*.*

Definition 3.53 (Outer derivations). *we define this derivations in direct analogy to the definitions of the outer automorphisms*

$$\mathrm{Ext}(\mathcal{A}) \simeq \mathrm{Der}(\mathcal{A})/\mathrm{Int}(\mathcal{A}) \tag{3.19}$$

Lemma 3.54 (Involution on inner derivations). *The involution acts on inner derivation as*

$$\mathrm{ad}_a^* = \mathrm{ad}_{-a^*}. \tag{3.20}$$

Proof.

$$\mathrm{ad}_a^*(b) = (\mathrm{ad}_a(b^*))^* = [a, b^*]^* = (ab^* - b^*a)^* = ba^* - a^*b = -[a^*, b] = [-a^*, b] \tag{3.21}$$

$$= \mathrm{ad}_{-a^*}(b) \tag{3.22}$$

Therefore a real inner derivation is provided by antihermitian elements of the algebra.

Proposition 3.55 (The inner derivations are an ideal). $\mathrm{Int}(\mathcal{A})$ *consitute an ideal in* $\mathrm{Der}(\mathcal{A})$*; given* $\mathcal{X} \in \mathrm{Der}(\mathcal{A})$ *and* $\mathrm{ad}_a \in \mathrm{Int}(\mathcal{A})$ *then, under the natural product that the Lie bracket provides we have*

$$[\mathcal{X}, \mathrm{ad}_a] \in \mathrm{Int}(\mathcal{A}) \tag{3.23}$$

Proof. Take any $b \in \mathcal{A}$

$$[\mathcal{X}, \mathrm{ad}_a](b) = \mathcal{X}(\mathrm{ad}_a(b)) - \mathrm{ad}_a(\mathcal{X}(b)) = \mathcal{X}([a, b]) - [a\mathcal{X}, (b)] \tag{3.24}$$

$$= [a, \mathcal{X}(b)] + [\mathcal{X}(a), b] - [a, \mathcal{X}(b)] = [\mathcal{X}(a), b] \tag{3.25}$$

$$= \mathrm{ad}_{\mathcal{X}(a)}(b), \tag{3.26}$$

and therefore $[\mathcal{X}, \mathrm{ad}_a] = \mathrm{ad}_{\mathcal{X}(a)}$ *which of course belongs to* $\mathrm{Int}(\mathcal{A})$*, in particular if we choose* $\mathcal{X} = \mathrm{ad}_a$ *then we have*

$$[\mathrm{ad}_a, \mathrm{ad}_b] = \mathrm{ad}_{\mathrm{ad}_a(b)} = \mathrm{ad}_{[a,b]} \tag{3.27}$$

3.6 Connections

Definition 3.56 (Connection). *A connection is a mapping between the derivations of an algebra* $\nabla_{\mathcal{X}} : \mathrm{Der}(\mathcal{A}) \to \mathrm{Der}(\mathcal{A})$ *that is* $\mathcal{Z}(\mathcal{A})$*-linear in* $\mathcal{X} \in \mathrm{Der}(\mathcal{A})$*, i.e.:*

$$\nabla_{z\mathcal{X}}(\mathcal{Y}) = z\nabla_{\mathcal{X}}(\mathcal{Y}) \qquad \text{with } z \in \mathcal{Z}(\mathcal{A}), \tag{3.28}$$

where $\mathcal{Y} \in \mathrm{Der}(\mathcal{A})$*. Moreover it obeys the Leibniz rule*

$$\nabla_{\mathcal{X}}(z\mathcal{Y}) = \mathcal{X}(z)\mathcal{Y} + z\nabla_{\mathcal{X}}(\mathcal{Y}), \tag{3.29}$$

a final remark: a connection is said to be real if $(\nabla_{\mathcal{X}}\mathcal{Y})^* = \nabla_{\mathcal{X}^*}\mathcal{Y}^*$.

Lemma 3.57 (Connection on $\mathrm{Int}(\mathcal{A})$**).** *The connection* ∇_{ad_a} *is* $\mathcal{Z}(\mathcal{A})$*-linear.*

Proof.

$$\nabla_{\mathrm{ad}_a}(z\mathcal{Y}) = \underbrace{\mathrm{ad}_a(z)\mathcal{Y}}_{=0} + z\nabla_{\mathrm{ad}_a}(\mathcal{Y}) = z\nabla_{\mathrm{ad}_a}(\mathcal{Y}). \tag{3.30}$$

Definition 3.58 (Connection on $\Omega^1_{\mathrm{Der}}(\mathcal{A})$**).** *Let* $a, b \in \mathcal{A}$ *and* $\omega \in \Omega^1_{\mathrm{Der}}(\mathcal{A})$*, then the connection over the algebra-valued one-forms satisfies the Leibniz rule*

$$\nabla_{\mathcal{X}}^{\Omega^1_{\mathrm{Der}}(\mathcal{A})}(a\omega b) = \mathcal{X}(a)\omega b + a\left(\nabla_{\mathcal{X}}^{\Omega^1_{\mathrm{Der}}(\mathcal{A})}\omega\right)b + a\omega\mathcal{X}(b). \tag{3.31}$$

Proposition 3.59 (Link between connections). *Given a connection* $\nabla^{\Omega^1_{\mathrm{Der}}(\mathcal{A})}$ *on* $\Omega^1_{\mathrm{Der}}(\mathcal{A})$ *there exists a correspondent connection* $\nabla^{\mathrm{Der}(\mathcal{A})}$ *on* $\mathrm{Der}(\mathcal{A})$ *and they are related in the following fashion*

$$\omega\left(\nabla_{\mathcal{X}}^{\mathrm{Der}}\mathcal{Y}\right) + \left(\nabla_{\mathcal{X}}^{\Omega^1_{\mathrm{Der}}(\mathcal{A})}\omega\right)(\mathcal{Y}) = \mathcal{X}(\omega(\mathcal{Y})) \tag{3.32}$$

Proof. For the sake of simplicity consider we are given $\nabla^{\Omega^1_{\mathrm{Der}}(\mathcal{A})}$*, this means that we need to prove that* ∇^{Der} *is a connection. Clearly* $\nabla^{\mathrm{Der}}_{\mathcal{X}}$ *is* $Z(\mathcal{A})$*-linear in* $\mathcal{X}$ *for it must obey the Leibniz rule. Now, for* $\omega \in \Omega^1_{\mathrm{Der}}(\mathcal{A})$*,* $\mathcal{X}, \mathcal{Y} \in Der(\mathcal{A}), z \in Z(\mathcal{A})$ *we arrive to the following result*

$$\omega(\nabla_{\mathcal{X}}^{\mathrm{Der}} z\mathcal{Y}) = \mathcal{X}(\omega(z\mathcal{Y})) - (\nabla_{\mathcal{X}}^{\Omega^1_{\mathrm{Der}}(\mathcal{A})}\omega)(z\mathcal{Y}),$$

since =ω *also is* $Z(\mathcal{A})$*-linear, we have*

$$\begin{aligned}\omega(\nabla_{\mathcal{X}}^{\mathrm{Der}} z\mathcal{Y}) &= \mathcal{X}(z)(\omega(\mathcal{Y})) + z\mathcal{X}(\omega(\mathcal{Y})) - z(\nabla_{\mathcal{X}}^{\Omega^1_{\mathrm{Der}}(\mathcal{A})}\omega)(\mathcal{Y}) \\ &= \mathcal{X}(z)(\omega(\mathcal{Y})) + z\omega(\nabla_{\mathcal{X}}^{\mathrm{Der}}\mathcal{Y}) \\ &= \omega(\mathcal{X}(z)\mathcal{Y} + z\nabla_{\mathcal{X}}^{\mathrm{Der}}\mathcal{Y}).\end{aligned}$$

Therefore ∇^{Der} *is a connection; the converse statement for* $\nabla^{\Omega^1_{\mathrm{Der}}(\mathcal{A})}$ *is proven in a analogous fashion.*

Part III

Curvature for a Lie algebraic space-time

4

Curvature for a Lie-algebraic space-time

In 1994 Majid and Brueggs calculated the curvature for a metric that is a central element of the Kappa-Minkowski space-time by using the generalized braiding formalism [MR94]. In this text we generalize their program giving explicit formulas to calculate the Riemann tensor and the Riemann scalar for a non-commutative space-time of general Lie-algebraic type. The quest for a full-fledged theory of quantum gravity has been a defying and extensive one. It is considered to be the most intricate question posed since the formal establishment of modern theoretical physics [Ros30, Dir67]. Since the nature of this theory has been eluding us for more than 80 years, different approaches have been tried. Among them there is a particular interesting approach: non-commutative geometry.

Besides being a generalization of the commutative framework, non-commutative geometry is a novel approach that arises from a deep and meaningful conceptual juxtaposition amidst general relativity and quantum mechanics. Venturing into smaller scales brings us to the point where our description of space-time as a continuum is neither physically nor mathematically well-defined, it just stops making any sense at all. This is readily exposed in the geometrical measurement problem [DFR95]. The argument tells us that imposing a continuum space-time is not possible at microscopic scales, dor this it takes into consideration the uncertainty principle and the Schwarzschild radius which in turn implies that the measurement of a space-time point with arbitrary precision is not possible since the energy involved in this process would spawn a black hole. Hence space-time around the Planck length does not have a continuous structure.

In other words, from a physical viewpoint, the introduction of a non-commutative structure, so-called quantized space-time is strongly justified. We would like to point out that the geometric problem and its solution are ontologically equivalent to solving the motion of the electron in the atom by using a quantized version of the phase space. When we deal we Quantum Mechanics we promote the observables (namely the position and the momentum) to operators on a Hilbert space. An equivalent path is to deform the product of the algebra of functions on phase space, [Lan14]. However, in the approach to a non-commutative space-time we take the path of choosing a non-commutative algebra to obtain a generalized geometry.

In particular, we restrict our research to the case of a non-commutative space-time of Lie algebraic type. This means that the classical coordinates are to be considered generators of an algebra $\mathcal{A}$ whose commutator is of the form

$$[x^\mu, x^\nu] = C^{\mu\nu}_{\ \ \lambda} x^\lambda.$$

We chose this case because there is evidence given in [BM14] where quantum corrections up to first order (of Planck length) lead to meaningful physical results. Moreover, the real interest in studying simple models as the Lie-algebraic type, is that of calculating corresponding non-commutative geometric entities that are associated with Einstein field equations.

Our main motivation is to obtain a general formula for the quantum-corrected connection for any Lie-algebra. This is achieved by enforcing a requirement that arises in all the before-mentioned examples: the line element has to be a central element of the algebra. Up to date, central bi-modules (see [Lan14, Chapter 6-9], [MP96], [MT88] and references therein) play a fundamental role in formulating geometric quantities in the non-commutative geometry approach. Moreover, the centrality of the metric tensor is closely related to keeping some of its tensorial features in the classical sense, and also allows us to invert it without any ambiguity, see [BM14]. Having an expression for the connection in a non-commutative Lie-algebraic space-time we are able to construct geometrical quantities of interest, mainly: the Riemann, Ricci and Einstein tensors. They are defined in an analogue fashion to their classical counterparts and allow us to give some physical predictions of our space-time up to first order in Planck's length.

Having established the geometrical sector we proceed to explore the dynamics of matter in a non-commutative space-time. The interest w.r.t. matter comes from the argument that the non-commutative geometry may give origin to matter [CC96].

Our framework can be synthesized in the following procedure: first of all we need to choose a non-commutative algebra for our space-time, for now we shall restrict ourselves to work with a Lie algebra type. In principle any algebra stands on equal footing, however there is strong evidence that it must respect the symmetries of the classical space-time we are quantizing, by doing this the calculations are simplified greatly. After this we define a differential calculus where an arbitrary quantity arises, this is due to the fact that a symmetric factor may be added if we consider the sum of two commutators; however, we deal with this ambiguity by the end of our procedure. The third step is to demand that our line element to be in the center of the algebra.

The centrality condition along with the symmetries play a prominent role through our work, in this sense our procedure is analogous to some classical techniques where the metric is chosen a priori and then they analyze the physical properties of the energy-momentum tensor, but in our formalism it also takes care of the ambiguities that arose when the differential calculus was defined.

The next step is to follow the algebraic formulation of connections made by Koszul [Kos86] where we take advantage of the generalized braiding to obtain quantum correction for the connection. This renders one of our main results, which is a formula for the quantum-corrected Christoffel symbols up to first order for any algebra, its associated differential calculus and any metric; nevertheless the only way to make it operational is to fix the symmetric quantities as it was stated before. From this point we follow the definitions that are analogues of their counterparts in classical Riemannian geometry; we obtain the Riemann, Ricci and Einstein tensors, with quantum corrections up to first order.

4.1 Differential calculus

Definition 4.1 (Space-time). *Consider a Lie-algebraic space-time, i.e. the coordinates are generators of a non-commutative, associative and unital algebra that fulfill the following commutation relations*

$$[x^\mu, x^\nu] = C^{\mu\nu}{}_\lambda x^\lambda. \tag{4.1}$$

This definition serves as a starting point for defining differential calculus, but it needs to be complemented with the notion of a universal differential algebra.

Definition 4.2 (Universal differential algebra). *Consider an associative algebra $\mathcal{A}$ with unit over $\mathbb{C}$, we define the universal differential algebra of forms (c.f. [Lan14, Chapter 7, Section 1] and [Con95, Chapter 3, Section 1]) which is denoted by $\Omega(\mathcal{A}) = \bigoplus_p \Omega^p(\mathcal{A})$ as:*

For $p = 0$ it is the algebra itself, i.e. $\Omega^0(\mathcal{A}) = \mathcal{A}$. The space $\Omega^1(\mathcal{A})$ of one-forms is generated, as a left $\mathcal{A}$-module by a $\mathbb{C}$-linear operator $d : \mathcal{A} \to \Omega^1(\mathcal{A})$, called the universal differential, which satisfies the relations,

$$d^2 = 0, \quad d(ab) = (da)b + adb, \quad \forall a, b \in \mathcal{A}. \tag{4.2}$$

If $\Omega^1(\mathcal{A})$ is a left (right) $\mathcal{A}$-module we can induce a right (left) $\mathcal{A}$-module structure via the universal differential given in Equation (4.2), which makes $\Omega^1(\mathcal{A})$ a bimodule. With this notion we are ready to build the $\Omega^p(\mathcal{A})$-space as

$$\Omega^p(\mathcal{A}) = \underbrace{\Omega^1(\mathcal{A}) \cdots \Omega^1(\mathcal{A})}.$$

An immediate consequence of our definition is that the differential algebra of forms is graded.

Proposition 4.3 (Deformation of the differential structure). *Let the non-commutative, associative and unital algebra $\mathcal{A}$ be defined by the Relations (4.1). Then, the application of the universal differential (4.2) on the algebra has the following solution,*

$$[dx^\mu, x^\nu] = \left(\frac{1}{2} C^{\mu\nu}{}_\lambda + S^{\mu\nu}{}_\lambda\right) dx^\lambda =: D^{\mu\nu}{}_\lambda dx^\lambda, \tag{4.3}$$

where the constant tensor components $S^{\mu\nu}{}_\lambda$ are symmetric in μ, ν.

Proof. *We act with the universal differential on the commutator, the left hand side renders commutators each of them contain a one-form basis and a generator, the right hand side is just the structure constants of the Lie-algebra contracted with a one form-basis,*

$$\begin{aligned} d[x^\mu, x^\nu] &= [dx^\mu, x^\nu] + [x^\mu, dx^\nu] \\ &= \left(\frac{1}{2} C^{\mu\nu}{}_\lambda + S^{\mu\nu}{}_\lambda\right) dx^\lambda - \left(\frac{1}{2} C^{\nu\mu}{}_\lambda + S^{\nu\mu}{}_\lambda\right) dx^\lambda \\ &= C^{\mu\nu}{}_\lambda dx^\lambda. \end{aligned}$$

4.2 Centrality condition

In the introduction we stated and explained that in addition to the differential calculus, the centrality (w.r.t. the algebra) of the metric is an important requirement. In this section the implications of the centrality requirement are investigated. We begin by defining the line element as a tensor product of two one-forms.

Definition 4.4 (Line element). *Let the metric g be defined as a two-form, i.e. $g \in \Omega^2(\mathcal{A})$. The expression for the metric in terms of the basis, that is a tensor product of two one forms, is given by*

$$g = g_{\mu\nu} dx^\mu \otimes_{\mathcal{A}} dx^\nu,$$

where we assume symmetry for the metric components $g_{\mu\nu} = g_{\nu\mu}$. For the rest of this text we omit the subscript on the tensor product.

Proposition 4.5 (Centrality condition). *Let the two-form $g \in \Omega^2(\mathcal{A})$ be the line element. Then, the requirement of centrality for the metric tensor, i.e. $g \in \mathcal{Z}(\mathcal{A})$, has the following solution,*

$$[x^\lambda, g_{\mu\nu}] = D^{\alpha\lambda}{}_\mu g_{\alpha\nu} + D^{\alpha\lambda}{}_\nu g_{\alpha\mu} \tag{4.4}$$

Proof. Demanding centrality means that $[x^\lambda, g] = 0$, thus

$$\begin{aligned}
0 &\stackrel{!}{=} [x^\lambda, g] = [x^\lambda, g_{\mu\nu} dx^\mu \otimes_{\mathcal{A}} dx^\nu] \\
&= [x^\lambda, g_{\mu\nu}] dx^\mu \otimes_{\mathcal{A}} dx^\nu + g_{\mu\nu} [x^\lambda, dx^\mu \otimes_{\mathcal{A}} dx^\nu] \\
&= [x^\lambda, g_{\mu\nu}] dx^\mu \otimes_{\mathcal{A}} dx^\nu - D^{\mu\lambda}{}_{\alpha} g_{\mu\nu} dx^\alpha \otimes_{\mathcal{A}} dx^\nu - D^{\nu\lambda}{}_{\alpha} g_{\mu\nu} dx^\mu \otimes_{\mathcal{A}} dx^\alpha
\end{aligned}$$

where in the last lines we used the Leibnitz rule and the solution of the commutator relation between the algebra and the differentials given in Equation (4.3).

Remark 4.6. If we choose the metric and the algebra (choose structure constants), then the symmetric term can be automatically found. Although it is also possible to choose the structure constants and the symmetric term in order to find the metric, this is a rather unusual path. This is due to the fact that we intend to respect the classical space-time and its symmetries by implementing those symmetries into the algebra.

4.3 Quantum connection

In this section we introduce the concept of the connection for non-commutative algebras and use it to obtain first order corrections to the covariant derivative, Christoffel symbols and the curvature quantities. In the end of this section we write down, as well, the Einstein tensor plus the quantum corrections that we obtain by using the concept of the bi-modular map.

As for the concept of the connection which has to be understood as the generalization of the Koszul formula [Kos86], see [MT88] and [MP96].

Definition 4.7 (Connection). *The connection ∇ is a linear map that acts on one-forms in the following fashion*

$$\begin{aligned}
\nabla : \Omega^1 &\to \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \\
\nabla(a\omega) &= da \otimes_{\mathcal{A}} \omega + a\nabla(\omega) \\
\nabla(\omega a) &= (\nabla\omega)a + \sigma(\omega \otimes_{\mathcal{A}} da)
\end{aligned}$$

where $a \in \mathcal{A}$ and $\omega \in \mathcal{E}$, the module and the symbol σ is a bi-modular map known as generalized braiding.

Next, in order to induce quantum corrections, i.e. corrections in orders of magnitude of the structure constants we use the definition of the covariant derivatives and the bi-modular map.

Proposition 4.8 (Generalised braiding). *The bi-modular map σ is obtained by using the expressions for the left and right covariant derivatives, shown in the Introduction, from where it follows that*

$$\sigma(\omega \otimes_{\mathcal{A}} da) = \nabla(\omega a) - (\nabla\omega)a = \nabla([\omega, a]) - [(\nabla\omega), a] - \nabla(a\omega) - a\nabla(\omega)$$
$$= da \otimes_{\mathcal{A}} \omega + [a, (\nabla\omega)] + \nabla([\omega, a]) \tag{4.5}$$

Also, using the definition in [BM14] for the covariant derivative.

Definition 4.9 (Covariant derivative). *The covariant derivative up to first order in the structure constants is given by*

$$\nabla(dx^\mu) = \frac{1}{2}(\mathbb{I} + \sigma) \circ \nabla_0(dx^\mu), \tag{4.6}$$

where the zero order (in the structure constants) of the covariant derivative has been denoted by ∇_0.

By using Equation (4.6) we calculate the explicit outcome of the covariant derivative for our algebra (4.1).

Theorem 4.10. *[General formula] The* ***covariant derivative*** *(see Equation (4.6)) for the most general Lie-algebraic type of non-commutative space-time, up to first-order in the structure constants, is given in terms of the zero-order connection as follows,*

$$\nabla(dx^\mu) = -\Bigg(\Gamma^\mu_{\rho\sigma} + \frac{1}{2}(\Gamma^\mu_{\alpha\beta}(D^{\lambda\beta}{}_\rho \Gamma^\alpha_{\lambda\sigma} + D^{\lambda\beta}{}_\sigma \Gamma^\alpha_{\rho\lambda} - D^{\alpha\beta}{}_\lambda \Gamma^\lambda_{\rho\sigma})$$
$$+ \Gamma^\mu_{\alpha\beta}[x^\beta, \Gamma^\alpha_{\rho\sigma}]\Bigg) dx^\rho \otimes dx^\sigma \tag{4.7}$$

where here Γ *denotes the connection of zero-order (in the structure constants).*

Proof. See appendix.

Remark 4.11. In the rest of the paper we refer to Equation (4.7) as the **general formula**.

This is the formula for the covariant derivative of the most general Lie-algebraic type of a non-commutative space-time. If one sets the deformation constants, i.e. the structure constants, equal to zero one obtains the classical case. For the quantum terms being unequal to zero there is one term remaining, the last term, that depends on the specific form of the connection Γ that depends on the algebra. However, in the following we give specific expressions for the term in regards to special cases. In this spirit we highlight a result that we obtained in the proof of the former proposition.

Proposition 4.12. *The commutator of the generators of the algebra* $x^\mu \in \mathcal{A}$ *and the covariant derivative of the differential of the algebra* $\nabla(dx^\nu) \in \Omega^2(\mathcal{A})$ *is given by*

$$[x^\mu, \nabla(dx^\nu)] = (D^{\lambda\mu}{}_\rho \Gamma^\nu_{\lambda\sigma} + D^{\lambda\mu}{}_\sigma \Gamma^\nu_{\rho\lambda} - [x^\mu, \Gamma^\nu_{\rho\sigma}]) dx^\rho \otimes dx^\sigma, \tag{4.8}$$

and it holds to all orders in the structure constants. Therefore if $[x^\mu, \nabla(dx^\nu)] = 0$ *we have*

$$[x^\mu, \Gamma^\nu_{\rho\sigma}] = D^{\lambda\mu}{}_\rho \Gamma^\nu_{\lambda\sigma} + D^{\lambda\mu}{}_\sigma \Gamma^\nu_{\rho\lambda}. \tag{4.9}$$

Moreover, let the connection be a central element, i.e. $\nabla_0(dx^\nu) \in \mathcal{Z}(\mathcal{A})$*, then the general formula reduces to*

$$\nabla(dx^\mu) = -\left(\Gamma^\mu_{\rho\sigma} - \frac{1}{2} D^{\alpha\beta}{}_\lambda \Gamma^\mu_{\alpha\beta} \Gamma^\lambda_{\rho\sigma}\right) dx^\rho \otimes dx^\sigma$$

Proof. The calculation is straight-forward and uses the specific form of the covariant derivative and the solution of the Commutator (4.3)

$$\begin{aligned}
[x^\mu, \nabla(dx^\nu)] &= -[x^\mu, \Gamma^\nu_{\rho\sigma} dx^\rho \otimes dx^\sigma] \\
&= -[x^\mu, \Gamma^\nu_{\rho\sigma}] dx^\rho \otimes dx^\sigma - \Gamma^\nu_{\rho\sigma} [x^\mu, dx^\rho \otimes dx^\sigma] \\
&= (D^{\lambda\mu}{}_\rho \Gamma^\nu_{\lambda\sigma} + D^{\lambda\mu}{}_\sigma \Gamma^\nu_{\rho\lambda} - [x^\mu, \Gamma^\nu_{\rho\sigma}]) dx^\rho \otimes dx^\sigma,
\end{aligned}$$

where in the commutator we omitted terms that are of order one since the commutator would generate a higher order in the structure constants. The second part follows immediately from the former proposition.

Next, we give the definition of the Riemann tensor in the context of non-commutative geometry. It is given as a combination of the exterior derivative d, the wedge product $\wedge$, which maps the tensor product of two elements of the algebra of forms to the skew-symmetric product, and finally the covariant derivative.

Definition 4.13 (Riemann tensor). *Let* $\omega^\mu \in \Omega^1(\mathcal{A})$ *and* ∇ *be the connection, then the Riemann tensor is given as*

$$R(\omega^\mu) := (d \otimes \mathbb{I} - (\wedge \otimes \mathbb{I}) \circ (\mathbb{I} \otimes \nabla)) \nabla(\omega^\mu). \tag{4.10}$$

By using the former definition and the explicit formula for the covariant derivative for a general Lie-algebraic non-commutative space-time we calculate the Riemann tensor explicitly. Note that this formula, as well, holds in general.

Proposition 4.14. *[Curvature on a coordinated basis] If consider Equation (4.10) acting on the coordinated basis up to first order in the structure constants, it becomes the following expression*

$$\tilde{R}^\mu{}_{\beta\alpha\lambda} = \tilde{\Gamma}^\mu_{\lambda\beta\alpha} - \tilde{\Gamma}^\mu_{\alpha\beta\lambda} + \tilde{\Gamma}^\mu_{\alpha\sigma} \tilde{\Gamma}^\sigma_{\lambda\beta} - \tilde{\Gamma}^\mu_{\lambda\sigma} \tilde{\Gamma}^\sigma_{\alpha\beta} + \Gamma^\mu_{\rho\sigma} (\Sigma^{\rho\sigma}{}_{\lambda\beta\alpha} - \Sigma^{\rho\sigma}{}_{\alpha\beta\lambda}), \tag{4.11}$$

where we introduced the new symbols $\tilde{\Gamma}, d_q\Gamma$ *and* Σ*; the first is a decomposition of the Christoffel symbol into its purely classical and purely quantum parts. The second is this symbol differentiated and expanded into a one-form basis while the third is the commutator of it with the one-form coordinated* *basis. It should be noted that since our calculation is up to first order, taking the classical Christoffel symbol suffices.*

$$\tilde{\Gamma}^{\mu}_{\rho\sigma} = \Gamma^{\mu}_{\rho\sigma} + {}_{q}\Gamma^{\mu}_{\rho\sigma}, \qquad d\Gamma^{\mu}_{\rho\sigma} =: \Gamma^{\mu}_{\rho\sigma\lambda}dx^{\lambda}$$
$$d\,{}_{q}\Gamma^{\mu}_{\rho\sigma} =: {}_{q}\Gamma^{\mu}_{\rho\sigma\lambda}dx^{\lambda}, \qquad \Sigma^{\sigma\mu}{}_{\rho\sigma\lambda}dx^{\lambda} := [dx^{\sigma}, \Gamma^{\mu}_{\rho\sigma}]. \tag{4.12}$$

In terms of the classical Riemann tensor plus corrections from the quantum part of the Christoffel symbol, the new Riemann tensor reads,

$$\tilde{R}^{\mu}{}_{\sigma\alpha\rho} = R^{\mu}_{\sigma\alpha\rho} + {}_{q}\Gamma^{\mu}_{\rho\sigma\alpha} - {}_{q}\Gamma^{\mu}_{\alpha\sigma\rho} + \Gamma^{\mu}_{\alpha\lambda}\,{}_{q}\Gamma^{\lambda}_{\rho\sigma} + {}_{q}\Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\mu}_{\rho\lambda}\,{}_{q}\Gamma^{\lambda}_{\alpha\sigma} - {}_{q}\Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\alpha\sigma}$$
$$+ \Gamma^{\mu}_{\lambda\beta}(\Sigma^{\lambda\beta}{}_{\rho\sigma\alpha} - \Sigma^{\lambda\beta}{}_{\alpha\sigma\rho}).$$

Proof. The action of the curvature upon the coordinated basis of one-forms is given in terms of its defining Equation (4.10)

$$R(dx^{\mu}) :=(d \otimes \mathbb{I} - (\wedge \otimes \mathbb{I}) \circ (\mathbb{I} \otimes \nabla))\nabla(dx^{\mu}),$$

by inserting the equation $\nabla(dx^{\mu}) = -\tilde{\Gamma}^{\mu}_{\rho\sigma}dx^{\rho} \otimes dx^{\sigma}$ *into the Riemann tensor one has*

$$\begin{aligned} &= -\,(d \otimes \mathbb{I} - (\wedge \otimes \mathbb{I}) \circ (\mathbb{I} \otimes \nabla))(\tilde{\Gamma}^{\mu}_{\rho\sigma}dx^{\rho} \otimes dx^{\sigma}) \\ &= -\,d(\tilde{\Gamma}^{\mu}_{\rho\sigma}dx^{\rho}) \otimes dx^{\sigma} + (\tilde{\Gamma}^{\mu}_{\rho\sigma}dx^{\rho} \wedge \nabla(dx^{\sigma})) \\ &= -\,d(\tilde{\Gamma}^{\mu}_{\rho\sigma}) \wedge dx^{\rho} \otimes dx^{\sigma} - \tilde{\Gamma}^{\mu}_{\rho\sigma}dx^{\rho} \wedge \tilde{\Gamma}^{\sigma}_{\alpha\beta}dx^{\alpha} \otimes dx^{\beta}. \end{aligned}$$

Since we want a 3-form, we have to take all elements of the algebra to the left, c.f. [Lan14, Chapter 7, Eq. (7.12)]. In order to do this we need to commute a Christoffel symbol with an element of the basis of one-forms. Given that our calculation is up to first order, we only consider the classical part of the Christoffel symbol multiplying the commutator

$$\simeq -\,d\tilde{\Gamma}^{\mu}_{\rho\sigma} \wedge dx^{\rho} \otimes dx^{\sigma} - \tilde{\Gamma}^{\mu}_{\rho\sigma}\tilde{\Gamma}^{\sigma}_{\alpha\beta}dx^{\rho} \wedge dx^{\alpha} \otimes dx^{\beta} - \Gamma^{\mu}_{\rho\sigma}[dx^{\rho}, \Gamma^{\sigma}_{\alpha\beta}] \wedge dx^{\alpha} \otimes dx^{\beta},$$

rearranging indices and using the definition of the commutator of a 1*-form with a classical Christoffel symbol given in the third expression of Equations (4.12) we get*

$$\begin{aligned} R(dx^{\mu}) &= -\,(\tilde{\Gamma}^{\mu}_{\alpha\beta\lambda} + \tilde{\Gamma}^{\mu}_{\lambda\sigma}\tilde{\Gamma}^{\sigma}_{\alpha\beta} + \Gamma^{\mu}_{\rho\sigma}\Sigma^{\rho\sigma}{}_{\alpha\beta\lambda})dx^{\lambda} \wedge dx^{\alpha} \otimes dx^{\beta} \\ &= -\,(\tilde{\Gamma}^{\mu}_{\alpha\beta\lambda} + \tilde{\Gamma}^{\mu}_{\lambda\sigma}\tilde{\Gamma}^{\sigma}_{\alpha\beta} + \Gamma^{\mu}_{\rho\sigma}\Sigma^{\rho\sigma}{}_{\alpha\beta\lambda} - \tilde{\Gamma}^{\mu}_{\lambda\beta\alpha} - \tilde{\Gamma}^{\mu}_{\alpha\sigma}\tilde{\Gamma}^{\sigma}_{\lambda\beta} \\ &\quad - \Gamma^{\mu}_{\rho\sigma}\Sigma^{\rho\sigma}{}_{\lambda\beta\alpha})dx^{\lambda} \otimes dx^{\alpha} \otimes dx^{\beta}. \end{aligned}$$

Therefore, the components of the Riemann tensor are

$$\tilde{R}^{\mu}{}_{\beta\alpha\lambda} = \tilde{\Gamma}^{\mu}_{\lambda\beta\alpha} - \tilde{\Gamma}^{\mu}_{\alpha\beta\lambda} + \tilde{\Gamma}^{\mu}_{\alpha\sigma}\tilde{\Gamma}^{\sigma}_{\lambda\beta} - \tilde{\Gamma}^{\mu}_{\lambda\sigma}\tilde{\Gamma}^{\sigma}_{\alpha\beta} + \Gamma^{\mu}_{\rho\sigma}(\Sigma^{\rho\sigma}{}_{\lambda\beta\alpha} - \Sigma^{\rho\sigma}{}_{\alpha\beta\lambda}).$$

Writing down the quantum corrected Christoffel symbol as its classical plus quantum parts we arrive to our result

$$\tilde{R}^{\mu}{}_{\sigma\alpha\rho} = R^{\mu}_{\sigma\alpha\rho} + {}_{q}\Gamma^{\mu}_{\rho\sigma\alpha} - {}_{q}\Gamma^{\mu}_{\alpha\sigma\rho} + \Gamma^{\mu}_{\alpha\lambda}\,{}_{q}\Gamma^{\lambda}_{\rho\sigma} + {}_{q}\Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\mu}_{\rho\lambda}\,{}_{q}\Gamma^{\lambda}_{\alpha\sigma} - {}_{q}\Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\alpha\sigma}$$
$$+ \Gamma^{\mu}_{\lambda\beta}(\Sigma^{\lambda\beta}{}_{\rho\sigma\alpha} - \Sigma^{\lambda\beta}{}_{\alpha\sigma\rho}).$$

In order to calculate the Einstein-tensor we need the non-commutative analogs of the Ricci tensor and scalar. In order to avoid certain ambiguities encountered by the original authors, [BM14], we define these quantities as one does in usual geometry. The motivation therein is two-fold. The first reason is, as already pointed out, to avoid certain ambiguities. The second reason comes from the core of the deformation quantization argument. Namely, up to zero order in the deformation parameters, which in our case are represented by the structure constants, the observables or quantities at hand are the classical ones. Hence, since we consider the obtained Christoffel symbols as the classical ones plus quantum corrections, the Ricci tensor and scalar that are obtained classically by the trace have to be in the quantum case up to first order obtainable in the same manner. Moreover, the metric itself has no deformation parameter explicitly. Hence, the (left) inverse metric should obey the same property. Therefore, we give in the following the definition of the Ricci tensor and scalar.

Definition 4.15 (Left-inverse of the metric). *Let $g^{\mu\nu}$ be the components of the left inverse metric*

Definition 4.16 (Ricci tensor and scalar). *The Ricci tensor is the trace of the Riemann tensor over the first and third indices and the Ricci scalar is the trace over the two indices of the Ricci tensor, i.e.*

$$\tilde{R}_{\beta\lambda} := \tilde{R}^{\mu}{}_{\beta\mu\lambda}, \qquad \tilde{R} := g^{\mu\nu}\tilde{R}_{\mu\nu}.$$

Next, we give the explicit formulas for the Ricci tensor and scalar by using the general formula for the covariant derivative and in particular by using the result of the Riemann-tensor (see Proposition 4.14).

Proposition 4.17. *[Ricci tensor and Ricci scalar] By using the explicit result of the Riemann-tensor (see Proposition 4.14) and the former definition the Ricci tensor is given in terms of the Christoffel symbols Γ, ${}_q\Gamma$ and $\tilde{\Gamma}$ as follows,*

$$\tilde{R}_{\beta\lambda} = \tilde{\Gamma}^{\mu}_{\lambda\beta\mu} - \tilde{\Gamma}^{\mu}_{\mu\beta\lambda} + \tilde{\Gamma}^{\mu}_{\mu\sigma}\tilde{\Gamma}^{\sigma}_{\lambda\beta} - \tilde{\Gamma}^{\mu}_{\lambda\sigma}\tilde{\Gamma}^{\sigma}_{\mu\beta} + \Gamma^{\mu}_{\rho\sigma}(\Sigma^{\rho\sigma}{}_{\lambda\beta\mu} - \Sigma^{\rho\sigma}{}_{\mu\beta\lambda}).$$

If we decide to write down once more the quantum corrected Riemann symbol as a classical part plus a purely quantum one, we obtain

$$\begin{aligned}\tilde{R}_{\sigma\rho} = R_{\sigma\rho} + {}_q\Gamma^{\mu}_{\rho\sigma\mu} - {}_q\Gamma^{\mu}_{\mu\sigma\rho} + \Gamma^{\mu}_{\mu\lambda}\,{}_q\Gamma^{\lambda}_{\rho\sigma} + {}_q\Gamma^{\mu}_{\mu\lambda}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\mu}_{\rho\lambda}\,{}_q\Gamma^{\lambda}_{\mu\sigma} - {}_q\Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\mu\sigma} \\ + \Gamma^{\mu}_{\lambda\beta}(\Sigma^{\lambda\beta}{}_{\rho\sigma\mu} - \Sigma^{\lambda\beta}{}_{\mu\sigma\rho}) \qquad (4.13)\end{aligned}$$

tracing once again leads us to the Ricci scalar that reads

$$\begin{aligned}\tilde{R} = R + g^{\rho\sigma}\Big({}_q\Gamma^{\mu}_{\rho\sigma\mu} - {}_q\Gamma^{\mu}_{\mu\sigma\rho} + \Gamma^{\mu}_{\mu\lambda}\,{}_q\Gamma^{\lambda}_{\rho\sigma} + {}_q\Gamma^{\mu}_{\mu\lambda}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\mu}_{\rho\lambda}\,{}_q\Gamma^{\lambda}_{\mu\sigma} - {}_q\Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\mu\sigma} \\ + \Gamma^{\mu}_{\lambda\beta}(\Sigma^{\lambda\beta}{}_{\rho\sigma\mu} - \Sigma^{\lambda\beta}{}_{\mu\sigma\rho})\Big)\end{aligned}$$

By using the former results of the Ricci tensor and scalar we finally turn our attention to the observable that is of real physical importance, i.e. the Einstein tensor. Hence, in the following we give the explicit expression for the Einstein tensor for a general Lie-algebraic space-time.

Theorem 4.18 (Einstein tensor in a coordinated bass). *The Einstein tensor is defined analogously to the classical by taking the classical and quantum parts of the Ricci tensor and scalar (see former Proposition), i.e.*

$$\tilde{G}_{\sigma\rho} := \tilde{R}_{\sigma\rho} - \frac{1}{2}\tilde{R} g_{\sigma\rho}.$$

Hence, the explicit Einstein tensor reads

$$\begin{aligned}\tilde{G}_{\sigma\rho} =& G_{\sigma\rho} + {}_q\Gamma^{\mu}_{\rho\sigma\mu} - {}_q\Gamma^{\mu}_{\mu\sigma\rho} + \Gamma^{\mu}_{\mu\lambda}\, {}_q\Gamma^{\lambda}_{\rho\sigma} + {}_q\Gamma^{\mu}_{\mu\lambda}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\mu}_{\rho\lambda}\, {}_q\Gamma^{\lambda}_{\mu\sigma} - {}_q\Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\mu\sigma} \\ &+ \Gamma^{\mu}_{\lambda\beta}({\Sigma^{\lambda\beta}}_{\rho\sigma\mu} - {\Sigma^{\lambda\beta}}_{\mu\sigma\rho}) - \frac{1}{2}g^{\alpha\beta}\Big({}_q\Gamma^{\mu}_{\alpha\beta\mu} - {}_q\Gamma^{\mu}_{\mu\beta\alpha} + \Gamma^{\mu}_{\mu\lambda}\, {}_q\Gamma^{\lambda}_{\alpha\beta} + {}_q\Gamma^{\mu}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\beta} \\ &- \Gamma^{\mu}_{\alpha\lambda}\, {}_q\Gamma^{\lambda}_{\mu\beta} - {}_q\Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\beta} + \Gamma^{\mu}_{\lambda\beta}({\Sigma^{\lambda\tau}}_{\alpha\beta\mu} - {\Sigma^{\lambda\tau}}_{\mu\beta\alpha})\Big) g_{\sigma\rho}. \end{aligned} \tag{4.14}$$

Proof. It is immediate from Proposition 4.17.

Here it is important to observe that this definition for the non-commutative Einstein tensor arises from analogy. It is not a consequence of a non-commutative equivalent of the Ricci identities nor is a consequence of the variation of an action. Is a proposal that in the classical limit matches with Einstein's theory, in the next session we are going to explore its validity and usefulness in several physical scenarios.

5

Physical examples

In this section we study some examples. The Bicrossproduct model allows us to check the validity of our formula since it can be compared with Majid's calculation. Then we study a two-dimensional conformal space-time, which later is generalized to four dimensions in order to have results that can be interpreted in the usual sense of gravity. Finally we show an application in a cosmological context, mainly in the FRW model.

5.1 Bicrossproduct model

First, since we proved that we generalized the results in BM14, a nice consistency check is to see if the general formula reproduces the results that were obtained by the original authors. The algebra of the so called Bicrossproduct model is defined in the following.

Definition 5.1 (Bicrossproduct model algebra). *The Lie algebra for the bicrossproduct model os space-time is*

$$[x,t] = \lambda x \quad [x,dt] = \lambda dx \quad [t,dt] = \lambda dt.$$

Which readily implies that

$$D_0^{00} = -\lambda \quad D_1^{01} = -\lambda,$$

Definition 5.2 (Classical Christoffel symbols in the bicrossproduct model). *The classical Christoffel symbols in this case are given by*

$${}_0\Gamma^0_{\mu\nu} = \begin{pmatrix} -2bt & \frac{1+2bt^2}{x} \\ \frac{1+2bt^2}{x} & -2t\frac{1+bt^2}{x^2} \end{pmatrix} \quad {}_0\Gamma^1_{\mu\nu} = \begin{pmatrix} -2bx & 2bt \\ 2bt & -2b\frac{t^2}{x} \end{pmatrix},$$

Next, we introduce the following central quantity

$$v = xdt - tdx,$$

this enables us to write the classical connections as

$$\nabla_0(dx) = \frac{2b}{x} v \otimes v \quad \nabla_0(v) = -\frac{2}{x} v \otimes dx,$$

which are not expressed in a coordinated basis. we want to observe

$$\nabla_0(v) = \nabla_0(xdt - tdx) = dx \otimes dt - dt \otimes dx + x\nabla_0(dt) - t\nabla_0(dx)$$

Lemma 5.3 (Classical connection on dt). *The classical connection acting on dt is*

$$\nabla_0(dt) = \frac{1}{x^2}\left(-v \otimes dx + 2btv \otimes v - dx \otimes v\right)$$

Proof.

$$\begin{aligned}
\nabla_0(dt) =& \frac{1}{x}(\nabla_0(v) + t\nabla_0(dx) - dx \otimes dt + dt \otimes dx) \\
=& \frac{1}{x}(-\frac{2}{x} v \otimes dx + \frac{2bt}{x} v \otimes v - dx \otimes dt + dt \otimes dx) \\
=& \frac{1}{x^2}\left(-2v \otimes dx + 2btv \otimes v - dx \otimes (v + tdx) + (v + tdx) \otimes dx\right) \\
=& \frac{1}{x^2}\left(-v \otimes dx + 2btv \otimes v - dx \otimes v - dx \otimes tdx + tdx \otimes dx\right) \\
=& \frac{1}{x^2}\left(-v \otimes dx + 2btv \otimes v - dx \otimes v\right)
\end{aligned}$$

It is of our interest to calculate

$$[t, \nabla_0(dt)] \quad [x, \nabla_0(dt)] \quad [t, \nabla_0(dx)] \quad [x, \nabla_0(dx)],$$

Proposition 5.4. *[Commutators between coordinates and classical connections] The commutators between all the coordinates and the classical connections are*

$$\begin{aligned}
[t, \nabla_0(dt)] =& \frac{2\lambda}{x^2}\left(2btx^2 dt \otimes dt - x(1 + 2bt^2)(dt \otimes dx + dx \otimes dt) + 2t(1 + bt^2) dx \otimes dx\right) \\
=& 2\lambda\nabla_0(dt) \\
[x, \nabla_0(dt)] =& \frac{2b\lambda}{x} v \otimes v = \lambda\nabla_0(dx) \\
[t, \nabla_0(dx)] =& \frac{2b\lambda}{x} v \otimes v = \lambda\nabla_0(dx) \\
[x, \nabla_0(dx)] =& 0
\end{aligned}$$

The result clearly matches the general formula.

5.2 2-dimensional Lie algebra

Consider the most general Lie algebra in 2-dimensions

$$[x,t] = \alpha x + \beta t, \quad \alpha, \beta \in \mathbb{C}/\mathbb{R}$$

Lemma 5.5 (The $\Lambda, \tilde{\Lambda}$ quantities). *Let us define*

$$\Lambda =: \alpha + \beta x^{-1} t \quad \tilde{\Lambda} =: \beta + \alpha t^{-1} x,$$

then the following equalities hold for every $k \in \mathbb{N}$

$$(x-\beta)^k \Lambda = \Lambda x^k, \quad (t-\alpha)^k \tilde{\Lambda} = \tilde{\Lambda} t^k \tag{5.1}$$

Proof. First we have to calculate the following commutators

$$[\Lambda, x] = \beta x^{-1}[t,x] = -\beta\Lambda, \quad [\tilde{\Lambda}, t] = \alpha t^{-1}[x,t] = \alpha\tilde{\Lambda},$$

and now we proof by induction for the first equality, the procedure for the second is analogous. The case $k=1$ renders

$$(x-\beta)\Lambda = \Lambda(x-\beta) + [x,\Lambda] = \Lambda(x-\beta) + \beta\Lambda = \Lambda x,$$

we suppose for $k=n$ and analyse when $k=n+1$

$$(x-\beta)^{n+1}\Lambda = (x-\beta)\Lambda x^n = \Lambda(x-\beta)x^n + [x,\Lambda]x^n = \Lambda x^{n+1} - \beta\Lambda x^n + \beta\Lambda x^n = \Lambda x^{n+1}$$

Lemma 5.6 (Commutator of the coordinates and their integer powers). *The commutators between the generators of the algebra and any integer power of them has a closed form and it is*

$$[t, x^n] = \frac{x}{\beta}\left((x-\beta)^n - x^n\right)\Lambda \quad [x, t^n] = -\frac{t}{\alpha}\left((t-\alpha)^n - t^n\right)\tilde{\Lambda}.$$

Proof. Please note that

$$[x,t] = x\Lambda = t\tilde{\Lambda}$$

Consider the commutator and expand it

$$\begin{aligned}
[t,x^n] =& tx^n - x^n t = tx^n - x^{n-1}tx - x^n\Lambda = tx^n - x^{n-2}tx^2 - x^{n-1}\Lambda x - x^n\Lambda \\
=& tx^n - x^{n-2}tx^2 - x^{n-1}(x-\beta)\Lambda - x^n\Lambda \\
=& tx^n - x^{n-3}tx^3 - x^{n-2}\Lambda x^2 - x^{n-1}(x-\beta)\Lambda - x^n\Lambda \\
=& tx^n - x^{n-3}tx^3 - x^{n-2}(x-\beta)^2\Lambda - x^{n-1}(x-\beta)\Lambda - x^n\Lambda \\
&\vdots \\
=& -x(x-\beta)^{n-1}\Lambda - x^2(x-\beta)^{n-1\Lambda} - \cdots - x^{n-2}(x-\beta)^2\Lambda - x^{n-1}(x-\beta)\Lambda \\
=& -\left(\sum_{k=1}^{n} x^k (x-\beta)^{n-k}\right)\Lambda = -(x-\beta)^n\left(\sum_{k=1}^{n} x^k(x-\beta)^{-k}\right)\Lambda \\
=& -(x-\beta)^n\left(\sum_{k=0}^{n} x^k(x-\beta)^{-k} - 1\right)\Lambda.
\end{aligned}$$

Now we make use of the closed form of the geometric series

$$\begin{aligned}[t,x^n] =& -(x-\beta)^n\left([1-x^{n+1}(x-\beta)^{-(n+1)}][1-x(x-\beta)^{-1}]^{-1}-1\right)\Lambda \\ =& -(x-\beta)^n\left([1-x^{n+1}(x-\beta)^{-(n+1)}]-[1-x(x-\beta)^{-1}]\right)[1-x(x-\beta)^{-1}]^{-1}\Lambda,\end{aligned}$$

note that

$$1-x(x-\beta)^{-1}=(x-\beta-x)(x-\beta)^{-1}=-\beta(x-\beta)^{-1},$$

consequently we obtain

$$\begin{aligned}[t,x^n] =&\beta^{-1}(x-\beta)^n\left(-x^{n+1}(x-\beta)^{-(n+1)}+x(x-\beta)^{-1}\right)(x-\beta)\Lambda \\ =&\frac{x}{\beta}(x-\beta)^n\left(-x^n(x-\beta)^{-n}+1\right)\Lambda \\ =&\frac{x}{\beta}\left((x-\beta)^n-x^n\right)\Lambda\end{aligned}$$

The procedure for the second commutator is the same.

5.3 2-dimensional conformally flat space-time

Let us consider the following non-commutative metric over a two-dimensional Lie algebraic space-time

$$g_{\mu\nu}=e^{a_0t}e^{a_1x}\eta_{\mu\nu}$$

Lemma 5.7 (Commutator between the algebra and the exponentials). *We claim that*

$$\begin{aligned}[t,e^{a_1x}] =& -\frac{2}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t) \\ [x,e^{a_0t}] =&\frac{2}{\alpha}e^{a_0(t-\alpha/2)}\sinh\left(\frac{a_0\alpha}{2}\right)(\alpha x+\beta t).\end{aligned}$$

Proof. Consider the power series expansion for the exponential

$$e^A=\sum_{k=0}^{\infty}\frac{A^k}{k!},$$

then we need to investigate the non-trivial commutators of both generators of the algebra with powers of it

$$\begin{aligned}[t,e^{a_1x}] =&\frac{x}{\beta}\sum_{k=1}^{\infty}\frac{(a_1)^k}{k!}\left((x-\beta)^k-x^k\right)\Lambda=\frac{x}{\beta}\left(e^{a_1(x-\beta)}-1-(e^{a_1x}-1)\right)\Lambda \\ =&\frac{x}{\beta}e^{a_1x}\left(e^{-a_1\beta}-1\right)\Lambda=-2\frac{x}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)\Lambda \\ =& -\frac{2}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t)\end{aligned}$$

the other commutator is immediate to find.

Proposition 5.8 (Differential structure from centrality condition). *The centrality condition imposes some constraints that lead to*

$$D^{00}{}_0 = 0, \quad D^{00}{}_1 = -\beta, \quad D^{01}{}_0 = -\beta, \quad D^{10}{}_0 = 0$$
$$D^{01}{}_1 = 0, \quad D^{10}{}_1 = \alpha, \quad D^{11}{}_0 = \alpha, \quad D^{11}{}_1 = 0.$$

Also, in order to hold consistency we have

$$a_0 = \frac{n}{\tilde{\alpha}}\pi \quad a_1 = \frac{m}{\tilde{\beta}}\pi \quad m, n \in \mathbb{Z},$$

where $\alpha = i\tilde{\alpha}, \ \ \beta = i\tilde{\beta}$.

Proof. See appendix.

Proposition 5.9 (Classical Christoffel symbols). *The classical Christoffel symbols are*

$${}_0\Gamma^\lambda_{\mu\nu} = \frac{\eta^{\lambda\kappa}}{2}\left(\eta_{\mu\kappa}a_\nu + \eta_{\nu\kappa}a_\mu - \eta_{\mu\nu}a_\kappa\right) = \frac{1}{2}(\delta^\lambda_\mu a_\nu + \delta^\lambda_\nu a_\mu - \eta_{\mu\nu}a^\lambda),$$

written in matrix form, this is

$${}_0\Gamma^0_{\mu\nu} = \frac{1}{2}\begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix}, \qquad {}_0\Gamma^1_{\mu\nu} = \frac{1}{2}\begin{pmatrix} a_1 & a_0 \\ a_0 & a_1 \end{pmatrix},$$

Proof. Let us take the exterior derivative of the metric

$$\begin{aligned} dg_{\mu\nu} =& \eta_{\mu\nu} d(e^{a_0 t} e^{a_1 x}) = \eta_{\mu\nu}(a_0 e^{a_0 t} dt e^{a_1 x} + a_1 e^{a_0 t} e^{a_1 x} dx) \\ =& \eta_{\mu\nu} e^{a_0 t} e^{a_1 x}(a_0 dt + a_1 dx) + \eta_{\mu\nu} a_0 [dt, e^{a_1 x}], \end{aligned}$$

to calculate the commutator we note that due to the consistency conditions of proposition 5.8 we have

$$[t, e^{a_1 x}] = 0, \quad [x, e^{a_0 t}] = 0,$$

if we differentiate the first commutator we get

$$\begin{aligned} [dt, e^{a_1 x}] =& -[t, d(e^{a_1 x})] = -a_1[t, e^{a_1 x} dx] = -a_1 \underbrace{[t, e^{a_1 x}]}_{=0} dx - a_1 e^{a_1 x} \underbrace{[t, dx]}_{-\alpha dx} \\ =& a_1 \alpha e^{a_1 x} dx, \end{aligned}$$

which we substitute in the exterior derivative of the metric

$$dg_{\mu\nu} = \eta_{\mu\nu} e^{a_0 t} e^{a_1 x}(a_0 dt + a_1 dx) + \eta_{\mu\nu} a_0 a_1 \alpha e^{a_1 x} dx,$$

disregarding the quantum terms and defining $h_{\mu\nu\lambda} dx^\lambda := dg_{\mu\nu}$ we have

$$h_{\mu\nu\lambda} = \eta_{\mu\nu} a_\lambda e^{a_0 t} e^{a_1 x}.$$

The classical Christoffel symbols are defined as

$$_0\Gamma^\lambda_{\mu\nu} = \frac{g^{\lambda\kappa}}{2}\left(h_{\mu\kappa\nu} + h_{\nu\kappa\mu} - h_{\mu\nu\kappa}\right),$$

in the present case we have

$$_0\Gamma^\lambda_{\mu\nu} = \frac{\eta^{\lambda\kappa}}{2}\left(\eta_{\mu\kappa} a_\nu + \eta_{\nu\kappa} a_\mu - \eta_{\mu\nu} a_\kappa\right),$$

Or in matrix form

$$_0\Gamma^0_{\mu\nu} = \frac{1}{2}\begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix}, \qquad _0\Gamma^1_{\mu\nu} = \frac{1}{2}\begin{pmatrix} a_1 & a_0 \\ a_0 & a_1 \end{pmatrix}$$

Proposition 5.10. *[Quantum Christoffel symbols for the conformally flat space-time] The quantum-corrected Christoffel symbols are*

$$\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} + \frac{\alpha a_0 a_1}{4}\begin{pmatrix} \frac{a_1}{a_0} & 1 \\ 1 & \frac{a_0}{a_1} \end{pmatrix} - \frac{(a_1)^2}{8}\begin{pmatrix} \alpha & \beta \\ \beta & 2\beta\frac{a_0}{a_1} - \alpha \end{pmatrix} - \frac{(a_0)^2}{8}\begin{pmatrix} -\alpha + 2\beta\frac{a_1}{a_0} & \beta \\ \beta & \alpha \end{pmatrix}$$

$$\tilde{\Gamma}^1_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_1 & a_0 \\ a_0 & a_1 \end{pmatrix} + \frac{(a_1)^2}{8}\begin{pmatrix} \beta & \alpha \\ \alpha & 2\alpha\frac{a_0}{a_1} - \beta \end{pmatrix} + \frac{(a_0)^2}{8}\begin{pmatrix} 2\alpha\frac{a_1}{a_0} - \beta & \alpha \\ \alpha & \beta \end{pmatrix} - \frac{\beta a_0 a_1}{4}\begin{pmatrix} \frac{a_1}{a_0} & 1 \\ 1 & \frac{a_0}{a_1} \end{pmatrix}.$$

Proposition 5.11 (Riemann tensor). *The Riemann tensor for the conformally flat metric is identically zero. Which means that up to $\mathcal{O}(D^2)$ the space-time is flat and is a vacuum solution.*

Proof. *Consider the formula for the Riemann tensor*

$$R^\mu{}_{\beta\alpha\lambda} = \tilde{\Gamma}^\mu_{\alpha\sigma}\tilde{\Gamma}^\sigma_{\lambda\beta} - \tilde{\Gamma}^\mu_{\lambda\sigma}\tilde{\Gamma}^\sigma_{\alpha\beta},$$

where we have dropped three terms due to the fact that the Christoffel symbols are constant, consider μ, β fixed, we only have one component

$$R^\mu{}_{\beta 01} = \tilde{\Gamma}^\mu_{00}\tilde{\Gamma}^0_{1\beta} - \tilde{\Gamma}^\mu_{10}\tilde{\Gamma}^0_{0\beta} + \tilde{\Gamma}^\mu_{01}\tilde{\Gamma}^1_{1\beta} - \tilde{\Gamma}^\mu_{11}\tilde{\Gamma}^1_{0\beta},$$

which renders four cases, the first is

$$\begin{aligned}
R^0{}_{001} =& \tilde{\Gamma}^0_{00}\tilde{\Gamma}^0_{10} - \tilde{\Gamma}^0_{10}\tilde{\Gamma}^0_{00} + \tilde{\Gamma}^0_{01}\tilde{\Gamma}^1_{10} - \tilde{\Gamma}^0_{11}\tilde{\Gamma}^1_{00} = \underbrace{[\tilde{\Gamma}^0_{00}, \tilde{\Gamma}^0_{10}]}_{=0} + \tilde{\Gamma}^0_{01}\tilde{\Gamma}^1_{10} - \tilde{\Gamma}^0_{11}\tilde{\Gamma}^1_{00} \\
=& \tilde{\Gamma}^0_{01}\tilde{\Gamma}^1_{10} - \tilde{\Gamma}^0_{11}\tilde{\Gamma}^1_{00} \\
=& \frac{1}{4}\left(a_1 + \alpha\frac{a_0 a_1}{2} - \beta\frac{(a_0)^2 + (a_1)^2}{4}\right)\left(a_0 - \beta\frac{a_0 a_1}{2} + \alpha\frac{(a_0)^2 + (a_1)^2}{4}\right) \\
&- \frac{1}{4}\left(a_0 + \alpha\frac{(a_0)^2 + (a_1)^2}{4} - \beta\frac{a_0 a_1}{2}\right)\left(a_1 - \beta\frac{(a_1)^2 + (a_0)^2}{4} + \alpha\frac{a_0 a_1}{2}\right) \\
=& 0,
\end{aligned}$$

The second is

$$
\begin{aligned}
R^0{}_{101} =&\tilde{\Gamma}^0_{00}\tilde{\Gamma}^0_{11} - \tilde{\Gamma}^0_{10}\tilde{\Gamma}^0_{01} + \tilde{\Gamma}^0_{01}\tilde{\Gamma}^1_{11} - \tilde{\Gamma}^0_{11}\tilde{\Gamma}^1_{01}\\
=&\frac{1}{4}\left(a_0 + \alpha\frac{(a_0)^2+(a_1)^2}{4} - \beta\frac{a_0a_1}{2}\right)^2 - \frac{1}{4}\left(a_1 + \alpha\frac{a_0a_1}{2} - \beta\frac{(a_0)^2+(a_1)^2}{4}\right)^2\\
&+\frac{1}{4}\left(a_1 + \alpha\frac{a_0a_1}{2} - \beta\frac{(a_0)^2+(a_1)^2}{4}\right)^2 - \frac{1}{4}\left(a_0 + \alpha\frac{(a_0)^2+(a_1)^2}{4} - \beta\frac{a_0a_1}{2}\right)^2\\
=&0
\end{aligned}
$$

The third

$$
\begin{aligned}
R^1{}_{001} =&\tilde{\Gamma}^1_{00}\tilde{\Gamma}^0_{10} - \tilde{\Gamma}^1_{10}\tilde{\Gamma}^0_{00} + \tilde{\Gamma}^1_{01}\tilde{\Gamma}^1_{10} - \tilde{\Gamma}^1_{11}\tilde{\Gamma}^1_{00}\\
=&\frac{1}{4}\left(a_1 - \beta\frac{(a_1)^2+(a_0)^2}{4} + \alpha\frac{a_0a_1}{2}\right)^2 - \frac{1}{4}\left(a_0 + \alpha\frac{(a_0)^2+(a_1)^2}{4} - \beta\frac{a_0a_1}{2}\right)^2\\
&+\frac{1}{4}\left(a_0 + \alpha\frac{(a_0)^2+(a_1)^2}{4} - \beta\frac{a_0a_1}{2}\right)^2 - \frac{1}{4}\left(a_1 + \alpha\frac{a_0a_1}{2} - \beta\frac{(a_0)^2+(a_1)^2}{4}\right)^2\\
=&0.
\end{aligned}
$$

The fourth and last one is

$$
\begin{aligned}
R^1{}_{101} =&\tilde{\Gamma}^1_{00}\tilde{\Gamma}^0_{11} - \tilde{\Gamma}^1_{10}\tilde{\Gamma}^0_{01} + \tilde{\Gamma}^1_{01}\tilde{\Gamma}^1_{11} - \tilde{\Gamma}^1_{11}\tilde{\Gamma}^1_{01} = \tilde{\Gamma}^1_{00}\tilde{\Gamma}^0_{11} - \tilde{\Gamma}^1_{10}\tilde{\Gamma}^0_{01} + \underbrace{[\tilde{\Gamma}^1_{01}, \tilde{\Gamma}^1_{11}]}_{=0}\\
=&\frac{1}{4}\left(a_1 - \beta\frac{(a_1)^2+(a_0)^2}{4} + \alpha\frac{a_0a_1}{2}\right)\left(a_0 + \alpha\frac{(a_0)^2+(a_1)^2}{4} - \beta\frac{a_0a_1}{2}\right)\\
&-\frac{1}{4}\left(a_0 + \alpha\frac{(a_0)^2+(a_1)^2}{4} - \beta\frac{a_0a_1}{2}\right)\left(a_1 - \beta\frac{(a_1)^2+(a_0)^2}{4} + \alpha\frac{a_0a_1}{2}\right)\\
=&0.
\end{aligned}
$$

This result was expected of course.

5.4 4-dimensional conformally flat space-time

Now let us take a look for the 4-dimensional case keeping the non-commutativity just between time and one spatial coordinate and with the following metric

$$g_{\mu\nu} = e^{a_0 t} e^{a_1 x} e^{a_{\hat{i}} x^{\hat{i}}} \eta_{\mu\nu},$$

where $\hat{i} = 23$ labels the commutative coordinates.

Proposition 5.12 (Deformation of the differential structure for the n-dimensional case). *The centrality condition implies*

$$D^{\mu\nu}{}_{\nu} = 0 \quad (\textit{ for fixed } \nu) \quad D^{00}{}_{i} = -\beta\delta_{1i} = D^{0i}{}_{0} \quad D^{01}{}_{\hat{j}} = -D^{0\hat{j}}{}_{1} := \gamma_{\hat{j}}$$
$$D^{0\hat{i}}{}_{\hat{j}} = -D^{0\hat{j}}{}_{\hat{i}} := \phi_{ij} \quad D^{10}{}_{1} = \alpha = D^{11}{}_{0} \quad D^{1\hat{i}}{}_{1} = -D^{11}{}_{\hat{i}} = \xi_{\hat{i}}$$
$$D^{1\hat{i}}{}_{\hat{j}} = -D^{1\hat{j}}{}_{\hat{i}} = -D^{\hat{i}\hat{j}}{}_{1} =: \chi_{ij} \quad D^{\hat{i}\hat{\ell}}{}_{1} = -D^{\hat{i}1}{}_{\hat{\ell}} =: \zeta^{\hat{i}}_{\hat{\ell}} \quad D^{\hat{i}\hat{\ell}}{}_{\hat{j}} = -D^{\hat{i}\hat{j}}{}_{\hat{\ell}} =: \omega^{\hat{i}}_{\hat{\ell}\hat{j}}.$$

However from now on ***we choose*** *to make zero most of them, so the only non-vanishing quantities are*

$$D^{00}{}_{i} = -\beta\delta_{1i} = D^{0i}{}_{0} \quad D^{10}{}_{1} = \alpha = D^{11}{}_{0}.$$

Proof. See appendix.

Proposition 5.13 (Classical Christoffel symbols for the n-dimensional case). *For the n-dimensional case the Christoffel symbols are*

$${}_0\Gamma^0_{\mu\nu} = \frac{1}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & 0 & a_0 & 0 & \cdots \\ a_3 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad {}_0\Gamma^1_{\mu\nu} = \frac{1}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_2 & -a_1 & 0 & \cdots \\ 0 & a_3 & 0 & -a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$${}_0\Gamma^2_{\mu\nu} = \frac{1}{2}\begin{pmatrix} a_2 & 0 & a_0 & 0 & \cdots \\ 0 & -a_2 & a_1 & 0 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_3 & -a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad {}_0\Gamma^3_{\mu\nu} = \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a_0 & \cdots \\ 0 & -a_3 & 0 & a_1 & \cdots \\ 0 & 0 & -a_3 & a_2 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. It is immediate to check using

$${}_0\Gamma^\lambda_{\mu\nu} = \frac{1}{2}(\delta^\lambda_\mu a_\nu + \delta^\lambda_\nu a_\mu - \eta^{\lambda\kappa}\eta_{\mu\nu}a_\kappa)$$

Remark 5.14 (Christoffel symbol for commutative generators). We want to draw attention on a fact contained in the last proposition

$${}_0\Gamma^{\hat{i}}_{0\nu} = a_{\hat{i}}\delta_{0\nu} + a_0\delta_{\hat{i}\nu} \quad {}_0\Gamma^{\hat{i}}_{1\nu} = -a_{\hat{i}}\delta_{1\nu} + a_1\delta_{\hat{i}\nu} \quad {}_0\Gamma^{\hat{i}}_{01} = 0 = {}_0\Gamma^{\hat{i}}_{10}$$

Proposition 5.15 (Quantum Christoffel symbols for the 4-dimensional case). *The quantum-corrected Christoffel symbols are*

$$\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix} - \frac{(a_0)^2}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} - \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} - \frac{(a_1)^2}{8}\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & 2\beta\frac{a_0}{a_1} - \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}$$
$$- \frac{(a_2)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ 0 & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_3)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & 0 & \frac{\beta a_0 - \alpha a_1}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 \end{pmatrix} + \frac{\alpha a_0 a_1}{4}\begin{pmatrix} \frac{a_1}{a_0} & 1 & 0 & 0 \\ 1 & \frac{a_0}{a_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\Gamma}^1_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix} + \frac{(a_0)^2}{8}\begin{pmatrix} 2\alpha\frac{a_1}{a_0} - \beta & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} + \frac{(a_1)^2}{8}\begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix}$$

$$- \frac{(a_2)^2}{8}\begin{pmatrix} 2\beta & \alpha & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ \alpha & 0 & 0 & 0 \\ \frac{\beta a_0 - \alpha a_1}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_3)^2}{8}\begin{pmatrix} 2\beta & \alpha & 0 & \frac{\beta a_0 - \alpha a_1}{a_3} \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 & 0 \end{pmatrix} - \frac{\beta a_0 a_1}{4}\begin{pmatrix} \frac{a_1}{a_0} & 1 & 0 & 0 \\ 1 & \frac{a_0}{a_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\Gamma}^2_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_2 & 0 & a_0 & 0 \\ 0 & -a_2 & a_1 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_3 & -a_2 \end{pmatrix} - \frac{\beta a_1 a_2}{8}\begin{pmatrix} 1 & 0 & \frac{a_0}{a_2} - \frac{\alpha}{\beta}\frac{a_1}{a_2} & 0 \\ 0 & 1 - 2\frac{\alpha}{\beta}\frac{a_0}{a_1} & 0 & 0 \\ \frac{a_0}{a_2} - \frac{\alpha}{\beta}\frac{a_1}{a_2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$- \frac{\alpha a_0 a_2}{8}\begin{pmatrix} 1 - 2\frac{\beta}{\alpha}\frac{a_1}{a_0} & 0 & 0 & 0 \\ 0 & 1 & \frac{\beta}{\alpha}\frac{a_0}{a_2} - \frac{a_1}{a_2} & 0 \\ 0 & \frac{\beta}{\alpha}\frac{a_0}{a_2} - \frac{a_1}{a_2} & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\tilde{\Gamma}^3_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a_0 \\ 0 & -a_3 & 0 & a_1 \\ 0 & 0 & -a_3 & a_2 \\ a_0 & a_1 & a_2 & a_3 \end{pmatrix} - \frac{\beta a_1 a_3}{8}\begin{pmatrix} 1 & 0 & 0 & \frac{a_0}{a_3} - \frac{\alpha}{\beta}\frac{a_1}{a_3} \\ 0 & 1 - 2\frac{\alpha}{\beta}\frac{a_0}{a_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{a_0}{a_3} - \frac{\alpha}{\beta}\frac{a_1}{a_3} & 0 & 0 & 1 \end{pmatrix}$$

$$- \frac{\alpha a_0 a_3}{8}\begin{pmatrix} 1 - 2\frac{\beta}{\alpha}\frac{a_1}{a_0} & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\beta}{\alpha}\frac{a_0}{a_3} - \frac{a_1}{a_3} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{\beta}{\alpha}\frac{a_0}{a_3} - \frac{a_1}{a_3} & 0 & -1 \end{pmatrix}$$

Proof. As in the last case, we consider the general formula

$$\tilde{\Gamma}^{\mu}_{\rho\sigma} = {}_0\Gamma^{\mu}_{\rho\sigma} + \frac{1}{2}\,{}_0\Gamma^{\mu}_{\alpha\beta}(D^{\lambda\beta}{}_{\rho}\,{}_0\Gamma^{\alpha}_{\lambda\sigma} + D^{\lambda\beta}{}_{\sigma}\,{}_0\Gamma^{\alpha}_{\rho\lambda} - D^{\alpha\beta}{}_{\lambda}\,{}_0\Gamma^{\lambda}_{\rho\sigma}).$$

And highlight once more that the last term is not present because in our case due to the fact $[x^{\alpha},\ {}_0\Gamma^{\beta}_{\rho\sigma}] = 0$. *Allow us to study the general case*

$$\tilde{\Gamma}^{\mu}_{\rho\sigma} = {}_0\Gamma^{\mu}_{\rho\sigma} + \frac{{}_0\Gamma^{\mu}_{00}}{2}(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{0}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{0}_{\lambda\rho} - \underbrace{D^{00}{}_{1}}_{=-\beta}\,{}_0\Gamma^{1}_{\rho\sigma}) + \frac{{}_0\Gamma^{\mu}_{01}}{2}(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{0}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{0}_{\rho\lambda} - \underbrace{D^{01}{}_{0}}_{=-\beta}\,{}_0\Gamma^{0}_{\rho\sigma})$$
$$+ \frac{{}_0\Gamma^{\mu}_{0\hat{i}}}{2}(\underbrace{D^{\lambda\hat{i}}{}_{\rho}}_{=0}\,{}_0\Gamma^{0}_{\lambda\sigma} + \underbrace{D^{\lambda\hat{i}}{}_{\sigma}}_{=0}\,{}_0\Gamma^{0}_{\rho\lambda} - \underbrace{D^{0\hat{i}}{}_{k}}_{=0}\,{}_0\Gamma^{k}_{\rho\sigma}) + \frac{{}_0\Gamma^{\mu}_{10}}{2}(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{1}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{1}_{\rho\lambda} - \underbrace{D^{10}{}_{1}}_{=\alpha}\,{}_0\Gamma^{1}_{\rho\sigma})$$
$$+ \frac{{}_0\Gamma^{\mu}_{\hat{i}0}}{2}(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{\hat{i}}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{\hat{i}}_{\rho\lambda} - \underbrace{D^{\hat{i}0}{}_{k}}_{=0}\,{}_0\Gamma^{k}_{\rho\sigma}) + \frac{{}_0\Gamma^{\mu}_{11}}{2}(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{1}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{1}_{\rho\lambda} - \underbrace{D^{11}{}_{0}}_{=\alpha}\,{}_0\Gamma^{0}_{\rho\sigma})$$
$$+ \frac{{}_0\Gamma^{\mu}_{1\hat{i}}}{2}(\underbrace{D^{\lambda\hat{i}}{}_{\rho}}_{=0}\,{}_0\Gamma^{1}_{\lambda\sigma} + \underbrace{D^{\lambda\hat{i}}{}_{\sigma}}_{=0}\,{}_0\Gamma^{1}_{\rho\lambda} - \underbrace{D^{1\hat{i}}{}_{\lambda}}_{=0}\,{}_0\Gamma^{\lambda}_{\rho\sigma}) + \frac{{}_0\Gamma^{\mu}_{\hat{i}1}}{2}(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{\hat{i}}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{\hat{i}}_{\rho\lambda} - \underbrace{D^{\hat{i}1}{}_{\lambda}}_{=0}\,{}_0\Gamma^{\lambda}_{\rho\sigma})$$
$$+ \frac{{}_0\Gamma^{\mu}_{\hat{i}\hat{j}}}{2}(\underbrace{D^{\lambda\hat{j}}{}_{\rho}}_{=0}\,{}_0\Gamma^{\alpha}_{\lambda\sigma} + \underbrace{D^{\lambda\hat{j}}{}_{\sigma}}_{=0}\,{}_0\Gamma^{\alpha}_{\rho\lambda} - \underbrace{D^{\hat{i}\hat{j}}{}_{\lambda}}_{=0}\,{}_0\Gamma^{\lambda}_{\rho\sigma}),$$

which ends up in

$$\tilde{\Gamma}^{\mu}_{\rho\sigma} = {}_0\Gamma^{\mu}_{\rho\sigma} + \frac{{}_0\Gamma^{\mu}_{00}}{2}(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{0}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{0}_{\lambda\rho} + \beta\,{}_0\Gamma^{1}_{\rho\sigma}) + \frac{{}_0\Gamma^{\mu}_{01}}{2}(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{0}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{0}_{\rho\lambda} + \beta\,{}_0\Gamma^{0}_{\rho\sigma})$$
$$+ \frac{{}_0\Gamma^{\mu}_{10}}{2}(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{1}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{1}_{\rho\lambda} - \alpha\,{}_0\Gamma^{1}_{\rho\sigma}) + \frac{{}_0\Gamma^{\mu}_{\hat{i}0}}{2}(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{\hat{i}}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{\hat{i}}_{\rho\lambda})$$
$$+ \frac{{}_0\Gamma^{\mu}_{11}}{2}(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{1}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{1}_{\rho\lambda} - \alpha\,{}_0\Gamma^{0}_{\rho\sigma}) + \frac{{}_0\Gamma^{\mu}_{\hat{i}1}}{2}(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{\hat{i}}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{\hat{i}}_{\rho\lambda}),$$

before proceeding we shall write down the form of the $D^{\lambda\mu}{}_{\rho}$ *matrices with* μ *fixed*

$$D^{\lambda 0}{}_{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D^{\lambda 1}{}_{\rho} = \begin{pmatrix} -\beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

as in the last case we shall study each term of the sum. The first one being

$$D^{\lambda 0}{}_{\rho\,0}\Gamma^0_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma\,0}\Gamma^0_{\lambda\rho} + \beta\,{}_0\Gamma^1_{\rho\sigma}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha a_1 - \beta a_0 & \alpha a_0 - \beta a_1 & -\beta a_2 & -\beta a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \alpha a_1 - \beta a_0 & 0 & 0 \\ 0 & \alpha a_0 - \beta a_1 & 0 & 0 \\ 0 & -\beta a_2 & 0 & 0 \\ 0 & -\beta a_3 & 0 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \alpha a_1 - \beta a_0 & 0 & 0 \\ \alpha a_1 - \beta a_0 & 2\alpha a_0 - 2\beta a_1 & -\beta a_2 & -\beta a_3 \\ 0 & -\beta a_2 & 0 & 0 \\ 0 & -\beta a_3 & 0 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix}$$

$$= \frac{a_1}{2}\begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix},$$

while the second renders

$$D^{\lambda 1}{}_{\rho\,0}\Gamma^0_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma\,0}\Gamma^0_{\rho\lambda} + \beta\,{}_0\Gamma^0_{\rho\sigma}$$

$$= \frac{1}{2}\begin{pmatrix} \alpha a_1 - \beta a_0 & \alpha a_0 - \beta a_1 & -\beta a_2 & -\beta a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \alpha a_1 - \beta a_0 & 0 & 0 & 0 \\ \alpha a_0 - \beta a_1 & 0 & 0 & 0 \\ -\beta a_2 & 0 & 0 & 0 \\ -\beta a_3 & 0 & 0 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2\alpha a_1 - 2\beta a_0 & \alpha a_0 - \beta a_1 & -\beta a_2 & -\beta a_3 \\ \alpha a_0 - \beta a_1 & 0 & 0 & 0 \\ -\beta a_2 & 0 & 0 & 0 \\ -\beta a_3 & 0 & 0 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix}$$

$$= \frac{a_0}{2}\begin{pmatrix} 2\alpha\frac{a_1}{a_0} - \beta & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}.$$

The third is quite similar

$$D^{\lambda 0}{}_{\rho\,0}\Gamma^1_{\lambda\sigma}+D^{\lambda 0}{}_{\sigma\,0}\Gamma^1_{\rho\lambda}-\alpha\,{}_0\Gamma^1_{\rho\sigma}$$
$$=\frac{1}{2}\begin{pmatrix}0&0&0&0\\ \alpha a_0-\beta a_1&\alpha a_1-\beta a_0&\alpha a_2&\alpha a_3\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&\alpha a_0-\beta a_1&0&0\\ 0&\alpha a_1-\beta a_0&0&0\\ 0&\alpha a_2&0&0\\ 0&\alpha a_3&0&0\end{pmatrix}-\frac{\alpha}{2}\begin{pmatrix}a_1&a_0&0&0\\ a_0&a_1&a_2&a_3\\ 0&a_2&-a_1&0\\ 0&a_3&0&-a_1\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}0&\alpha a_0-\beta a_1&0&0\\ \alpha a_0-\beta a_1&2\alpha a_1-2\beta a_0&\alpha a_2&\alpha a_3\\ 0&\alpha a_2&0&0\\ 0&\alpha a_3&0&0\end{pmatrix}-\frac{\alpha}{2}\begin{pmatrix}a_1&a_0&0&0\\ a_0&a_1&a_2&a_3\\ 0&a_2&-a_1&0\\ 0&a_3&0&-a_1\end{pmatrix}$$
$$=-\frac{a_1}{2}\begin{pmatrix}\alpha&\beta&0&0\\ \beta&2\beta\frac{a_0}{a_1}-\alpha&0&0\\ 0&0&-\alpha&0\\ 0&0&0&-\alpha\end{pmatrix},$$

the fourth is simpler, but it splits in two cases

$$D^{\lambda 0}{}_{\rho\,0}\Gamma^2_{\lambda\sigma}+D^{\lambda 0}{}_{\sigma\,0}\Gamma^2_{\rho\lambda}=\frac{1}{2}\begin{pmatrix}0&0&0&0\\ -\beta a_2&-\alpha a_2&\alpha a_1-\beta a_0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&-\beta a_2&0&0\\ 0&-\alpha a_2&0&0\\ 0&\alpha a_1-\beta a_0&0&0\\ 0&0&0&0\end{pmatrix}$$
$$=-\frac{a_2}{2}\begin{pmatrix}0&\beta&0&0\\ \beta&2\alpha&\frac{\beta a_0-\alpha a_1}{a_2}&0\\ 0&\frac{\beta a_0-\alpha a_1}{a_2}&0&0\\ 0&0&0&0\end{pmatrix}$$
$$D^{\lambda 0}{}_{\rho\,0}\Gamma^3_{\lambda\sigma}+D^{\lambda 0}{}_{\sigma\,0}\Gamma^3_{\rho\lambda}=\frac{1}{2}\begin{pmatrix}0&0&0&0\\ -\beta a_3&-\alpha a_3&0&\alpha a_1-\beta a_0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&-\beta a_3&0&0\\ 0&-\alpha a_3&0&0\\ 0&0&0&0\\ 0&\alpha a_1-\beta a_0&0&0\end{pmatrix}$$
$$=-\frac{a_3}{2}\begin{pmatrix}0&\beta&0&0\\ \beta&2\alpha&0&\frac{\beta a_0-\alpha a_1}{a_3}\\ 0&0&0&0\\ 0&\frac{\beta a_0-\alpha a_1}{a_3}&0&0\end{pmatrix}.$$

For the fifth we proceed as usual

$$D^{\lambda 1}{}_{\rho\, 0}\Gamma^{1}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma\, 0}\Gamma^{1}_{\rho\lambda} - \alpha\, {}_{0}\Gamma^{0}_{\rho\sigma}$$

$$= \frac{1}{2}\begin{pmatrix} \alpha a_0 - \beta a_1 & \alpha a_1 - \beta a_0 & \alpha a_2 & \alpha a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \alpha a_0 - \beta a_1 & 0 & 0 & 0 \\ \alpha a_1 - \beta a_0 & 0 & 0 & 0 \\ \alpha a_2 & 0 & 0 & 0 \\ \alpha a_3 & 0 & 0 & 0 \end{pmatrix} - \frac{\alpha}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2\alpha a_0 - 2\beta a_1 & \alpha a_1 - \beta a_0 & \alpha a_2 & \alpha a_3 \\ \alpha a_1 - \beta a_0 & 0 & 0 & 0 \\ \alpha a_2 & 0 & 0 & 0 \\ \alpha a_3 & 0 & 0 & 0 \end{pmatrix} - \frac{\alpha}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix}$$

$$= -\frac{a_0}{2}\begin{pmatrix} 2\beta\frac{a_1}{a_0} - \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}.$$

The sixth and last also contains two possibilities

$$D^{\lambda 1}{}_{\rho\, 0}\Gamma^{2}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma\, 0}\Gamma^{2}_{\rho\lambda} = \frac{1}{2}\begin{pmatrix} -\beta a_2 & -\alpha a_2 & \alpha a_1 - \beta a_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\beta a_2 & 0 & 0 & 0 \\ -\alpha a_2 & 0 & 0 & 0 \\ \alpha a_1 - \beta a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= -\frac{a_2}{2}\begin{pmatrix} 2\beta & \alpha & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ \alpha & 0 & 0 & 0 \\ \frac{\beta a_0 - \alpha a_1}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D^{\lambda 1}{}_{\rho\, 0}\Gamma^{3}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma\, 0}\Gamma^{3}_{\rho\lambda} = \frac{1}{2}\begin{pmatrix} -\beta a_3 & -\alpha a_3 & 0 & \alpha a_1 - \beta a_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\beta a_3 & 0 & 0 & 0 \\ -\alpha a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha a_1 - \beta a_0 & 0 & 0 & 0 \end{pmatrix}$$

$$= -\frac{a_3}{2}\begin{pmatrix} 2\beta & \alpha & 0 & \frac{\beta a_0 - \alpha a_1}{a_3} \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 & 0 \end{pmatrix}.$$

Now we may , for $\mu = 0$ we have

$$\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix} + \frac{a_0 a_1}{8}\begin{pmatrix} 2\alpha\frac{a_1}{a_0} & 2\alpha & 0 & 0 \\ 2\alpha & 2\alpha\frac{a_0}{a_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_1)^2}{8}\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & 2\beta\frac{a_0}{a_1} - \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}$$
$$- \frac{(a_2)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ 0 & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_3)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & 0 & \frac{\beta a_0 - \alpha a_1}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 \end{pmatrix}$$
$$- \frac{(a_0)^2}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} - \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}.$$

For $\mu = 1$ we have

$$\tilde{\Gamma}^1_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix} + \frac{(a_1)^2}{8}\begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix} + \frac{(a_0)^2}{8}\begin{pmatrix} 2\alpha\frac{a_1}{a_0} - \beta & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$
$$- \frac{a_0 a_1}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} & 2\beta & 0 & 0 \\ 2\beta & 2\beta\frac{a_0}{a_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_2)^2}{8}\begin{pmatrix} 2\beta & \alpha & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ \alpha & 0 & 0 & 0 \\ \frac{\beta a_0 - \alpha a_1}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$- \frac{(a_3)^2}{8}\begin{pmatrix} 2\beta & \alpha & 0 & \frac{\beta a_0 - \alpha a_1}{a_3} \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 & 0 \end{pmatrix}.$$

For $\mu = 2$ we obtain

$$\tilde{\Gamma}^2_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_2 & 0 & a_0 & 0 \\ 0 & -a_2 & a_1 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_3 & -a_2 \end{pmatrix} + \frac{a_1 a_2}{8}\begin{pmatrix} -\beta & 0 & -\frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ 0 & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ -\frac{\beta a_0 - \alpha a_1}{a_2} & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix}$$
$$+ \frac{a_0 a_2}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} - \alpha & 0 & 0 & 0 \\ 0 & -\alpha & -\frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ 0 & -\frac{\beta a_0 - \alpha a_1}{a_2} & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

And finally, for $\mu = 3$ we reach

$$\tilde{\Gamma}^3_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a_0 \\ 0 & -a_3 & 0 & a_1 \\ 0 & 0 & -a_3 & a_2 \\ a_0 & a_1 & a_2 & a_3 \end{pmatrix} + \frac{a_1 a_3}{8}\begin{pmatrix} -\beta & 0 & 0 & -\frac{\beta a_0 - \alpha a_1}{a_3} \\ 0 & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ -\frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 & -\beta \end{pmatrix}$$
$$+ \frac{a_0 a_3}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} - \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & -\frac{\beta a_0 - \alpha a_1}{a_3} \\ 0 & 0 & \alpha & 0 \\ 0 & -\frac{\beta a_0 - \alpha a_1}{a_3} & 0 & \alpha \end{pmatrix}.$$

It is evident that any calculation involving these Christoffel symbols is very difficult, that is why we consider the homogeneous case

Proposition 5.16 (Homogeneous case). *Let $a_0 = a_1 =: a$, $\alpha = \beta =: \Theta$ and $(a_2)^2 + (a_3)^2 =: b^2$, then the 4-dimensional Christoffel symbols become*

$$\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & a_2 & a_3 \\ a & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_3 & 0 & 0 & a \end{pmatrix} - \theta\frac{b^2}{8}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} a & a - \theta\frac{b^2}{4} & a_2 & a_3 \\ a - \theta\frac{b^2}{4} & a - \theta\frac{b^2}{2} & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_3 & 0 & 0 & a \end{pmatrix}$$
$$\tilde{\Gamma}^1_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & 0 & 0 \\ a & a & a_2 & a_3 \\ 0 & a_2 & -a & 0 \\ 0 & a_3 & 0 & -a \end{pmatrix} - \theta\frac{b^2}{8}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} a - \theta\frac{b^2}{2} & a - \theta\frac{b^2}{4} & 0 & 0 \\ a - \theta\frac{b^2}{4} & a & a_2 & a_3 \\ 0 & a_2 & -a & 0 \\ 0 & a_3 & 0 & -a \end{pmatrix}$$
$$\tilde{\Gamma}^2_{\rho\sigma} = {}_0\Gamma^2_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_2 & 0 & a & 0 \\ 0 & -a_2 & a & 0 \\ a & a & a_2 & a_3 \\ 0 & 0 & a_3 & -a_2 \end{pmatrix}$$
$$\tilde{\Gamma}^3_{\rho\sigma} = {}_0\Gamma^3_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a \\ 0 & -a_3 & 0 & a \\ 0 & 0 & -a_3 & a_2 \\ a & a & a_2 & a_3 \end{pmatrix}.$$

Proof. *First let us see what happens when $a_0 = a_1 =: a$*

$$
\begin{aligned}
\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & a_2 & a_3 \\ a & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_3 & 0 & 0 & a \end{pmatrix} - \frac{a^2}{8}\begin{pmatrix} 2\beta-\alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} - \frac{a^2}{8}\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & 2\beta-\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix} \\
- \frac{(a_2)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & a\frac{\beta-\alpha}{a_2} & 0 \\ 0 & a\frac{\beta-\alpha}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_3)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & 0 & a\frac{\beta-\alpha}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & a\frac{\beta-\alpha}{a_3} & 0 & 0 \end{pmatrix} + \frac{\alpha a^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
= \frac{1}{2}\begin{pmatrix} a & a & a_2 & a_3 \\ a & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_3 & 0 & 0 & a \end{pmatrix} + (\alpha-\beta)\frac{a^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_2)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & a\frac{\beta-\alpha}{a_2} & 0 \\ 0 & a\frac{\beta-\alpha}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
- \frac{(a_3)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & 0 & a\frac{\beta-\alpha}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & a\frac{\beta-\alpha}{a_3} & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Gamma}^1_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & 0 & 0 \\ a & a & a_2 & a_3 \\ 0 & a_2 & -a & 0 \\ 0 & a_3 & 0 & -a \end{pmatrix} + \frac{a^2}{8}\begin{pmatrix} 2\alpha-\beta & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} + \frac{a^2}{8}\begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 2\alpha-\beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix} \\
- \frac{(a_2)^2}{8}\begin{pmatrix} 2\beta & \alpha & a\frac{\beta-\alpha}{a_2} & 0 \\ \alpha & 0 & 0 & 0 \\ a\frac{\beta-\alpha}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_3)^2}{8}\begin{pmatrix} 2\beta & \alpha & 0 & a\frac{\beta-\alpha}{a_3} \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\frac{\beta-\alpha}{a_3} & 0 & 0 & 0 \end{pmatrix} - \frac{\beta a^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
= \frac{1}{2}\begin{pmatrix} a & a & 0 & 0 \\ a & a & a_2 & a_3 \\ 0 & a_2 & -a & 0 \\ 0 & a_3 & 0 & -a \end{pmatrix} + (\alpha-\beta)\frac{a^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_2)^2}{8}\begin{pmatrix} 2\beta & \alpha & a\frac{\beta-\alpha}{a_2} & 0 \\ \alpha & 0 & 0 & 0 \\ a\frac{\beta-\alpha}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
- \frac{(a_3)^2}{8}\begin{pmatrix} 2\beta & \alpha & 0 & a\frac{\beta-\alpha}{a_3} \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\frac{\beta-\alpha}{a_3} & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$\tilde{\Gamma}^2_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_2 & 0 & a & 0 \\ 0 & -a_2 & a & 0 \\ a & a & a_2 & a_3 \\ 0 & 0 & a_3 & -a_2 \end{pmatrix} - \frac{\beta a a_2}{8}\begin{pmatrix} 1 & 0 & \frac{a}{a_2} - \frac{\alpha}{\beta}\frac{a}{a_2} & 0 \\ 0 & 1 - 2\frac{\alpha}{\beta} & 0 & 0 \\ \frac{a}{a_2} - \frac{\alpha}{\beta}\frac{a}{a_2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$- \frac{\alpha a a_2}{8}\begin{pmatrix} 1 - 2\frac{\beta}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & \frac{\beta}{\alpha}\frac{a}{a_2} - \frac{a}{a_2} & 0 \\ 0 & \frac{\beta}{\alpha}\frac{a}{a_2} - \frac{a}{a_2} & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\tilde{\Gamma}^3_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a \\ 0 & -a_3 & 0 & a \\ 0 & 0 & -a_3 & a_2 \\ a & a & a_2 & a_3 \end{pmatrix} - \frac{\beta a a_3}{8}\begin{pmatrix} 1 & 0 & 0 & \frac{a}{a_3} - \frac{\alpha}{\beta}\frac{a}{a_3} \\ 0 & 1 - 2\frac{\alpha}{\beta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{a}{a_3} - \frac{\alpha}{\beta}\frac{a}{a_3} & 0 & 0 & 1 \end{pmatrix}$$

$$- \frac{\alpha a a_3}{8}\begin{pmatrix} 1 - 2\frac{\beta}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\beta}{\alpha}\frac{a}{a_3} - \frac{a}{a_3} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{\beta}{\alpha}\frac{a}{a_3} - \frac{a}{a_3} & 0 & -1 \end{pmatrix}.$$

Now we shall set $\alpha = \beta =: \theta$

$$\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & a_2 & a_3 \\ a & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_3 & 0 & 0 & a \end{pmatrix} - \theta\frac{(a_2)^2 + (a_3)^2}{8}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\Gamma}^1_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & 0 & 0 \\ a & a & a_2 & a_3 \\ 0 & a_2 & -a & 0 \\ 0 & a_3 & 0 & -a \end{pmatrix} - \theta\frac{(a_2)^2 + (a_3)^2}{8}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\Gamma}^2_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_2 & 0 & a & 0 \\ 0 & -a_2 & a & 0 \\ a & a & a_2 & a_3 \\ 0 & 0 & a_3 & -a_2 \end{pmatrix} - a\theta\frac{a_2}{8}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - a\theta\frac{a_2}{8}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\tilde{\Gamma}^3_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a \\ 0 & -a_3 & 0 & a \\ 0 & 0 & -a_3 & a_2 \\ a & a & a_2 & a_3 \end{pmatrix} - a\theta\frac{a_3}{8}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - a\theta\frac{a_3}{8}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Proposition 5.17 (Ricci tensor for the 4-dimensional case). *The Ricci tensor is*

$$\tilde{R}_{\mu\nu} = \frac{n^2\pi^2}{\tilde{\theta}^2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

With $\theta = i\tilde{\theta}$.

Proof. *Recalling the first order expansion for the Ricci tensor we use it in our case, where the first terms are missing due to the fact that the Christoffel symbols are constant*

$$\tilde{R}_{\sigma\rho} = R_{\sigma\rho} + \varGamma^{\mu}_{\mu\lambda}\, {}_q\varGamma^{\lambda}_{\rho\sigma} + {}_q\varGamma^{\mu}_{\mu\lambda}\varGamma^{\lambda}_{\rho\sigma} - \varGamma^{\mu}_{\rho\lambda}\, {}_q\varGamma^{\lambda}_{\mu\sigma} - {}_q\varGamma^{\mu}_{\rho\lambda}\varGamma^{\lambda}_{\mu\sigma}.$$

For the classical Ricci we have

$$\begin{aligned} R_{\sigma\rho} &= {}_0\varGamma^{\mu}_{\mu\lambda}\, {}_0\varGamma^{\lambda}_{\rho\sigma} - {}_0\varGamma^{\mu}_{\rho\lambda}\, {}_0\varGamma^{\lambda}_{\mu\sigma} \\ &= \frac{1}{4}\left(\delta^{\mu}_{\mu}a_{\lambda} + \delta^{\mu}_{\lambda}a_{\mu} - \eta^{\mu\kappa}\eta_{\mu\lambda}a_{\kappa}\right)\left(\delta^{\lambda}_{\rho}a_{\sigma} + \delta^{\lambda}_{\sigma}a_{\rho} - \eta^{\lambda\tau}\eta_{\rho\sigma}a_{\tau}\right) \\ &\quad - \frac{1}{4}\left(\delta^{\mu}_{\rho}a_{\lambda} + \delta^{\mu}_{\lambda}a_{\rho} - \eta^{\mu\kappa}\eta_{\rho\lambda}a_{\kappa}\right)\left(\delta^{\lambda}_{\mu}a_{\sigma} + \delta^{\lambda}_{\sigma}a_{\mu} - \eta^{\lambda\tau}\eta_{\mu\sigma}a_{\tau}\right) \\ &= \frac{1}{4}\left(\eta^{\lambda\tau}\eta_{\rho\sigma}a_{\lambda}a_{\tau} - 4a_{\rho}a_{\sigma}\right) = \frac{1}{4}\left(\eta_{\rho\sigma}(-a^2 + a^2 + b^2) - 4a_{\rho}a_{\sigma}\right) \\ &= \frac{1}{4}\left(\eta_{\rho\sigma}b^2 - 4a_{\rho}a_{\sigma}\right). \end{aligned}$$

Which means that the classical Ricci scalar is zero, as can be seen below

$$\begin{aligned} R = g^{\sigma\rho}R_{\sigma\rho} &= \frac{1}{4}e^{-a_{\hat{i}}x^{\hat{i}}}e^{-ax}e^{-at}\eta^{\sigma\rho}\left(\eta_{\rho\sigma}b^2 - 4a_{\rho}a_{\sigma}\right) = \frac{1}{4}e^{-a_{\hat{i}}x^{\hat{i}}}e^{-ax}e^{-at}\left(4b^2 - 4(-a^2 + a^2 + b^2)\right) \\ &= 0 \end{aligned}$$

Since we don't have quantum corrections for Christoffel symbols with upstairs index $\mu = 2, 3$ we have

$$\begin{aligned} \tilde{R}_{\sigma\rho} =& R_{\sigma\rho} + {}_0\varGamma^{\mu}_{\mu 0}\, {}_q\varGamma^{0}_{\rho\sigma} + {}_0\varGamma^{\mu}_{\mu 1}\, {}_q\varGamma^{1}_{\rho\sigma} + {}_q\varGamma^{0}_{01}\, {}_0\varGamma^{1}_{\rho\sigma} + {}_q\varGamma^{1}_{10}\, {}_0\varGamma^{0}_{\rho\sigma} - {}_0\varGamma^{0}_{\rho 0}\, {}_q\varGamma^{0}_{0\sigma} - {}_0\varGamma^{1}_{\rho 0}\, {}_q\varGamma^{0}_{1\sigma} \\ &- {}_0\varGamma^{0}_{\rho 1}\, {}_q\varGamma^{1}_{0\sigma} - {}_0\varGamma^{1}_{\rho 1}\, {}_q\varGamma^{1}_{1\sigma} - {}_q\varGamma^{0}_{\rho 0}\, {}_0\varGamma^{0}_{0\sigma} - {}_q\varGamma^{0}_{\rho 1}\, {}_0\varGamma^{1}_{0\sigma} - {}_q\varGamma^{1}_{\rho 0}\, {}_0\varGamma^{0}_{1\sigma} - {}_q\varGamma^{1}_{\rho 1}\, {}_0\varGamma^{1}_{1\sigma} \\ =& \frac{1}{4}\left(\eta_{\rho\sigma}b^2 - 4a_{\rho}a_{\sigma}\right) + 2a\left({}_q\varGamma^{0}_{\rho\sigma} + {}_q\varGamma^{1}_{\rho\sigma}\right) - \theta\frac{b^2}{16}\left((\delta^0_{\rho} + \delta^1_{\rho})a_{\sigma} + (\delta^0_{\sigma} + \delta^1_{\sigma})a_{\rho}\right) \\ &+ \theta\frac{b^2}{16}\left(\left(\delta^0_{\rho}a + a_{\rho} + \eta_{\rho 0}a\right)\delta_{\sigma 1} + \left(\delta^0_{\sigma}a + a_{\sigma} + \eta_{\sigma 0}a\right)\delta_{\rho 1}\right) - \frac{a}{2}\left(\left(\delta^1_{\rho} - \eta_{\rho 0}\right){}_q\varGamma^{0}_{1\sigma} + \left(\delta^1_{\sigma} - \eta_{\sigma 0}\right){}_q\varGamma^{0}_{\rho 1}\right) \\ &- \frac{a}{2}\left(\left(\delta^0_{\rho} + \eta_{\rho 1}\right){}_q\varGamma^{1}_{0\sigma} + \left(\delta^0_{\sigma} + \eta_{\sigma 1}\right){}_q\varGamma^{1}_{0\rho}\right) + \theta\frac{b^2}{16}\left(\left(\delta^1_{\rho}a + a_{\rho} - \eta_{\rho 1}a\right)\delta_{0\sigma} + \left(\delta^1_{\sigma}a + a_{\sigma} - \eta_{\sigma 1}a\right)\delta_{\rho 0}\right). \end{aligned}$$

Let us calculate all the components

$$\begin{aligned}
\tilde{R}_{00} &= -\frac{1}{4}\left(b^2+4a^2\right) - a\theta\frac{b^2}{8} \\
\tilde{R}_{01} &= -a^2 - a\theta\frac{b^2}{8} \\
\tilde{R}_{02} &= -aa_2 - a_2\theta\frac{b^2}{16} \\
\tilde{R}_{03} &= -aa_3 - a_3\theta\frac{b^2}{16} \\
\tilde{R}_{11} &= \frac{1}{4}\left(b^2-4a^2\right) - a\theta\frac{b^2}{8} \\
\tilde{R}_{12} &= -aa_2 \\
\tilde{R}_{13} &= -aa_3 \\
\tilde{R}_{22} &= -(a_2)^2 \\
\tilde{R}_{23} &= -a_2a_3 \\
\tilde{R}_{33} &= -(a_3)^2.
\end{aligned}$$

In matrix notation this is

$$\tilde{R}_{\mu\nu} = \begin{pmatrix} -\frac{1}{4}\left(b^2+4a^2\right) - a\theta\frac{b^2}{8} & -a^2 - a\theta\frac{b^2}{8} & -aa_2 - a_2\theta\frac{b^2}{16} & -aa_3 - a_3\theta\frac{b^2}{16} \\ -a^2 - a\theta\frac{b^2}{8} & \frac{1}{4}\left(b^2-4a^2\right) - a\theta\frac{b^2}{8} & -aa_2 & -aa_3 \\ -aa_2 - a_2\theta\frac{b^2}{16} & -aa_2 & -(a_2)^2 & -a_2a_3 \\ -aa_3 - a_3\theta\frac{b^2}{16} & -aa_3 & -a_2a_3 & -(a_3)^2 \end{pmatrix}$$

If we require this tensor to be Hermitian, then it must be invariant under flipping of the tensor factors and complex conjugation of each coefficient, which leads us to the condition

$$a\theta b^2 = a\theta^* b^2 \Rightarrow a\theta b^2 = 0. \tag{5.2}$$

The discretization constraints are $a\theta = n\pi$ for some $n \in \mathbb{Z}$, this along the hermiticity leads to either $b^2 = 0$ or $n = 0$, we discard the second option since that is the commutative case or flat space and is of no interest to us. Since $b^2 = (a_2)^2 + (a_3)^2$ where $a_2, a_3 \in \mathbb{R}$ we conclude that to fulfill this condition both a_2 and a_3 have to be zero.

$$\tilde{R}_{\mu\nu} = \frac{n^2\pi^2}{\tilde{\theta}^2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Where $\theta = i\tilde{\theta}$. This tensor is purely quantum mechanical and it is not a vacuum solution since that would imply that $n = 0$.

Proposition 5.18 (Ricci Scalar). *The Ricci scalar is*

$$\tilde{R} = 0$$

Proof. *By direct calculation, we find*

$$\tilde{R} = g^{\mu\nu} R_{\mu\nu} = e^{-ax} e^{-at} (-R_{00} + R_{11}) = \frac{n^2 \pi^2}{2\tilde{\theta}^2} e^{-ax} e^{-at} (-1+1) = 0$$

Proposition 5.19 (Einstein tensor). *The Einstein tensor is*

Proof. *By definition, the Einstein tensor is*

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} g_{\mu\nu} = \frac{n^2 \pi^2}{\tilde{\theta}^2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.3}$$

Proposition 5.20 (Stress-energy tensor). *The stress-energy tensor is*

$$T_{\mu\nu} = \frac{n^2 \pi}{8\tilde{\theta}^2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which makes a contribution to energy, momentum and pressure; we note that it is traceless, which might indicate that it describes a field invariant under conformal transformations.

5.5 FRW space-time

In this section we consider flat FRW space-time along the following algebra

$$[x^i, x^j] = \gamma \varepsilon^{ijk} x^k \quad r^2 =: x^i x^i.$$

The metric is given by

$$g = -dt \otimes dt + g_{ij} dx^i \otimes dx^j, \;\; g_{ij} = a^2(t) h_{ij}, \quad h_{ij} := \left(\delta_{ij} + K \frac{x^i x^j + x^j x^i}{2(1 - Kr^2)} \right)$$

Proposition 5.21 (Centrality condition). *The centrality condition fixes the deformation of the differential structure as follows*

$$D^{00}{}_{\nu} = 0, \quad D^{0i}{}_{j} = 0, \quad D^{k0}{}_{0} = 0, \quad D^{kj}{}_{0} = 0, \quad D^{k\ell}_{i} = -\gamma \varepsilon^{k\ell}_{i}.$$

Proof. *The centrality condition for time is*

$$0 = [t, g_{00}] = -2D^{00}{}_{0} \Rightarrow D^{00}{}_{0} = 0 \tag{5.4}$$

$$0 = [t, g_{0i}] = D^{0j}{}_{0} g_{ji} - D^{00}{}_{i} \Rightarrow D^{0j}{}_{0} = 0 \text{ and } D^{00}{}_{i} = 0 \tag{5.5}$$

$$0 = [t, g_{ij}] = D^{0\ell}{}_{i} g_{\ell j} + D^{0\ell}{}_{j} g_{\ell i} = D^{0\ell}{}_{i} a^2(t) \left(\delta_{\ell j} + K \frac{x^\ell x^j + x^j x^\ell}{2(1 - Kr^2)} \right) \tag{5.6}$$

$$+ D^{0\ell}{}_{j} a^2(t) \left(\delta_{\ell i} + K \frac{x^\ell x^i + x^i x^\ell}{2(1 - Kr^2)} \right) \tag{5.7}$$

$$\Rightarrow D^{0j}{}_{i} + D^{0i}{}_{j} = 0 \Rightarrow 0 = D^{0\ell}{}_{i} \left(x^\ell x^j + x^j x^\ell \right) + D^{0\ell}{}_{j} \left(x^\ell x^i + x^i x^\ell \right) \Rightarrow D^{0\ell}{}_{i} = 0 \tag{5.8}$$

Now let us consider an spatial generator with $k = 1, 2, 3$,

$$0 = [x^k, g_{00}] = -2D^{k0}{}_0 \Rightarrow D^{k0}{}_0 = 0$$
$$0 = [x^k, g_{0i}] = -D^{k0}{}_i + D^{kj}{}_0 g_{ij} \Rightarrow D^{k0}{}_i = 0 \text{ and } D^{kj}{}_0 = 0$$
$$[x^k, g_{ij}] = D^{k\ell}{}_i a^2(t) \left(\delta_{\ell j} + K\frac{x^\ell x^j + x^j x^\ell}{2(1-Kr^2)}\right) + D^{k\ell}{}_j a^2(t) \left(\delta_{\ell i} + K\frac{x^\ell x^i + x^i x^\ell}{2(1-Kr^2)}\right),$$

but on the other hand, we have

$$\begin{aligned}[x^k, g_{ij}] &= \left[x^k, a^2(t)\left(\delta_{ij} + K\frac{x^i x^j + x^j x^i}{2(1-Kr^2)}\right)\right] = \frac{a^2(t)K}{2(1-Kr^2)}\left[x^k, x^i x^j + x^j x^i\right] \\ &= \gamma\frac{a^2(t)K}{1-Kr^2}(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell.\end{aligned}$$

Equating both expressions we get

$$\gamma\frac{K}{1-Kr^2}(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell = D^{k\ell}{}_i \left(\delta_{\ell j} + K\frac{x^\ell x^j + x^j x^\ell}{2(1-Kr^2)}\right) + D^{k\ell}{}_j \left(\delta_{\ell i} + K\frac{x^\ell x^i + x^i x^\ell}{2(1-Kr^2)}\right),$$

we search for a solution demanding

$$D^{k\ell}{}_i \delta_{\ell j} + D^{k\ell}{}_j \delta_{\ell i} = 0 \Rightarrow D^{kj}{}_i + D^{ki}{}_j = 0,$$

which takes us to

$$\begin{aligned}&\gamma(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell = x^\ell\left(D^{k\ell}{}_i x^j + D^{k\ell}{}_j x^i\right) \\ &\frac{\gamma}{2}x^\ell(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j) + \frac{\gamma}{2}(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell + \mathcal{O}(\gamma^2) \\ =&\frac{x^\ell}{2}\left(D^{k\ell}{}_i x^j + D^{k\ell}{}_j x^i\right) + \left(D^{k\ell}{}_i x^j + D^{k\ell}{}_j x^i\right)\frac{x^\ell}{2} \\ \Rightarrow& D^{k\ell}{}_i = -\gamma\varepsilon^{k\ell}{}_i\end{aligned}$$

Definition 5.22 (Classical Christoffel symbols for the FRW space-time). *The classical Christoffel symbols for the FRW metric are*

$${}_0\Gamma^0_{ij} = a\dot{a}h_{ij}, \quad {}_0\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta_{ij}, \quad {}_0\Gamma^i_{j\ell} = Kh_{jl}x^i$$

Proposition 5.23 (Quantum Christoffel symbols). *The components of the connection quantum-corrected up to first order are identical to the classical ones*

Proof. *Consider the formula for the quantum correction*

$$\tilde{\Gamma}^\mu_{\rho\sigma} = {}_0\Gamma^\mu_{\rho\sigma} + \frac{{}_0\Gamma^\mu_{\alpha\beta}}{2}((D^{\lambda\beta}{}_\rho\, {}_0\Gamma^\alpha_{\lambda\sigma} + D^{\lambda\beta}{}_\sigma\, {}_0\Gamma^\alpha_{\rho\lambda} - D^{\alpha\beta}{}_\lambda\, {}_0\Gamma^\lambda_{\rho\sigma}) - [x^\beta, {}_0\Gamma^\alpha_{\rho\sigma}]),$$

We observe that

$$[x^\beta,\, {}_0\Gamma^\alpha_{00}] = 0, \quad [x^\beta,\, {}_0\Gamma^\alpha_{0j}] = 0.$$

And proceed to calculate

$$\begin{aligned}
\Rightarrow \tilde{\Gamma}^\mu_{00} =& 0\\
\Rightarrow \tilde{\Gamma}^\mu_{0i} =& {}_0\Gamma^\mu_{0i} - \frac{{}_0\Gamma^\mu_{kj}}{2}\frac{\dot{a}}{a}\left(\gamma\varepsilon^{kj}{}_i - \gamma\varepsilon^{kj}{}_i\right) = {}_0\Gamma^\mu_{0i}\\
\tilde{\Gamma}^\mu_{ij} =& {}_0\Gamma^\mu_{ij} + \frac{{}_0\Gamma^\mu_{mk}}{2}K\left(\gamma\left(\varepsilon^{\ell k}{}_i h_{\ell j}x^m + \varepsilon^{\ell k}{}_j h_{i\ell}x^m - \varepsilon^{mk}{}_\ell h_{ij}x^\ell\right) + [x^k, h_{ij}x^m]\right)\\
=& {}_0\Gamma^\mu_{ij} + \frac{{}_0\Gamma^\mu_{mk}}{2}K\left(\gamma\left(\varepsilon^{\ell k}{}_i\left(\cancel{\delta_{\ell j}} + K\frac{x^\ell x^j + x^j x^\ell}{2(1-Kr^2)}\right)x^m\right.\right.\\
&\left.\left.+\varepsilon^{\ell k}{}_j\left(\cancel{\delta_{\ell i}} + K\frac{x^\ell x^i + x^i x^\ell}{2(1-Kr^2)}\right)x^m - \varepsilon^{mk}{}_\ell h_{ij}x^\ell\right) + h_{ij}[x^k, x^m] + [x^k, h_{ij}]x^m\right)\\
=& {}_0\Gamma^\mu_{ij} + \frac{{}_0\Gamma^\mu_{mk}}{2}K\left(\gamma\left(\varepsilon^{\ell k}{}_i\left(K\frac{x^\ell x^j + x^j x^\ell}{2(1-Kr^2)}\right)x^m + \varepsilon^{\ell k}{}_j\left(K\frac{x^\ell x^i + x^i x^\ell}{2(1-Kr^2)}\right)x^m\right.\right.\\
&\left.\left.-\varepsilon^{mk}{}_\ell h_{ij}x^\ell\right) + \gamma h_{ij}\varepsilon^{km}{}_\ell x^\ell + \gamma\frac{K}{1-Kr^2}(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell x^m\right)\\
=& {}_0\Gamma^\mu_{ij} + \frac{{}_0\Gamma^\mu_{mk}}{2}K\left(\frac{\gamma K}{2(1-Kr^2)}\left(\varepsilon^{\ell k}{}_i(x^\ell x^j + x^j x^\ell) + \varepsilon^{\ell k}{}_j(x^\ell x^i + x^i x^\ell)\right)x^m\right.\\
&\left.+2\gamma h_{ij}\varepsilon^{km}{}_\ell x^\ell + \frac{\gamma K}{1-Kr^2}(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell x^m\right)\\
=& {}_0\Gamma^\mu_{ij} + \gamma K\, {}_0\Gamma^\mu_{mk}\left(h_{ij}\cancel{\varepsilon^{km}{}_\ell}x^\ell + \frac{K}{2(1-Kr^2)}\left((\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell\right.\right.\\
&\left.\left.+x^\ell(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)\right)x^m + \mathcal{O}(\gamma^2)\right).
\end{aligned}$$

We dropped a term due to the contraction of symmetric quantity with an antisymmetric one. We arrive to

$$\begin{aligned}
\tilde{\Gamma}^\mu_{ij} =& {}_0\Gamma^\mu_{ij} + \frac{\gamma K^2}{2(1-Kr^2)}\,{}_0\Gamma^\mu_{mk}\left((\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell + x^\ell(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)\right)x^m\\
\Rightarrow \tilde{\Gamma}^0_{ij} =& {}_0\Gamma^0_{ij}\\
\Rightarrow \tilde{\Gamma}^n_{ij} =& {}_0\Gamma^n_{ij} + \frac{\gamma K^3}{2(1-Kr^2)}h_{mk}x^n\left((\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell + x^\ell(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)\right)x^m\\
=& {}_0\Gamma^n_{ij} + \frac{\gamma K^3}{2(1-Kr^2)}h_{mk}x^m x^n\left((\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell + x^\ell(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)\right) + \mathcal{O}(\gamma^2),
\end{aligned}$$

we draw our attention to

$$\begin{aligned}
h_{mk}x^m =& \left(\delta_{mk} + K\frac{x^m x^k + x^k x^m}{2(1-Kr^2)}\right)x^m = \left(1 + K\frac{r^2}{1-Kr^2}\right)x^k = \left(1 + K\frac{r^2}{1-Kr^2}\right)x^k\\
=& \frac{x^k}{1-Kr^2},
\end{aligned}$$

putting this into previous calculation yields

$$\begin{aligned}\tilde{\Gamma}^n_{ij} &= {}_0\Gamma^n_{ij} + \frac{\gamma K^3}{2(1-Kr^2)^2} \underbrace{x^k x^n \left((\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)x^\ell + x^\ell(\varepsilon^{kj\ell}x^i + \varepsilon^{ki\ell}x^j)\right)}_{\mathcal{O}(\gamma)} \\ &= {}_0\Gamma^n_{ij}\end{aligned}$$

Where we have exploited the fact that for any antisymmetric quantity F_{ij} we reach

$$F_{ij}x^i x^j = \frac{1}{2}(F_{ij} + F_{ij})x^i x^j = \frac{F_{ij}}{2}x^i x^j - \frac{F_{ji}}{2}x^i x^j = \frac{F_{ij}}{2}[x^i, x^j],$$

Definition 5.24 (Classical Ricci tensor). *The classical Ricci tensor for an FRW space-time is*

$$R_{00} = 3\frac{\ddot{a}}{a}, \quad R_{0k} = 0, \quad R_{mn} = -(2K + 2\dot{a}^2 + a\ddot{a})h_{mn}$$

Proposition 5.25 (Quantum corrected Ricci tensor). *The Ricci tensor is given by*

$$\tilde{R}_{00} = R_{00}, \quad \tilde{R}_{0n} = R_{0n}, \quad \tilde{R}_{mn} = R_{mn} + \frac{\gamma K \varepsilon^{in}{}_m x^i}{(1-Kr^2)}\left(\dot{a}^2 + \frac{K^2 r^2}{1-Kr^2}\right) \tag{5.9}$$

Proof. The definition Ricci tensor is

$$\tilde{R}_{\beta\lambda} = R_{\beta\lambda} + {}_0\Gamma^\mu_{\rho\sigma}(\Sigma^{\rho\sigma}{}_{\lambda\beta\mu} - \Sigma^{\rho\sigma}{}_{\mu\beta\lambda}),$$

since ${}_0\Gamma = {}_q\Gamma$. We analyse the last term for $\beta = 0,\ \lambda = 0$

$${}_0\Gamma^\mu_{i\sigma}(\underbrace{\Sigma^{i\sigma}{}_{00\mu}}_{=0} - \Sigma^{i\sigma}{}_{\mu 00}) = -\ {}_0\Gamma^\mu_{ij}\underbrace{\Sigma^{ij}{}_{k00}}_{=0} = 0,$$

therefore

$$\tilde{R}_{00} = R_{00}.$$

for $\beta = n,\ \lambda = 0$

$${}_0\Gamma^\mu_{ij}(\Sigma^{ij}{}_{0n\mu} - \Sigma^{ij}{}_{\mu n0}) = 0,$$

while for $\beta = 0,\ \lambda = n$ we have the same result, thus

$$\tilde{R}_{0i} = \tilde{R}_{i0} = R_{0i}.$$

finally, for for $\beta = n,\ \lambda = m$

$$ {}_0\Gamma^0_{i\sigma}(\Sigma^{i\sigma}{}_{mn0} - \Sigma^{i\sigma}{}_{0nm}) + {}_0\Gamma^\ell_{i\sigma}(\Sigma^{i\sigma}{}_{mn\ell} - \Sigma^{i\sigma}{}_{\ell nm}) $$

we study several cases, first

$$ \begin{aligned} [dx^i, {}_0\Gamma^0_{mn}] =& a\dot{a}[dx^i, h_{mn}] = \frac{Ka\dot{a}}{2(1-Kr^2)}[dx^i, x^m x^n + x^n x^m] \\ =& \frac{-\gamma K a\dot{a}}{2(1-Kr^2)}\left(\varepsilon^{im}{}_p dx^p x^n + \varepsilon^{in}{}_p x^m dx^p + \varepsilon^{in}{}_p dx^p x^m + \varepsilon^{im}{}_p x^n dx^p\right) \\ =& \frac{-\gamma K a\dot{a}}{1-Kr^2}\left(\varepsilon^{in}{}_p x^m + \varepsilon^{im}{}_p x^n\right) dx^p + \mathcal{O}(\gamma^2) \\ \Rightarrow \Sigma^{i0}{}_{mnp} =& -\frac{\gamma K a\dot{a}}{1-Kr^2}\left(\varepsilon^{in}{}_p x^m + \varepsilon^{im}{}_p x^n\right), \end{aligned} $$

now

$$ \begin{aligned} [dx^i, {}_0\Gamma^j_{mn}] =& K[dx^i, h_{mn}]x^j + K h_{mn}[dx^i, x^j] \\ =& -\frac{\gamma K^2}{1-Kr^2}\left(\varepsilon^{in}{}_p x^m + \varepsilon^{im}{}_p x^n\right) x^j dx^p - K\gamma h_{mn}\varepsilon^{ij}{}_p dx^p \\ \Sigma^{ij}{}_{mnp} =& -\frac{\gamma K^2}{1-Kr^2}\left(\varepsilon^{in}{}_p x^m + \varepsilon^{im}{}_p x^n\right) x^j - K\gamma h_{mn}\varepsilon^{ij}{}_p. \end{aligned} $$

From these two last quantities we conclude that $\Sigma^{i\sigma}{}_{mn0} = 0$. *Finally we find*

$$ [dx^i, {}_0\Gamma^j_{0n}] = 0 \Rightarrow \Sigma^{ij}{}_{0n\mu} = 0 $$

Therefore

$$ \begin{aligned} & {}_0\Gamma^0_{i\sigma}\underbrace{(\Sigma^{i\sigma}{}_{mn0} - \Sigma^{i\sigma}{}_{0nm})}_{=0} + {}_0\Gamma^\ell_{i\sigma}(\Sigma^{i\sigma}{}_{mn\ell} - \Sigma^{i\sigma}{}_{\ell nm}) = {}_0\Gamma^\ell_{i\sigma}(\Sigma^{i\sigma}{}_{mn\ell} - \Sigma^{i\sigma}{}_{\ell nm}) \\ =& {}_0\Gamma^\ell_{i0}(\Sigma^{i0}{}_{mn\ell} - \Sigma^{i0}{}_{\ell nm}) + {}_0\Gamma^\ell_{ij}(\Sigma^{ij}{}_{mn\ell} - \Sigma^{ij}{}_{\ell nm}) \\ =& -\frac{\gamma K\dot{a}^2}{1-Kr^2}\delta_{\ell i}(\cancel{\varepsilon^{in}{}_\ell x^m} - \varepsilon^{in}{}_m x^\ell + 2\cancel{\varepsilon^{im}{}_\ell x^n}) \\ & -\gamma K^2 h_{ij}x^\ell\left(\frac{K}{1-Kr^2}\left(\varepsilon^{in}{}_\ell x^m - \varepsilon^{in}{}_m x^\ell + 2\varepsilon^{im}{}_\ell x^n\right) x^j + h_{mn}\cancel{\varepsilon^{ij}{}_\ell} - h_{\ell n}\cancel{\varepsilon^{ij}{}_m}\right) \end{aligned} $$

where we have dropped the last two terms due to the fact that they will render a commutator along with h_{ij} *and thus are second order contributions, we arrive to*

$$
\begin{aligned}
&{}_0\Gamma^0_{i\sigma}(\Sigma^{i\sigma}{}_{mn0} - \Sigma^{i\sigma}{}_{0nm}) + {}_0\Gamma^\ell_{i\sigma}(\Sigma^{i\sigma}{}_{mn\ell} - \Sigma^{i\sigma}{}_{\ell nm}) \\
=&\frac{\gamma K\dot{a}^2}{1-Kr^2}\varepsilon^{in}{}_m x^i - \gamma x^\ell \frac{K^3 h_{ij}x^j}{1-Kr^2}\left(\varepsilon^{in}{}_\ell x^m - \varepsilon^{in}{}_m x^\ell + 2\varepsilon^{im}{}_\ell x^n\right) + \mathcal{O}(\gamma^2) \\
=&\frac{\gamma K\dot{a}^2}{1-Kr^2}\varepsilon^{in}{}_m x^i - \gamma \frac{K^3 x^\ell x^i}{(1-Kr^2)^2}\left(\varepsilon^{in}{}_\ell x^m - \varepsilon^{in}{}_m x^\ell + 2\varepsilon^{im}{}_\ell x^n\right) \\
=&\frac{\gamma K\dot{a}^2}{1-Kr^2}\varepsilon^{in}{}_m x^i + \frac{\gamma K^3 r^2}{(1-Kr^2)^2}\varepsilon^{in}{}_m x^i + \mathcal{O}(\gamma^2) \\
=&\frac{\gamma K\varepsilon^{in}{}_m x^i}{(1-Kr^2)}\left(\dot{a}^2 + \frac{K^2r^2}{1-Kr^2}\right).
\end{aligned}
$$

Therefore, the spatial Ricci tensor is

$$\tilde{R}_{mn} = R_{mn} + \frac{\gamma K\varepsilon^{in}{}_m x^i}{(1-Kr^2)}\left(\dot{a}^2 + \frac{K^2r^2}{1-Kr^2}\right)$$

Proposition 5.26 (Inverse metric). *The inverse metric is*

$$g^{00} = -1, \quad g^{0i} = 0, \quad g^{ij} = \frac{h^{ij}}{a^2(t)}, \quad h^{ij} := \delta^{jk} - \frac{K}{2}\left(x^j x^k + x^k x^j\right) \tag{5.10}$$

Proof. The only thing we need to check is the spatial part

$$h_{ij}h^{jk} = h_{ij}\left(\delta^{jk} - \frac{K}{2}\left(x^j x^k + x^k x^j\right)\right) = h_{ik} - K\frac{x^i x^k + x^k x^i}{2(1-kr^2)} \tag{5.11}$$

$$= \delta_{ik} + K\frac{x^i x^k + x^k x^i}{2(1-kr^2)} - K\frac{x^i x^k + x^k x^i}{2(1-kr^2)} = \delta_{ik} \tag{5.12}$$

Proposition 5.27 (Ricci scalar). *The Ricci scalar is*

$$\tilde{R} = R = -\frac{6}{a^2}(K + \dot{a}^2 + a\ddot{a})$$

Proof. Taking the Ricci tensor we obtain the scalar

$$
\begin{aligned}
\tilde{R} =& g^{\mu\nu}\tilde{R}_{\mu\nu} = -R_{00} + \frac{1}{a^2(t)}h^{jk}\tilde{R}_{jk} \\
=& -3\frac{\ddot{a}}{a} + \frac{h^{jk}}{a^2(t)}\left(-(2K + 2\dot{a}^2 + a\ddot{a})h_{jk} + \frac{\gamma K\varepsilon^{ik}{}_j x^i}{(1-Kr^2)}\left(\dot{a}^2 + \frac{K^2r^2}{1-Kr^2}\right)\right) \\
=& -3\frac{\ddot{a}}{a} + \frac{1}{a^2(t)}\left(-3(2K + 2\dot{a}^2 + a\ddot{a})\right) + \mathcal{O}(\gamma^2) \\
=& -\frac{6}{a^2}(K + \dot{a}^2 + a\ddot{a}) = R.
\end{aligned}
$$

Proposition 5.28 (Einstein tensor). *The Einstein tensor is*

$$
\begin{aligned}
\tilde{G}_{00} =& -\frac{3}{a^2}(K+\dot{a}^2)\\
\tilde{G}_{0i} =& 0\\
\tilde{G}_{ij} =& (K+\dot{a}^2+2a\ddot{a})h_{ij} - \frac{\gamma K \varepsilon^{ijk}x^k}{1-Kr^2}\left(\dot{a}^2+\frac{K^2r^2}{1-Kr^2}\right)
\end{aligned}
$$

Proof. We use the definition of the Einstein tensor

$$
\begin{aligned}
\tilde{G}_{\mu\nu} =& \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}g_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\\
\tilde{G}_{00} =& \tilde{R}_{00} + \frac{1}{2}R = 3\frac{\ddot{a}}{a} - \frac{3}{a^2}(K+\dot{a}^2+a\ddot{a}) = -\frac{3}{a^2}(K+\dot{a}^2)\\
\tilde{G}_{0i} =& 0\\
\tilde{G}_{ij} =& \tilde{R}_{ij} - \frac{a^2}{2}Rh_{ij}\\
=& -(2K+2\dot{a}^2+a\ddot{a})h_{ij} + \frac{\gamma K \varepsilon^{kj}_{\ i}x^k}{(1-Kr^2)}\left(\dot{a}^2+\frac{K^2r^2}{1-Kr^2}\right) + 3(K+\dot{a}^2+a\ddot{a})h_{ij}\\
=& (K+\dot{a}^2+2a\ddot{a})h_{ij} - \frac{\gamma K \varepsilon^{ijk}x^k}{1-Kr^2}\left(\dot{a}^2+\frac{K^2r^2}{1-Kr^2}\right)
\end{aligned}
$$

Definition 5.29 (Energy-Stress tensor). *We define the energy stress tensor as*

$$
T_{\mu\nu} =: \frac{G_{\mu\nu}}{8\pi}
$$

Proposition 5.30 (Energy-Stress tensor for non-commutative FRW). *The energy-stress tensor for the FRW space-time is the same as in the classical case plus an additional term that is an anomaly.*

Proof. We write explicitly the components

$$
\begin{aligned}
G_{00} =& -\frac{3}{a^2}(K+\dot{a}^2) =: 8\pi\rho(t)\\
G_{ij} =& (K+\dot{a}^2+2a\ddot{a})h_{ij} - \frac{\gamma K \varepsilon^{ijk}x^k}{1-Kr^2}\left(\dot{a}^2+\frac{K^2r^2}{1-Kr^2}\right) =: 8\pi a^2(t)p(t)h_{ij} + 8\pi\, {}^{A}T_{ij}
\end{aligned}
$$

Where ${}^{A}T_{ij} = -{}^{A}T_{ji}$, thus we conclude that a Lorentz anomaly arises; this anomaly occurs when general covariance is broken in the quantum theory [Ber96, BZ, AGG85, Mat85]. Furthermore this tells us that the Friedmann equations are still valid plus an additional term. If $K=1$, we choose $r=\sin(\chi)$ and the anomalous term becomes

$$
-\frac{\gamma\varepsilon^{ijk}x^k}{1-\sin^2(\chi)}\left(\dot{a}^2+\frac{\sin^2(\chi)}{1-\sin^2(\chi)}\right) = -\gamma\varepsilon^{ijk}x^k\sec^2(\chi)\left(\dot{a}^2+\tan^2(\chi)\right), \tag{5.13}
$$

which is never zero. Alternatively we may choose $K = -1$ *and* $r = \sinh(\chi)$, *which renders*

$$\frac{\gamma\varepsilon^{ijk}x^k}{1+\sinh^2(\chi)}\left(\dot{a}^2 + \frac{\sinh^2(\chi)}{1+\sinh^2(\chi)}\right) = \gamma\varepsilon^{ijk}x^k\,\mathrm{sech}^2(\chi)\left(\dot{a}^2 + \tanh^2(\chi)\right), \tag{5.14}$$

this expression vanishes for both $\chi = \pm\infty$.

6

Conclusions and Outlook

The main purpose of our endeavor was to generalize Majid's formalism to any space-time of the lie-algebraic type. Since it relies extensively on the universal differential calculus that is chosen along with the centrality condition on the line element, we used the latter in addition with some symmetry considerations to fix our differential structure.

This differs from the usual procedure, and in our opinion it offers a consistent method that renders the differential calculus instead of choosing it. From this we could generalize the Koszul formula for any Lie-algebra associated to a microscopic space-time, expanded up to first order in Planck's length, that it is to be deemed as a deformation parameter that encodes the non-commutativity features of our depiction.

The quantum-corrected connection is algebra-valued, this gives origin to several paths towards building geometrical entities. However we choose the simplest one that will allow us to recover a classical limit: perform an ontological and mathematical analogy of each concept we want to generalize into the non-commutative realm just by considering it to be algebra-valued; this is to be taken as a definition that will always make contact with the classical limit. Research upon clear and meaningful physical scenarios should shed some light upon whether it is correct or not.

In this sense, we are claiming that most of the relevant data of our non-commutative space-time has to be in the connection. Clearly this disregards some aspects such as topological considerations or global properties. The study of this features is an active research subject, and most of this efforts are being driven through K-theory. However, the connection is not able to encode global properties, and this should always be considered through our work; this was to be expected of course given that non-commutative geometry is a generalization of a local description. It may be argued that it is non-local, this can de readily seen from the commutators of coordinates. Although this is true, that does not implies that the resulting theory is going to have global properties of the space.

The examples afore-mentioned serve several purposes. Although the conformal cases do not take into account the symmetries of the classical space-time, they contain a rather interesting result: discretization of the conformal factors of the exponential function within the metric (see Eq. **??**), which could be interpreted as the introduction of a minimal length that depends on the structure constants of the algebra, among other interpretations. However we regard this case as a toy model because it is not sensitive to the symmetries of the classical space-time.

To fulfill the symmetry requirements of our program we present the FRW space-time along an $SO(3)$ algebra for our space-time, since this algebra preserves the rotational symmetry inherent to the model; furthermore its Casimir, r is thoroughly used in all the calculations. One of the main results is the appearance of an anomaly, namely of the Lorentz kind (c.f. [Ber96, p. 508-509]) which is related to the antisymmetry of the stress-energy tensor in the quantum treatment of general relativity. Such anomaly usually occurs in the context of calculating the expectation value for a given stress-energy tensor, via the path integral using a Liouville-measure; in other words, it is a phenomena related to the quantum behavior of matter.

However, in Eq. (5.9) it can be seen that this anomaly comes from the non-commutative effects, i.e. from the quantum behavior of geometry, which is a new result. In this case it is mandatory that we get rid of the anomaly, which demands that $K = 0$ as it is proven in both (5.13) and (5.14) by exploring $K = \pm$. This matches well-known experimental results for the geometry of the observable universe [Per99].

The emergence of the anomaly is crucial. without a doubt is a quantum feature, however its origin is not clear. It may arise due to our definitions regarding curvature, where we basically mimic the classical entities but considering that they are now algebra valued. From Eq. (4.11) it can be seen that although the connection contains no quantum corrections, the Riemann tensor will still carry quantum contributions that come directly from the universal differential calculus we are considering.

This might be related to an idea posed by Connes and Chamseddine in [CC12] where they use non-commutative geometry also in the context of an FRW space-time and the calculate the spectral action associated to the Dirac operator. They recover the usual Einstein-Hilbert action plus some other terms: a constant a_0 that is linked to dark energy, a_2 the Einstein-Hilbert term and additional contributions a_{2n} with $n > 1$ that account for non-commutative effects. It is yet to be seen if we can produce such terms from the Riemann and Ricci tensors we get in our FRW space-time treatment.

This is what happens in our FRW space-time, where the anomaly comes purely from the differential calculus in our non-commutative algebra. This clearly matches the usual result in the classical limit, however it may be argued that another definition could be used for the curvature and Einstein tensors.

For the moment we are dealing with other classical space-times with well established symmetries, since applying our framework is an straightforward procedure, a remarkable example among our research is the Schwarszchild space-time. Also, we are working on the non-commutative sphere, where the representations are known and we can make a direct comparison between our method and calculating the expectation value from the metric and then doing geometry with Kac's quantum differential calculus [KC15].

However, if we want to proceed in the same philosophical track of General Relativity, we need to define an action functional. Connes has achieved this, however in our framework it is quite challenging to calculate the Dirac operator spectra, which is essential to write down Connes' spectral action for gravity. We could also stop at a middle point proposing an alternative action, however most of the evidence points to the fact that instead of using integration we would need to use the Dixmier trace, which relies extensively on representation theory, which, for our differential forms is yet to be explored. Or, if we do not want to build an action principle an proceed just by geometric means, then we should find the non-commutative generalizations of Bianchi identities, and find out if they imply the existence of a divergence-less entity; this entity would be deemed as the non-commutative Einstein tensor that we should use as one side of the quantum Einstein equations.

Furthermore, another interesting branch of research lies within direct exploitation of Koszul's formula to obtain higher-order corrections, and in some cases an exact result. This would provide the full quantum theory of geometry and thus a fundamental step towards quantum gravity.

A

Appendix

All the lengthy proofs are enclosed within this section.

Proof of Proposition 4.10

Proof. The Koszul formula for the connection involves the classical connection and a generalized braiding acting upon it

$$\nabla(dx^{\mu}) = \frac{1}{2}\,{}_0\nabla(dx^{\mu}) + \frac{1}{2}\sigma({}_0\nabla(dx^{\mu})) = -\frac{1}{2}\Gamma^{\mu}_{\rho\sigma}dx^{\rho}\otimes dx^{\sigma} - \frac{1}{2}\sigma(\Gamma^{\mu}_{\rho\sigma}dx^{\rho}\otimes dx^{\sigma}),$$

for the sake of simplicity we are going to focus on the second term, which is precisely the generalized braiding 4.5

$$\sigma(\Gamma^{\mu}_{\rho\sigma}dx^{\rho}\otimes dx^{\sigma}) = dx^{\sigma}\Gamma^{\mu}_{\rho\sigma}\otimes dx^{\rho} + [x^{\sigma},\,{}_0\nabla(\Gamma^{\mu}_{\rho\sigma}dx^{\rho})] + {}_0\nabla[\Gamma^{\mu}_{\rho\sigma}dx^{\rho}, x^{\sigma}],$$

now we make use of the third expression in Equations (4.12) to pull $\Gamma^{\mu}_{\rho\sigma}$ through dx^{σ}

$$\begin{aligned}
=&\Gamma^{\mu}_{\rho\sigma}dx^{\sigma}\otimes dx^{\rho} + \Sigma^{\sigma\mu}{}_{\rho\sigma\lambda}dx^{\lambda}\otimes dx^{\rho} + [x^{\sigma}, d(\Gamma^{\mu}_{\rho\sigma})\otimes dx^{\rho} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\rho}_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta}]\\
&+ {}_0\nabla(\Gamma^{\mu}_{\rho\sigma}D^{\rho\sigma}{}_{\lambda}dx^{\lambda} + [\Gamma^{\mu}_{\rho\sigma}, x^{\sigma}]dx^{\rho})\\
=&(\Gamma^{\mu}_{\sigma\rho} + \Sigma^{\lambda\mu}{}_{\sigma\lambda\rho})dx^{\rho}\otimes dx^{\sigma} + [x^{\sigma}, d(\Gamma^{\mu}_{\rho\sigma})\otimes dx^{\rho}] - [x^{\sigma}, \Gamma^{\mu}_{\rho\sigma}\Gamma^{\rho}_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta}]\\
&+ D^{\rho\sigma}{}_{\lambda}(d(\Gamma^{\mu}_{\rho\sigma})\otimes dx^{\lambda} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\lambda}_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta}) + d([\Gamma^{\mu}_{\rho\sigma}, x^{\sigma}])\otimes dx^{\rho}\\
&- [\Gamma^{\mu}_{\rho\sigma}, x^{\sigma}]\Gamma^{\rho}_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta},
\end{aligned}$$

where the last four terms come from applying the covariant derivative keeping in mind the Leibniz rule and the difference between operating it on a zero-form and a one-form. Our calculation of the generalized braiding becomes

$$
\begin{aligned}
=&(\Gamma^{\mu}_{\sigma\rho}+\Sigma^{\lambda\mu}{}_{\sigma\lambda\rho})dx^{\rho}\otimes dx^{\sigma}-d(\Gamma^{\mu}_{\rho\sigma})D^{\rho\sigma}{}_{\lambda}dx^{\lambda}+[x^{\sigma},d(\Gamma^{\mu}_{\rho\sigma})]\otimes dx^{\rho}\\
&-[x^{\sigma},\Gamma^{\mu}_{\rho\sigma}]\Gamma^{\rho}_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta}-\Gamma^{\mu}_{\rho\sigma}[x^{\sigma},\Gamma^{\rho}_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta}]\\
&+D^{\rho\sigma}{}_{\lambda}(d(\Gamma^{\mu}_{\rho\sigma})\otimes dx^{\lambda}-\Gamma^{\mu}_{\rho\sigma}\Gamma^{\lambda}_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta})+[d(\Gamma^{\mu}_{\rho\sigma}),x^{\sigma}]\otimes dx^{\rho}\\
&+\underbrace{[\Gamma^{\mu}_{\rho\sigma},dx^{\sigma}]\otimes dx^{\rho}}_{=-\Sigma^{\sigma\mu}{}_{\rho\sigma\lambda}dx^{\lambda}}-[\Gamma^{\mu}_{\rho\sigma},x^{\sigma}]\Gamma^{\rho}_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta},
\end{aligned}
$$

after canceling out some terms we get the following expression

$$
=(\Gamma^{\mu}_{\sigma\rho}-\Gamma^{\mu}_{\alpha\beta}[x^{\beta},\Gamma^{\alpha}_{\rho\sigma}dx^{\rho}\otimes dx^{\sigma}]-D^{\alpha\beta}{}_{\lambda}\Gamma^{\mu}_{\alpha\beta}\Gamma^{\lambda}_{\rho\sigma}dx^{\rho}\otimes dx^{\sigma}
$$

the remaining commutator renders two terms, one of which results in $[x^{\beta},dx^{\rho}\otimes dx^{\sigma}]=-(D^{\rho\beta}{}_{\lambda}dx^{\lambda}\otimes dx^{\sigma}+D^{\sigma\beta}{}_{\lambda}dx^{\rho}\otimes dx^{\lambda})$, if we rearrange indices in such a way the two form basis is a common factor for the whole expression we arrive to the final result for the generalized braiding

$$
\begin{aligned}
\sigma(\Gamma^{\mu}_{\rho\sigma}dx^{\rho}\otimes dx^{\sigma})=\Big(&\Gamma^{\mu}_{\sigma\rho}+\Gamma^{\mu}_{\alpha\beta}(D^{\lambda\beta}{}_{\rho}\Gamma^{\alpha}_{\lambda\sigma}+D^{\lambda\beta}{}_{\sigma}\Gamma^{\alpha}_{\rho\lambda}-D^{\alpha\beta}{}_{\lambda}\Gamma^{\lambda}_{\rho\sigma})\\
&-\Gamma^{\mu}_{\alpha\beta}[x^{\beta},\Gamma^{\alpha}_{\rho\sigma}]\Big)dx^{\rho}\otimes dx^{\sigma}.
\end{aligned}
$$

Therefore, the quantum-corrected connection is

$$
\begin{aligned}
&=-\frac{1}{2}\Gamma^{\mu}_{\rho\sigma}dx^{\rho}\otimes dx^{\sigma}-\frac{1}{2}(\Gamma^{\mu}_{\sigma\rho}+\Gamma^{\mu}_{\alpha\beta}(D^{\lambda\beta}{}_{\rho}\Gamma^{\alpha}_{\lambda\sigma}+D^{\lambda\beta}{}_{\sigma}\Gamma^{\alpha}_{\rho\lambda}-D^{\alpha\beta}{}_{\lambda}\Gamma^{\lambda}_{\rho\sigma})\\
&\quad-\Gamma^{\mu}_{\alpha\beta}[x^{\beta},\Gamma^{\alpha}_{\rho\sigma}])dx^{\rho}\otimes dx^{\sigma}\\
\nabla(dx^{\mu})&=-\frac{1}{2}(\Gamma^{\mu}_{\rho\sigma}+\Gamma^{\mu}_{\sigma\rho})dx^{\rho}\otimes dx^{\sigma}-\frac{1}{2}(\Gamma^{\mu}_{\alpha\beta}(D^{\lambda\beta}{}_{\rho}\Gamma^{\alpha}_{\lambda\sigma}+D^{\lambda\beta}{}_{\sigma}\Gamma^{\alpha}_{\rho\lambda}-D^{\alpha\beta}{}_{\lambda}\Gamma^{\lambda}_{\rho\sigma})\\
&\quad-\Gamma^{\mu}_{\alpha\beta}[x^{\beta},\Gamma^{\alpha}_{\rho\sigma}])dx^{\rho}\otimes dx^{\sigma},
\end{aligned}
$$

Proof of Lemma ??

Proof. Please note that

$$
[x,t]=x\Lambda=t\tilde{\Lambda}
$$

We consider the commutator and expand it, our goal is to commute n times so we can cancel the tx^{n} term, the first step gives

$$
\begin{aligned}
[t,x^{n}]=&tx^{n}-x^{n}t\\
=&tx^{n}-x^{n-1}tx-x^{n}\Lambda,
\end{aligned}
$$

while the second yields the following,

$$
\begin{aligned}
=&tx^{n}-x^{n-2}tx^{2}-x^{n-1}\Lambda x-x^{n}\Lambda\\
=&tx^{n}-x^{n-2}tx^{2}-x^{n-1}(x-\beta)\Lambda-x^{n}\Lambda,
\end{aligned}
$$

a pattern emerges when we analyze the third step, which leads us to

$$
\begin{aligned}
=&tx^n - x^{n-3}tx^3 - x^{n-2}\Lambda x^2 - x^{n-1}(x-\beta)\Lambda - x^n\Lambda \\
=&tx^n - x^{n-3}tx^3 - x^{n-2}(x-\beta)^2\Lambda - x^{n-1}(x-\beta)\Lambda - x^n\Lambda \\
&\vdots \\
=&-x(x-\beta)^{n-1}\Lambda - x^2(x-\beta)^{n-1\Lambda} - \cdots - x^{n-2}(x-\beta)^2\Lambda - x^{n-1}(x-\beta)\Lambda,
\end{aligned}
$$

summarizing the line above leads us to

$$
\begin{aligned}
=&-\left(\sum_{k=1}^{n} x^k(x-\beta)^{n-k}\right)\Lambda = -(x-\beta)^n\left(\sum_{k=1}^{n} x^k(x-\beta)^{-k}\right)\Lambda \\
=&-(x-\beta)^n\left(\sum_{k=0}^{n} x^k(x-\beta)^{-k} - 1\right)\Lambda.
\end{aligned}
$$

Now we make use of the closed form of the geometric series and factorize to the right a term

$$
\begin{aligned}
[t, x^n] =&-(x-\beta)^n\Big([1 - x^{n+1}(x-\beta)^{-(n+1)}][1 - x(x-\beta)^{-1}]^{-1} - 1\Big)\Lambda \\
=&-(x-\beta)^n\Big([1 - x^{n+1}(x-\beta)^{-(n+1)}] - [1 - x(x-\beta)^{-1}]\Big)[1 - x(x-\beta)^{-1}]^{-1}\Lambda,
\end{aligned}
$$

note that the term we isolated is

$$
1 - x(x-\beta)^{-1} = (x - \beta - x)(x-\beta)^{-1} = -\beta(x-\beta)^{-1},
$$

consequently we obtain

$$
\begin{aligned}
[t, x^n] =&\beta^{-1}(x-\beta)^n\Big(-x^{n+1}(x-\beta)^{-(n+1)} + x(x-\beta)^{-1}\Big)(x-\beta)\Lambda \\
=&\frac{x}{\beta}(x-\beta)^n\left(-x^n(x-\beta)^{-n} + 1\right)\Lambda \\
=&\frac{x}{\beta}\left((x-\beta)^n - x^n\right)\Lambda
\end{aligned}
$$

The procedure for the second commutator is the same.

Proof of Proposition 5.8

Proof. The centrality conditions, given in Equation (4.4) for the generators of the algebra are given by the following equations

$$
\begin{aligned}
\eta_{\mu\nu}[t, e^{a_1 x}] =&e^{a_1 x}(D^{0\kappa}{}_{\mu}\eta_{\kappa\nu} + D^{0\kappa}{}_{\nu}\eta_{\kappa\mu}) \\
\eta_{\mu\nu}[x, e^{a_0 t}] =&e^{a_0 t}(D^{1\kappa}{}_{\mu}\eta_{\kappa\nu} + D^{1\kappa}{}_{\nu}\eta_{\kappa\mu})
\end{aligned}
$$

it is more easy to perform the calculations in matrix form, which is

$$e^{a_1 x}\begin{pmatrix} -2D^{00}{}_0 & D^{01}{}_0 - D^{00}{}_1 \\ D^{01}{}_0 - D^{00}{}_1 & 2D^{01}{}_1 \end{pmatrix} = -\frac{2}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.1}$$

$$e^{a_0 t}\begin{pmatrix} -2D^{10}{}_0 & D^{11}{}_0 - D^{10}{}_1 \\ D^{11}{}_0 - D^{10}{}_1 & 2D^{11}{}_1 \end{pmatrix} = \frac{2}{\alpha}e^{a_0(t-\alpha/2)}\sinh\left(\frac{a_0\alpha}{2}\right)(\alpha x+\beta t)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.2}$$

For the fist commutator A.1, we take into account that $D^{00}{}_0 = D^{01}{}_1$ we equal both to their r.h.s, which promptly yields the following relation

$$D^{01}{}_1 = D^{00}{}_0 = -\frac{1}{\beta}e^{-a_1\beta/2}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t),$$

for $D^{00}{}_0$ we know that $C^{00}{}_0$, thus

$$S^{00}{}_0 = -\frac{1}{\beta}e^{-a_1\beta/2}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t),$$

while for $D^{01}{}_1$ the structure constant in it is $C^{01}{}_1/2 = -\alpha/2$, we pass it to the r.h.s

$$S^{01}{}_1 = \frac{\alpha}{2} - \frac{1}{\beta}e^{-a_1\beta/2}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t),$$

proceeding analogously for A.2 we get similar results

$$\begin{aligned} D^{10}{}_0 = D^{11}{}_1 &= \frac{1}{\alpha}e^{-a_0\alpha/2}\sinh\left(\frac{a_0\alpha}{2}\right)(\alpha x+\beta t) \\ S^{11}{}_1 &= \frac{1}{\alpha}e^{-a_0\alpha/2}\sinh\left(\frac{a_0\alpha}{2}\right)(\alpha x+\beta t) \\ S^{10}{}_0 &= -\frac{\beta}{2} + \frac{1}{\alpha}e^{-a_0\alpha/2}\sinh\left(\frac{a_0\alpha}{2}\right)(\alpha x+\beta t). \end{aligned}$$

So far we have only considered the diagonal terms for both expressions, from the non-diagonal part we realize that $D^{01}{}_0 = D^{00}{}_1$ and $D^{11}{}_0 = D^{10}{}_1$. Given that both $C^{00}{}_1 = 0$ and $C^{11}{}_0$ we get

$$D^{01}{}_0 = D^{00}{}_1 \Rightarrow -\frac{\beta}{2} + S^{01}{}_0 = S^{00}{}_1 \Rightarrow S^{00}{}_1 = -\beta + \frac{1}{\alpha}e^{-a_0\alpha/2}\sinh\left(\frac{a_0\alpha}{2}\right)(\alpha x+\beta t)$$

$$D^{11}{}_0 = D^{10}{}_1 \Rightarrow S^{11}{}_0 = \frac{\alpha}{2} + S^{10}{}_1 \Rightarrow S^{11}{}_0 = \alpha - \frac{1}{\beta}e^{-a_1\beta/2}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t).$$

We need the S to be constant so it will satisfy the Jacobi identities cite the stability stuff, this is easily fulfilled if we note that $\alpha, \beta \in \mathbb{C}/\mathbb{R}$ so we can write them as $\alpha = i\tilde{\alpha}, \quad \beta = i\tilde{\beta}$; due to this, the hyperbolic sine turns out to be a sine given the known relation $\sinh(ix) = i\sin(x)$. Then it is evident that the following holds for $n, m \in \mathbb{Z}$

$$a_0 = \frac{n}{\tilde{\alpha}}\pi \quad a_1 = \frac{m}{\tilde{\beta}}\pi.$$

Therefore, the symmetric part of the deformation of the differential structure is

$$S^{00}{}_0 = 0, \quad S^{00}{}_1 = -\beta, \quad S^{10}{}_0 = -\frac{\beta}{2}, \quad S^{01}{}_1 = \frac{\alpha}{2}, \quad S^{11}{}_0 = \alpha, \quad S^{11}{}_1 = 0.$$

Which leads us to

$$\begin{aligned} &D^{00}{}_0 = 0, \ D^{00}{}_1 = -\beta, \ D^{01}{}_0 = -\beta, \ D^{10}{}_0 = 0, \\ &D^{01}{}_1 = 0, \ D^{10}{}_1 = \alpha, \ D^{11}{}_0 = \alpha, \ D^{11}{}_1 = 0. \end{aligned}$$

Proof of Proposition 5.12

Proof. The centrality condition splits into three cases, the first with the temporal generator, the second with the spatial generator and the third with the commutative generators

$$\begin{aligned} \eta_{\mu\nu}[t, e^{a_1 x}] =& e^{a_1 x}(D^{0\kappa}{}_\mu \eta_{\kappa\nu} + D^{0\kappa}{}_\nu \eta_{\kappa\mu}) \\ \eta_{\mu\nu}[x, e^{a_0 t}] =& e^{a_0 t}(D^{1\kappa}{}_\mu \eta_{\kappa\nu} + D^{1\kappa}{}_\nu \eta_{\kappa\mu}) \\ 0 =& D^{\hat{i}\kappa}{}_\mu \eta_{\kappa\nu} + D^{\hat{i}\kappa}{}_\nu \eta_{\kappa\mu} \end{aligned}$$

The first two equations in matrix notation are

$$\begin{aligned} &e^{a_1 x}\begin{pmatrix} -2D^{00}{}_0 & D^{01}{}_0 - D^{00}{}_1 & D^{02}{}_0 - D^{00}{}_2 & D^{03}{}_0 - D^{00}{}_3 \\ D^{01}{}_0 - D^{00}{}_1 & 2D^{01}{}_1 & D^{02}{}_1 + D^{01}{}_2 & D^{03}{}_1 + D^{01}{}_3 \\ D^{02}{}_0 - D^{00}{}_2 & D^{01}{}_2 + D^{02}{}_1 & 2D^{02}{}_2 & D^{03}{}_2 + D^{02}{}_3 \\ D^{03}{}_0 - D^{00}{}_3 & D^{03}{}_1 + D^{01}{}_3 & D^{03}{}_2 + D^{02}{}_3 & 2D^{03}{}_3 \end{pmatrix} \\ =& -\frac{2}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x + \beta t)\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ &e^{a_0 t}\begin{pmatrix} -2D^{10}{}_0 & D^{11}{}_0 - D^{10}{}_1 & D^{12}{}_0 - D^{10}{}_2 & D^{13}{}_0 - D^{10}{}_3 \\ D^{11}{}_0 - D^{10}{}_1 & 2D^{11}{}_1 & D^{12}{}_1 + D^{11}{}_2 & D^{13}{}_1 + D^{11}{}_3 \\ D^{12}{}_0 - D^{10}{}_2 & D^{12}{}_1 + D^{11}{}_2 & 2D^{12}{}_2 & D^{12}{}_3 + D^{13}{}_2 \\ D^{13}{}_0 - D^{10}{}_3 & D^{13}{}_1 + D^{11}{}_3 & D^{12}{}_3 + D^{13}{}_2 & 2D^{13}{}_3 \end{pmatrix} \\ =& \frac{2}{\alpha}e^{a_0(t-\alpha/2)}\sinh\left(\frac{a_0\alpha}{2}\right)(\alpha x + \beta t)\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

which generalizes the two-dimensional case in the following fashion, for the diagonal we have

$$
\begin{aligned}
&D^{00}{}_0 = D^{01}{}_1 = D^{02}{}_2 = D^{03}{}_3 = -\frac{1}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t)\\
&\Rightarrow S^{00}{}_0 = -\frac{1}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t)\\
&\Rightarrow S^{01}{}_1 = \frac{\alpha}{2}-\frac{1}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t)\\
&\Rightarrow S^{02}{}_2 = -\frac{1}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t)\\
&\Rightarrow S^{03}{}_3 = -\frac{1}{\beta}e^{a_1(x-\beta/2)}\sinh\left(\frac{a_1\beta}{2}\right)(\alpha x+\beta t)
\end{aligned}
$$

as in the last case, we require S to be a constant, we get the same conditions for a_0 and a_1 and end up with

$$
\begin{aligned}
&D^{00}{}_0 = D^{01}{}_1 = D^{02}{}_2 = \cdots = 0\\
\Rightarrow &S^{00}{}_0 = 0 \quad S^{01}{}_1 = \frac{\alpha}{2} \quad S^{0\hat{i}}{}_{\hat{i}} = 0 \quad \text{(for fixed } \hat{i})\\
\Rightarrow &D^{00}{}_0 = 0 \quad D^{01}{}_1 = 0 \quad D^{0\hat{i}}{}_{\hat{i}} = 0 \quad \text{(for fixed } \hat{i})\\
\Rightarrow &D^{10}{}_1 = \alpha = D^{11}{}_0.
\end{aligned}
$$

For the second centrality condition we have

$$
\begin{aligned}
&D^{10}{}_0 = D^{11}{}_1 = D^{12}{}_2 = D^{13}{}_3 = 0\\
\Rightarrow &S^{10}{}_0 = -\frac{\beta}{2} \quad S^{11}{}_1 = 0 \quad S^{1\hat{i}}{}_{\hat{i}} = 0 \quad \text{(for fixed } \hat{i})\\
\Rightarrow &D^{10}{}_0 = 0 \quad D^{11}{}_1 = 0 \quad D^{1\hat{i}}{}_{\hat{i}} = 0 \quad \text{(for fixed } \hat{i})\\
\Rightarrow &D^{01}{}_0 = -\beta = D^{00}{}_1,
\end{aligned}
$$

and for the third

$$0 = D^{\hat{i}\mu}{}_\mu \Rightarrow 0 = S^{\hat{i}\mu}{}_\mu \text{(for fixed } \mu)$$

For the off-diagonal terms we have for the first matrix in the mixed part

$$D^{0i}{}_0 = D^{00}{}_i \Rightarrow \frac{1}{2}C^{0i}{}_0 + S^{0i}{}_0 = S^{00}{}_i \Rightarrow -\beta\delta_{1i} = S^{00}{}_i \Rightarrow D^{00}{}_i = -\beta\delta_{1i} = D^{0i}{}_0,$$

for the purely spatial part

$$0 = D^{0i}{}_j + D^{0j}{}_i \Rightarrow D^{0i}{}_1 = -D^{01}{}_i \quad \text{and } D^{0i}{}_{\hat{j}} = -D^{0\hat{j}}{}_i.$$

which gives us four cases

$$D^{01}{}_{1} = -D^{01}{}_{1} = 0 \quad D^{01}{}_{\hat{j}} = -D^{0\hat{j}}{}_{1} := \gamma_{\hat{j}}$$

and

$$D^{01}{}_{\hat{j}} = -D^{0\hat{j}}{}_{1} = \gamma_{\hat{j}} \quad D^{0\hat{i}}{}_{\hat{j}} = -D^{0\hat{j}}{}_{\hat{i}} := \phi^{\hat{j}}_{\hat{i}}$$

while for the second

$$D^{1i}{}_{0} = D^{10}{}_{i} \;\; \Rightarrow \;\; S^{11}{}_{0} = \alpha \quad \text{and} \quad D^{1\hat{i}}{}_{0} = D^{10}{}_{\hat{i}} =: \gamma_{\hat{i}},$$

and its spatial part

$$0 = D^{1i}{}_{j} + D^{1j}{}_{i} \;\; \Rightarrow D^{11}{}_{j} = -D^{1j}{}_{1} \quad \text{and} \quad D^{1\hat{i}}{}_{j} = -D^{1j}{}_{\hat{i}},$$

analogously to the last case, four sub-cases arise

$$D^{11}{}_{1} = -D^{11}{}_{1} = 0 \quad D^{11}{}_{\hat{j}} = -D^{1\hat{j}}{}_{1} =: \xi_{\hat{j}}$$

and

$$D^{1\hat{i}}{}_{1} = -D^{11}{}_{\hat{i}} = \xi_{\hat{i}} \quad D^{1\hat{i}}{}_{\hat{j}} = -D^{1\hat{j}}{}_{\hat{i}} = \chi_{ij}$$

for the third we have the following cases for the off-diagonal terms

$$0 = D^{\hat{i}j}{}_{0} - D^{\hat{i}0}{}_{j} \Rightarrow D^{\hat{i}0}{}_{1} = D^{\hat{i}1}{}_{0} =: \eta_{\hat{i}} \quad \text{and} \quad D^{\hat{i}0}{}_{\hat{j}} = D^{\hat{i}\hat{j}}{}_{0} =: \theta_{\hat{i}\hat{j}}$$

also for the purely spatial part we have for $\ell \neq j$

Proof of the first part of Proposition ??

Proof. As in the last case, we consider the general formula

$$\tilde{\Gamma}^{\mu}_{\rho\sigma} = {}_0\Gamma^{\mu}_{\rho\sigma} + \frac{1}{2}\, {}_0\Gamma^{\mu}_{\alpha\beta}(D^{\lambda\beta}{}_{\rho}\, {}_0\Gamma^{\alpha}_{\lambda\sigma} + D^{\lambda\beta}{}_{\sigma}\, {}_0\Gamma^{\alpha}_{\rho\lambda} - D^{\alpha\beta}{}_{\lambda}\, {}_0\Gamma^{\lambda}_{\rho\sigma}).$$

And highlight once more that the last term is not present because in our case due to the fact $[x^{\alpha},\ {}_0\Gamma^{\beta}_{\rho\sigma}] = 0$. Allow us to study the general case

$$
\begin{aligned}
\tilde{\Gamma}^{\mu}_{\rho\sigma} = {}_0\Gamma^{\mu}_{\rho\sigma} &+ \frac{{}_0\Gamma^{\mu}_{00}}{2}\Big(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{0}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{0}_{\lambda\rho} - \underbrace{D^{00}{}_{1}}_{=-\beta}\,{}_0\Gamma^{1}_{\rho\sigma}\Big) + \frac{{}_0\Gamma^{\mu}_{01}}{2}\Big(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{0}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{0}_{\rho\lambda} - \underbrace{D^{01}{}_{0}}_{=-\beta}\,{}_0\Gamma^{0}_{\rho\sigma}\Big) \\
&+ \frac{{}_0\Gamma^{\mu}_{0\hat{i}}}{2}\Big(\underbrace{D^{\lambda\hat{i}}{}_{\rho}}_{=0}\,{}_0\Gamma^{0}_{\lambda\sigma} + \underbrace{D^{\lambda\hat{i}}{}_{\sigma}}_{=0}\,{}_0\Gamma^{0}_{\rho\lambda} - \underbrace{D^{0\hat{i}}{}_{k}}_{=0}\,{}_0\Gamma^{k}_{\rho\sigma}\Big) + \frac{{}_0\Gamma^{\mu}_{10}}{2}\Big(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{1}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{1}_{\rho\lambda} - \underbrace{D^{10}{}_{1}}_{=\alpha}\,{}_0\Gamma^{1}_{\rho\sigma}\Big) \\
&+ \frac{{}_0\Gamma^{\mu}_{\hat{i}0}}{2}\Big(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{\hat{i}}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{\hat{i}}_{\rho\lambda} - \underbrace{D^{\hat{i}0}{}_{k}}_{=0}\,{}_0\Gamma^{k}_{\rho\sigma}\Big) + \frac{{}_0\Gamma^{\mu}_{11}}{2}\Big(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{1}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{1}_{\rho\lambda} - \underbrace{D^{11}{}_{0}}_{=\alpha}\,{}_0\Gamma^{0}_{\rho\sigma}\Big) \\
&+ \frac{{}_0\Gamma^{\mu}_{1\hat{i}}}{2}\Big(\underbrace{D^{\lambda\hat{i}}{}_{\rho}}_{=0}\,{}_0\Gamma^{1}_{\lambda\sigma} + \underbrace{D^{\lambda\hat{i}}{}_{\sigma}}_{=0}\,{}_0\Gamma^{1}_{\rho\lambda} - \underbrace{D^{1\hat{i}}{}_{\lambda}}_{=0}\,{}_0\Gamma^{\lambda}_{\rho\sigma}\Big) + \frac{{}_0\Gamma^{\mu}_{\hat{i}1}}{2}\Big(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{\hat{i}}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{\hat{i}}_{\rho\lambda} - \underbrace{D^{\hat{i}1}{}_{\lambda}}_{=0}\,{}_0\Gamma^{\lambda}_{\rho\sigma}\Big) \\
&+ \frac{{}_0\Gamma^{\mu}_{\hat{i}\hat{j}}}{2}\Big(\underbrace{D^{\lambda\hat{j}}{}_{\rho}}_{=0}\,{}_0\Gamma^{\alpha}_{\lambda\sigma} + \underbrace{D^{\lambda\hat{j}}{}_{\sigma}}_{=0}\,{}_0\Gamma^{\alpha}_{\rho\lambda} - \underbrace{D^{\hat{i}\hat{j}}{}_{\lambda}}_{=0}\,{}_0\Gamma^{\lambda}_{\rho\sigma}\Big),
\end{aligned}
$$

which ends up in

$$
\begin{aligned}
\tilde{\Gamma}^{\mu}_{\rho\sigma} = {}_0\Gamma^{\mu}_{\rho\sigma} &+ \frac{{}_0\Gamma^{\mu}_{00}}{2}\Big(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{0}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{0}_{\lambda\rho} + \beta\,{}_0\Gamma^{1}_{\rho\sigma}\Big) + \frac{{}_0\Gamma^{\mu}_{01}}{2}\Big(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{0}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{0}_{\rho\lambda} + \beta\,{}_0\Gamma^{0}_{\rho\sigma}\Big) \\
&+ \frac{{}_0\Gamma^{\mu}_{10}}{2}\Big(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{1}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{1}_{\rho\lambda} - \alpha\,{}_0\Gamma^{1}_{\rho\sigma}\Big) + \frac{{}_0\Gamma^{\mu}_{\hat{i}0}}{2}\Big(D^{\lambda 0}{}_{\rho}\,{}_0\Gamma^{\hat{i}}_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma}\,{}_0\Gamma^{\hat{i}}_{\rho\lambda}\Big) \\
&+ \frac{{}_0\Gamma^{\mu}_{11}}{2}\Big(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{1}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{1}_{\rho\lambda} - \alpha\,{}_0\Gamma^{0}_{\rho\sigma}\Big) + \frac{{}_0\Gamma^{\mu}_{\hat{i}1}}{2}\Big(D^{\lambda 1}{}_{\rho}\,{}_0\Gamma^{\hat{i}}_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma}\,{}_0\Gamma^{\hat{i}}_{\rho\lambda}\Big),
\end{aligned}
$$

before proceeding we shall write down the form of the $D^{\lambda\mu}{}_{\rho}$ matrices with μ fixed

$$
D^{\lambda 0}{}_{\rho} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D^{\lambda 1}{}_{\rho} = \begin{pmatrix} -\beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

as in the last case we shall study each term of the sum. The first one being

$$
\begin{aligned}
&D^{\lambda 0}{}_{\rho\,0}\Gamma^0_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma\,0}\Gamma^0_{\lambda\rho} + \beta\,{}_0\Gamma^1_{\rho\sigma} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha a_1 - \beta a_0 & \alpha a_0 - \beta a_1 & -\beta a_2 & -\beta a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \alpha a_1 - \beta a_0 & 0 & 0 \\ 0 & \alpha a_0 - \beta a_1 & 0 & 0 \\ 0 & -\beta a_2 & 0 & 0 \\ 0 & -\beta a_3 & 0 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & \alpha a_1 - \beta a_0 & 0 & 0 \\ \alpha a_1 - \beta a_0 & 2\alpha a_0 - 2\beta a_1 & -\beta a_2 & -\beta a_3 \\ 0 & -\beta a_2 & 0 & 0 \\ 0 & -\beta a_3 & 0 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix} \\
&= \frac{a_1}{2}\begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix},
\end{aligned}
$$

while the second renders

$$
\begin{aligned}
&D^{\lambda 1}{}_{\rho\,0}\Gamma^0_{\lambda\sigma} + D^{\lambda 1}{}_{\sigma\,0}\Gamma^0_{\rho\lambda} + \beta\,{}_0\Gamma^0_{\rho\sigma} \\
&= \frac{1}{2}\begin{pmatrix} \alpha a_1 - \beta a_0 & \alpha a_0 - \beta a_1 & -\beta a_2 & -\beta a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \alpha a_1 - \beta a_0 & 0 & 0 & 0 \\ \alpha a_0 - \beta a_1 & 0 & 0 & 0 \\ -\beta a_2 & 0 & 0 & 0 \\ -\beta a_3 & 0 & 0 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 2\alpha a_1 - 2\beta a_0 & \alpha a_0 - \beta a_1 & -\beta a_2 & -\beta a_3 \\ \alpha a_0 - \beta a_1 & 0 & 0 & 0 \\ -\beta a_2 & 0 & 0 & 0 \\ -\beta a_3 & 0 & 0 & 0 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix} \\
&= \frac{a_0}{2}\begin{pmatrix} 2\alpha\frac{a_1}{a_0} - \beta & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}.
\end{aligned}
$$

The third is quite similar

$$
\begin{aligned}
&D^{\lambda 0}{}_{\rho\,0}\Gamma^1_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma\,0}\Gamma^1_{\rho\lambda} - \alpha\,{}_0\Gamma^1_{\rho\sigma} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha a_0 - \beta a_1 & \alpha a_1 - \beta a_0 & \alpha a_2 & \alpha a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \alpha a_0 - \beta a_1 & 0 & 0 \\ 0 & \alpha a_1 - \beta a_0 & 0 & 0 \\ 0 & \alpha a_2 & 0 & 0 \\ 0 & \alpha a_3 & 0 & 0 \end{pmatrix} - \frac{\alpha}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & \alpha a_0 - \beta a_1 & 0 & 0 \\ \alpha a_0 - \beta a_1 & 2\alpha a_1 - 2\beta a_0 & \alpha a_2 & \alpha a_3 \\ 0 & \alpha a_2 & 0 & 0 \\ 0 & \alpha a_3 & 0 & 0 \end{pmatrix} - \frac{\alpha}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix} \\
&= -\frac{a_1}{2}\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & 2\beta\frac{a_0}{a_1} - \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix},
\end{aligned}
$$

the fourth is simpler, but it splits in two cases

$$
\begin{aligned}
D^{\lambda 0}{}_{\rho\,0}\Gamma^2_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma\,0}\Gamma^2_{\rho\lambda} &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\beta a_2 & -\alpha a_2 & \alpha a_1 - \beta a_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -\beta a_2 & 0 & 0 \\ 0 & -\alpha a_2 & 0 & 0 \\ 0 & \alpha a_1 - \beta a_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= -\frac{a_2}{2}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ 0 & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
D^{\lambda 0}{}_{\rho\,0}\Gamma^3_{\lambda\sigma} + D^{\lambda 0}{}_{\sigma\,0}\Gamma^3_{\rho\lambda} &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\beta a_3 & -\alpha a_3 & 0 & \alpha a_1 - \beta a_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -\beta a_3 & 0 & 0 \\ 0 & -\alpha a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha a_1 - \beta a_0 & 0 & 0 \end{pmatrix} \\
&= -\frac{a_3}{2}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & 0 & \frac{\beta a_0 - \alpha a_1}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

For the fifth we proceed as usual

$$
\begin{aligned}
&D^{\lambda 1}{}_{\rho\,0}\Gamma^{1}_{\lambda\sigma}+D^{\lambda 1}{}_{\sigma\,0}\Gamma^{1}_{\rho\lambda}-\alpha\,{}_{0}\Gamma^{0}_{\rho\sigma}\\
=&\frac{1}{2}\begin{pmatrix}\alpha a_0-\beta a_1 & \alpha a_1-\beta a_0 & \alpha a_2 & \alpha a_3\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}\alpha a_0-\beta a_1 & 0&0&0\\ \alpha a_1-\beta a_0&0&0&0\\ \alpha a_2&0&0&0\\ \alpha a_3&0&0&0\end{pmatrix}-\frac{\alpha}{2}\begin{pmatrix}a_0&a_1&a_2&a_3\\ a_1&a_0&0&0\\ a_2&0&a_0&0\\ a_3&0&0&a_0\end{pmatrix}\\
=&\frac{1}{2}\begin{pmatrix}2\alpha a_0-2\beta a_1 & \alpha a_1-\beta a_0 & \alpha a_2 & \alpha a_3\\ \alpha a_1-\beta a_0&0&0&0\\ \alpha a_2&0&0&0\\ \alpha a_3&0&0&0\end{pmatrix}-\frac{\alpha}{2}\begin{pmatrix}a_0&a_1&a_2&a_3\\ a_1&a_0&0&0\\ a_2&0&a_0&0\\ a_3&0&0&a_0\end{pmatrix}\\
=&-\frac{a_0}{2}\begin{pmatrix}2\beta\frac{a_1}{a_0}-\alpha & \beta & 0 & 0\\ \beta&\alpha&0&0\\ 0&0&\alpha&0\\ 0&0&0&\alpha\end{pmatrix}.
\end{aligned}
$$

The sixth and last also contains two possibilities

$$
\begin{aligned}
D^{\lambda 1}{}_{\rho\,0}\Gamma^{2}_{\lambda\sigma}+D^{\lambda 1}{}_{\sigma\,0}\Gamma^{2}_{\rho\lambda}=&\frac{1}{2}\begin{pmatrix}-\beta a_2 & -\alpha a_2 & \alpha a_1-\beta a_0 & 0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-\beta a_2&0&0&0\\ -\alpha a_2&0&0&0\\ \alpha a_1-\beta a_0&0&0&0\\ 0&0&0&0\end{pmatrix}\\
=&-\frac{a_2}{2}\begin{pmatrix}2\beta & \alpha & \frac{\beta a_0-\alpha a_1}{a_2} & 0\\ \alpha&0&0&0\\ \frac{\beta a_0-\alpha a_1}{a_2}&0&0&0\\ 0&0&0&0\end{pmatrix}\\
D^{\lambda 1}{}_{\rho\,0}\Gamma^{3}_{\lambda\sigma}+D^{\lambda 1}{}_{\sigma\,0}\Gamma^{3}_{\rho\lambda}=&\frac{1}{2}\begin{pmatrix}-\beta a_3 & -\alpha a_3 & 0 & \alpha a_1-\beta a_0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-\beta a_3&0&0&0\\ -\alpha a_3&0&0&0\\ 0&0&0&0\\ \alpha a_1-\beta a_0&0&0&0\end{pmatrix}\\
=&-\frac{a_3}{2}\begin{pmatrix}2\beta & \alpha & 0 & \frac{\beta a_0-\alpha a_1}{a_3}\\ \alpha&0&0&0\\ 0&0&0&0\\ \frac{\beta a_0-\alpha a_1}{a_3}&0&0&0\end{pmatrix}.
\end{aligned}
$$

Now we may , for $\mu = 0$ we have

$$\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & 0 & 0 \\ a_2 & 0 & a_0 & 0 \\ a_3 & 0 & 0 & a_0 \end{pmatrix} + \frac{a_0 a_1}{8}\begin{pmatrix} 2\alpha\frac{a_1}{a_0} & 2\alpha & 0 & 0 \\ 2\alpha & 2\alpha\frac{a_0}{a_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_1)^2}{8}\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & 2\beta\frac{a_0}{a_1} - \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}$$

$$- \frac{(a_2)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ 0 & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_3)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & 0 & \frac{\beta a_0 - \alpha a_1}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 \end{pmatrix}$$

$$- \frac{(a_0)^2}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} - \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}.$$

For $\mu = 1$ we have

$$\tilde{\Gamma}^1_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_2 & -a_1 & 0 \\ 0 & a_3 & 0 & -a_1 \end{pmatrix} + \frac{(a_1)^2}{8}\begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix} + \frac{(a_0)^2}{8}\begin{pmatrix} 2\alpha\frac{a_1}{a_0} - \beta & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

$$- \frac{a_0 a_1}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} & 2\beta & 0 & 0 \\ 2\beta & 2\beta\frac{a_0}{a_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_2)^2}{8}\begin{pmatrix} 2\beta & \alpha & \frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ \alpha & 0 & 0 & 0 \\ \frac{\beta a_0 - \alpha a_1}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \frac{(a_3)^2}{8}\begin{pmatrix} 2\beta & \alpha & 0 & \frac{\beta a_0 - \alpha a_1}{a_3} \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 & 0 \end{pmatrix}.$$

For $\mu = 2$ we obtain

$$\tilde{\Gamma}^2_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_2 & 0 & a_0 & 0 \\ 0 & -a_2 & a_1 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_3 & -a_2 \end{pmatrix} + \frac{a_1 a_2}{8}\begin{pmatrix} -\beta & 0 & -\frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ 0 & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ -\frac{\beta a_0 - \alpha a_1}{a_2} & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix}$$

$$+ \frac{a_0 a_2}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} - \alpha & 0 & 0 & 0 \\ 0 & -\alpha & -\frac{\beta a_0 - \alpha a_1}{a_2} & 0 \\ 0 & -\frac{\beta a_0 - \alpha a_1}{a_2} & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

And finally, for $\mu = 3$ we reach

$$\tilde{\Gamma}^3_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a_0 \\ 0 & -a_3 & 0 & a_1 \\ 0 & 0 & -a_3 & a_2 \\ a_0 & a_1 & a_2 & a_3 \end{pmatrix} + \frac{a_1 a_3}{8}\begin{pmatrix} -\beta & 0 & 0 & -\frac{\beta a_0 - \alpha a_1}{a_3} \\ 0 & 2\alpha\frac{a_0}{a_1} - \beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ -\frac{\beta a_0 - \alpha a_1}{a_3} & 0 & 0 & -\beta \end{pmatrix}$$

$$+ \frac{a_0 a_3}{8}\begin{pmatrix} 2\beta\frac{a_1}{a_0} - \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & -\frac{\beta a_0 - \alpha a_1}{a_3} \\ 0 & 0 & \alpha & 0 \\ 0 & -\frac{\beta a_0 - \alpha a_1}{a_3} & 0 & \alpha \end{pmatrix}.$$

Next, we investigate the special case $a_0 = a_1 =: a$

$$\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & a_2 & a_3 \\ a & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_3 & 0 & 0 & a \end{pmatrix} - \frac{a^2}{8}\begin{pmatrix} 2\beta - \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} - \frac{a^2}{8}\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & 2\beta - \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix}$$

$$- \frac{(a_2)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & a\frac{\beta-\alpha}{a_2} & 0 \\ 0 & a\frac{\beta-\alpha}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_3)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & 0 & a\frac{\beta-\alpha}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & a\frac{\beta-\alpha}{a_3} & 0 & 0 \end{pmatrix} + \frac{\alpha a^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} a & a & a_2 & a_3 \\ a & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_3 & 0 & 0 & a \end{pmatrix} + (\alpha - \beta)\frac{a^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_2)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & a\frac{\beta-\alpha}{a_2} & 0 \\ 0 & a\frac{\beta-\alpha}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$- \frac{(a_3)^2}{8}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 2\alpha & 0 & a\frac{\beta-\alpha}{a_3} \\ 0 & 0 & 0 & 0 \\ 0 & a\frac{\beta-\alpha}{a_3} & 0 & 0 \end{pmatrix}$$

$$
\begin{aligned}
\tilde{\Gamma}^1_{\rho\sigma} = & \frac{1}{2}\begin{pmatrix} a & a & 0 & 0 \\ a & a & a_2 & a_3 \\ 0 & a_2 & -a & 0 \\ 0 & a_3 & 0 & -a \end{pmatrix} + \frac{a^2}{8}\begin{pmatrix} 2\alpha-\beta & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} + \frac{a^2}{8}\begin{pmatrix} \beta & \alpha & 0 & 0 \\ \alpha & 2\alpha-\beta & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{pmatrix} \\
& - \frac{(a_2)^2}{8}\begin{pmatrix} 2\beta & \alpha & a\frac{\beta-\alpha}{a_2} & 0 \\ \alpha & 0 & 0 & 0 \\ a\frac{\beta-\alpha}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_3)^2}{8}\begin{pmatrix} 2\beta & \alpha & 0 & a\frac{\beta-\alpha}{a_3} \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\frac{\beta-\alpha}{a_3} & 0 & 0 & 0 \end{pmatrix} - \frac{\beta a^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
= & \frac{1}{2}\begin{pmatrix} a & a & 0 & 0 \\ a & a & a_2 & a_3 \\ 0 & a_2 & -a & 0 \\ 0 & a_3 & 0 & -a \end{pmatrix} + (\alpha-\beta)\frac{a^2}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{(a_2)^2}{8}\begin{pmatrix} 2\beta & \alpha & a\frac{\beta-\alpha}{a_2} & 0 \\ \alpha & 0 & 0 & 0 \\ a\frac{\beta-\alpha}{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& - \frac{(a_3)^2}{8}\begin{pmatrix} 2\beta & \alpha & 0 & a\frac{\beta-\alpha}{a_3} \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\frac{\beta-\alpha}{a_3} & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Gamma}^2_{\rho\sigma} = & \frac{1}{2}\begin{pmatrix} a_2 & 0 & a & 0 \\ 0 & -a_2 & a & 0 \\ a & a & a_2 & a_3 \\ 0 & 0 & a_3 & -a_2 \end{pmatrix} - \frac{\beta a a_2}{8}\begin{pmatrix} 1 & 0 & \frac{a}{a_2}-\frac{\alpha}{\beta}\frac{a}{a_2} & 0 \\ 0 & 1-2\frac{\alpha}{\beta} & 0 & 0 \\ \frac{a}{a_2}-\frac{\alpha}{\beta}\frac{a}{a_2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
& - \frac{\alpha a a_2}{8}\begin{pmatrix} 1-2\frac{\beta}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & \frac{\beta}{\alpha}\frac{a}{a_2}-\frac{a}{a_2} & 0 \\ 0 & \frac{\beta}{\alpha}\frac{a}{a_2}-\frac{a}{a_2} & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Gamma}^3_{\rho\sigma} = & \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a \\ 0 & -a_3 & 0 & a \\ 0 & 0 & -a_3 & a_2 \\ a & a & a_2 & a_3 \end{pmatrix} - \frac{\beta a a_3}{8}\begin{pmatrix} 1 & 0 & 0 & \frac{a}{a_3}-\frac{\alpha}{\beta}\frac{a}{a_3} \\ 0 & 1-2\frac{\alpha}{\beta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{a}{a_3}-\frac{\alpha}{\beta}\frac{a}{a_3} & 0 & 0 & 1 \end{pmatrix} \\
& - \frac{\alpha a a_3}{8}\begin{pmatrix} 1-2\frac{\beta}{\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\beta}{\alpha}\frac{a}{a_3}-\frac{a}{a_3} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{\beta}{\alpha}\frac{a}{a_3}-\frac{a}{a_3} & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

Now we shall set $\alpha = \beta =: \theta$

$$
\tilde{\Gamma}^0_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & a_2 & a_3 \\ a & a & 0 & 0 \\ a_2 & 0 & a & 0 \\ a_3 & 0 & 0 & a \end{pmatrix} - \theta\frac{(a_2)^2+(a_3)^2}{8}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$\tilde{\Gamma}^1_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a & a & 0 & 0 \\ a & a & a_2 & a_3 \\ 0 & a_2 & -a & 0 \\ 0 & a_3 & 0 & -a \end{pmatrix} - \theta\frac{(a_2)^2 + (a_3)^2}{8}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\Gamma}^2_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_2 & 0 & a & 0 \\ 0 & -a_2 & a & 0 \\ a & a & a_2 & a_3 \\ 0 & 0 & a_3 & -a_2 \end{pmatrix} - a\theta\frac{a_2}{8}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - a\theta\frac{a_2}{8}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\tilde{\Gamma}^3_{\rho\sigma} = \frac{1}{2}\begin{pmatrix} a_3 & 0 & 0 & a \\ 0 & -a_3 & 0 & a \\ 0 & 0 & -a_3 & a_2 \\ a & a & a_2 & a_3 \end{pmatrix} - a\theta\frac{a_3}{8}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - a\theta\frac{a_3}{8}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

References

[AGG85] Luis Alvarez-Gaumé and Paul Ginsparg. The structure of gauge and gravitational anomalies. *Annals of Physics*, 161(2):423–490, 1985.

[Ber96] Reinhold A. Bertlmann. *Anomalies in quantum field theory.* Clarendon Press, 1996.

[Ble05] David Bleecker. *Gauge theory and variational principles.* Dover Publ., 2005.

[BM14] Edwin J. Beggs and Shahn Majid. Gravity induced from quantum spacetime. *Class. Quant. Grav.*, 31:035020, 2014.

[BZ] William A. Bardeen and Bruno Zumino. Consistent and covariant anomalies in gauge and gravitational theories. *Current Algebra and Anomalies.*

[CBDDB82] Yvonne Choquet-Bruhat, Witt-Morette Cecile De, and Margaret Dillard-Bleick. *Analysis, manifolds and physics.* Elsevier, 1982.

[CC96] Ali H. Chamseddine and Alain Connes. Universal formula for noncommutative geometry actions: Unification of gravity and the standard model. *Physical Review Letters*, 77(24):4868–4871, Sep 1996.

[CC12] Ali H. Chamseddine and Alain Connes. Spectral action for robertson-walker metrics. *Journal of High Energy Physics*, 2012(10):101, 2012.

[Con95] A. Connes. *Noncommutative Geometry.* Elsevier Science, 1995.

[DB86] Robert S. Doran and Victor A. Belfi. *Characterizations of C-algebras: the Gelfand-Naimark theorems.* Dekker, 1986.

[DFR95] Sergio Doplicher, Klaus Fredenhagen, and John E. Roberts. The Quantum structure of space-time at the Planck scale and quantum fields. *Commun.Math.Phys.*, 172:187–220, 1995.

[Dir67] Paul A. M. Dirac. *Lectures on quantum mechanics.* Belfer graduate school of Science, Yeshiva University, 1967.

[Dix82] Jacques Dixmier. *C*-algebras.* North-Holland, 1982.

[KC15] Victor G. Kac and Pokman Cheung. *Quantum calculus.* Springer, 2015.

[Kha13] Masoud Khalkhali. *Basic noncommutative geometry.* European Mathematical Society, 2013.

[Kos86] J. L. Koszul. *Lectures on fibre bundles and differential geometry.* Springer-Vlg., 1986.

[Lan14] Giovanni Landi. *An Introduction to Noncommutative Spaces and Their Geometries.* Springer Berlin, 2014.

[Mad99] J. Madore. *An introduction to noncommutative differential geometry and its physical applications.* Cambridge University Press, 1999.

[Mat85] Takayuki Matsuki. Lorentz anomaly in arbitrary dimensions. *Zeitschrift für Physik C Particles and Fields*, 29(1):103–105, 1985.

[MP96] Dubois-Violette M. and Michor P.W. Connnections on central bimodules in noncommutative differential geometry. *Journal of Geometry and Physics*, 20:218–232, 1996.

[MR94] S. Majid and H. Ruegg. Bicrossproduct structure of κ-poincaré group and non-commutative geometry. *Physics Letters B*, 334(3-4):348–354, 1994.

[MT88] Dubois-Violette M. and Masson T. Derivations and non-commutative differential calculus. *Acad. Sci. Paris*, 307:403–408, 1988.

[Nak05] Mikio Nakahara. *Geometry, topology, and physics.* Inst. of Physics Publishing, 2005.

[Per99] S. et. al. Perlmutter. Measurements of Omega and Lambda from 42 high redshift supernovae. *Astrophys. J.*, 517:565–586, 1999.

[Ros30] L. Rosenfeld. Über die gravitationswirkungen des lichtes. *Zeitschrift für Physik*, 65(9-10):589–599, 1930.