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GEOMETRIC AND VISCOSITY SOLUTIONS FOR THE CAUCHY PROBLEM OF FIRST ORDER

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Geometric and viscosity solutions for the Cauchy problem of first order

por

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A mi esposa, Anita,
por estar a mi lado,
por iluminar mi vida,
por su cariño y su amor.
Juliho Castillo

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Prólogo

Consideremos el problema de Cauchy

$$\begin{cases} \partial_t u(t,x) + H(t,x,\partial_x u(t,x), u(t,x)) = 0, & t \in (0,T] \\ u(0,t) = v(x), & x \in \mathbb{R}^k. \end{cases} \tag{HJ}$$

El método clásico (sección 2.2) para resolver este problema, con $v \in C^2$ y en un tiempo corto de tiempo, consiste en: (i) Resolver las ecuaciones características

$$\dot{x} = \partial_y H \tag{H1}$$

$$\dot{y} = -\partial_x H - y\partial_z H \tag{H2}$$

$$\dot{z} = y\partial_y H - H; \tag{H3}$$

(ii) obtener las líneas características

$$\Phi(t) \doteq (t, x(t), y(t), z(t))$$

generadas por las ecuaciones características; y (iii) finalmente construir la solución $u(t,x)$ del problema de Cauchy de la siguiente manera: haciendo $u_t(x) \doteq u(t,x)$, para t fija y su 1-jet $j^1 u_t(x) = (x, Du_t(x), u_t(x))$, la imagen de $j^1 u_t$ es la sección al tiempo t de la unión

$$\mathcal{L} = \bigcup_{t\in[0,T]} \Phi(t, j^1 v) = \left\{ \Phi(t) \middle| \, t \in [0,T], \Phi(0) \in j^1 v \right\}$$

donde $j^1 v = \{(x, Dv(x), v(x))\}$.

Sin embargo, en general este procedimiento no nos da una solución global del problema en todo el intervalo $[0,T]$, ya que la solución geométrica $\mathcal{L}$ no siempre es el conjunto $\{(t, j^1 u_t(x))\}$ para alguna función $u(t,x)$. En otras palabras, el *frente de onda*

$$\mathcal{F} = \bigcup_{t\in[0,T]} \pi_{t,x,z}(\mathcal{L}),$$

(donde $\pi_{t,x,z}(t,x,y,z) = (t,x,z)$) obtenido en el espacio (t,x,y,z) al resolver la ecuación

$$dz = -H(t,x,y,z)dt + ydx$$

restringido a $(t,x,y,z) \in \mathcal{L}$, no siempre es la gráfica de una función: las características proyectadas

$$\pi_{t,x,z}(\Phi(t,q_0)) = (t, x(t,q_0), z(t,q_0)),\ q_0 \in j^1 v$$

pueden cruzarse después de algún tiempo.

Mientras que en algunas aplicaciones, como a la óptica geométrica, el frente de onda $\mathcal{F}$ pueden ser considerado como una solución del problema físico, nosotros estamos interesados en encontrar una solución univaluada $u(t,x)$.

Supongamos que la proyección de $\mathcal{F}$ sobre el espacio (t,x) es sobreyectiva. En tal caso, uno puede construir tal solución como una sección del frente de onda eligiendo una única u sobre cada (t,x). Cuando la función H es suficientemente convexa respecto a y (y v no es demasiado "salvaje" en el infinito), tal "selector" consiste en escoger como $u(t,x)$ el más pequeño de los valores para u con $(t,x,u) \in \mathcal{F}$.

Estas soluciones *min* resultan ser la "solución de viscosidad" la cual fueron primero introducidas como el límite de viscosidad $\epsilon \to 0^+$ de las soluciones del problema de Cauchy para la ecuación viscosa

$$\partial_t u(t,x) + H(t,x,\partial_x u, u) = \epsilon \triangle_x u(t,x),$$

y después se obtuvo una definición general para ecuaciones diferenciales parciales no lineales de primer orden en el trabajo de Crandall, Evans y Lions M.G. Crandall [1984] M.G. Crandall [1984]; M. Bardi [1997].

En el caso no convexo la solución de viscosidad puede no ser la sección de un frente de onda (véase por ejemplo Chenciner [1974]). Por otro lado, Chaperon introdujo en Chaperon [1995] soluciones débiles cuya gráfica es una sección del frente de onda, obtenida por un procedimiento "minimax" el cuál generaliza el mínimo considerado en caso de convexo y depende de la existencia de una familia generadora adecuada para la solución geométrica.

Expliquemos con más detalle este procedimiento: Primero consideremos la *subvariedad Legendriana* $\Lambda = \varphi^t(j^1 v) \subset J^1\mathbb{R}^k$, para algún $\mathbb{R}^k$, donde φ^t es el flujo generado por (H1)-(H3) de manera que $\Phi(t) = (t, \varphi^t)$. Resulta que para tal subvariedad existe una *función generadora cuadrática al infinito*

$$S : [0,T] \times \mathbb{R}^k \times \mathbb{R}^q, (t,x,\xi) \to S^t_{H,v}(x,\xi)$$

(donde $\mathbb{R}^q$ es una familia de parametros ξ) tal que

$$\Lambda = \left\{ \left(x, \partial_x S^t_{H,v}(x,\xi), S^t_{H,v}(x,\xi)\right) \Big| \, \partial_\xi S^t_{H,v}(x,\xi) = 0 \right\}. \tag{0-1}$$

Ahora bien, se puede definir un *selector minimax* tal que

$$u(t,x) = \text{infmax}_\xi\, S^t_{H,v}(x,\xi)$$

es una solución generalizada del problema de Cauchy (HJ), la cual es llamada *solución minimax.* Pero en general, esta no es una solución clásica y de hecho $u \in C^{Lip}$.

Aunque hemos estado considerando $v \in C^2(\mathbb{R}^k)$, en el contexto más general del *cálculo de Clarke,* podemos considerar $v \in C^{Lip}(\mathbb{R}^k)$ y en ese caso, para un $H \in C^2(J^1\mathbb{R}^k)$ dado, obtenemos un operador

$$R^{0,t}_H : C^{Lip}(\mathbb{R}^k_H) \to C^{Lip}(\mathbb{R}^k),\ v(\cdot) \mapsto \text{infmax}_\xi\, S^t_{H,v}(\cdot,\xi).$$

Por supuesto, podemos considerar otra condición de frontera $u(s,x) = v(x)$ con tiempo inicial $t_0 = s \neq 0$.

Uno puede tratar de encontrar una solución como el límite obtenido al dividir una tiempo inicial dado en pequeños pedazos e iterar el procedimiendo anterior paso a paso. Nuestro objetivo es mostrar que cuando el tamaño de los intervalos de tiempo se va a cero, se obtiene la solución de viscosidad como límite:

Theorem 1 (Main Theorem) *Supongamos* $H \in C^2_c([0,T] \times J^1(\mathbb{R}^k))$, $v \in C^{Lip}(\mathbb{R}^k)$. *Entonces la solución de viscosidad es el límite de soluciones minimax iteradas para el problema* (HJ) *en* $[0,T]$.

De hecho, *el proposito de este trabajo de doctorado es dar una prueba completa de estos resultados.*

Antes de continuar con una explicación más detallada del significado del teorema anterior, es importante comentar la hipótesis sobre la compacidad del soporte del hamiltoniano. En principio, uno puede considerar hamiltonianos $H \in C^2(J^1M)$ con M una variedad compacta, pero gracias al así llamado *truco de Chekanov* [véase Ferrand, 1997, prop. 5], existe $q(M) \in \mathbb{N}$ tal que $M \hookrightarrow \mathbb{R}^q$ de manera que toda isotopía de contacto en M generada por H puede extenderse a una isotopía de contacto en $\mathbb{R}^q$ generada por algún otro hamiltoniano $\tilde{H} \in C^2_c\left(J^1\mathbb{R}^q\right)$.

Ahora, expliquemos el significado del teorema 1 con más detalle. Tomemos una partición de $[0,T]$:

$$0 = t_0 < t_1 < ... < t_N = T,$$

y para $i = 0,...,N-1$, definamos una *solución minimax iterada*

$$\begin{cases} u_0(t,x) = v(x) \\ u_{i+1}(t,x) = R^{t_i,t_{i+1}}_H\left(u_i(t,x)\right). \end{cases}$$

Nuestro resultado principal establece que $u_N(t,x)$ converge uniformemente en (subconjuntos) compactos a la solución de viscosidad del problema de Cauchy (HJ) cuando

$$\sup\left\{t_{i+1} - t_i\right\} \to 0.$$

Esto extiende el resultado obtenido por Q. Wei Wei [2014] del contexto simpléctico al de contacto. De hecho, la Dra. Wei fue estudiante del Prof. Marc Chaperon, quien en Chaperon [1995] había ya definido familias generadoras y soluciones minimax en el caso de contacto. Pero de la manera que las soluciones fueron definidas en aquel artículo, no eran adecuadas para construir una familia generadora similar a la que Wei logró construir.

Sin embargo, en su tesis doctoral Bhupal [1998], el Prof. Mohan Bhupal definió una función generadora para contactomorphismos en una manera más conveniente, y aprovechamos los resultados de su trabajo para desarrollar una teoría análoga a la de Wei [2014], pero en el contexto de contacto.

Como el prof. Chaperon nos comentó en una breve plática hace algunos años, resolver este problema es importante para entender la geometría de las soluciones de viscosidad. Aunque el teorema en el caso simpléctico era ya de por sí importante por su relación con mecánica clásica, nuestra generalización permite estudiar una clase más general de problemas relacionados con ecuaciones diferenciales parciales no lineales de primer orden.

Para conveniencia del lector, daremos una breve guía de esta tesis. Para comenzar, en el capítulo 1, daremos las definiciones y resultados más comunes en topología de contacto y ecuaciones diferenciales parciales no lineales, tales como subvariedades legendrianas, el método de las características y contactomorphismos.

Posteriormente, en el capítulo 2, deduciremos una fórmula para familias generatrices de subvariedades legendrianas de la forma $\varphi^t(j^1 v)$, y generalizaremos esta fórmula al contexto del cálculo de Clarke para poder construir soluciones minimax iteradas.

Finalmente, en el capítulo 3, definiremos el *selector minimax*, construiremos las soluciones minimax iteradas y demostraremos las validez de la relación enunciada anteriormente con la solución de viscosidad. En este capítulo, seguiremos muy de cerca los métodos de la Dra. Wei para obtener la generalización deseada.

Al final de este trabajo, hay dos apéndices: El primero es acerca del cálculo generalizado de Clarke y el segundo, acerca del principio minimax. Un tratamiento más detallado de cada tema pueden ser consultado en Clarke [2013] and Struwe [2008], respectivamente.

Geometric and viscosity solutions for the Cauchy problem of first order

by

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Abstract

There are two kinds of solutions of the Cauchy problem of first order, the viscosity solution and the more geometric minimax solution and in general they are different. The aim of this thesis is to show how they are related: iterating the minimax procedure during shorter and shorter time intervals one approaches the viscosity solution. This can be considered as an extension to the contact framework of the result of Q. Wei Wei [2014] in the symplectic case.

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Resumen

Existe dos tipos de soluciones del problema de Cauchy de primer orden, la solución de viscosidad y la solución minimax que es más geométrica y en general estas dos son muy diferentes. El propósito de esta tesis es mostrar como es que estas están relacionadas: Iterando el procedimiento minimax durante periodos más cortos de tiempo, uno aproxima la solución de viscosidad. Esto puede ser considerado como una extensión al contexto de contacto del resultado de Q. Wei Wei [2014] en el caso simpléctico.

Chapter 1

Preliminaries

As noticed in Vladimir I. Arnold [2004], due to a more complex geometry than ordinary differential equations (`ode's`), there not exists a unified theory for partial differential equations (`pde's`).

For an `ode`, we can always consider locally integrable vector fields, that is, there are always integral curves for them. However, even an hyperplane field on $\mathbb{R}^3$ is not always integrable.

For example, let us examine the hyperplane field given by equation

$$\alpha := dz - ydx = 0.$$

Now, consider $v \in \ker \alpha|_{(x,y,z)}$, $v = (v_x, v_y, v_z)$. Thus, $\alpha|_{(x,y,z)}$ is a linear form induced with associated matrix $R_\alpha = [-y, 0, 1]$. The last equation states that $v_z = yv_x$.

This hyperplane field is not integrable (that is, there is no submanifold N such that at each $p \in N$, T_pN is on the hyperplane field at p) because of *Frobenius integrability condition* ($\alpha \wedge d\alpha = 0$) does not hold at α inasmuch as

$$\alpha \wedge d\alpha = (dz - ydx) \wedge (-dy \wedge dx) = dx \wedge dy \wedge dz \neq 0,$$

that is, $a \wedge d\alpha$ is a *volume form* on $\mathbb{R}^3$.

In this chapter, we shall give definitions and results from contact geometry and theory of first-order p.d.e's in order to make clear statements of subsequent results in this thesis.

1.1 First-Order Partial Differential Equations

Any `pde` of first order can be written as

$$F(x_1, ..., x_n, \partial_{x_1} u(x), ..., \partial_{x_n} u(x), u(x)) = 0, \tag{F=0}$$

where $x = (x_1, ..., x_n)$. So any `pde` of first order can be regarded as a hypersurface Θ in $J^1(\mathbb{R}^n) \simeq T^*\mathbb{R}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$, and a solution of this `pde` as a function $u : M \to \mathbb{R}$ whose $1-$graph

$$j^1 u = \left\{ j^1_x u | x \in M \right\}.$$

(where $j^1_x u := (x, \partial_x u(x), u(x))$) lies in Θ.

Instead of $\mathbb{R}^n$, one can consider an $n-$ dimensional manifold M, and in that case we ontain the space $J^1 M \simeq T^* M \times \mathbb{R}$.

Indeed, a $1-$graph is a section over M. Therefore $J^1 M$ is also a vector bundle over M with projection

$$\pi_M : J^1(M) \to M, (x, y, z) \mapsto x.$$

1.2 Legendrian Submanifolds

$J^1(M)$ is not just a differentiable manifold, but has an analogous strucutre to the symplectic structure of the cotangent bundle T^*M. Now, we will explain some details about this contact struture. At the end of the section, we will define a very important concept, namely *contactomorphism*, that is, transformations preserving this structure. For example, every solution Θ of (F=0) is a legendrian submanifold of $J^1 M$ and the flow φ^t generated by the characteristic equations consists of contactomorphisms.

Definition 1.2.1 *A* contact structure *on* Ξ *is a field of hyperplanes* $\xi = \ker\alpha \subset T\Xi$ *where* α *is a* $1-$*form satisfying* $\alpha \wedge (d\alpha)^n \neq 0$. *We say that* α *is a* contact form, *and the pair* (Ξ, ξ) *is a* contact manifold.

Example 1.2.1 (Standar structure) *For* $J^1\mathbb{R}^n \simeq T^*\mathbb{R}^n \times \mathbb{R}$, *we can choose* $\alpha = dz - ydx$ *in local coordinates* (x, y, z). *It is straightforward to verify that* $J^1(M)$ *is a contact manifold and such* α *is a contact form: Consider the* $1-$*form* α *in local coordinates, i.e.,*

$$\alpha = dz - ydx = dz - \sum_{i=1}^{n} y_i dx_i. \tag{1-1}$$

Then

$$d\alpha = -dy \wedge dx = -\sum dy_i \wedge dx_i,$$

and therefore

$$(d\alpha)^n = (-1)^n n! \bigwedge_{i=1}^{n} (dy_i \wedge dx_i).$$

Thus

$$\alpha \wedge (d\alpha)^n = (-1)^n n! dz \bigwedge_{i=1}^{n} (dy_i \wedge dx_i) \neq 0.$$

Remark 2 *Indeed, along this work we will just consider the contact structure in* $J^1\mathbb{R}^n$ *induced by* $\alpha = dz - ydx$. *In the next paragraphs we will explain why it is enough to choose this contact structure.*

Notice also that if α is a contact form, then $\alpha \wedge (d\alpha)^n$ is a *volume form* and therefore, Ξ is orientable. However, some authors define $\xi = \ker\alpha$ just locally and in that case Ξ is not necessary an orientable manifold anymore.

If there were another form such that $\xi = \ker\beta$, then we would have $\beta = \lambda\alpha$, for some function $\lambda \in C^\infty(\Xi, \mathbb{R}\setminus\{0\})$, because

$$\operatorname{codim}\xi = \dim \operatorname{Im}\alpha = 1.$$

Since

$$\beta \wedge (d\beta)^n = (\lambda\alpha) \wedge (d(\lambda\alpha))^n = \lambda\alpha \wedge (\lambda d\alpha + d\lambda \wedge \alpha)^n = \lambda^{n+1}(\alpha \wedge (d\alpha)^n) \neq 0$$

the definition of a contact structure ξ does not depend on a particular choice of α.

Claim 3 *An equivalent definition of $\xi \subset T\Xi$ as a contact structure over a manifold of dimension $(2n+1)$ is as follows: For any local $1-$ form α with $\xi = \ker\alpha$, $(d\alpha)^n|_\xi$ is non-degenerate, that is, $(\xi_p, d\alpha|_{\xi_p})$ is a symplectic space for every $p \in \Xi$.*

Proof. Choose a local trivialization

$$\{e_1, f_1, ..., e_n, f_n, r\}$$

of $T\Xi = \xi \oplus \xi'$, such that $\xi = \ker\alpha = \langle e_1, f_1, ..., e_n, f_n\rangle$ y $\xi' = \langle r\rangle$. Thus

$$(\alpha \wedge (d\alpha)^n)(e_1, f_1, ..., e_n, f_n, r) = \alpha(r) \cdot (d\alpha)^n(e_1, f_1, ..., e_n, f_n),$$

with $\alpha(r) \neq 0$ and therefore

$$\alpha \wedge (d\alpha)^n \neq 0 \iff (d\alpha)^n \neq 0 \iff d\alpha\Big|_H \text{ is non-degenerate.}$$

■

As in symplectic topology, the following result states that there are not local invariants in contact topology:

Theorem 4 (Darboux) *[Geiges, 2008, Th. 2.5.1] Let (Ξ, ξ) be a contact structure and $p \in \Xi$. Thus there exists a coordinate system*

$$(U, x_1, y_1, ..., x_n, y_n, z)$$

centered around p such that in U and a diffeomorphism $f : U \to f(U) \subset J^1\mathbb{R}^n$ such that

$Df\,(\xi) = \ker \alpha$ *by defining*

$$\alpha = dz - ydx. \tag{1-2}$$

A submanifold is called integral if the tangent plane at each point is a subspace of the contact plane. For example, a $1-$graph is always an integral submanifold of $J^1(M)$, of the same dimension as M. For $\sigma(x) = (x, \sigma_y(x), \sigma_z(x))$:

$$\begin{aligned} \sigma = j^1\varphi \Rightarrow \sigma_z(x) = \varphi(x),\ \sigma_y(x)dx = d\varphi(x) \\ \Rightarrow \sigma^*\alpha = 0. \end{aligned}$$

Since $\alpha\,(D\sigma(x)(v)) = 0$, we have

$$\operatorname{Im} D\sigma(x) \subset \ker \alpha|_{\sigma(x)}\,, \tag{1-3}$$

that is, every tangent plane of every $1-$graph at a given point lies in the same hyperplane.

Indeed, there are not integral submanifolds of higher dimension.

Definition 1.2.2 *Let (Ξ, ξ) be a contact manifold. A submanifold L of (Ξ, ξ) is called* isotropic *if $T_pL \subset \xi_p$ for every $p \in L$.*

Proposition 5 *[Geiges, 2008, Prop. 1.5.12] Let (Ξ, ξ) be a contact manifold of dimension $2n+1$ and $L \subset (\Xi, \xi)$ a isotropic submanifold. Then $\dim L \leq n$.*

Proof. Let $i : L \hookrightarrow \Xi$ be an injection and α a contact form defining ξ, at least locally. Thus isotropicity condition can be restated as $i^*\alpha = 0$. So that $i^*d\alpha = 0$. In particular, $T_pL \subset \xi_p$ is an isotropic subspace of the symplectic $2n-$dimensional vector space $(\xi_p, d\alpha|_{\xi_p})$. By well-known results from linear symplectic algebra [see Geiges, 2008, remark 1.3.6] $\dim T_pL \leq (\dim \xi_p)/2 = n$. ■

Definition 1.2.3 *An isotropic submanifold $L \subset (\Xi, \xi)$, $\dim \Xi = 2n+1$ of maximal dimension n is called Legendrian.*

Claim 6 *Let* $(\Xi, \xi) = (J^1\mathbb{R}^n, \ker\alpha)$*. It follows form* (1-3) *and the fact that* $\dim(j^1 f) = \dim(M)$ *that* $1-$*graphs are Legendrian submanifolds of* $J^1(M)$*.*

1.3 Characteristic Equations

We return to our analysis of geometric solutions of (F=0). Our aim is to obtain the characteristic equations (H1)-(H3) from the geometry of a solution Θ of (F=0). In this section, we follow closely [Vladimir I. Arnold, 2004, lecture 2]. For more details, [see Vladimir I. Arnold, 2004, lecture 1].

At each point $Q \in J^1\mathbb{R}^n$, there is a contact plane K defined in local coordinates by $\alpha = 0$. At points of Θ, the tangent plane to Θ intersect the contact plane. If the intersection is $(2n-1)-$dimensional, the point is regular; otherwise it is singular. In general position singular points are isolated. At regular points Q, we obtain a distribution of $(2n-1)-$dimensional planes $T_Q\Theta \cap K_Q$, which are subspaces of the contact planes K_Q.

We know from claim 3 that each contact plane K is a symplectic space, whith symplectic structure is given by $\omega = d\alpha|_K$. This is a non-degenerate $2n-$form which defines a skew-scalar product in K, and the *skew-orthogonal complement* of K

$$K^\omega \doteq \{v | \, \forall u \in K : \omega(u, v) = 0\}$$

has complementary dimension.

In particular, the skew-orthogonal complement to the $(2n-1)-$dimensional intersection $T_Q\Theta \cap K_Q$ is one.dimensional and it is called *characteristic direction* for (F=0). Indeed, as a consequence of the following lemma, the *characteristic direction* lies in $T_Q\Theta \cap K_Q$.

Lemma 7 *The skew-orthogonal complement* W^ω *of a hyperplane* W *on a* $2n-$*dimensional subspace* $V < J^1(\mathbb{R}^n)$ *is a line which lies in the hyperplane itself.*

Proof. Consider a symplectic base for V, namely

$$\{e_1, ..., e_n, f_1, ..., f_n\},$$

so that

$$\begin{cases} \omega(e_i, e_j) = \omega(f_i, f_j) = 0 \\ \omega(e_i, f_j) = \delta_{ij}. \end{cases}$$

Without loss of generality, suppose that the hyperplane W is generated by $\{e_2, ..., e_n, f_1, ..., f_n\}$, and

$$v \in W^\omega, v = \sum_{i=1}^{n} v_i e_i + v'_i f_i.$$

Thus

$$\begin{cases} 0 = \omega(e_i, v) = v'_i & 2 \leq i \leq n \\ 0 = \omega(f_i, v) = v'_i & 1 \leq i \leq n, \end{cases}$$

so that $v = v'_1 f_1$. Therefore $W^\omega = \langle f_1 \rangle \subset W$. ∎

Because of previous consideration, we can give the following:

Definition 1.3.1 *A* characteristic direction *in a contact plane is the skewsymmetric complement to the intersection of the plane itself with the tangent plane to the solution θ of* (F=0) *at a regular point. The integral curves which pass through this direction are called characteristic.*

Remark 8 *The characteristic directions depend only on the contact structure and not on the contact form α chosen: If α is multiplied by λ, so is ω.*

For a solution $u : \mathbb{R}^n \to \mathbb{R}$ of the first-order p.d.e. (F=0), we can obtain a set of o.d.e.'s which characterize the solution, at least in a interval of time small enough.

Let (x, y, z) be the coordinates in $T_Q J^1\mathbb{R}^n$. The characteristic direction must be tangent to Θ. By differentiating (F=0), we obtain

$$\partial_x F(x, \partial u(x), u(x))\,\dot{x} + \partial_y F(x, \partial u(x), u(x))\,\dot{y} + \partial_z F(x, \partial u(x), u(x))\,\dot{z} = 0, \qquad (1\text{-}4)$$

and since the characteristic vector lies in contact plane K, we obtain a second condition:

$$\dot{z} = y\dot{x}. \tag{1-5}$$

From above equations, we have

$$\left(\partial_x F\left(x, \partial u(x), u(x)\right) + y\partial_z F\left(x, \partial u(x), u(x)\right)\right)\dot{x} + \partial_y F\left(x, \partial u(x), u(x)\right)\dot{y} = 0. \tag{1-6}$$

That said, we must find the skew-orthogonal complement K. For example, in the case $n = 1$, we obtain

$$dy \wedge dx\left((\dot{x}, \dot{y}), (x', y')\right) = x'\dot{y} - \dot{x}y'.$$

Indeed, last expression holds if we consider vector with scalar product, so that

$$\omega\left((\dot{x}, \dot{y}), (x', y')\right) = x'\dot{y} - \dot{x}y'. \tag{1-7}$$

Comparing equations (1-5), (1-6) and (1-7), we obtain that the direction of the characteristic field is given by equations

$$\begin{cases} \dot{x} = \partial_y F \\ \dot{y} = -(\partial_x F + y\partial_z F) \\ \dot{z} = y\partial_y F \end{cases} \tag{1-8}$$

Example 1.3.1 *Suppose that F is independent from z. The corresponding equation is called* (stationary) Hamilton-Jacobi equation:

$$H(x, \partial_x u(x)) = 0.$$

The related characteristic equations are

$$\begin{cases} \dot{x} = \partial_y H \\ \dot{y} = -\partial_x H \\ \dot{z} = y \partial_y H \end{cases} \tag{1-9}$$

We can deduce characteristic equations (H1)-(H3):

$$\dot{x} = \partial_y H, \ \dot{y} = -\partial_x H - y\partial_z H, \ \dot{z} = y\partial_y H - H;$$

from (1-8). At first, let's define

$$F(q, p, z) = p_0 + H\left(q, y, z\right),$$

where

$$q = (\underbrace{q_0}_{=t}; \underbrace{q_1, ..., q_n}_{=x}), p = (p_0; \underbrace{p_1, ..., p_n}_{=y}).$$

For

$$F(q, \partial_q u(q), u(q)) = \partial_t u(t, x) + H\left(t, x, \partial_y u(t, x), u(t, x)\right) = 0, \tag{1-10}$$

the characteristic direction are given by

$$\dot{q} = \partial_p F \tag{1-11}$$

$$\dot{p} = -(\partial_q F + p\partial_z F) \tag{1-12}$$

$$\dot{z} = p \cdot \partial_p F. \tag{1-13}$$

Notice that

$$\partial_p F = \left(1, \partial_y H\right), \ \partial_q F = \left(\partial_t H, \partial_x H\right), \ \partial_x F = \partial_z F.$$

First, (1-11) become $\left(\dot{t}, \dot{x}\right) = (1, \partial_y H)$, and we obtain (H1). Next, (1-12) become

$$(\dot{p_0}, \dot{y}) = -(\partial_t H, \partial_x H) - (\partial_z H)(p_0, y) = (-\partial_t H - \partial_z H p_0, \partial_x H - \partial_z H y)$$

and in this manner we obtain (H2). Finally (1-13) become

$$\dot{z}(t) = (1, \partial_y H) \cdot (p_0, y) = p_0 + y\partial_y H = -H + y\partial_y H$$

because of (1-10), and so we deduce (H3).

1.4 Contactomorphisms

Another two important concepts in the study of geometry of our Cauchy problem are *contatomorphism* and *contact isotopy.* The first one is a diffeomorphism preserving contact structures and the second one is a family of contactomorphisms varying in the time. Our most important example of a contact isotopy is the flow generated by (H1)-(H3). At the end of the section, we will give a result about how to recovery a classical solution for (HJ) from this flow. From here and because Theorem 4, we will just consider contact manifolds $(J^1\mathbb{R}^n, \ker\alpha)$ with (x, y, z) local coordinates for $J^1\mathbb{R}^n \simeq T^*\mathbb{R}^n \times \mathbb{R}$ and $\alpha = dz - ydx$.

Let $\xi = \ker\alpha$ a contact structure for contact manifold J^1M. A diffeomorphism $\psi : J^1M \to J^1M$ is called a *contactomorphism* if ψ preserves the oriented hyperplane field ξ. This is equivalent to the condition

$$\psi^*\alpha = e^h\alpha$$

for some function $h : J^1M \to \mathbb{R}$. A contact isotopy is a smooth family $\psi_t : J^1M \to J^1M$ of contactomorphism such that

$$\psi_t^*\alpha = e^{h_t}\alpha.$$

Suppose these contactomorphisms are generated by smooth vector field $X : J^1M \to TJ^1M$ via

$$\frac{d}{dt}\psi_t = X \circ \psi_t,\ \psi_0 = \mathrm{Id}\,.$$

Let $\mathcal{L}$ be the *Lie derivative*. Recall that if the flow ψ_t is generated by vector field X, then for a given $k-$form

$$\mathcal{L}_X\omega = \left.\frac{d}{dt}\right|_{t=0} \psi_t^*\omega,$$

[see Morita, 2001, prop. 2.13]. The following calculation

$$\begin{aligned}
\psi_{t_0}^*\mathcal{L}_X\alpha &= \psi_{t_0}^*\left(\left.\frac{d}{dt}\right|_{t=0} \psi_t^*\alpha\right) \\
&= \left.\frac{d}{dt}\right|_{t=t_0} \psi_t^*\alpha \\
&= \left(\left.\frac{d}{dt}\right|_{t=t_0} h_t\right)\left(e^{h_{t_0}}\alpha\right) \\
&= \left(\left.\frac{d}{dt}\right|_{t=t_0} h_t\right)\psi_{t_0}^*\alpha \\
&= \psi_{t_0}^*\left(g_{t_0}\alpha\right).
\end{aligned}$$

with $g_{t_0} = \left(\left(d/dt|_{t=t_0}\right) h_t\right) \circ \psi_{t_0}^{-1}$.

In particular, when $t_0 = 0$ we have $\mathcal{L}_X\alpha = g\alpha$ with $g := g_t|_{t=0}$ and this identity shows that

$$\mathcal{L}_X\alpha = g\alpha.$$

Conversely, if X satisfies this condition then the diffeomorphism $\psi_t : J^1M \to J^1M$ generated by X as above determine a contact isotopy with

$$h_t = \int_0^t g_s \circ \psi_s ds.$$

A vector field $X : J^1M \to TJ^1M$ which satisfies $\mathcal{L}_X\alpha = g\alpha$ for some function $g : J^1M \to \mathbb{R}$ is called a *contact vector field*.

For every contact form α, we define the *Reeb vector field* R_α as the unique one for which the following equations hold

$$\begin{cases} d\alpha(R_a, v) = 0, \quad v \in T_pJ^1M \\ \alpha(R_\alpha) = 1. \end{cases} \tag{Reeb}$$

Such field exists because for every $p \in J^1M$, $\ker d\alpha|_{T_pJ^1M}$ is one-dimensional, and R_α is defines except for a rescaling and the second condition allows us choose it uniquely. The importance of this vector field is given by the following result:

Lemma 9 *[Dusa McDuff, 1999, lemma 3.49] Let* $(\Xi, \xi = \ker\alpha)$ *be a contact structure with Reeb field* R_α. *Thus:*

(i) $X : \Xi \to T\Xi$ *is a contact vector field if and only if there exist a function* $H : \Xi \to \mathbb{R}$ *such that*

$$\begin{cases} i(X)\alpha = -H, \\ i(X)d\alpha = dH - (i(R_\alpha)dH)\,\alpha. \end{cases} \tag{1-14}$$

(ii) For every function $H : \Xi \to \mathbb{R}$, *there exists an unique contact vector field* $X_H : \Xi \to T\Xi$ *which satisfies* (1-14)

Proof. If (1-14) holds, then $\mathcal{L}_X\alpha = g\alpha$, where $g = -i(R_\alpha)dH$. Conversly, suppose that $\mathcal{L}_X\alpha = g\alpha$ and define $H = -i(X)\alpha$. Thus

$$i(X)d\alpha = \mathcal{L}_X\alpha - di(X)\alpha = dH + g\alpha.$$

Evaluating this 1-form at R_α, we obtain $i(R_\alpha)dH + g = 0$, and in this way, we have proof the first statement.

Now, consider a given function $H : \Xi \to \mathbb{R}$. Then, there exists a unique vector field $Z : \Xi \to T\Xi$, such that $Z \in \xi = \ker\alpha$ and

$$i(Z)d\alpha\Big|_\xi = dH\Big|_\xi. \tag{1-15}$$

By Darboux' Theorem 4, at least locally we can choose a local coordinates such that

$$\Xi \simeq J^1\mathbb{R}^n, \ \dim \Xi = 2n+1$$

and $\alpha = dz - ydx$. So w.o.l.g. let's take

$$Z = (Z_x, Z_y, Z_z) \in \ker \alpha,$$

such that $Z_z - yZ_x = 0$. Hence

$$Z = (Z_x, Z_y, yZ_x)\,.$$

Let $W = (W_x, W_y, W_z)$ be another arbitrary vector on $\ker \alpha$, such that using (1-15), we have

$$-dy \wedge dx(Z, W) = dH(W),$$

and therefore

$$W_x(-Z_y) + W_y(Z_x) = W_x\left(\partial_x H + y\partial_z H\right) + W_y(\partial_y H),$$

from which we deduce

$$\begin{cases} Z_x = \partial_y H \\ Z_y = -\partial_x H - y\partial_z H. \end{cases}$$

The vector field which we are looking for is defined by

$$X_H = Z - HR_\alpha.$$

■

Example 1.4.1 *The Reeb vector field for the standard contact* $1-$*form* $\alpha = dz - ydx$ *on*

$J^1\mathbb{R}^n$ is ∂_z. Let's verify this claim: If $R = (R_x, R_y, R_z)$, then

$$0 = i(R)d\alpha = -R_y dx + R_x dy,$$

implies $R_x = R_y = 0$. Finally

$$1 = \alpha(R) = R_z - yR_x = R_z.$$

We conclude that $R = (0, 0, 1) = \partial_z$. So

$$X_H = Z - HR_\alpha = (Z_x, Z_y, Z_z) - H(0,0,1) = (\partial_y H, -\partial_x H - y\partial_z H, y\partial_y H - H),$$

and therefore, the flow generated by X_H is given precisely by (H1)*,* (H2) *and* (H3)*.*

For a given function $H \in C_c^2\left([0,T] \times J^1\mathbb{R}^k\right)$, the following statement relates solutions $S : [0,T]\times\mathbb{R}^k \to \mathbb{R}$ for *Cauchy problem of First Order* (HJ) (being $S_0 : \mathbb{R}^k \to \mathbb{R}$ the initial condition) and solutions for characteristic equations (H1)-(H3) starting at the $1-$graph of S_0.

Proposition 10 *[Bhupal, 1998, Prop. 1.3] Suppose that $S \in C^2([0,T]\times\mathbb{R}^k)$ is a solution for the given Cauchy problem* (HJ)*. Then, if a solution for the equation*

$$\dot{x} = \partial_y H\left(t, x, \partial_x S(t,x), S(t,x)\right), \tag{1-16}$$

on $[0,T]$ is given, the curve defined by

$$(x(t), y(t), z(t)) = \left(x(t), \partial_x S(t, x(t)), S(t, x(t))\right). \tag{1-17}$$

is a solution for characteristic equations (H1)-(H3)*.*

Conversely, $S_0 \in C^1(M)$, T is small enough and

$$\left(X_{(t,x)}, Y_{(t,x)}, Z_{(t,x)}\right) : [0,T] \to J^1\mathbb{R}^k$$

is the unique solution for characteristic equations (H1)-(H3) *with boundary conditions*

$$\begin{cases} X_{(t,x)}(t) = x, \\ Y_{(t,x)}(0) = \partial_x S_0 X_{(t,x)}(0)), \\ Z_{(t,x)}(0) = S_0(X_{(t,x)}(0)), \end{cases}$$

then

$$S(t,x) = Z_{(t,x)}(t) \tag{1-18}$$

defines the solution for the Cauchy problem with initial condition $S(0,x) = S_0(x)$.

Proof.

[⇒] Since S satisfies HJ, then for all $(t,x) \in I \times \mathbb{R}^k$, where $I = [0,T]$ we obtain

$$\partial_t S(t,x) = -H\left(t, j^1_x S_t\right). \tag{1-19}$$

Let $x : I \to \mathbb{R}^k$ be a solution for (1-16), and define

$$\begin{cases} \gamma : I \to I \times \mathbb{R}^k, & t \mapsto (t, x(t)) \\ \Gamma : I \to I \times J^1\mathbb{R}^k, & t \mapsto \left(t, j^1_{x(t)} S_t\right). \end{cases}$$

Differentiating in both sides of (1-19), it follows that

$$\begin{aligned} &\partial_{xt} S(t,x)dx + \partial_{tt} S(t,x)dt = \\ &\qquad - \partial_x H(t, j^1_x S_t)dx - \partial_y H(t, j^1_x S_t)dy - \partial_z H(t, j^1_x S_t)dz - \partial_t H(t, j^1_x S_t)dt, \end{aligned} \tag{1-20}$$

where for the sake of brevity

$$dy = \partial_{tx}S(t,x)dt + \partial_{xx}S(t,x)dx \tag{1-21}$$

$$dz = \partial_{t}S(t,x)dt + \partial_{x}S(t,x)dx; \tag{1-22}$$

by replacing (1-21) and (1-22) in (1-20), and comparing both sides of the equation, we come to the following equations

$$\partial_{xt}S(t,x) = -\partial_x H(t,j^1_x S_t) - \partial_y H(t,j^1_x S_t)\,\partial_{xx}S(t,x) - \partial_z H(t,j^1_x S_t)\,\partial_x S(t,x) \tag{1-23}$$

$$\partial_{tt}S(t,x) = -\partial_t H(t,j^1_x S_t) - \partial_y H(t,j^1_x S_t)\,\partial_{tx}S(t,x) - \partial_z H(t,j^1_x S_t)\,\partial_t S(t,x). \tag{1-24}$$

For (1-17), it follows that

$$\dot{z}(t) = \partial_t S(\gamma(t)) + \partial_x S(\gamma(t))\dot{x}(t) \tag{1-25}$$

$$\dot{y}(t) = \partial_{tx} S(\gamma(t)) + \partial_{xx} S(\gamma(t))\dot{x}(t). \tag{1-26}$$

Therefore, by (1-17), (1-19) and (1-25), we deduce

$$\dot{z}(t) = -H\left(t, j^1_{x(t)}S_t\right) + y(t)\dot{x}(t). \tag{1-27}$$

By rewriting (1-23), we have

$$\partial_{xt}S(t,x) + \partial_y H(t,j^1_x S_t)\left(\partial_{xx}S(t,x)\right) = -\partial_x H(t,j^1_x S_t) - \partial_z H(t,j^1_x S_t)\left(\partial_x S(t,x)\right). \tag{1-28}$$

Finally, evaluating the identity above in the solution for (1-16), because of (1-26) y (1-17), it follows that

$$\dot{y}(t) = -\partial_x H(\Gamma(t)) - \partial_z H(\Gamma(t))y(t). \tag{1-29}$$

[$\Leftarrow$] In the opposite direction, we will show that the function $S : [0,T]\times\mathbb{R}^k \to \mathbb{R}$ defined by (1-18) satisfies

$$Y_{(t,x)}(t) = \partial_x S(t,x) \tag{1-30}$$

for all $t \in [0,T]$. At first, fix some $t \in [0,T]$ and consider

$$F : (-\epsilon,\epsilon) \to \mathbb{R}, F(s) = S(t, x + s\xi),$$

where $\xi \in \mathbb{R}^k$ is fixed, and denote by

$$(x_s, y_s, z_s) : [0,t] \to J^1\mathbb{R}^k$$

the unique solution for the given characteristic equations (H1)-(H3) with boundary conditions

$$\begin{cases} x_s(t) = X_{(t,x+s\xi)}(t) = x + s\xi \\ y_s(0) = \partial_x S_0(x_s(0)) \\ z_s(0) = S_0(x_s(0)). \end{cases} \tag{1-31}$$

Therefore

$$S(t, x+s\xi) = z_s(t) = S_0(x_s(0)) + \int_0^t \left(y_s\dot{x}_s - H(\tau, x_s, y_s, z_s)\right) d\tau.$$

Claim 11 *Differetiating respect s and evaluating at $s = 0$ we obtain $\partial_x S(t,x)\cdot\xi = Y_{(t,x)}(t)\cdot\xi$; hence $\partial_x S(t,x) = Y_{(t,x)}(t)$. In such case, S satisfies* (HJ)*.*

To verify such claim 11, consider the function

$$g_s(t) = (\partial_s z_s(t)) - y_s(t)\left(\partial_s x_s(t)\right),$$

and using characteristic equations for (x_s, y_s, z_s), we deduce

$$\frac{d}{dt}\left(g_s(t)\right) = \left(-\varrho(t)\right)\left(g_s(t)\right), \tag{1-32}$$

where $\varrho(t) = \partial_z H\left(x_s(t), y_s(t), z_s(t)\right)$, so that

$$g_s(t) = g_s(0)e^{-\int \varrho(t)}.$$

Now then (1-31):

$$\begin{aligned} g_s(0) &= (\partial_s z_s(0)) - y_s(0)(\partial_s x_s(0)) \\ &= \partial_s\left(S_0(x_s(0))\right) - y_s(0)\partial_s x_s(0) \\ &= (\partial_x S_0)(x_s(0))(\partial_s x_s(0)) - y_s(0)\left(\partial_s x_s(0)\right) \\ &= y_s(0)\partial_s x_s(0) - y_s(0)\partial_s x_s(0) \\ &= 0; \end{aligned}$$

so we conclude that for evey $(s,t) : g_s(t) = 0$, that is,

$$\partial_s z_s(t) - y_s(t)\partial_s x_s(t) = 0; \tag{1-33}$$

and evaluating at $s = 0$, we obtain

$$\partial_s z_s(t)|_{s=0} = Y_{(t,x)}(t)\xi.$$

Finally, recall that S satisfies (HJ) because

$$\begin{aligned}
Y_{(t,x)}\dot{X}_{(t,x)} - H(X_{(t,x)}, Y_{(t,x)}, Z_{(t,x)}) &= \dot{Z}_{(t,x)} \\
&= \frac{d}{dt}[S(t,x)] \\
&= \partial_t S(t,x) + \partial_x S(t,x)\dot{X}_{(t,x)} \\
&= \partial_t S(t,x) + Y_{(t,x)}\dot{X}_{(t,x)},
\end{aligned}$$

and this proves our claim.

Therefore

$$-H(X_{(t,x)}, Y_{(t,x)}, Z_{(t,x)}) = \partial_t S(t,x). \tag{1-34}$$

■

If (g_t) is the isotopy related to H that is,

$$\frac{d}{dt} g_t = X_H(g_t); \tag{1-35}$$

proposition 10 says that if $S(x,t) = S_t(x)$ is a solution for (HJ), then

$$j^1 S_t = g_t\left(j^1 S_0\right), \tag{1-36}$$

and conversely, if t is small enough, then the solution for the Cauchy problem (HJ) with initial condition S_0 is defined by (1-36).

However, there could happen some issues; first of all, contact transformation g_t could no longer be well-defined at each point of $j^1 S_0$. In addition, although $g_t(j^1 S_0)$ was well-defined, it could happen that it is not a section of $\pi : J^1\mathbb{R}^k \to \mathbb{R}^k,\ (x,y,z) \to x$ anymore.

Chapter 2

Generating Families

The main goal of this chapter is to construct *generating families for Legendrian submanifolds* $\varphi^t(j^1v)$, where φ^t is the flow generated by (H1)-(H3). As a previous step, we have to demonstrate the existence of *generating functions* for contactomorphism φ^t.

These constructions are fundamental for our work, because they will allow us to generalize in a very natural way those formulas given in Wei [2014]. Following that work, we will be able to define explicitly *iterate minimax solutions* in the next section, but in the more general setting of contact topology, that is, for Hamiltonians depending on $(t, x, \partial_x u, u)$.

At the end of the section, we extend the concept of generating families using Clarke calculus (see appendix A), allowing us to consider initial conditions $v \in C^{Lip}$. This step is crucial because minimax solutions are not differentiable anymore but only Lipschitz, and we will iterate solutions of this type.

2.1 Generating Functions

Consider $J^1(M) = T^*M \times \mathbb{R}$ endowed with the natural contact structure $\ker \alpha$ given in local coordinates (x, y) for T^*M by $\alpha = dz - ydx$. We denote by $\pi : J^1(M) \to M$ the canonical projection. Recall that a submanifold L of $J^1(M)$ is called Legendrian if

$T_pL \subset \ker \alpha|_p$ for any $p \in L$ and dim $L =$ dim M.

Suppose $S \in C^2(M \times \mathbb{R}^q)$ has fiber derivative $\partial_\xi S$ transversal to 0. Then

$$\Lambda := \{(x, \partial_x S(x,\xi), S(x,\xi)) | \partial_\xi S(x,\xi) = 0\}$$

is a Legendrian submanifold of J^1M and we say that S a *generating family* of Λ.

Definition 2.1.1 (g.f.q.i.) *A generating family $S \in C^2(M \times \mathbb{R}^q)$ is* quadratic at infinity *(g.f.q.i. for short) if there exists a* non degenerate quadratic form Q*, such that for any compact $K \subset M$,*

$$|\partial_\xi(S(x,\xi) - Q(\xi))|$$

is bounded on $K \times \mathbb{R}^q$.

Consider the sub-level sets $S_x^a := \{\xi : S(x,\xi) \leq a\}$, for a large enough the homotopy type of (S_x^a, S_x^{-a}) does not depend on a and coincides with the homotopy type of (Q^a, Q^{-a}), so we may write it as $(S_x^\infty, S_x^{-\infty})$. If the Morse index of Q is k, then

$$H_i(S_x^\infty, S_x^{-\infty}; \mathbb{Z}_2) = H_i(Q^\infty, Q^{-\infty}; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & i = k, \\ 0, & i \neq k. \end{cases}$$

For more details, [see Cardin, 2015, section 7.2.3]

Definition 2.1.2 *The minimax function is defined as*

$$R_S(x) := \inf_{[\sigma]=A} \max_{\xi \in |\sigma|} S(x,\xi),$$

where A is a generator of the homology group $H_k(S_x^\infty, S_x^{-\infty}; \mathbb{Z}_2)$ and $|\sigma|$ denotes the image of the relative singular homology cycle σ.

Remark 12 *In Wei [2013], Q. Wei proved that the minimax values of a function quadratic nondegenerate at infinity are equal when defined in homology or cohomology with coefficients in a field. However, by an example of F. Laudenbach, this is not always true for coefficients in a ring and, even in the case of a field, the minimax-maximin depends on the field.*

Although this is the most common manner to describe the minimax function*, we will give a more concrete description of the minimax in section 3.*

The function R_S is determined by Λ and does not depend on the particular choice of the `g.f.q.i.` S according to the following result

Theorem 13 *[Theret, 1999, prop 2.12] The* `g.f.q.i.` *of a Legendrian submanifold contact isotopic to the zero section is unique up to the following operations relating S_1 to S_2.*

Stabilitation $S_2(x, \xi, \eta) = S_1(x, \xi) + q(\eta)$ *with q a non degenerate quadratic form.*

Diffeomorphism $S_2(x, \xi) = S_1(x, \psi(x, \xi))$ *with $\psi(x, \cdot)$ a diffeomorphism $\forall x \in M$.*

The following definition is common in the literature

Definition 2.1.3 (strict `g.f.q.i.`) *A generating function is* strictly quadratic at infinity *if there is a non degenerate quadratic form Q such that $S(x, \xi) = Q(\xi)$ for (x, ξ) outside some compact set (we will say strict* `g.f.q.i.` *for short).*

This definition is more appropriate to work with hamiltonians $H \in C^1([0, T] \times J^1M)$, for M a compact manifold. However under some restrictions, both definitions of `g.f.q.i.` are equivalent

Proposition 14 *[Viterbo, 2006, prop. 1.6]*

Suppose there is a constante C suh that

(a) $\|\nabla(S - Q)\|_{C^0} < C$

(b) $\sup\left\{|S-Q|\,|x\in\mathbb{R}^k, |\xi|\le r\right\} < Cr$

Then L_S *has a strict* **g.f.q.i.**.

Recall that a diffeomorphism $\varphi : J^1M \to J^1M$ is called *contactomorphism* if $D\varphi(\ker\alpha) = \ker\alpha$ or equivalently $\varphi^*\alpha = g\alpha$ with $g \in C^1(J^1M, \mathbb{R}-\{0\})$.

From here, we follow the construction given in [Bhupal, 2001, section 6].

Definition 2.1.4 *Let* $\varphi : J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ *be a contactomorphism. A* generating function *for* φ *is a function* $\Phi : \mathbb{R}^{2k+1} \to \mathbb{R}$ *such that* $1 - \partial_z\Phi(x, Y, z)$ *never vanishes and the set of equalities*

$$X - x = \partial_Y\Phi(x, Y, z) \tag{cx}$$

$$Y - y = -\partial_x\Phi(x, Y, z) - y\partial_z\Phi(x, Y, z) \tag{cy}$$

$$Z - z = (X - x)Y - \Phi(x, Y, z) \tag{cz}$$

is equivalent to $\varphi(x, y, z) = (X, Y, Z)$.

Remark 15 *The contactomorphism* φ *has compact support if and only if* Φ *does.*

Proposition 16 *[Bhupal, 2001, prop. 6.1]*

(i) A contactomorphism $\varphi : J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ *with* $\|\mathbb{1} - d\varphi(p)\| < \frac{1}{2}$ *for all* $p \in J^1\mathbb{R}^k$ *has a unique generating function.*

(ii) If $\Phi \in C_c^\infty(\mathbb{R}^{2k+1})$ *has sufficiently small first and second derivatives, there exists a unique contactomorphism* $\varphi : J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ *having* Φ *as generating function*

We need the following result

Lemma 17 *A continuous differantiable map*

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

has a continuous differentiable inverse map if

$$\|\mathbb{1} - dF(x)\| \leq \frac{1}{2}$$

for all $x \in \mathbb{R}^n$.

Proof. Because of *Theorem of inverse function,* it is enough to show that F is bijective. First, we will show that such F is inyective, proving that for all $x, y \in \mathbb{R}^n$:

$$2\|F(y) - F(x)\| \geq \|y - x\|.$$

If we define $G(x) = F(x) - x$, then by hipothesis $\|G'(x)\| \leq \frac{1}{2}$. Now, because of *mean value Theorem,* there exists $c \in [0, 1]$ such that

$$\begin{aligned} \|G(y) - G(x)\| &\leq \|G'\left((1-c)x + cy\right)\|\|y - x\| \\ &= \frac{1}{2}\|y - x\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|F(x) - F(y)\| &= \|G(x) + x - G(y) - y\| \\ &\geq \|x - y\| - \|G(x) - G(y)\| \\ &\geq \|x - y\| - \frac{1}{2}\|x - y\| \\ &\geq \frac{1}{2}\|x - y\|. \end{aligned}$$

Now, let's prove that F is onto. We need this classical result

Theorem 18 (Banach Fixed Point) *Let* (E, d) *be a non-empty metric space, with a contraction* $T : E \to E$, *that is,*

$$\exists q \in (0, 1) \forall x, y \in E : d(T(x), T(y)) \leq q d(x, y).$$

Then, T has a unique fixed point $x^* \in E$. Moreover, if $x_n = T(x_{n-1})$ with $x \in E$ arbitrary, then $x_n \to x^*$.

Now, consider $G_y(x) - G(x)$, such that

$$\begin{aligned}\|G_y(x) - G_y(x')\| &= \|G(x') - G(x)\| \\ &\leq \frac{1}{2}\|x' - x\|.\end{aligned}$$

Therefore therefore there exists a unique $x^* \in \mathbb{R}^n$ such that $G_y(x^*) = x^*$, or equivalently $y = F(x^*)$, that is, F is onto. ■

Proof of prop. 16.

(i) Writing $\varphi(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$, Lemma 17 implies that if $\|\mathbb{1} - d\varphi(p)\| < 1$ for all $p \in J^1\mathbb{R}^k$ then the map

$$\tilde{v} : \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}, (x, y, z) \mapsto (x, v(x, y, z), z)$$

has a C^1 inverse and so there is a function $f : \mathbb{R}^{2k+1} \to \mathbb{R}^k$ such that

$$y = f(x, v(x, y, z), z).$$

We claim that $\Phi : \mathbb{R}^{2k+1} \to \mathbb{R}$ defined by

$$\Phi(x, Y, z) = Y \cdot (u(x, f(x, Y, z), z) - x) - w(x, f(x, Y, z), z) + z \tag{2-1}$$

is a generating function for φ. Indeed, since φ is a contactomorphism

$$dw - vdu = g(dz - ydx)$$

for $g : \mathbb{R}^{2k+1} \to \mathbb{R} - \{0\}$. Then

$$\partial_y w - v\partial_y u = 0, \quad \partial_x w - v\partial_x u = -gy = -(\partial_z w - v\partial_z u)y \tag{2-2}$$

From the definition of F

$$\begin{aligned}\partial_z \Phi &= Y \cdot (\partial_z u + (\partial_y u)(\partial_z f)) - \partial_z w - (\partial_y w)(\partial_z f) + 1 \\ &= -\partial_z w + Y\partial_z u + 1 \\ &= 1 - g\end{aligned}$$

$$\begin{aligned}\partial_x \Phi &= Y \cdot (\partial_x u + (\partial_y u)(\partial_x f) - 1) - (\partial_x w + (\partial_y w)(\partial_x f)) \\ &= -\partial_x f \cdot (\partial_y w - Y(\partial_y u)) - (\partial_x w - Y\partial_x u) - Y \\ &= -(\partial_x w - Y\partial_x u) - Y \\ &= (1 - \partial_z \Phi) f(x, Y, z) - Y, \end{aligned} \tag{2-3}$$

$$\partial_Y \Phi = u - x + Y \cdot (\partial_y u)(\partial_Y f) - (\partial_y w)(\partial_Y f) = u - x. \tag{2-4}$$

Suppose that $(X, Y, Z) = \varphi(x, y, z)$, then $y = f(x, Y, z)$ and (2-1), (2-3), (2-4) give (cx), (cy), (cz).

Now suppose (x, y, z, X, Y, Z) satisfy (cx), (cy), (cz). Then $Y = v(x, f(x, Y, z), z)$ and (cx), (2-4) give $X = u(x, f(x, Y, z), z)$, which together with (cz) and (2-1) imply $Z = w(x, f(x, Y, z), z)$. By (cy), (2-4) we have $(1 - \partial_z \Phi)y = (1 - \partial_z \Phi) f(x, Y, z)$ and so $y = f(x, Y, z)$.

(ii) Define $\mu, \nu : \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$ by

$$\begin{cases} \mu(x,Y,z) = (x, \dfrac{Y + \partial_x \Phi(x,Y,z)}{1 - \partial_z \Phi(x,Y,z)}, z) \\ \nu(x,Y,z) = (x + \partial_Y \Phi(x,Y,z), Y, z + Y \cdot \partial_Y \Phi(x,Y,z) - \Phi(x,Y,z)) \end{cases}$$

By Lemma 17 and the assumptions on Φ we have that μ, ν are diffeomorphisms. Define $\varphi = \nu \circ \mu^{-1} = (u, v, w)$. Notice that $\mu^{-1}(x,y,z) = (x, v(x,y,z), z)$, so that $\mu(x, v(x,y,z), z) = (x,y,z)$ and then

$$v + \partial_x \Phi(x,v,z) = (1 - \partial_z \Phi(x,v,z))y$$

We now verify that φ is a contactomorphism:

$$\begin{aligned} dw - vdu &= (1 - \partial_z \Phi(x,v,z))dz - (\partial_x \Phi(x,v,z) + v)dx \\ &= (1 - \partial_z \Phi(x,v,z))dz - (1 - \partial_z \Phi(x,v,z))ydx \\ &= (1 - \partial_z \Phi(x,v,z))(dz - ydx). \end{aligned}$$

■

2.2 Method of Characteristics

For $H \in C^2([0,T] \times J^1\mathbb{R}^k)$ let X_H be the associated time-dependent contact vector field given by (H1)-(H3), and $t \mapsto \varphi^t(q)$ be the integral curve with $\varphi^0(q) = q$. One calls φ^t the *contact isotopy* defined by H. We define $\varphi^{s,t} = \varphi^t \circ (\varphi^s)^{-1}$. Suppose that H has compact support cotained in the set

$$\left\{ (x,y,z) \in J^1\mathbb{R}^k : |y| \leq a \right\}$$

and let $c_H = \sup \{|DH_t(x,y,z)|, |D^2 H_t(x,y,z)|\}$, then $\max_t \|X_H\|_{\text{Lip}} \leq (2+a)c_H$.

Lemma 19 *If $\delta_H = \log 2 / ((2+a)c_H)$, for $0 < t - s < \delta_H$ there is a generating function*

$\Phi^{s,t} : J^1\mathbb{R}^k \to \mathbb{R}$ *for* $\varphi^{s,t}$*. Let* $q = (x, y, z)$*,* $r = (X, Y, Z)$ *and for* $s \leq \tau \leq t$*, define* $\varphi^{s,\tau}(q) = (x(\tau, q), y(\tau, q), z(\tau, q))$*,* $\varphi^{t,\tau}(r) = (\bar{x}(\tau, r), \bar{y}(\tau, r), \bar{z}(\tau, r))$*. Then*

$$\Phi^{s,t}(x, y(t,q), z) = \int_s^t (\dot{x}(\tau, q)(y(t,q) - y(\tau, q)) + H(\tau, \varphi^{s,\tau}(q)))d\tau, \tag{2-5}$$

$$\partial_t\Phi^{s,t}(x, y(t,q), z) = H(t, \varphi^{s,t}(q)), \tag{2-6}$$

$$\partial_s\Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r)) = H(t, \varphi^{t,s}(r))(\partial_z\Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r)) - 1) \tag{2-7}$$

Proof. From the general theory of differential equations, $\|\mathbb{1} - d\varphi^{s,t}(p)\| < 1$ for $0 < t - s < \delta_H$ and $p \in J^1\mathbb{R}^k$. By Proposition 16, $\varphi^{s,t}$ has a generating function $\Phi^{s,t}$ so that

$$\begin{aligned} x(t,q) - x &= \partial_y\Phi^{s,t}(x, y(t,q), z) \\ y(t,q) - y &= -\partial_x\Phi^{s,t}(x, y(t,q), z) - y\partial_z\Phi^{s,t}(x, y(t,q), z) \\ z(t,q) - z &= y(t,q)(x(t,q) - x) - \Phi^{s,t}(x, y(t,q), z). \end{aligned} \tag{2-8}$$

Then

$$\begin{aligned} \Phi^{s,t}(x, y(t,q), z) = (x(t,q) - x)y(t,q) - (z(t,q) - z) &= \int_s^t \dot{x}(\tau, q)y(t,q) - \dot{z}(\tau, q))d\tau \\ &= \int_s^t (\dot{x}(\tau, q)(y(t,q) - y(\tau, q)) + \dot{x}(\tau, q)y(\tau, q) - \dot{z}(\tau, q))d\tau \\ &= \int_s^t (\dot{x}(\tau, q)(y(t,q) - y(\tau, q)) + H(\tau, \varphi^{s,\tau}(q)))d\tau \end{aligned}$$

Differentiating respect to t

$$\begin{aligned} \frac{d}{dt}(\Phi^{s,t}(x, y(t,q), z)) &= \dot{y}(t,q)(x(t,q) - x) + H(t, \varphi^{s,t}(q)) \\ &= \dot{y}(t,q)\partial_y\Phi^{s,t}(x, y(t,q), z) + H(t, \varphi^{s,t}(q)). \end{aligned}$$

On the other hand

$$\frac{d}{dt}(\Phi^{s,t}(x, y(t,q), z)) = \partial_y\Phi^{s,t}(x, y(t,q), z)\dot{y}(t,q) + \partial_t\Phi^{s,t}(x, y(t,q), z).$$

Comparing these expressions we obtain (2-6). Similarly we have

$$\Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r)) = \int_s^t (\dot{\bar{x}}(\tau,r)(Y - \bar{y}(\tau,r)) + H(\tau, \varphi^{t,\tau}(r)))d\tau$$

Differentiating respect to s

$$\frac{d}{ds}(\Phi^{s,t}(x, y(t,q), z)) = (\bar{y}(s,r) - Y)\dot{\bar{x}}(s,r) - H(t, \varphi^{t,s}(r))$$

On the other hand

$$\begin{aligned}\frac{d}{ds}(\Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r)) &= \partial_x \Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r))\dot{\bar{x}}(t,r) \\ &+ \partial_z \Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r))\dot{\bar{z}}(t,r) + \partial_s \Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r)) \\ &= (y(s,r) - Y)\dot{\bar{x}}(t,r) - \partial_z \Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r))H(t, \varphi^{t,s}(r)) \\ &\qquad + \partial_s \Phi^{s,t}(\bar{x}(s,r), Y, \bar{z}(s,r))\end{aligned}$$

Comparing these expressions we obtain (2-7). ■

2.3 Generating families for Legendrian submanifols

Let $H \in C_c^2([0,T] \times J^1\mathbb{R}^k)$ and φ^t be the contact isotopy defined by H. Following Bhupal [1998] we will construct a *generating families* for Legendrian submanifols.

Let $s = t_0 < t_1 < \cdots < t_N = t$ be a partition such that $|t_{i+1} - t_i| < \delta_H$ so that

$$\varphi^{s,t} = \varphi^{t_{N-1},t_N} \circ \cdots \circ \varphi^{t_0,t_1}.$$

Proposition 20 *Let $0 \le s = t_0 < t_1 < \cdots < t_N = t \le T$ be a partition such that $|t_{i+1} - t_i| < \delta_H$ and $\Phi^{t_i,t_{i+1}} : \mathbb{R}^{2k+1} \to \mathbb{R}$ be the generating function of $\varphi^{t_i,t_{i+1}}$ given by Proposition 16. For $v \in C^2(\mathbb{R}^k)$ we have* *(a) One can define a generating family* $S^{s,t}:\mathbb{R}^k\times\mathbb{R}^{2kN}\to\mathbb{R}$ *of* $\varphi^{s,t}(j^1v)$ *by*

$$S^{s,t}(x;\xi)=v(x_0)+\sum_{j=1}^{N}y_j(x_j-x_{j-1})-\Phi^{t_{j-1},t_j}(x_{j-1},y_j,z_{j-1}) \tag{2-9}$$

where $x=x_N$, $\xi=(x_0,\dots,x_{N-1},y_1,\dots,y_N)$, *and* $z_0,\dots,z_{N-1}$ *are defined inductively by*

$$\begin{aligned} z_0&=v(x_0)\\ z_j&=z_{j-1}+(x_j-x_{j-1})y_j-\Phi_{j-1}(x_{j-1},y_j,z_{j-1})\quad 0<j\le N. \end{aligned} \tag{2-10}$$

Notice that z_j *depends only on* $(x_0,..,x_j,y_1,\dots,y_j)$.

(b) One can define a C^2 *function* $S^{s,t}:[s,t]\times\mathbb{R}^k\times\mathbb{R}^{2kN}\to\mathbb{R}$, *such that each* $S^{s,t}(\tau,\cdot)$ *is a generating family of* $\varphi_H^{s,\tau}(j^1v)$, *as follows: let* $\tau_j=s+(\tau-s)\dfrac{t_j-s}{t-s}$ *and*

$$S^{s,t}(\tau,x;\xi)=v(x_0)+\sum_{j=1}^{N}y_j(x_j-x_{j-1})-\Phi^{\tau_{j-1},\tau_j}(x_{j-1},y_j,\bar z_{j-1}). \tag{2-11}$$

where $x=x_N$, $\xi=(x_0,\dots,x_{N-1},y_1,\dots,y_N)$, *and* $\bar z_0,\dots,\bar z_{N-1}$ *are defined inductively as before*

(c) For each critical point ξ *of* $S^{s,t}(\tau,x;\cdot)$ *we have*

$$S^{s,t}(\tau,x,\xi)=\int_s^\tau\Big(\dot x(\sigma,j^1v(x_0))y(\sigma,j^1v(x_0))-H\Big(\sigma,\varphi^{s,\sigma}(j^1v(x_0))\Big)\Big)d\sigma \tag{2-12}$$

where $\varphi^{s,\sigma}(p)=(x(\sigma,p),y(\sigma,p),z(\sigma,p))$, $x(\tau,j^1v(x_0))=x$.

Proof. We have that the generating function $\Phi_i:=\Phi^{t_i,t_{i+1}}:\mathbb{R}^{2k+1}\to\mathbb{R}$ satisfies 1 −

$\partial_z \Phi_i \neq 0$ and that conditions

$$\begin{cases} x_{i+1} - x_i = \partial_y \Phi_i(x_i, y_{i+1}, z_i) \\ y_{i+1} - y_i = -\partial_x \Phi_i(x_i, y_{i+1}, z_i) - y_i \partial_z \Phi_i(x_i, y_{i+1}, z_i) \\ z_{i+1} - z_i = (x_{i+1} - x_i) y_{i+1} - \Phi_i(x_i, y_{i+1}, z_i) \end{cases} \tag{2-13}$$

hold if and only if $\phi^{t_i, t_{i+1}}(x_i, y_i, z_i) = (x_{i+1}, y_{i+1}, z_{i+1})$. We have

$$\partial_x S^{s,t}(x;\xi) = y_N. \tag{2-14}$$

Let $i = 0, \ldots, N-1$. For $i < j-1$ we have

$$\begin{aligned} \partial_{x_i} z_j &= \partial_{x_i}(z_{j-1} + (x_j - x_{j-1}) y_j - \Phi_{j-1}(x_{j-1}, y_j, z_{j-1})) \\ &= \partial_{x_i} z_{j-1} - \partial_{z_{j-1}} \Phi_{j-1} \partial_{x_i} z_{j-1} = (1 - \partial_{z_{j-1}} \Phi_{j-1}) \partial_{x_i} z_{j-1} \end{aligned}$$

and since $\partial_{x_i} z_i = y_i$ we get

$$\partial_{x_i} z_{i+1} = y_i - y_{i+1} - \partial_{x_i} \Phi_i - y_i \partial_{z_i} \Phi_i.$$

As $S^{s,t}(x, \xi) = z_N$,for $0 < i < N$ we obtain

$$\partial_{x_i} S^{s,t}(x, \xi) = (1 - \partial_{z_{N-1}} \Phi_{N-1}) \cdots (1 - \partial_{z_{i+1}} \Phi_{i+1})(y_i - y_{i+1} - \partial_{x_i} \Phi_i - y_i \partial_{z_i} \Phi_i), \tag{2-15}$$

$$\partial_{x_0} S^{s,t}(x;\xi) = (1 - \partial_{z_{N-1}} \Phi_{N-1}) \cdots (1 - \partial_{z_1} \Phi_1)(dv(x_0) - y_1 - \partial_{x_0} \Phi_0 - \partial_{z_0} \Phi_0 dv(x_0)) \tag{2-16}$$

For $i < j \le N$

$$\begin{aligned}
\partial_{y_i} z_j &= \partial_{y_i}(z_{j-1} + (x_j - x_{j-1})y_j - \Phi_{j-1}(x_{j-1}, y_j, z_{j-1})) \\
&= \partial_{y_i} z_{j-1} - \partial_{z_{j-1}}\Phi_{j-1}\partial_{y_i} z_{j-1} = (1 - \partial_{z_{j-1}}\Phi_{j-1})\partial_{y_i} z_{j-1}, \\
\partial_{y_i} z_i &= \partial_{y_i}(z_{i-1} + (x_i - x_{i-1})y_i - \Phi_{i-1}(x_{i-1}, y_i, z_{i-1})) \\
&= x_i - x_{i-1} - \partial_{y_i}\Phi_{i-1},
\end{aligned}$$

so we get

$$\partial_{y_i} S^{s,t}(x,\xi) = (1 - \partial_{z_{N-1}}\Phi_{N-1}) \cdots (1 - \partial_{z_i}\Phi_i)(x_i - x_{i-1} - \partial_{y_i}\Phi_{i-1}). \tag{2-17}$$

From (2-9), (2-10), (2-13) and equations (2-15), (2-16) (2-17) we have that the system $\partial_\xi S(x;\xi) = 0$, (2-14) is equivalent to

$$\begin{aligned}
\varphi^{s,t_1}(x_0, dv(x_0), v(x_0)) &= (x_1, y_1, z_1), \\
\varphi^{t_i,t_{i+1}}(x_i, y_i, z_i) &= (x_{i+1}, y_{i+1}, z_{i+1}), \quad i = 1, \dots, N-2, \qquad (2\text{-}18) \\
\varphi^{t_{N-1},t}(x_{N-1}, y_{N-1}, z_{N-1}) &= (x, \partial_x S^{s,t}(x;\xi), S^{s,t}(x;\xi)). \qquad (2\text{-}19)
\end{aligned}$$

Letting $q_i = (x_i, y_i, \bar{z}_i)$ we have from Lemma 19

$$\Phi^{\tau_i,\tau_{i+1}}(x_i, y_{i+1}, \bar{z}_i) = y_i(x_{i+1} - x_i) - \int_{\tau_i}^{\tau_{i+1}} \dot{x}(\sigma, q_i)y(\sigma, q_i) - H(\sigma, \varphi^{s,\sigma}(q_i)))d\sigma$$

from which item (c) follows. ■

Defining

$$Q(\xi) = -y_N x_{N-1} + \sum_{i=1}^{N-1} y_i(x_i - x_{i-1}), \tag{2-20}$$

$$W^{s,t}(\tau, x, \xi) = v(x_0) + x \cdot y_N - \sum_{j=1}^{N} \Phi^{t_{j-1},t_j}(x_{j-1}, y_j, \bar{z}_{j-1}), \tag{2-21}$$

we see that for $v \in C^{2,Lip}(\mathbb{R}^k)$, $S^{s,t}(\tau, x, \xi)$ is a `g.f.q.i.`

2.4 Generalized generating families

We consider the Cauchy problem (HJ) with $H \in C_c^2([0,T] \times J^1\mathbb{R}^k)$ and $v \in C^{Lip}$.

Proposition 21 *Suppose that in the Cauchy problem* (HJ) *v is locally Lipschitz and let $\partial v = \{(x, y, v(x)) : y \in \partial v(x)\}$. The generating family $S^{s,t}$ given by* (2-11) *generated $L^\tau = \varphi_H^{s,\tau}(\partial v)$ in the sense that*

$$L^\tau = \left\{(x, \partial_x S^{s,\tau}(\tau, x, \xi), S^{s,t}(\tau, x, \xi)) : 0 \in \partial_\xi S^{s,t}(\tau, x, \xi)\right\}, \tag{2-22}$$

where ∂ denotes Clarke's generalized derivative and $\partial v = \{(x, y, z) \mid y \in \partial v(x), z = v(x)\}$.

Proof. The condition $0 \in \partial_\xi S(x, \xi)$ means that there exists $y_0 \in \partial v(x_0)$ such that

$$y_0 - y_1 = \partial_x \Phi^{s,t_1}(x_0, y_1, v(x_0)) + y_0 \partial_z \Phi^{s,t_1}(x_0, y_1, v(x_0)) \tag{c1}$$

$$y_i - y_{i+1} = \partial_{x_i} \Phi^{t_i,t_{i+1}}(x_i, y_{i+1}, z_i) + \partial_{z_i} \Phi^{t_i,t_{i+1}}(x_i, y_{i+1}, z_i) y_i, \quad 0 < i < N \tag{c2}$$

$$x_i - x_{i-1} = \partial_{y_i} \Phi^{t_{i-1},t_i}(x_{i-1}, y_i, z_{i-1}), \quad 0 < i \leq N. \tag{c3}$$

Since $\partial_x S^{s,t}(x;\xi) = y_N$, we have that $\varphi^{s,t_1}(x_0, y_0, v(x_0)) = (x_1, y_1, z_1)$, (2-18) and (2-19) hold, and using (2-9) give $\varphi^{s,t}(x_0, y_0, v(x_0)) = (x, \partial_x S^{s,t}(x;\xi), S^{s,t}(x;\xi))$. ■

Proposition 22 *Let $H \in C_c^2([0,T] \times J^1\mathbb{R}^k)$, $v \in C^{Lip}(\mathbb{R}^k)$. Write $S^{s,t} : [s,t] \times \mathbb{R}^k \times \mathbb{R}^q \to \mathbb{R}$ given by* (2-11) *as*

$$S^{s,t}(\tau, x, \xi) = W^{s,t}(\tau, x, \xi) + Q(\xi),$$

with Q, $W^{s,t}$ defined in (2-20), (2-21) *For each compact subset K of $\mathbb{R}^k$, the family of functions $\{W^{s,t}(\tau, x, \cdot)\}_{\tau \in [s,t], x \in K}$ is uniformly Lipschitz. Moreover for any $\theta \in C_c(\mathbb{R}^q, [0,T])$ identically 1 in a neighborhhod of the origin with $\|D\theta\| < 1$, there exists a constant*

$a_K > 1$ *such that for* $\tau \in [s,t]$,

$$(x,\xi) \mapsto S_K^{s,t}(\tau,x,\xi) = \theta\Big(\frac{\xi}{a_K}\Big)W^{s,t}(x,\xi) + Q(\xi) \qquad (2\text{-}23)$$

is a **g.f.q.i.** *for*

$$L_K^\tau = L^\tau \cap \pi^{-1}(K) = \left\{(x, \partial_x S_K^{s,t}(\tau,x,\xi), S_K^{s,t}(\tau,x,\xi)|0 \in \partial_\xi S_K^{s,t}(\tau,x,\xi))\right\},$$

where $\pi : J^1\mathbb{R}^k \to \mathbb{R}^k, (x,y,z) \to x$.

Proof. For a fixed compact K, let $c_K = \max\{\|W^{s,t}(\tau,x,\cdot)\|_{\mathrm{Lip}} : \tau \in [s,t], x \in K\}$. Writing $Q(\xi) = \frac{1}{2}\langle B\xi,\xi\rangle$

$$\partial_\xi S_K^{s,t}(x,\xi) \subset \frac{1}{a_K} D\theta\Big(\frac{\xi}{a_K}\Big)W^{s,t}(\tau,x,\xi) + \theta\Big(\frac{\xi}{a_K}\Big)\partial_\xi W^{s,t}(\tau,x,\xi) + B\xi.$$

Defining $b_K = \max\{|W(\tau,x,0)| : \tau \in [s,t], x \in K\}$ we have

$$\Big|W^{s,t}(\tau,x,\xi)\Big| \le \Big|W^{s,t}(\tau,x,0)\Big| + \Big|W^{s,t}(\tau,x,\xi) - W^{s,t}(\tau,x,0)\Big| \le b_K + c_K\|\xi\|.$$

Thus, if a_K, b_K are sufficiently large, for $\|\xi\| \ge b_K$ and any $w \in \partial_\xi W^{s,t}(\tau,x,\xi)$ we have

$$\begin{aligned}\left|\frac{1}{a_K} D\theta\Big(\frac{\xi}{a_K}\Big)W^{s,t}(\tau,x,\xi) + \theta\Big(\frac{\xi}{a_K}\Big)w\right| \le \frac{1}{a_K}(b_K + c_K\|\xi\|) + c_K &\le \frac{1}{2}\|B^{-1}\|^{-1}\|\xi\| \\ &< \|B\xi\|.\end{aligned}$$

We can choose a_K sufficiently large so that $\theta\Big(\dfrac{\xi}{a_K}\Big) = 1$ if $\|\xi\| \le b_K$. Thus $S = S_k$, for $\|\xi\| \le b_K$, and $0 \notin \partial_\xi S_k(x,\xi)$ for $\|\xi\| \ge b_K$. Therefore

$$L_K^\tau = \left\{(x, \partial_x S_K^{s,t}(\tau,x,\xi), S_K^{s,t}(\tau,x,\xi) : 0 \in \partial_\xi S_K^{s,t}(\tau,x,\xi)\right\}.$$

■

Chapter 3

Generalized Solutions of the Cauchy problem

In this last chapter, we will prove our main result, Theorem 1. First, we will define *minimax solutions* and give some of their basic properties. For this goal, we need a *minimax principle,* which basic definitions and results are given in appendix B. Most of analytic results that we have obtain in this section are generalizations for those in Wei [2014]. Next, using those results, we finally give a demostration of Theorem 1. Constructions obtained in previous section allows us to follow very closely methods in [Wei, 2014, section 3] to achieve our goal. At the end of this section, we present an example. Although it is very simple, this example shows that it is posible to use our results in order to study Hamiltonians with non-compact support.

3.1 Minimax Selector

Let $K \subset \mathbb{R}^k$ be a compact set, $S_K^{s,t} \in C^1([s,t] \times \mathbb{R}^k \times \mathbb{R}^q)$ be `g.f.q.i.` given as in (2-23) and $Q(\xi) = \dfrac{1}{2}\langle P\xi, \xi\rangle$ be the associated quadratic form. As $S_K^{s,t} = Q$ outside a compact set, the critical levels of $S_K^{s,t}$ are bounded. There is $R(K) < 0$ such that for $R' < R(K)$, the sub–level set

$$(S_K^{s,t})_{\tau,x}^{R'} = \left\{ \xi \in \mathbb{R}^q | S_k^{s,t}(\tau, x; \xi) < R' \right\}$$

is identical to the sub–level $Q^{R'}$.

Definition 3.1.1 *Let j be the Morse index of Q and $a > 0$ large. We define $\mathfrak{G}_a$ as the set of continuous maps $\sigma : B_j \to \mathbb{R}^q$, of the unit ball B_j of dimension j, such that*

$$\sigma(\partial B_j) \subset Q^{-1}(-\infty, -a).$$

Lemma 23 *Let $v \in C^{Lip}(\mathbb{R}^k)$ $H \in C_c^2([0,T] \times J^1\mathbb{R}^k)$. Let $S_K^{s,t} \in C^1([s,t] \times \mathbb{R}^k \times \mathbb{R}^q)$ be as in* (2-23)*. Let $K \subset \mathbb{R}^k$ be compact, $a > -R(K)$. The function*

$$(\tau, x) \in [s,t] \times \mathbb{R}^k \mapsto R_{H,K}^{s,\tau} v(x) = \inf_{\sigma \in \mathfrak{G}_a} \max_{e \in B_j} S_K^{s,t}(\tau, x, \sigma(e)), \tag{3-1}$$

has the following properties

(a) $R_{H,K}^{s,\tau} v(x)$ is a critical value of $\xi \to S_K^{s,t}(\tau, x, \xi)$;

(b) it is a Lipschitz function and therefore differentiable almost everywhere a.e.*;*

(c) $j^1 R_{H,K}^{s,\tau} v$ is an a.e. *section of the wave front*

$$\left\{ (\tau, x, \partial_x S_K^{s,t}(\tau, x, \xi), S_K^{s,t}(\tau, x, \xi)) : 0 \in \partial_\xi S_K^{s,t}(\tau, x, \xi) \right\}.$$

Proof. Since $S_K^{s,t}(\tau, x, \xi) = Q(\xi)$ outside a compact set and Q is non-degenerate, we have that $\Sigma_K^{s,t} = \left\{ (\tau, x, \xi) | \partial_\xi S_K^{s,t}(\tau, x, \xi) = 0 \right\}$ is compact and $S_K^{s,t}(x, \tau, \cdot) : \mathbb{R}^q \to \mathbb{R}$ satisfies the Palais-Smale condition.

Let $\pi : \Sigma_K^{s,t} \to \mathbb{R} \times \mathbb{R}^k$ be the projection $(\tau, x, \xi) \mapsto (\tau, x)$. By Sard's Theorem the set of critival values of π has null measure.

(a) Apply the minimax principle (Theorem 49) with the invariant family $\mathfrak{G}_a$ of Definition 3.1.1.

(b) Given $\epsilon > 0$, there exists $\sigma_0 \in \mathfrak{G}_a$ such that

$$R^{s,\tau_0}_{H,K}v(x_0) \geq \max_{e \in B_j} S^{s,t}_K(\tau_0, x_0, \sigma_0(e)) - \epsilon \geq S^{s,t}_K(\tau_0, x_0, \sigma_0(e)) - \epsilon,$$

for any $e \in B_j$. Let $\max_{e \in B_j} S^{s,t}_K(\tau_1, x_1, \sigma_0(e)) = S^{s,t}_K(\tau_1, x_1, \xi_1)$, then

$$\begin{aligned} R^{s,\tau_1}_{H,K}v(x_1) - R^{s,t}_{H,K}v(\tau_0, x_0) &\leq S^{s,t}_K(\tau_1, x_1, \xi_1) - S^{s,t}_K(\tau_0, x_0, \xi_1) + \epsilon \\ &= \theta\Big(\frac{\xi_1}{a_K}\Big)\Big(W^{s,t}(\tau_1, x_1, \xi_1) - W^{s,t}(\tau_0, x_0, \xi_1)\Big) + \epsilon \\ &\leq A_K(|\tau_1 - \tau_0| + \|x_1 - x_0\|) + \epsilon. \end{aligned}$$

Letting $\epsilon \to 0$ and exchanging (τ_0, x_0) and (τ_1, x_1), we get

$$\Big|R^{s,\tau_1}_{H,K}v(x_1) - R^{s,t}_{H,K}v(\tau_0, x_0)\Big| \leq A_K(|\tau_1 - \tau_0| + \|x_1 - x_0\|.$$

(c) Let $x_0 \in \mathbb{R}^k$ be a regular value of $\pi : \Sigma_t \to \mathbb{R}^k$. There is a neighborhood U of x and difeomorphisms $\phi_i : V_i \to U$, $i = 1, \ldots, m$ such that

$$\pi^{-1}(U) = \bigcup_{i=1}^{m} \phi_i(U).$$

For each $x \in U$ there is $i = 1, \ldots, m$ such that

$$R^{s,\tau}_{H,K}v(x) = S^{s,t}_K(\tau, x; \phi_i(x)) \tag{3-2}$$

Let $x \in U$ be differentiability point of $R^{s,\tau}_{H,K}v$. Proving that there is $i = 1, \ldots, m$ such that

$$dR^{s,\tau}_{H,K}v(x) = dS^{s,t}_K(\tau, \cdot; \phi_i(\cdot))(x) \tag{3-3}$$

will finish the proof. Indeed, as $\partial_\xi S_K^{s,t}(\tau, x, \phi_i(x)) = 0$ we have

$$\begin{aligned} dS_K^{s,t}(\tau, \cdot; \phi_i(\cdot))(x) &= \partial_x S_K^{s,t}(\tau, x; \phi_i(x)) + \partial_\xi S_K^{s,t}(\tau, x; \phi_i(x)) d\phi_i(x) \\ &= \partial_x S_K^{s,t}(\tau, x; \phi_i(x)). \end{aligned}$$

and so

$$j^1 R_{H,K}^{s,\tau} v(x) \in \{(x, \partial_x S_K^{s,t}(\tau, x; \xi), S_K^{s,t}(\tau, x; \xi)) : \partial_\xi S_K^{s,t}(\tau, x; \xi) = 0\}.$$

To prove (3-3) it suffices to show that there is i such that for any unit vector h

$$dR_{H,K}^{s,\tau} v(x) \cdot h = dS_K^{s,t}(\tau, \cdot; \phi_i(\cdot))(x) \cdot h \tag{3-4}$$

and for that it is enough to show that any unit vector h there is $i = 1, \ldots, m$ such that (3-4) holds, because in such a case there is $i = 1, \ldots, m$ such that (3-4) holds for a base of unit vectors. Now, there is $\epsilon > 0$ such that for any unit vector h and $|s| < \epsilon$ $x + sh \in U$ and so there is $i = i(h, s)$ such that $R_{H,K}^{s,\tau} v(x + sh) = S_K^{s,t}(x + sh; \phi_i(x + sh))$. For h fixed there is $i = i(h)$ for which and a sequence s_k converging to zero such that

$$R_{H,K}^{s,\tau} v(x + s_k h) = S_K^{s,t}(\tau, x + s_k h; \phi_i(x + s_k h))$$

which implies (3-4). ■

Corollary 3.1.1 *If $v \le w$ then $R_{H,K}^{s,\tau} v \le R_{H,K}^{s,\tau} w$*

Proof. This is clear from (2-11), (2-23) and (3-1). ■

Proposition 24 *Let $K, K' \subset \mathbb{R}^k$ be compact. If $x \in K \cap K'$, $\tau \in [s, t]$ then $R_{H,K'}^{s,\tau} v(x) = R_{H,K}^{s,\tau} v(x)$.*

Proof. This follows from the fact for $a > -R(K), a' > -R(K')$, any $\sigma \in \mathfrak{G}_a$, $\sigma' \in \mathfrak{G}_{a'}$ can be deformed into an $\sigma'' \in \mathfrak{G}_{a''}$, $a'' > a, a'$, with

$$\max_{e \in B_j} S_K^{s,t}(\tau, x, \sigma''(e)) = \max_{e \in B_j} S_{K'}^{s,t}(\tau, x, \sigma''(e)),$$

by using the gradient flow de Q, suitable truncated. ■

Propositions 22 and 24 allow one to define

$$R_H^{s,\tau} v(x) = \inf_{\sigma \in \mathfrak{G}_a} \max_{e \in B_j} S^{s,t}(\tau, x; \sigma(e)). \tag{3-5}$$

From Lemma 23 we obtain

Theorem 25 *Function $(\tau, x) \in [s,t] \times \mathbb{R}^k \mapsto R_H^{s,\tau} v(x)$ has the following properties*

(a) $R_H^{s,\tau} v(x)$ is a critical value of $\xi \to S^{s,t}(\tau, x, \xi)$;

(b) it is a Lipschitz function and therefore differentiable almost everywhere a.e..

(c) $j^1 R_H^{s,\tau} v$ is an a.e. *section of the wave front*

$$\left\{(\tau, x, \partial_x S^{s,t}(\tau, x, \xi), S^{s,t}(\tau, x, \xi)) : 0 \in \partial_\xi S^{s,t}(\tau, x, \xi)\right\}.$$

Proposition 26 *The definition of $R_H^{s,\tau} v(x)$ is independent of the partition of $[0,T]$ used to define S.*

Proof. First assume $t - s < \delta_H$; and let $\tau \in (s,t)$. Consider the family of partitions $\zeta_\mu = \{s \le s + \mu(\tau - s) < t\}$, $\mu \in [0,1]$, and the corresponding generating families

$$\begin{aligned} S_\mu^{s,t}(\tau, x; x_0, y_1, x_1, y_2) = v(x_0) + y_1(x_1 - x_0) - \varPhi^{s,s+\mu(\tau-s)}(x_0, y_1, z_0) \\ + y_2(x - x_1) - \varPhi^{s+\mu(\tau-s),t}(x_1, y_2, z_1) \end{aligned} \tag{3-6}$$

Function S_μ is continuous in μ and the minimax $R_{S_\mu}^{s,t}(\tau, x)$ is a critical value of the map $\eta \mapsto S_\mu^{s,t}(\tau, x; \eta)$. By (2-12) the set of such critical values is independependient of μ, and by Sard's Theorem, it has measure zero. Therefore $R_{S_\mu}^{s,t}$ is constant for $\mu \in [0,1]$. Letting $x' = x_0 - x_1$ y $y' = y_2 - y_1$, we obtain

$$S_0^{s,t}(\tau, x; x_0, y_1, x_1, y_2) = v(x_0) - \varPhi^{s,t}(x_1, y_2, z_1) + (x - x_1)\, y_2 + x'y'.$$

One gets this `g.f.q.i.`adding the quadratic form $x'y'$ to the `g.f.q.i.`

$$S^{s,t}(\tau,x;x_0,y_1,x_1,y_2)=v(x_0)-\Phi^{s,t}(x_1,y_2,z_1)+(x-x_1)\,y_2,$$

so that

$$R_S^{s,t}v(x)=R_{S_0}^{s,t}v(x)=R_{S_1}^{s,t}v(x).$$

In general, given two partitions ζ',ζ'' of $[s,t]$ with $|\zeta'|,|\zeta'|<\delta_H$, let

$$\zeta=\zeta'\cup\zeta''=\{s=t_0<\cdots<t_n=t\}\,,$$

be the (smallest) common refinement of ζ',ζ''. If t_j does not belong to ζ', consider the family of partitions

$$\zeta_\mu(j)=\{t_0<t_{j-1}\leq t_{j-1}+\mu(t_j-t_{j-1})<t_{j+1}<\cdots<t_n\}\,,\ \mu\in[0,1]$$

The argument given at the begining shows that the minimax relative to $\zeta_0(j)$ and $\zeta_1(j)$ coincide. Continuing this process, we obtaing that the minimax relative to ζ' and ζ coincide, and so do the minimax relative to ζ'' and ζ as well as the minimax relative to ζ' and ζ''. ■

Proposition 27 *The critical levels*

$$C(\tau,x):=\left\{\eta:0\in\partial_\eta S^{s,t}(\tau,x,\eta),S^{s,t}(\tau,x,\eta)=R_H^{s,\tau}v(x)\right\}$$

are compact and the set-valued correspondence $(\tau,x)\to C(\tau,x)$ is upper semicontinuous, i.e. for every convergent sequence $(\tau_j,x_j,\eta_j)\to(\tau,x,\eta)$ with $\eta_j\in C(\tau_j,x_j)$, one has $\eta\in C(\tau,x)$. In other words the graph $\{(\tau,x,\eta)|\eta\in C(\tau,x)\}$ of the correspondence is closed.

Proof. Let $(\tau_j,x_j,\eta_j)\to(\tau,x,\eta)$ with $\eta_j\in C(\tau_j,x_j)$. Since $S^{s,t}$ is C^1 with respect to x, one has $\partial S^{s,t}=\partial_x S^{s,t}\times\partial_\eta S^{s,t}$, which is upper semicontinuous (Theorem

44). It follows that the limit $(\partial_x S^{s,t}(\tau_j, x, \eta), 0)$ of the sequence $\partial_x S^{s,t}(\tau_j, x_j, \eta_j), 0 \in \partial S^{s,t}(\tau_j, x_j, \eta_j)$ belongs to $\partial S^{s,t}(\tau, x, \eta)$, hence, $0 \in \partial S^{s,t}(\tau, x, \eta)$. As $S^{s,t}$ and $R_H^{s,t}$ are continuous, $S^{s,t}(\tau_j, x_j, \eta_j) \to S^{s,t}(\tau, x, \eta)$, $R_H^{s,\tau_j} v(x_j) \to R_H^{s,\tau} v(x)$, and therefore $\eta \in C(\tau, x)$.
■

Lemma 28 *Given $\delta > 0$, there exists $\epsilon > 0$ such that*

$$R_H^{s,\tau} v(x) = \inf_{\sigma \in \Sigma_\epsilon} \max\{S^{s,t}(\tau, x, \sigma(e)) : \sigma(e) \in C_\delta(x)\} \tag{3-7}$$

where $\Sigma_\epsilon = \left\{ \sigma \in \mathfrak{G}_a : \max_{e \in B_j} S^{s,t}(\tau, x, \sigma(e)) \leq R_H^{s,\tau} v(x) + \epsilon \right\}$ *and* $C_\delta(x) = B_\delta(C(\tau, x))$ *denotes the $\delta-$ neighborhood of the critical set $C(\tau, x)$.*

Proof. We apply to $S_{\tau,x}^{s,t}(\cdot) = S^{s,t}(\tau, x, \cdot)$ the following result:

Lemma 29 (Deformation Lemma) *[Struwe, 2008, Theorem 3.4] Suppose f satisfies the Palais-Smale condition. If $c \in \mathbb{R}$ is a critical value of f and N any neighbourhood of $K_c :=$ Crit$(f) \cap f^{-1}(c)$, then there exist $\epsilon > 0$ and a bounded smooth vector field V equal to 0 off $f^{c+2\epsilon} \setminus f^{c-2\epsilon}$, whose flow φ_V^t satisfies $\varphi_V^t(f^{c+2\epsilon} \setminus N) \subset f^{c-2\epsilon}$.*

For $\delta > 0$, and $c = R_H^{s,\tau} v(x)$, there exist $\epsilon > 0$ and V, a smooth vector field vanishing outside $(S_{\tau,x}^{s,t})^{c+2\epsilon} \setminus (S_{\tau,x}^{s,t})^{c-2\epsilon}$ such that

$$\varphi_V^1((S_{\tau,x}^{s,t})^{c+\epsilon} \setminus C_\delta(x)) \subset (S_{\tau,x}^{s,t})_x^{c-\epsilon}.$$

For $\sigma \in \mathfrak{G}_a$ we have $\sigma(B_j) \cap C_{\delta(x)} \neq \emptyset$, because otherwise

$$\max_{e \in B_j} S^{s,t}(\tau, x; \varphi_V^1(\sigma(e))) \leq R_H^{s,\tau} v(x) - \epsilon$$

which contradicts the definition of the minimax.

For any $r < c$, the complement of $(S_{\tau,x}^{s,t})^r$ is a neighborhood of $C(\tau, x)$. By the same argument one has that $\sigma(B_j) \cap C_{\delta(x)} \setminus (S_{\tau,x}^{s,t})^r \neq \emptyset$. Therefore, for any $r < c$ and $\sigma \in \Sigma_\epsilon$

one has

$$r \leq \max\{S^{s,t}(\tau, x, \sigma(e)) : \sigma(e) \in C_\delta(x)\} \leq \max_{e\in B_j} S^{s,t}(\tau, x, \sigma(e))$$

wich implies (3-7).

■

Proposition 30 *The generalized gradient of $R_H^{s,\tau} v$ satisfies*

$$\partial R_H^{s,\tau} v(x) \subset \operatorname{co}\left\{\partial_x S^{s,t}(\tau, x, \eta) : \eta \in C(\tau, x)\right\}, \tag{3-8}$$

where co *denotes the convex envelope.*

Proof. First we consider a point $\bar{x}$ where $R_H^{s,\tau} v$ is diferentiable and prove that

$$dR_H^{s,\tau} v(\bar{x}) \subset \operatorname{co}\left\{\partial_x S^{s,t}(\bar{x}, \eta) | \eta \in C(\tau, \bar{x})\right\}. \tag{3-9}$$

Take $\delta, \epsilon > 0$ for $\bar{x}$ as in Lemma 28. Consider $K = \overline{B_1(\bar{x})}$ and $S_K^{s,t}$ as in (2-23). Choose $B = B_\rho(\bar{x})$ with $\rho \in (0,1)$ sufficiently small such that for $x \in B$

$$\left| S_K^{s,t}(\tau, x, \cdot) - S_K^{s,t}(\tau, \bar{x}, \cdot) \right|_{C^0} < \frac{\epsilon}{4}. \tag{3-10}$$

Let $y \in \mathbb{R}^d, \lambda < 0$ such that $x_\lambda = \bar{x} + \lambda y \in B$, and $\lambda^2 < \frac{\epsilon}{4}$. By definition of $R_H^{s,t} v$, for each x_λ, there exists $\sigma_\lambda \in \mathfrak{G}_a$ such that

$$\max_{e\in B_j} S^{s,t}(\tau, x_\lambda, \sigma_\lambda(e)) \leq R_H^{s,\tau} v(x_\lambda) + \lambda^2, \tag{3-11}$$

then,

$$\max_{e\in B_j} S^{s,t}(\tau, \bar{x}, \sigma_\lambda(e)) \leq \max_{e\in B_j} S^{s,t}(\tau, x_\lambda, \sigma_\lambda(e)) + \frac{\epsilon}{4} \leq R_H^{s,\tau} v(x_\lambda) + \frac{\epsilon}{2} \leq R_H^{s,\tau} v(\bar{x}) + \frac{3\epsilon}{4}$$

On the other hand, there exists $\eta_\lambda \in \sigma_\lambda(B_j) \cap C_\delta(\bar{x})$ such that

$$R_H^{s,\tau} v(\bar{x}) \leq \max\{S^{s,t}(\tau, \bar{x}, \sigma_\lambda(e)) : \sigma_\lambda(e) \in C_\delta(\bar{x})\} = S^{s,t}(\tau, \bar{x}, \eta_\lambda), \tag{3-12}$$

that implies

$$\left(R_H^{s,\tau} v(x_\lambda) + \lambda^2\right) - R_H^{s,\tau} v(\bar{x}) \geq S^{s,t}(\tau, x_\lambda, \eta_\lambda) - S^{s,t}(\tau, \bar{x}, \eta_\lambda),$$

since $\lambda < 0$,

$$\begin{aligned} \frac{1}{\lambda}\left(R_H^{s,\tau} v(x_\lambda) - R_H^{s,\tau} v(\bar{x})\right) &\leq \frac{1}{\lambda}\left(S^{s,t}(\tau, x_\lambda, \eta_\lambda) - S^{s,t}(\tau, \bar{x}, \eta_\lambda)\right) - \lambda \\ &\in \langle \partial_x S^{s,t}(\tau, x'_\lambda, \eta_\lambda), y\rangle - \lambda, \end{aligned} \tag{3-13}$$

where the last belonging follows from the Mean Value Theorem (Theorem 42) for some x'_λ in the line segment between $\bar{x}$ and x_λ

Take lim sup in (3-13) and let $\lambda \to 0$, we get for all $y \in \mathbb{R}^d$:

$$\langle dR_H^{s,\tau} v(\bar{x}), y\rangle \leq \max_{\eta \in C(\bar{x})} \langle \partial_x S^{s,t}(\bar{x}, \eta), y\rangle. \tag{3-14}$$

Considering the convex function $f(y) = \max_{\eta \in C(\bar{x})} \langle \partial_x S^{s,t}(\bar{x}, \eta), y\rangle$, inequality (3-14) implies

$$dR_H^{s,\tau} v(\bar{x}) \in \partial f(0) = \mathrm{co}\left\{\partial_x S^{s,t}(\bar{x}, \eta) : \eta \in C(\tau, \bar{x})\right\},$$

In the general case

$$\begin{aligned} \partial R_H^{s,\tau} v(x) &= \mathrm{co}\left\{\lim_{x' \to x} dR_H^{s,\tau} v(x')\right\} \\ &\subset \mathrm{co}\left\{\mathrm{co}\left\{\lim_{x' \to x}\left\{\partial_x S^{s,t}(\tau, x', \eta') : \eta' \in C(\tau, x')\right\}\right\}\right\} \\ &\subset \mathrm{co}\left\{\partial_x S^{s,t}(\tau, x, \eta) : \eta \in C(\tau, x)\right\} \end{aligned}$$

by the upper semicontinuity of $(\tau, x) \to C(\tau, x)$ and the continuity of $\partial_x S$. ■

3.2 Viscosity solutions and iterated minimax

We recall the definition of *viscosity solution*

Definition 3.2.1 *Let $V \subset \mathbb{R}^k$ be open*

(a) A function $u \in C([0,T] \times V)$ is called a viscosity subsolution *(respectively* supersolution *of*

$$\partial_t u + H(t, x, \partial_x u, u) = 0, \tag{3-15}$$

if for any $\phi \in C^1(V \times [0,T])$ and any $(t_0, x_0) \in [0,T] \times V$ at which $u - \phi$ has a maximum (respectively minimum) one has

$$\partial_t \phi(t_0, x_0) + H(t_0, x_0, \partial_x \phi(t_0, x_0), u(t_0, x_0)) \leq 0 \text{ (respectively } \geq 0).$$

(b) The function u is a viscosity solution *if it is both a viscosity subsolution and a supersolution.*

Theorem 31 (M.G. Crandall [1986]) *If $v : \mathbb{R}^k \to \mathbb{R}$ is uniformly continuous and $H \in C_c^2([0,T] \times J^1\mathbb{R}^k)$, then there exists a unique uniformly continuous viscosity solution of the Cauchy problem* (HJ)*.*

Proposition 32 *Suppose that $H \in C_c^2([0,T] \times J^1\mathbb{R}^k)$, then the* minimax operator $R_H^{s,\tau}$: $C^{Lip}(\mathbb{R}^k) \to C^{Lip}(\mathbb{R}^k)$ *satisfies*

(i) For $v \in C^{Lip}(\mathbb{R}^k)$,

$$\|\partial(R_H^{s,t} v)\| \leq (\|\partial v\| + |t - s| \, \|\partial_x H\|) e^{|t-s| \|\partial_z H\|} \tag{3-16}$$

(ii) There is a constant $C(H) > 0$ such that for any $v \in C^{Lip}(\mathbb{R}^k)$,

$$\|R_H^{s,t} v - R_H^{s,\tau} v\| \leq |t - \tau| \, C(H) \|H\|. \tag{3-17}$$

(iii) If $v^0, v^1 \in C^{Lip}(\mathbb{R}^k)$, $K \subset \mathbb{R}^k$ compact, there exists a bounded $\tilde{K} \subset \mathbb{R}^k$, depending on K and $\|\partial v^i\|$, such that for all $0 \leq s < t \leq T$:

$$\|R_H^{s,t} v^0 - R_H^{s,t} v^1\|_K \leq \|v^0 - v^1\|_{\tilde{K}}. \tag{3-18}$$

Proof. First we assume that $|t - s| < \delta_H$ so that

$$S(\tau, x; x_s, y) = v(x_s) + (x - x_s)y - \Phi^{s,\tau}(x_s, y, v(x_s))$$

is a `g.f.q.i.` for $\varphi_H^{s,\tau}(\partial v)$.

(i) For $(x_s, y_t) \in C(t, x)$, there is $y_s \in \partial v(x_s)$ such that

$$\varphi_H^{s,t}(x_s, y_s, v(x_s)) = (x, y_t, S(t, x; x_s, y_t)) = (x, y_t, R_H^{s,t} v(x)).$$

As $\partial_x S(t, x; x_s, y_t) = y_t$, by (3-8) on has

$$\partial_x R_H^{s,t} v(x) \subset \operatorname{co}\Big\{y_t : y_s \in \partial v(x_s), \varphi_H^{s,t}(x_s, y_s, v(x_s)) = (x, y_t, S(t, x; x_s, y_t))\Big\}$$

Let $\gamma(\tau) = (x(\tau), y(\tau), z(\tau)) = \varphi_H^{s,\tau}(x_s, y_s, v(x_s))$, $y_s \in \partial v(x_s)$, then

$$y_t - y_s = \int_s^t \dot{y}(\tau) d\tau = \int_s^t (-\partial_x H(\gamma(\tau)) - y(\tau)\partial_z H(\gamma(\tau))) d\tau,$$

$$|y_t| \leq |y_s| + \int_s^t |\partial_x H(\gamma(\tau))|\, d\tau + \int_s^t |y(\tau)|\, |\partial_z H(\gamma(\tau))|\, d\tau.$$

Hence, by Grönwall's inequality

$$\begin{aligned} |y_t| &\leq \left(|y_s| + \int_s^t |\partial_x H(\gamma(\tau))|\, d\tau \right) \exp \int_s^t |\partial_z H(\gamma(\tau))|\, d\tau \\ &\leq (\|\partial v\| + |t - s|\, \|\partial_x H\|) e^{|t-s| \|\partial_z H\|} \end{aligned}$$

(ii) For $(x_s, y_\tau) \in C(\tau, x)$, there is $y_s \in \partial v(x_s)$ such that $\varphi_H^{s,\tau}(x_s, y_s, v(x_s)) = (x, y_\tau, S(\tau, x; x_s, y_\tau))$. By (2-6) we have $\partial_\tau S(\tau, x; x_s, y_\tau) = -H(\tau, x, y_\tau, S(\tau, x; x_s, y_\tau))$. Hence

$$\begin{aligned}\partial_\tau R_H^{s,t} v(x) \subset \ \text{co} \, \{&-H(\tau, x, y_\tau, S(\tau, x; x_s, y_\tau)) : y_s \in \partial v(x_s), \\ &\varphi_H^{s,t}(x_s, y_s, v(x_s)) = (x, y_\tau, S(t, x; x_s, y_\tau))\}\end{aligned}$$

By the mean value Theorem, (Theorem 42),

$$\left| R_H^{s,\tau} v(x) - R_H^{s,t} v(x) \right| \leq |\tau - t| \, \|H\|.$$

(iii) Consider $v^\lambda = (1-\lambda)v^0 + \lambda v^1$, $\lambda \in [0,1]$ and let $S_\lambda^{s,t}$ be the corresponding generating family, then $\partial_\lambda S_\lambda^{s,t}(t, x; x_s, y_t) = v^1(x_s) - v^0(x_s)$. For $(x_s, y_t) \in C^\lambda(t, x)$, there is $y_s^\lambda \in \partial v(x_s)$ such that $\varphi_H^{s,t}(x_s, y_s^\lambda, v(x_s)) = (x, y_t, S_\lambda^{s,t}(t, x; x_s, y_t))$. By a similar argument to the proof of (3-8) we have

$$\begin{aligned}\partial_\lambda R_H^{s,t} v^\lambda(x) &\subset \text{co} \left\{ \partial_\lambda S_\lambda^{s,t}(t, x; x_s, y_t) : (x_s, y_t) \in C^\lambda(t, x) \right\} \\ &\subset \text{co} \left\{ v^1(x_s) - v^0(x_s) : x_s \in \tilde{K} \right\}\end{aligned}$$

where

$$\tilde{K} = \left\{ x_s \in \mathbb{R}^k : \|x_s\| \leq \max_{\bar{x} \in K} \|\bar{x}\| + |t - s| \sup_{y \in Y} \|\partial_y H\| \right\}, \tag{3-19}$$

$$Y = \left\{ y \in \mathbb{R}^k : \|y\| \leq (\|\partial v\| + |t - s| \, \|\partial_x H\|) e^{|t-s| \|\partial_z H\|} \right\}. \tag{3-20}$$

Thus we obtain

$$\left| R_H^{s,t} v^0 - R_H^{s,t} v^1 \right|_K \leq \left| v^0 - v^1 \right|_{\tilde{K}}.$$

For the general cases fix $N > T/\delta_H$ and take the partition $t_0 < \cdots < t_N$, with $t_i = s + i(t-s)/N$.

(i) We have the generating family $S^{s,t}(x; \xi)$ given in (2-9) where $\xi = (x_0, \ldots, x_{N-1}, y_1, \ldots, y_N)$.

For $\xi \in C(t,x)$ there exists $y_0 \in \partial v(x_0)$ such that equations (c1)-(c3) are satisfied. Then we have that $\varphi^{s,t_i}(x_0, y_0, v(x_0)) = (x_i, y_i, z_i)$, $i < N$, $y_N = \partial_x S^{s,t}(x;\xi))$, $\varphi^{s,t}(x_0, y_0, v(x_0)) = (x, y_N, S^{s,t}(x;\xi))$. By Proposition 30

$$\partial_x R_H^{s,t} v(x) \subset \mathrm{co}\left\{ y_N : y_0 \in \partial v(x_0), \varphi_H^{s,t}(x_0, y_0, v(x_0)) = (x, y_N, S^{s,t}(x;\xi)) \right\}.$$

Writing $\gamma(\tau) = \varphi_H^{s,\tau}(x_0, y_0, v(x_0))$ we have

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} (-\partial_x H(\gamma(\tau)) - y(\tau)\partial_z H(\gamma(\tau)))d\tau.$$

By Grönwall's inequality we have as before

$$|y_{i+1}| \leq \left(|y_i| + \int_{t_i}^{t_{i+1}} |\partial_x H| \right) \exp\left(\int_{t_i}^{t_{i+1}} |\partial_x H| \right)$$

which imply by induction that

$$|y_i| \leq \left(|y_0| + \int_{t_0}^{t_i} |\partial_x H| \right) \exp\left(\int_{t_0}^{t_i} |\partial_z H| \right).$$

Indeed, the inductive step is given by the inequalities

$$\begin{aligned} |y_{i+1}| &\leq \left(\left(|y_0| + \int_{t_0}^{t_i} |\partial_x H| \right) \exp\left(\int_{t_0}^{t_i} |\partial_z H| \right) + \int_{t_i}^{t_{i+1}} |\partial_x H| \right) \exp\left(\int_{t_i}^{t_{i+1}} |\partial_z H| \right) \\ &= \left(|y_0| + \int_{t_0}^{t_i} |\partial_x H| \right) \exp\left(\int_{t_0}^{t_{i+1}} |\partial_z H| \right) + \left(\int_{t_i}^{t_{i+1}} |\partial_x H| \right) \exp\left(\int_{t_i}^{t_{i+1}} |\partial_z H| \right) \\ &\leq \left(|y_{t_0}| + \int_{t_0}^{t_{i+1}} |\partial_x H| \right) \exp\left(\int_{t_0}^{t_{i+1}} |\partial_z H| \right). \end{aligned}$$

Therefore we have

$$|y_N| \leq (\|\partial v\| + |t-s| \, \|\partial_x H\|) \, e^{|t-s| \|\partial_z H\|}.$$

(ii) Set $D = \sup\left\{ \|g^{\tau,\tau'}\| : |t-\tau| < \delta_H \right\}$ where $\varphi_H^{\tau,\tau'*}\alpha = g^{\tau,\tau'}\alpha$. We have the generating family $S^{s,t}(\tau, x;\xi)$ given in (2-11) with$\bar{z}_0 = v(x_0), \bar{z}_1, \ldots, \bar{z}_{N-1}$ defined inductively as in

(2-10). Thus

$$\frac{d\bar z_j}{d\tau} = -\left(\partial_{\tau_{j-1}}\Phi^{\tau_{j-1},\tau_j}(x_{j-1},y_j,\bar z_{j-1}) + \partial_{\tau_j}\Phi^{\tau_{j-1},\tau_j}(x_{j-1},y_j,\bar z_{j-1})\right)\frac{t_j-s}{t-s}$$
$$-\partial_z\Phi^{\tau_{j-1},\tau_j}(x_{j-1},y_j,\bar z_{j-1})\frac{d\bar z_{j-1}}{d\tau}$$

Using (2-6) and (2-7) one proves by induction that for $j=1,\dots,N$ one has

$$\frac{d\bar z_j}{d\tau} = \sum_{k=1}^{j-1}\prod_{i=k}^{j-1}(1-\partial_z\Phi^{\tau_i,\tau_{i+1}}(x_i,y_{i+1},\bar z_i))(H(x_k,y_{k+1},\bar z_k)-H(x_{k-1},y_k,\bar z_{k-1}))\frac{t_k-s}{t-s}$$
$$-H(x_{j-1},y_j,\bar z_{j-1})\frac{t_j-s}{t-s}$$

For $\xi\in C(\tau,x)$ there exists $y_0\in\partial v(x_0)$ such that equations (c1)-(c3) are satisfied. Then we have that $\varphi^{s,t_i}(x_0,y_0,v(x_0)) = (x_i,y_i,z_i)$, $i\le N$, $y_N=\partial_x S^{s,t}(\tau,x;\xi))$, $\bar z_N = S^{s,t}(\tau,x;\xi)$. We recall that

$$g^{\tau_i,\tau_{i+1}}(x_i,y_i,\bar z_i) = (1-\partial_z\Phi^{\tau_i,\tau_{i+1}})(x_i,y_{i+1}\bar z_i)$$

so that $\|1-\partial_z\Phi^{\tau_i,\tau_{i+1}}\|\le D$ for $i=0,\dots,N-1$ and then

$$\left|\partial_\tau S^{s,t}(\tau,x;\xi)\right| \le \frac{D^N-1}{D-1}2\|H\|$$

Since

$$\partial_\tau R_H^{s,t}v(x)\subset \text{ co }\{\partial_\tau S^{s,t}(\tau,x;\xi) : \varphi_H^{s,t}(x_0,y_0,v(x_0)) = (x,\partial_x S^{s,t}(\tau,x;\xi),S^{s,t}(t,x;\xi))\}$$

by the mean value Theorem, (Theorem 42),

$$\left|R_H^{s,\tau}v(x)-R_H^{s,t}v(x)\right| \le |\tau-t|\,\frac{D^N-1}{D-1}2\|H\|.$$

The proof of (iii) does not present changes in the general case. ■

Given a subdivision $\zeta = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0, T]$, we define its *norm* as $|\zeta| = \max |t_{i+1} - t_i|$, and the step function

$$\zeta(s) = \max\{t_i : t_i \leq s\}, \; s \in [0, T]$$

Definition 3.2.2 *The* iterated minimax operator *for the Cauchy problem* (HJ) *(with respect to ζ) is defined as follows: for $0 \leq s' < s \leq T$,*

$$R^{s',s}_{H,\zeta} v(x) = R^{t_j,s}_H \circ \cdots \circ R^{s',t_{i+1}}_H v(x) \tag{3-21}$$

where $t_j = \zeta(s), t_i = \zeta(s')$. When H is fixed, we omit the corresponding subscript.

Lemma 33 *Suppose that $(\zeta_n)_n$ is a sequence of partitions of $[0, T]$ such that $|\zeta_n| \to 0$. For $v \in C^{Lip}(\mathbb{R}^k)$, the sequence of functions $u_n(s, x) := R^{0,s}_{\zeta_n} v(x)$ is equi-Lipschitz and uniformly bounded on $[0, T] \times K$ for any compact $K \subset \mathbb{R}^k$.*

Proof. It follows from (3-16) that

$$\|\partial R^{0,s}_{\zeta_n} v\| \leq (\|\partial v\| + |s| \, \|\partial_x H\|) e^{|s| \|\partial_z H\|},$$

and from (3-17) that

$$\left| R^{0,t}_{\zeta_n} v(x) - R^{0,s}_{\zeta_n} v(x) \right| \leq |t - s| \, C(H) \|H\|,$$

so that in particular

$$\|R^{0,s}_{\zeta_n} v\|_K \leq \|v\|_K + T C(H) \|H\|.$$

The Lemma follows from these inequalities. ■

Proposition 34 *For any sequence (ζ_n) of subdivisions of $[0, T]$ with $|\zeta_n| \to 0$, and any compact set $K \subset \mathbb{R}^k$, the sequence $u_n := R^{0,s}_{\zeta_n} v(x)$ has a subsequence converging uniformly* *on* $[0,T]\times K$ *to the viscosity solution of the Cauchy problem* (HJ).

Proof. By lemma 33, we can apply Arzela-Ascoli Theorem to $(u_n)\subset C^0([0,T]\times K)$ to get a subsequence (u_{n_k}) converging uniformly to a function $\bar{R}^{0,s}v$. Define $\tilde{K}$ as in (3-19)-(3-20).

Claim 35 *For* $0\le s'<s\le T$ *one has*

$$\bar{R}^{0,s}v(x)=\lim_{n\to\infty}R^{s',s}_{\zeta_{n_k}}\circ\bar{R}^{0,s'}v(x) \tag{3-22}$$

Proof of claim 35.

Applying Arzela-Ascoli Theorem to $(u_{n_k})\subset C^0([0,T]\times\tilde{K})$, we can extract a subsequence converging uniformly in $[0,T]\times\tilde{K}$. To easy notation, when $s'=0$ we omit this superscript, and for the iterated minimax with respect to the partition ζ_n, we use the subscript n instead, and $(s)_n$ instead of $\zeta_n(s)$.

We first notice that for $0\le s\le T$, $x\in\tilde{K}$:

$$\bar{R}^{s}v(x)=\lim_{n\to\infty}R^{(s)_n}_{n}v(x) \tag{3-23}$$

because

$$\left|R^{s}_{n}v(x)-R^{(s)_n}_{n}v(x)\right|=\left|R^{(s)_n,s}\circ R^{(s)_n}_{n}v(x)-R^{(s)_n}_{n}v(x)\right|\le\|H\|(s-(s)_n)\le\|H\|\|\zeta_n\|.$$

Then for any $\epsilon>0$, there exists N such that if $i,j>N$, then

$$\forall s\in[0,T]:\|R^{(s)_i}_{i}v-R^{(s)_j}_{j}v\|_{\tilde{K}}<\epsilon.$$

Therefore,

$$\begin{aligned}\|R^{(s')_i,(s)_i}_{i}\circ R^{0,(s')_i}_{i}v-R^{(s)_i}_{i}v\|_K&=\|R^{(s')_i,(s)_i}_{i}\circ R^{(s')_j}_{i}v-R^{(s')_i,(s)_i}_{i}\circ R^{(s')_i}_{i}v\|_K\\&\le\|R^{(s')_j}_{j}v-R^{(s')_i}_{i}v\|_{\tilde{K}}<\epsilon\end{aligned}$$

Letting j go to ∞, we get

$$\|R_i^{(s')_i,(s)_i} \circ \bar{R}^{s'} v - R_i^{(s)_i} v\|_K < \epsilon,$$

thus, for any $x \in K$

$$\lim_{i \to \infty} R_i^{(s')_i,(s)_i} \circ \bar{R}^{s'} v(x) = \bar{R}^s v(x).$$

Similarly, we conclude that

$$\lim_{i \to \infty} R_i^{s',s} \circ \bar{R}^{s'} v(x) = \lim_{i \to \infty} R_i^{s',s} \circ \bar{R}^{s'} v(x).$$

■

We now prove that $\bar{R}^t v(x)$ is a viscosity subsolution of (3-15). Let ψ be a C^2 function with bounded second derivative defined in a neighborhood of $(t,x) \in \mathbb{R} \times K$, such that for s is close enough to t, $\psi(s,y) = \psi_s(y) \geq \bar{R}^s v(y)$, with equality at (t,x).

Suppose that $\tau \leq t$ is close enough t, so that the projections of the characteristics originating from

$$j^1(\psi_\tau)(x_\tau) = (x_\tau, d\psi_\tau(x_\tau), \psi_\tau(x_\tau))$$

do not intersect. Hence, the map $x_\tau \to x_t$ is a difeomorphism.

We conclude that

$$\psi_t(x) = \bar{R}^t v(x) = \lim_{k \to \infty} R_{n_k}^{\tau,t} \circ \bar{R}^\tau v(x) \leq \lim_{k \to \infty} R_{n_k}^{\tau,t} \psi_\tau(x) = R^{\tau,t} \psi_\tau(x). \tag{3-24}$$

The inequality is consequence of Corollary 3.1.1.

Also, when τ is close enough to t, iterated minimax will be the minimax ($N = 1$) which is a C^2 solution of (HJ) with initial condition ψ_τ, and thus

$$R^{\tau,t} \psi_\tau(x) = \psi_\tau(x) - \int_\tau^t H(\theta, j^1(\psi_\theta(x)))d\theta. \tag{3-25}$$

Subtracting (3-24) from (3-25), $\psi_t(x)$ to the right side, dividing both sides by $t - \tau$

and letting $\tau \to t$, we get

$$0 \leq -\partial_t \psi_t(x) - H(t, j^1(\psi_t(x))).$$

One proves that $\bar{R}^t v(x)$ is a viscosity supersolution of (3-15) in a similar way. ■

Given $H \in C_c^2([0,T] \times J^1(\mathbb{R}^k))$, $v \in C^{Lip}(\mathbb{R}^k)$, we say that a function $w : [s,t] \times \mathbb{R}^k \to \mathbb{R}$ is the limit of iterated minimax solutions for (HJ) on $[s,t]$, if for any sequence of subdivisions $\{\zeta_n\}_{n\in\mathbb{N}}$ of $[s,t]$ such that $|\zeta_n| \to 0$ as $n \to \infty$, the corresponding sequence of iterated minimax solutions

$$\left\{R^{s,\tau}_{H,\zeta_n} v(x)\right\}, (\tau, x) \in [s,t] \times \mathbb{R}^k$$

converges uniformly on compact subsets to w. We denote $w(\tau, x) := \bar{R}^{s,\tau}_H v(x)$.

We can now prove our main result

Proof of Main Theorem 1. Let $K \subset \mathbb{R}^k$ and (ζ_n) be any subsequence of subdivisions of $[0,T]$ such that $|\zeta_n| \to 0$. Denote $u_n(t,x) = R^{0,t}_{\zeta_n} v(x)$ and $u(t,x)$ the viscosity solution of the (HJ) problem. If u_n does not converge uniformly on $[0,T] \times K$, there exists a $\epsilon > 0$ and a subsequence n_k such that $|u_{n_k} - u| > \epsilon$. Note that ζ_{n_k} is itself a sequence of subdivions, this contradicts Proposition 34. ■

3.3 Example

Consider $H(x,y,z) = z + h(y)$ with h of compact support. The caracteristics equations are

$$\begin{cases} \dot{x} = dh(y) \\ \dot{y} = -y \\ \dot{z} = y dh(y) - z - h(y). \end{cases}$$

which can be integrated to obtain the flow

$$\varphi^t(x_0, y_0, z_0) = (x_0 + \int_0^t dh(y_0 e^{-s})ds, y_0 e^{-t}, -h(y_0 e^{-t}) + e^{-t}(h(y_0) + z_0).$$

Since that the map $(x_0, y_0, z_0) \mapsto (x_0, y_0 e^{-t}, z_0)$ is invertible, we can use (2-1) to define a generating function of φ^t

$$\begin{aligned} \Phi^t(x_0, y, z_0) &= y \int_0^t dh(y e^{t-s})ds + h(y) - e^{-t}h(e^t y) + z_0(1 - e^{-t}) \\ &= \int_0^t e^{-s}h(e^s y)ds + z_0(1 - e^{-t}) \end{aligned}$$

Thus the minimax solution of

$$\begin{cases} \partial_t u(t,x) + u(t,x) + h(\partial_x u(t,x)) = 0 \\ u(0,x) = v(x) \end{cases}$$

is given by

$$u(t,x) = \inf\max S_t(x, x_0, y),$$

where the generating function

$$S_t(x, x_0, y) = (x - x_0)y - \int_0^t e^{-s}h(e^s y)ds + e^{-t}v(x_0)$$

is quadratic at infinity because h has compact support. Indeed, $Q(x_0, y) = -x_0 y$ so that

$$S_t(x, x_0, y) - Q(x_0, y) = xy - \int_0^t e^{-s}h(e^s y)ds + e^{-t}v(x_0).$$

Since $S_t(x, x_0, y)$ is C^1 with respect to y, we have

$$\begin{aligned} \partial_{(x_0,y)}\left(S_t(x, x_0, y) - Q(x_0, y)\right) &= e^{-t}\partial v(x_0) \times \left\{x - \int_0^t dh(e^s y)ds\right\} \\ &= \left\{\left(e^{-t}p, x - \int_0^t dh(e^s y)ds\right) : p \in \partial v(x_0)\right\}. \end{aligned}$$

As h is compactly supported and v is a Lipschitz function,

$$\|\partial_{(x_0,y)}\left(S_{(t,x)} - Q\right)\| = \max_{(x_0,y)}\left\{\left\|\left(e^{-t}p, x - \int_0^t dh(e^s y)ds\right)\right\| : p \in \partial v(x_0)\right\}$$

is bounded, and therefore S_t is a `gfqi`.

Had we assumed instead that h was convex we would still had obtained a minimax

$$u(t,x) = \inf_{x_0}\max_{y}\left((x - x_0)y - \int_0^t e^{-s}h(e^s y)ds\right) + e^{-t}v(x_0)$$

with the $\max_y$ a Legendre transform being achieved when

$$x - x_0 = \int_0^t dh(e^s y)ds.$$

Letting l to be the Legendre transform of h, it is not hard to prove that

$$u(t,x) = \min_{y}\int_0^t e^{-s}l(dh(e^s y))ds + v\left(x - \int_0^t dh(e^s y)ds\right)$$

a formula that has appeared in the literature.

Appendix A

Clarke Calculus

In this chapter, we will show some results from *Clarke calculus* for *generalizaed gradients,* used in chapters above.

A.1 Definitions

We say that $\mathbb{R}^m \to \mathbb{R}$ *Lipschitz of rank* K near a given point $x \in \mathbb{R}^m$, if for some point $\epsilon > 0$, we have

$$|f(y) - f(z)| \leq K\|y - z\|,$$

for all $y, z \in B(x, \epsilon)$.

The *generalized directional derivative* of f at x in the direction v is defined as

$$f^o(x; v) := \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t},$$

for $y \in \mathbb{R}^m$ and $t > 0$.

A function g is called *positively homogeneous* if $g(\lambda v) = \lambda g(v)$ for $\lambda \geq 0$, and subadditive if for every v, w :

$$g(v + w) \leq g(v) + g(w).$$

Definition A.1.1 *A function* $F : X \to \mathbb{R}$ *is called* upper semicontinuous *if:*

$$v_i \in X \to v \in X \implies \limsup_{i\to\infty} F(v_i) \leq F(v).$$

Proposition 36 *[Clarke, 2013, Prop. 10.2] Let* f *be a Lipschitz function of rank* K *near* x*. Hence:*

(a) $v \mapsto f^o(x; v)$ *is finite, positively homogeneous and subadditive function on* $\mathbb{R}^m$*, and for all* $v \in \mathbb{R}^m$ *:*

$$|f^o(x; v)| \leq K\|v\|.$$

(b) For all $v \in \mathbb{R}^m$*, the map* $(u, w) \mapsto f^o(u; w)$ *is upper semicontinuous at* $(x; v)$*, and* $w \mapsto f^o(x; w)$ *is Lipschitz of rank* K *on* $\mathbb{R}^m$*.*

The following results allows us to prove the above one:

Theorem 37 *[Clarke, 2013, Theorem 4.25] Let* $g \in \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ *a lower semicontinuous, subadditive and positively homogeneous funtion such that* $g(0) = 0$*. Then there exists a unique convex subset* $\Sigma \subset \mathbb{R}^m$ *such that* g *is the support function* H_σ *of* Σ*, that is, for every* $x \in \mathbb{R}^m$ *:*

$$g(x) = H_\sigma(x) := \sup_{\sigma\in\Sigma} \langle \sigma, x\rangle.$$

The set Σ *is characterized by*

$$\Sigma = \{\eta \in \mathbb{R}^m | g(v) \geq \langle \eta, v\rangle, v \in \mathbb{R}^m\},$$

and it is a compact one if and only if g *is bounded on the unitary disc.*

Proof, prop. 36. *For the sake of brevity, we will not give a demonstration of the first statement.*

For the second one, whereas $v \to f^o(x; v)$ *is a lower semicontinuous, subadditive and positively homogeneous function, bounded on the unitary disc, because of Theorem 37, this is the support function of a convex subset on* $\mathbb{R}^m$*.* ■

Definition A.1.2 *The* generalized gradient *of a function f on x, denoted by $\partial f(x)$ in the unique compact convex non-empty subset of $\mathbb{R}^m$ whose support function is $f^o(x;\cdot)$. Therefore:*

$$\begin{cases} \zeta \in \partial f(x) \iff \forall v \in \mathbb{R}^m : f^o(x;v) \geq \zeta \cdot v \\ \forall v \in \mathbb{R}^m : f^o(x;v) = \max\{\zeta \cdot v | \zeta \in \partial f(x)\} \end{cases}$$

As corollary, the *generalized gradient* is well-defined.

A.2 Properties

Proposition 38 *Let f be a Lipschitz function of rank K near x. Then $\partial f(x) \subset B(0,K)$.*

Proof. For any $\eta \in \partial f(x)$, by proposition 36 we have that for all $v \in \mathbb{R}^m$

$$\langle \eta, v \rangle \leq K\|v\|.$$

■

If f is Lipschitz near x, and differentiable at x, then $f'(x) \subset \partial f(x)$, and the following result allows to generalize the concept of a critical point:

Lemma 39 (Fermat Rule) *Let $f : \mathbb{R}^m \to \mathbb{R}$ be locally Lipschitz:*

(a) If f has a local maximum or minimum at x, then $0 \in \partial f(x)$.

(b) If $0 \notin \partial f(x)$, there exists a decreasing direction v for f at x, that is,

$$\limsup_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t} < 0.$$

Conversely, a function $f : X \to \mathbb{R}$ which is continuously differentiable near x is locally Lipschitz near x, because the Mean Value Theorem. In this case, $\partial f(x)$ reduces to the classical derivative:

Proposition 40 *[Clarke, 2013, Theorem 10.8] If f is continuously differentiable near x, then $\partial f(x) = \{f'(x)\}$.*

Proof. At first, we will show that $f^o(x;v) = f'(x;v)$. Let $y_i \to x$ and $t_i \to 0$ be sequences such that $z_i \in [y_i, y_i + t_i v]$

$$\begin{aligned} f^o(x;v) &= \lim_{i\to\infty} \frac{f(y_i + t_i v) - f(y_i)}{t_i} \\ &= \lim_{i\to\infty} \langle f'(z_i), v\rangle \\ &= \langle f'(x), v\rangle = f'(x;v), \end{aligned}$$

because f' is continuous. ■

Proposition 41 (Sum Rule) *[Clarke, 2013, Theorem 10.13] Let f, g be Lipschitz near x. Then*

$$\partial(f+g)(x) \subset \partial f(x) + \partial g(x).$$

Proof. It can be shown ([see Clarke, 2013, cor. 3.13]) that if $\Sigma, \Delta \subset \mathbb{R}^m$ are closed convex subsets, then

$$\Sigma \subset \Delta \iff H_\Sigma \leq H_\Delta.$$

For the sake of brevity, we omit the proof of this claim.

Now it is straightforward to verify, by definition of generalized directional derivative, that for all $v \in X$:

$$(f+g)^o(x;v) \leq f^o(x;v) + g^o(x;v).$$

Since sets on both sides of the inclusion are closed and convex, it follows that $H_1(v) = (f+g)^o(x;v)$ and $H_2(v) = f^o(x;v) + g^o(x;v)$ are the support function of the corresponding sets and because of the above statements, we obtain the desired conclusion. ■

Corollary A.2.1 *If f is Lipschitz near x, and g es continuously differentiable, then*

$$\partial\left(f+g\right)(x) = \partial f(x) + \{g'(x)\}. \tag{A-1}$$

Similarly, we can obtain a version of the following classical result for generalized calculus:

Theorem 42 (Mean Value Theorem) *[Clarke, 2013, Theorem 10.17] Given* $x, y \in \mathbb{R}^m$ *such that* f *is Lipschitz in a neighborhood of interval*

$$[x, y] = \{\lambda x + (1 - \lambda) y, \lambda \in [0, 1]\}.$$

Hence. there exist a point at $(x, y) = \text{Interior}[x, y]$ *such that*

$$f(y) - f(x) \in \langle \partial f(z), y - x \rangle.$$

We will use the following special case of the *chain rule*:

Lemma 43 *Suppose that* f *is Lipschitz in a neighborhood of interval* $[x, y]$*. Define* $x_t = x + t(y - x), t \in [0, 1]$ *and*

$$g : [0, 1] \to \mathbb{R}, g(t) = f(x_t).$$

Then $\partial g(t) \subset \langle \partial f(x_t), y - x \rangle$.

Proof. Since both sets in the inclusion are indeed intervals on $\mathbb{R}$, it is enough to show that for every $v = \pm 1$:

$$\max\{\partial g(t) v\} \leq \max\{\langle \partial f(x_t), y - x \rangle v\}.$$

However, in this case $g^o(t;v) = \partial g(t)$. Hence

$$\begin{aligned} g^o(t;v) &= \limsup \left\{ \frac{g(s+\lambda v) - g(s)}{\lambda} \Big| s \to t, \lambda \downarrow 0 \right\} \\ &= \limsup_{s\to t, \lambda\downarrow 0} \left\{ \frac{f(x+[s+\lambda v](y-x)) - f(x+s(y-x))}{\lambda} \right\} \\ &\le \limsup \left\{ \frac{f(z+\lambda v(y-x)) - f(z)}{\lambda} \Big| z \to x_t, \lambda \downarrow 0 \right\} \\ &= f^o(x_t; v(y-x)) = \max\langle \partial f(x_t), v(y-x)\rangle. \end{aligned}$$

■

Proof, Theorem 42. Consider the function : $[0,1] \to \mathbb{R}$ defined by

$$\theta(t) = f(x_t) + t\left(f(x) - f(y)\right).$$

Note that $\theta(0) = \theta(1) = f(x)$, so that there exists $t^* \in [0,1]$ at which τ attains an extremal value. In this case, $0 \in \partial\theta(t^*)$, so that

$$0 \in f(x) - f(y) + \langle \partial f(x_{t^*}), y - x\rangle,$$

and we have proved our Theorem when $z = x_{t^*}$. ■

Finally. the following result says that the *generalized gradient* is closed:

Theorem 44 *[Wei, 2014, prop. A.2] Let f be Lipschitz of rank K near x and x_i, v_i sequences on $\mathbb{R}^m$ such that*

$$\begin{cases} x_i \to x \\ \zeta_i \in \partial f(x_i). \end{cases}$$

If $\zeta_i \to \zeta$, then $\zeta \in \partial f(x)$.

Proof. Fix $v \in \mathbb{R}^m$. For every i, we have that

$$f^o(x_i; v) \ge \zeta_i \cdot v.$$

The sequence $\zeta_i \cdot v$ is bounded on $\mathbb{R}$ and it contains term which are arbitrarly closed to $\zeta \cdot v$. Extract a subsequence of ζ_i (which we denote in the same way as the original one) such that $\zeta_i \cdot v \to \zeta \cdot v$. Taking the limit in the above inequality and whereas f^o is upper semicontinuous en x, we conclude that

$$f^o(x; v) \geq \zeta \cdot v.$$

Since v is arbitrary, it follows that $\zeta \in \partial f(x)$. ■

Appendix B

The Minimax Principle

B.1 Palais-Smale Condition

Definition B.1.1 *1. A sequence (u_m) in a manifold M is called of* Palais-Smale *for $E \in C^1(M)$ if $|E(u_m)| \leq c$ uniformly in m and*

$$\lim_{m\to\infty} \|DE(u_m)\| = 0.$$

2. We say that E satisfies Palais-Smale compactness condition (PS) *if every P-S sequence for E has a strongly convergent subsequence.*

Palais-Smale condition allows us to distinguish certain family of neighborhood of critical point of a given functional E; and hence, it will useful to characterize regular values of E.

For $\beta \in \mathbb{R}$, $\delta > 0$, $\rho > 0$ define

$$\begin{aligned} E_\beta &= \{u \in V : E(u) < \beta\} \\ K_\beta &= \{u \in V : E(u) = \beta, DE(u) = 0\} \\ N_{\beta,\delta} &= \{u \in V : |E(u) - \beta| < \delta, \|DE(u)\| < \delta\} \\ U_{\beta,\rho} &= \bigcup_{u \in K_\beta} \{v \in V : \|v - u\| < \rho\}. \end{aligned}$$

Proposition 45 *[Struwe, 2008, lemma 2.3] Suppose that E satisfies $(P.S)$. Then for every $\beta \in \mathbb{R}$, it follows that:*

1. *K_β is a compact subset;*
2. *both $\{U_{\beta,\rho}\}_{\rho>0}$ and $\{N_{\beta,\delta}\}_{\delta>0}$ are fundamental systems of neighborhoods for K_β.*

Remark 46 *In particular, if $K_\beta = \emptyset$ for some $\beta \in \mathbb{R}$ there exists $\delta > 0$ such that $N_{\beta,\delta} = \emptyset$; that is, the differential $DE(u)$ is uniformly bounded in norm for all $u \in V$, far enough from origin but with $E(u)$ close to β.*

B.2 A very general deformation lemma

Denote by

$$\tilde{V} = \{u \in V : DE(u) \neq 0\}$$

the set of regular points of E. Instead of a gradient, which requires the existence of well-defined inner product, we will use the following:

Definition B.2.1 *A* pseudogradient *vector field for E is a locally Lipschitz continuous one $v \in \tilde{V} \to V$ for which the following conditions hold:*

1. $\|v(u)\| < 2 \min \{\|DE(u)\|, 1\}$;

2. *for all* $u \in \tilde{V}$:

$$\langle v(u), DE(u)\rangle > \min\{\|DE(u)\|, 1\}\,\|DE(u)\|.$$

Lemma 47 *[Struwe, 2008, lemma 3.2] Any functional* $E \in C^1(V)$ *admits a pseudogradient vector field* $v : \tilde{V} \to V$.

Theorem 48 (Deformation Lemma) *[Struwe, 2008, Theorem 3.4] Suppose that* $E \in C^1(V)$ *satisfies* (PS). *Fix* $\beta \in \mathbb{R}$, $\bar{\epsilon} > 0$ *and* N *some neighborhood of* K_β. *Thus there exist* $\epsilon \in (0, \bar{\epsilon})$ *and a uniparametric continuous family of homeomorphisms*

$$\Phi(\cdot, t) : V \to V,\ 0 \le t < t$$

with the following properties:

1. $\Phi(u, t) = u$, *if* $t = 0$ *or* $DE(u) = 0$ *or* $|E(u) - \beta| \ge \bar{\epsilon}$;

2. $E(\Phi(u, t))$ *is non-decreasing on* t *for all* $u \in V$;

3. $\Phi(E_{\beta+\epsilon}\backslash N, 1) \subset E_{\beta-\epsilon}$, *and* $\Phi(E_{\beta+\epsilon}, 1) \subset E_{\beta-\epsilon} \cup N$.

Moreover $\Phi : V \times [0, \infty) \to V$ *has semigroup property, that is,*

$$\forall s, t \ge 0 :\ \Phi(\cdot, t) \circ \Phi(\cdot, t) = \Phi(\cdot, s + t).$$

Since $\Phi : V \times [0, \infty)$ is obtained integrating a truncated pseudogradient vector field in suitable manner, Φ is called local pseudogradient flow.

B.3 Minimax Principle

Definition B.3.1 *Let* $\Phi : M \times [0, \infty) \to M$ *be a semiflow in a manifold* M. *A family* $\mathcal{F}$ *of subsets of* M *is called positively* $\Phi-$*invariant if* $\Phi(F, t) \in \mathcal{F}$ *for every* $F \in \mathcal{F}, t \ge 0$.

Theorem 49 (Minimax Principle) *[Struwe, 2008, Theorem 4.2] Suppose that M is a complete Finsler manifold of class $C^{1,1}$ and $E \in C^1(M)$ satisfies the (PS) condition. Also suppose that $\mathcal{F} \subset \mathcal{P}(M)$ is a collection of subsets invariant with respect to any semiflow $\Phi : M \times [0, \infty) \to M$ such that $\Phi(\cdot, 0) = \mathrm{Id}$, $\Phi(\cdot, t)$ is a homeomorphism of M for every $t \geq 0$, y $E(\Phi(u, t))$ is non-decreasing in t for every $u \in M$. Therefore, if*

$$\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u)$$

is finite, β is a critical value of E.

Example B.3.1 *Let X be a topological space and $[X, M]$ the set of free-homotopy classes $[f]$ of continuous maps $f : X \to M$. For $[f] \in [X, M]$ define*

$$\mathcal{F} = \{g(X) | g \in [f]\}.$$

Since $[\Phi \circ f] = [f]$ for any homeomorphism Φ de M homotopic to identity, the family $\mathcal{F}$ is invariant under such maps Φ. Hence if

$$\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u)$$

is finite, β is a critical value.

B.4 Palais-Smale conditions in the Context of Clarke Calculus

Definition B.4.1 *A point $\xi \in X$ is called* critical *for f if $0 \in \partial f(\xi)$; the value $f(\xi)$ is called* critical *for f. Note that the* critical set *$crit(f)$ of f, consisting of every critical point is closed on X.*

Define

$$\lambda(\xi) = \min_{w \in \partial f(\xi)} \|w\|_{X^*}.$$

We say that f satisfies the Palais-Smale condition (PS) *if every subsequence (ξ_n) such that $f(\xi_n)$ is bounded and $\lambda(\xi_n) \to 0$ has a convergent subsequence whose limit is a critical point of f and thus there exists $y_n \in \partial f(\xi_n)$ such that $y_n \to 0$.*

Proposition 50 *[Wei, 2014, example A.4] (PS) condition holds where $\|f - Q\|_{\text{Lip}} \leq \infty$ for some non-degenerate quadratic form $Q : X \to \mathbb{R}$. In this case $crit(f)$ is* compact.

Proof. If we define $\psi = f - Q$, each subset $\partial f(\xi) = \partial\psi(\xi) + dQ(\xi)$ consists of vectors whose norm is at least $\|dQ(\xi)\| - \|\psi\|_{\text{Lip}}$, y hence

$$\lambda(x) \geq \|dQ(\xi)\| - \|\psi\|_{\text{Lip}},$$

so that $\lambda(x) \to \infty$ where $\|x\| \to \infty$. Therefore, there exists $R > 0$ such that each sequence (x_n) with $\lim \lambda(x) = 0$ satisface $\|x_n\| \leq R$ for n large enough, and this follows both (PS) condition and compactness of $crit(f)$. ∎

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