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CONTROL ADAPTABLE BASADO EN CONTRACCIÓN PARA SISTEMAS CON PARAMETRIZACIÓN NO LINEAL

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PRESENTA:
M. ING. ANAHÍ FLORES PÉREZ

TUTOR PRINCIPAL
DR. YU TANG XU, FACULTAD DE INGENIERÍA, UNAM
COMITÉ TUTOR
DR. LEONID FRIDMAN, FACULTAD DE INGENIERÍA, UNAM
DR. GERARDO RENÉ ESPINOSA PÉREZ, FACULTAD DE INGENIERÍA, UNAM

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JURADO ASIGNADO:

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1 er. Suplente:	DR. ÁLVAREZ ICAZA LONGORIA LUIS A.
2 do. Suplente:	DR. FRIDMAN LEONID

Lugar o lugares donde se realizó la tesis: FACULTAD DE INGENIERÍA, UNAM.

TUTOR DE TESIS:

DR. YU TANG XU

FIRMA

(Segunda hoja)

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Chapter 1

Introduction

1.1 Motivation

Adaptive control of dynamical systems with linear parameterizations has been extensively studied and its theoretical framework is formally well established [1, 2]. If the parameters of the system are non linear, it is often possible to introduce additional variables by means of nonlinear transformations in order to obtain linear uncertain systems. However, there is no systematic way to find such transformations and they may imply certain loss of robustness or can cause stricter convergence conditions [3]. Many control problems of practical importance ([3–6] and the references cited therein) and potential applications in interdisciplinary fields (*e.g.* bio-chemical processes [7, 8], biology and neuroscience [9, 10]) are written in terms of complex and nonlinearly parameterized dynamics. Adaptive control of such kind of systems is currently a challenging problem and an active area of research.

1.2 State of the art

When the nonlinear functions describing the system dynamics have some particular features in the parameters (*e.g.,* convexity/concavity or monotonicity), important progress in developing new algorithms for direct adaptive control has been accomplished. For concave/convex functions of the parameters, a Min-max optimization procedure combined with a switching mechanism between regions of convexity/concavity was presented in [11] relying on the basic ideas of [12, 13]. Such switching approach was recently removed in

[4–6], where monotonicity and general parameterizations were studied. These works share a common strategy to deal with nonlinear parameterizations: an additional *proportional* term, which serves as an interaction between the tracking dynamics and the adaptation law, is added to the conventional *integral* Lyapunov-based adaptive algorithm. The immersion and invariance methodology [4] relies upon the idea of system *projection* into an invariant asymptotically attractive manifold with specified properties. In this case, a mapping, directly related with the nonlinearity, is required to be monotone in the parameters or in a particular parameterization. The main goal of the proportional action is to make the off-manifold variable (basically the estimation error) asymptotically convergent. Based on the Lyapunov analysis of a 'linear-like' error model, [5] introduces the same degree of freedom in order to make a matrix, related to tracking and estimation error, negative definite. In [6] the scheme is founded on an operator formalism where monotonicity, w.r.t. the parameterization, is needed to satisfy certain growth nonlinear requirements.

If convex, concave or monotone conditions are not satisfied, it is possible to 'convexify' [8] or 'monotonize' [14] the parameterization. In [8] a transformation was built to impose a convex parameterization on a plant otherwise non-convex. In [14], derived from the results of [4], monotonicity is not assumed *a priori* but can be enforced by means of the aforementioned proportional term.

In contrast with the numerous results for direct adaptive control, reports of adaptive estimation (or indirect control) for nonlinearly parameterized systems are few in the literature. Regarding to general nonlinear parameterizations, a methodology based on a hierarchical strategy was proposed in [15]. In [14], direct and indirect methodologies are obtained from the same approach, *i.e.* the overall system is viewed and analyzed as the cascade of a convergent estimator and a perturbed controlled system. This means that, even in the direct version, a nonlinear condition of persistent excitation is implicitly involved but not clearly established. In recent results [16, 17], adaptive control and estimation have been considered for uncertainties within a nonlinear function of linear parameterizations.

On the other hand, nonlinear contraction theory [18, 19] has been recently developed as a nonlinear analysis and design tool (see [20, 21] and the references therein). Contraction is an incremental form of stability which studies convergence in terms of the proximity among trajectories of a nonlinear system. A particular concept in contraction analysis is the *partial contraction* introduced in [19]. In virtue of such concept, a key step in contraction based adaptive design is to find a *virtual* system which has as particular solutions the trajectories of the target and those of the closed-loop systems. Contraction of the virtual system implies that trajectories of the system under study, initialized within the contraction region, converge exponentially to those of the target system.

1.3 Objectives and methodology

The main objectives of this work are

- To develop a general scheme of adaptive control for systems with nonlinear parameterizations based on contraction.
- To generalize the strong (classical) and the existing relaxed conditions for contraction with the objective of formulating a stability analysis for adaptive problems completely immersed in the contraction framework.
- To establish relationships between the extended conditions for contraction and well known results of conventional adaptive control.
- To apply the algorithms to a wide variety of academic examples and practical applications and verify performance by means of numerical simulations.

These objectives are accomplished by proposing an *averaged* condition for contraction and by formulating a top-down approach of adaptive control which consists in two stages. At the top-level, the *actual system*, the closed-loop system formed by the tracking (or estimation) error dynamics and an adaptive algorithm, is proposed. At this stage, the adaptive algorithm may depend on the unknown parameters and hence is non-implementable. Based on the structure of the actual system and the design objectives written as a *desired system*, a *virtual system*, which has as particular solutions the trajectories of the underlined actual system and those of the desired system, is established. Averaged contraction properties of the virtual system are then searched through a proper selection of the design degrees of freedom. Once the desired contraction requirements are established, at the down-level, the adaptive algorithms are realized through a proportional-integral (PI) scheme. The main advantages of this methodology include the design flexibility in the proposal of the adaptive algorithms, the transparency in selecting the design degrees of freedom, and simplicity of the resulting adaptive control designs, as compared to those obtained by using Lyapunov techniques.

1.4 Contributions

The contributions of this work are summarized in the following:

- A contraction characterization which extends the strong (classical) results given in [18, 19] and strictly weaker than those relaxed conditions reported in [18, 22] is formulated, as far as we know, for the first time in this work. Such extension, called averaged condition for contraction, is instrumental for the convergence and stability analysis of the proposed methodology.

- A general adaptive control framework, composed by schemes of estimation and direct-indirect adaptive control, is derived for nonlinear systems with nonlinear parameterizations with stability and convergence analysis founded on averaged contraction conditions.

- A strategy to design adaptive controllers and identification schemes capable of dealing with a large class of systems with nonlinear parameterizations that is based on a top-down approach in which boundedness of trajectories is analyzed independently of the algorithm implementation. Therefore, design is modular, simpler and more transparent than those adaptive controllers developed under the Lyapunov techniques.

- A properly designed virtual system which allows to conclude exponential convergence of the tracking (or estimation) and parameters trajectories to the desired ones is also provided. Similarly to the problem of finding an appropriate Lyapunov function, the virtual system proposal is crucial to improve clearness when dealing with contraction based adaptive problems.

- Nonlinear parameters models with practical significance such as fermentation dynamics, nonlinear friction systems and electro hydraulic mechanisms are successfully controlled and/or identified by means of the proposed methodology. The main features of the scheme are shown for each case and performance is verified by means of numerical simulations.

1.5 Notation and thesis structure

The gradient of scalar functions $g(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} = [x_1\ x_2 \ldots x_n]^T \in \mathbb{R}^n$ is denoted by *row* vectors, *i.e.*

$$\frac{\partial g(\mathbf{x})}{\partial \mathbf{x}} = \left[\frac{\partial g(\mathbf{x})}{\partial x_1}\ \frac{\partial g(\mathbf{x})}{\partial x_2} \cdots \frac{\partial g(\mathbf{x})}{\partial x_n}\right].$$

The derivative of vector functions $\mathbf{h}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{h}(\mathbf{x}) = [h_1(\mathbf{x})\ h_2(\mathbf{x}) \ldots h_m(\mathbf{x})]^T$, $h_i(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R},\ i = 1, \ldots, m$ w.r.t a scalar variable x_i is a *column* vector given by

$$\frac{\partial \mathbf{h}(\mathbf{x})}{\partial x_i} = \begin{bmatrix} \frac{\partial h_1(\mathbf{x})}{\partial x_i} \\ \frac{\partial h_2(\mathbf{x})}{\partial x_i} \\ \vdots \\ \frac{\partial h_m(\mathbf{x})}{\partial x_i} \end{bmatrix}.$$

The $m \times n$ Jacobian matrix of $\mathbf{h}(\mathbf{x})$ is denoted by

$$\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}} = \left[\frac{\partial \mathbf{h}(\mathbf{x})}{\partial x_1}\ \frac{\partial \mathbf{h}(\mathbf{x})}{\partial x_2} \cdots \frac{\partial \mathbf{h}(\mathbf{x})}{\partial x_n}\right].$$

Arguments of functions are omitted when clear in the context. Throughout the work, $\mathbf{A}_s$, $\overline{\lambda}(\mathbf{A})$, $\underline{\lambda}(\mathbf{A})$ and $\sigma(\mathbf{A})$ stand for the symmetric part, largest eigenvalue, smallest eigenvalue, and largest singular value of matrix $\mathbf{A}$, respectively. $\mathbb{R}_{\geq t_0} = \{t \in \mathbb{R} |\ t \geq t_0 \geq 0\}$ and $\mathbf{I}$ represents the identity matrix. Symbol $\|\cdot\|$ will refer to the Euclidean norm of vectors and $\|\cdot\|_2$ will denote the induced 2-norm for transfer functions. A function dependent on virtual variables will be denoted by a subindex v, *i.e.* $f(\mathbf{x}, \boldsymbol{\theta}_v, t) \triangleq f_v$, $\mathbf{R}(\varphi_v, \varphi_d, \mathbf{x}, \boldsymbol{\theta}_v) \triangleq \mathbf{R}_v$, $\boldsymbol{\alpha}(\mathbf{x})^T \boldsymbol{\theta}_v \triangleq \xi_v$, $f_\alpha(\mathbf{x}, \xi_v, t) \triangleq f_{v_\alpha}$, $\boldsymbol{\alpha}_i(\mathbf{x})^T \boldsymbol{\theta}_{v_i} \triangleq \xi_{v_i}$, $f_i(\mathbf{x}, \xi_{v_i}, t) \triangleq f_{v_i}$, and $(\varphi_v, \boldsymbol{\theta}_v) \triangleq \mathbf{z}_v$. Symbol $\hat{(\cdot)}$ is used for estimated variables: $f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) \triangleq \hat{f}$, $\mathbf{R}(\varphi, \varphi_d, \mathbf{x}, \hat{\boldsymbol{\theta}}) \triangleq \hat{\mathbf{R}}$, $\boldsymbol{\alpha}(\mathbf{x})^T \hat{\boldsymbol{\theta}} \triangleq \hat{\xi}$, $f_\alpha(\mathbf{x}, \hat{\xi}, t) \triangleq \hat{f}_\alpha$, $\boldsymbol{\alpha}_i(\mathbf{x})^T \hat{\boldsymbol{\theta}}_i \triangleq \hat{\xi}_i$, $f_i(\mathbf{x}, \hat{\xi}_i, t) \triangleq \hat{f}_i$, and $(\varphi, \hat{\boldsymbol{\theta}}) \triangleq \hat{\mathbf{z}}$. For the desired variables the functions are referred as $f(\mathbf{x}, \boldsymbol{\theta}, t) \triangleq f$, $\mathbf{R}(\varphi_d, \varphi_d, \mathbf{x}, \boldsymbol{\theta}_d) \triangleq \mathbf{R}$, $\boldsymbol{\alpha}(\mathbf{x})^T \boldsymbol{\theta} \triangleq \xi$, $f_\alpha(\mathbf{x}, \xi, t) \triangleq f_\alpha$, $\boldsymbol{\alpha}_i(\mathbf{x})^T \boldsymbol{\theta}_i \triangleq \xi_i$, $f_i(\mathbf{x}, \xi_i, t) \triangleq f_i$, and $(\varphi_d, \boldsymbol{\theta}_d) \triangleq \mathbf{z}_d$.

The theoretical preliminaries necessary to develop the adaptive control algorithm for systems with nonlinear parameterizations and a formulation of the averaged condition for contraction are depicted on Chapter 2. Averaged condition is instrumental for analyzing the stability of the direct adaptive control scheme which is developed in Chapter 3 where the main features of the methodology are exposed. Through a parallel analysis to that done in Chapter 3, an identification framework is developed in Chapter 4 where parameterizations are restricted to those which motivate relationships with conventional results of adaptive control theory. For completing the adaptive control framework, the problem of indirect adaptive control is solved in Chapter 5. Finally, conclusions and future perspectives are outlined in Chapter 6.

Chapter 2

Theoretical preliminaries

2.1 Contraction theory

Consider a nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t),\ \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, t is the time, and $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ is a nonlinear vector field assumed continuously differentiable. Contraction theory focuses its analysis on infinitesimal displacements $\delta\mathbf{x}$ of the state $\mathbf{x}$ at fixed time. The time evolution of these displacements is obtained from (2.1) by computing its first variation, *i.e.*

$$\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \delta\mathbf{x}. \tag{2.2}$$

A state-dependent change of coordinates can be introduced by means of a nonsingular uniformly invertible matrix $\mathbf{P}(\mathbf{x}, t)$. New coordinates are defined by $\delta\mathbf{z} \triangleq \mathbf{P}(\mathbf{x}, t)\delta\mathbf{x}$ which lead to

$$\delta\dot{\mathbf{z}} = \mathbf{J}(\mathbf{x}(t), t)\delta\mathbf{z}, \tag{2.3}$$

where $\mathbf{J}$ is given in the following definition.

Definition 2.1. [18] The dynamical system (2.1) is said to be contracting if there exists a constant $\beta > 0$, called the contraction rate of (2.1), such that the generalized Jacobian is uniformly negative definite, *i.e.*

$$\mathbf{J} \triangleq \left(\dot{\mathbf{P}} + \mathbf{P}\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}}\right)\mathbf{P}^{-1} \leq -\beta \mathbf{I}_{n\times n},\ \forall \mathbf{x} \in \mathcal{X} \subset \mathbb{R}^n,\ \forall t \geq t_0, \tag{2.4}$$

where $\mathcal{X}$ is the contraction region. △

If $\mathcal{X} = \mathbb{R}^n$, the contraction is global and if $\beta = 0$, system (2.1) is semi-contracting. The following Lemma is the core of the concept of partial contraction.

Lemma 2.1. *[19] Consider a nonlinear system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t)$, and assume that the virtual system $\dot{\mathbf{x}}_v = \mathbf{f}(\mathbf{x}_v, \mathbf{x}, t)$ is contracting with respect to $\mathbf{x}_v$. If a particular solution of the virtual system verifies a smooth specific property, then all trajectories of the original $\mathbf{x}$ system will verify this property exponentially.*

By Definition 2.1, a system is contracting if the generalized Jacobian matrix is uniformly negative definite $\forall t \geq t_0$. However, in many problems concerning adaptive systems, contraction analysis needs to rely on weaker conditions [22]. The following lemma, proposed as part of the results of this work, gives a condition for contraction strictly weaker than those in [18, 22]. It states that the system (2.1) is contracting if the symmetric part of the generalized Jacobian matrix is uniformly negative definite in an averaged sense.

Lemma 2.2. *Suppose that $\mathbf{J}_s(\mathbf{x}(t), t)$ is uniformly semi-negative definite (USND) and assume that there exist $T > 0$ and $\lambda > 0$ such that*

$$\int_t^{t+T} \mathbf{J}_s(\mathbf{x}(\tau), \tau) d\tau \leq -\lambda \mathbf{I}_{n\times n}, \tag{2.5}$$

uniformly $\forall t \geq t_0$, and for all $\mathbf{x} \in \mathcal{X}$. Then (2.1) is contracting.

Proof. It will be shown that (2.1) is contracting by showing that, under the given assumptions, $\delta\mathbf{z}(t) = \mathbf{0}$ is a local exponentially stable trajectory of (2.3). By assumption $\mathbf{J}_s(\mathbf{x}(t), t)$ is USND then, from $\frac{d}{dt}\|\delta\mathbf{z}(t)\|^2 = \delta\mathbf{z}^T(t)\mathbf{J}_s(\mathbf{x}(t), t)\delta\mathbf{z}(t)$, we conclude that $\delta\mathbf{z}(t)$ and $\delta\dot{\mathbf{z}}(t)$ are uniformly bounded. Hence, uniform boundedness of $\mathbf{J}$ is obtained from (2.3). This argument allows us to use the approach of [23]. By [23], Theorem 1, the first variation (2.3) will be locally exponentially stable in the particular solution $\delta\mathbf{z}(t) = \mathbf{0}$ if there exists an increasing sequence of times t_k ($k \in \mathbb{Z}$), $t_k \to \infty$ as $k \to \infty$, where $t_{k+1} - t_k \leq T$, $T > 0$, such that the positive definite function $V(\delta\mathbf{z}(t)) = \frac{1}{2}\delta\mathbf{z}^T(t)\delta\mathbf{z}(t) \triangleq V(t)$ satisfies

$$V(t_{k+1}) - V(t_k) \leq -\nu\|\delta\mathbf{z}(t_k)\|^2, \tag{2.6}$$

for some $\nu > 0$, $\forall k \in \mathbb{Z}$, for all $\mathbf{x} \in \mathcal{X}$. The time derivative of $V(t)$ along trajectories of (2.3) is equal to $\dot{V}(t) = \delta\mathbf{z}^T(t)\mathbf{J}_s(\mathbf{x}(t), t)\delta\mathbf{z}(t)$. By integrating both sides of the previous equation in the time interval $[t_k, t_{k+1}]$, it is clear that (2.6) is equivalent to show that

$\int_{t_k}^{t_k+1} \delta\mathbf{z}^T(\tau)\mathbf{J}_s(\mathbf{x}(\tau),\tau)\delta\mathbf{z}(\tau)d\tau \leq -\nu\|\delta\mathbf{z}(t_k)\|^2$. By [23], Theorem 2, eq. (14) particularized to linear, not time scaled ($\alpha = 1$) systems (2.3), the previous inequality will hold if condition (2.5) is true. The result follows by taking $[t_k, t_{k+1}] = [t, t+T]$. □

The dynamics in (2.1) is said to be contracting toward a flow-invariant linear subspace $\mathcal{M} \subset \mathbb{R}^n$ if all its trajectories converge toward $\mathcal{M}$ exponentially. This is stated below.

Lemma 2.3. *[24] Let p be the dimension of $\mathcal{M}$ and $\mathbf{V}$ be an $(n-p)\times n$ matrix, whose rows are an orthonormal basis of $\mathcal{M}^\perp$. If $\mathbf{P}\mathbf{V}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{V}^T\mathbf{P}^{-1}$ is uniformly negative definite for some constant invertible matrix $\mathbf{P}$ in $\mathbb{R}^{n-p}$, then system (2.1) is contracting toward $\mathcal{M}$.*

2.2 Tools on matrix theory

When two contracting systems are feedback interconnected [19], the overall dynamics leads to a generalized Jacobian whose symmetric part can be written as

$$-\mathbf{J}_s = \begin{bmatrix} \mathbf{J}_{1_s} & \mathbf{G}^T \\ \mathbf{G} & \mathbf{J}_{2_s} \end{bmatrix}. \tag{2.7}$$

The right hand side of (2.7) is uniformly positive definite (UPD) if and only if $\mathbf{J}_{1_s}$ and $\mathbf{J}_{2_s}$ are UPD and $\mathbf{J}_{2_s} > \mathbf{G}^T\mathbf{J}_{1_s}^{-1}\mathbf{G}$ ([25], p. 472), this condition is satisfied if

$$\overline{\lambda}(\mathbf{J}_{1_s})\overline{\lambda}(\mathbf{J}_{2_s}) > \sigma^2(\mathbf{G}),\ \forall \mathbf{x} \in \mathcal{X} \subset \mathbb{R}^n,\ \forall t \geq t_0. \tag{2.8}$$

Let some matrix $\mathbf{M} \in \mathbb{R}^{p\times p}$, $p \geq 2$, be such that it can be partitioned as

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & d \end{bmatrix}, \tag{2.9}$$

where $\mathbf{A} \in \mathbb{R}^{(p-1)\times(p-1)}$, $\mathbf{b} \in \mathbb{R}^{p-1}$, $d \in \mathbb{R}$. Define $\mathbf{A}_0 = d\mathbf{A} - \mathbf{b}\mathbf{b}^T$. By means of the identity

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & d \end{bmatrix}\begin{bmatrix} d\mathbf{I}_{p-1\times p-1} & \mathbf{0} \\ -\mathbf{b}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{b} \\ \mathbf{0} & d \end{bmatrix},$$

it is clear that $\det(\mathbf{M}) = \frac{\det(\mathbf{A}_0)}{d^p}$, hence, if $d > 0$, $\det(\mathbf{M}) \geq 0$ if and only if $\det(\mathbf{A}_0) \geq 0$.

2.3 Persistent excitation

Let us make explicit the parameters dependence on system (2.1)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}, t), \;\; \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2.10}$$

where $\boldsymbol{\theta} \in \mathbb{R}^p$ is the vector of unknown parameters. The following definition is established.

Definition 2.2. [26] Let the function $\boldsymbol{\alpha}(\mathbf{x}, t) : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^p$ be bounded and globally Lipschitz in $\mathbf{x}$ uniformly in t. Define $\mathbf{x}(\zeta, t)$ as one solution of (2.10) where $\zeta = (\mathbf{x}_0, \boldsymbol{\theta}_0, t_0)$, and $\boldsymbol{\theta}_0 \in \mathbb{R}^p$ stands for initial conditions in tuning parameters. Function $\boldsymbol{\alpha}(\mathbf{x}(\zeta, t), t)$ is *uniformly* persistently exciting if and only if there exist $\epsilon_\alpha > 0$ and $\delta > 0$ such that for all $t \geq t_0, \;\; \mathbf{x}_0 \in \mathbb{R}^n, \;\; \boldsymbol{\theta}_0 \in \mathbb{R}^p$,

$$\int_t^{t+\delta} \boldsymbol{\alpha}(\mathbf{x}(\zeta, \tau), \tau) \boldsymbol{\alpha}^T(\mathbf{x}(\zeta, \tau), \tau) d\tau \geq \epsilon_\alpha \mathbf{I}_{p \times p}. \tag{2.11}$$

△

Chapter 3

Direct adaptive control

In this chapter, a direct adaptive control scheme for dynamical systems with nonlinear parameterizations is developed and the main features of such methodology are described. The design is based on a top-down approach in which establishment of the tracking error dynamics and proposal of the (non-implementable) adaptive algorithms are at the top stage of the procedure. Based on the resulting closed-loop (actual) system structure and the design objectives expressed as a desired system, a virtual system, whose particular solutions are the trajectories of the actual and desired dynamics, is proposed. Contractive properties of the virtual system are then looked for through an appropriate selection of the design degrees of freedom. At the down stage of the design, the adaptive algorithms are realized through a proportional-integral (PI) scheme. It will be shown that, when the parameterization is linear, the proposed controller recovers gradient-like adaptive algorithms with certain improved convergence property. Several nonlinearly parameterized models in practical applications and academic examples are simulated to illustrate theoretical features and performance of the methodology.

3.1 Motivating example

The following example from [5] motivates the design of the top-down methodology. Consider

$$\dot{x} = -x - (1 + x^2)\theta + x\theta^2 - \frac{1}{3}\theta^3 + u, \tag{3.1}$$

where $x \in \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input and the parameter $\theta \in \mathbb{R}$ belongs to an interval $[\underline{\theta}, \overline{\theta}]$ whose bounds are known. The main objective is to drive the tracking error

$\varphi(t) = x(t) - x_r(t)$ exponentially to zero, where $x_r(t)$ is a known smooth reference with $|x_r(t)| < \overline{x}_r,\ \overline{x}_r > 0$, and, if possible, to reconstruct the nominal parameter while all the closed-loop signals remain bounded. Suppose that over parameterization is not allowed so θ is the only parameter to be adapted. By taking the notation $f_p(x,u) = -x + u$ and $g(x,\theta) = (1+x^2)\theta - x\theta^2 + \frac{1}{3}\theta^3$, the control input $u(x,\hat{\theta}) = -\lambda_\varphi\varphi + g(x,\hat{\theta}) + \dot{x}_r + x,\ \lambda_\varphi > 0$, renders the tracking dynamics as

$$\dot{\varphi} = -\lambda_\varphi \varphi + g(x,\hat{\theta}) - g(x,\theta). \tag{3.2}$$

Consider, at the top-level of the design, the following (non-implementable) adaptive algorithm

$$\dot{\hat{\theta}} = R(\varphi,\varphi_d,x,\hat{\theta}) + \Gamma(x)\left(g(x,\hat{\theta}) - g(x,\theta)\right), \tag{3.3}$$

where φ_d denotes the *desired tracking error* of the controlled system, $\Gamma(x) : \mathbb{R} \to \mathbb{R}$ and $R(\varphi,\varphi_d,x,\hat{\theta}) : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, with $R(\varphi_d,\varphi_d,x,\theta_d) = 0$, are design functions. Subindex d will stand for desired trajectories, *i.e.* the behavior that the *actual* system (3.2)-(3.3) would ideally have in absence of the parameter mismatch. The rationale of the adaptive algorithm is that the adaptation of the parameter must be proportional to the the parameter mismatch and have sufficient freedom for meeting the desired (contraction) properties. Structure of (3.2)-(3.3) suggests that the *desired* dynamics can be written as

$$\dot{\varphi}_d = -\lambda_\varphi \varphi_d + g(x,\theta_d) - g(x,\theta), \tag{3.4a}$$

$$\dot{\theta}_d = R(\varphi_d,\varphi_d,x,\theta_d) + \Gamma(x)(g(x,\theta_d) - g(x,\theta)),\ \theta_d(0) = \theta. \tag{3.4b}$$

From Lemma 2.1, it follows that a virtual system common to the actual and the desired systems can be defined. The core of our methodology is based on rendering this virtual system contracting in the generalized sense of Lemma 2.2 which guarantees mutual exponential convergence among particular solutions. Towards this aim, the virtual system for the actual (3.2)-(3.3) and the desired (3.4) dynamics is found to be

$$\dot{\varphi}_v = -\lambda_\varphi \varphi_v + g(x,\theta_v) - g(x,\theta), \tag{3.5a}$$

$$\dot{\theta}_v = R(\varphi_v,\varphi_d,x,\theta_v) + \Gamma(x)\left(g(x,\theta_v) - g(x,\theta)\right). \tag{3.5b}$$

To study its contraction properties under the approach of Lemma 2.2, the time integral of the symmetric part of its Jacobian is calculated as (see (2.7))

$$-\int_t^{t+T} \mathbf{J}_s(\tau)d\tau = \int_t^{t+T} \begin{bmatrix} \lambda_\varphi & G(\tau) \\ G(\tau) & J_{2_s}(\tau) \end{bmatrix} d\tau,$$

where[1] $G = -\frac{1}{2}\left(\frac{\partial R_v}{\partial \varphi_v} + 1 + (x-\theta_v)^2\right)$ and $J_{2_s} = -\frac{\partial R_v}{\partial \theta_v} - \Gamma(x)\left(1 + (x-\theta_v)^2\right)$. The objective now is to choose R_v and $\Gamma(x)$ such that the sufficient conditions

$$J_{2_s} > \lambda_\theta,\ \lambda_\theta > 0, \tag{3.6}$$

$$\lambda_\varphi \lambda_\theta > G^2, \tag{3.7}$$

hold. In the following two different R_v functions, which show the flexibility of this methodology, are considered.

i) $R_v = 0$. With the choice $\Gamma(x) = -\gamma,\ \gamma > 0$, and by noticing that $1 + (x-\theta_v)^2 \geq 1$, (3.6) is satisfied with $\lambda_\theta = \gamma$. To verify (3.7) let $\mathbf{w}_v \triangleq [x\ \theta_v]^T$, and consider

$$\mathcal{B} = \{\mathbf{w}_v \in \mathbb{R}^2 \mid |x - \overline{x}_r| \leq r_x,\ |\theta_v - \overline{\theta}| \leq r_\theta\}, \tag{3.8}$$

where $r_x,\ r_\theta$ are positive constants. Condition (3.7) turns into $|x-\overline{x}_r| + |\theta_v - \overline{\theta}| < \sqrt{4\lambda_\varphi\gamma - 1} - k$, with $k = |\overline{x_r} - \overline{\theta}|$, which has to be fulfilled for $\mathbf{w}_v$ evaluated in $\mathbf{w}_d = [x\ \theta_d]^T$ and $\hat{\mathbf{w}} = [x\ \hat{\theta}]^T$. If $\mathbf{w}_v = \mathbf{w}_d$, the second term in the left hand side of the previous inequality is constant and initial conditions on the state x can be adjusted for satisfying it. When $\mathbf{w}_v = \hat{\mathbf{w}}$, the condition turns into

$$|x-\overline{x}_r| + |\hat{\theta} - \overline{\theta}| < \sqrt{4\lambda_\varphi\gamma - 1} - k, \tag{3.9}$$

which gives, in addition, the region of initial conditions for the estimate $\hat{\theta}$.

ii) $R_v = -(\varphi_v - \varphi_d)(1 + (x-\theta_v)^2)$. With this selection G is equal to zero. Hence only (3.6) has to be verified. By taking $\Gamma(x) = -\gamma,\ \gamma > 0$, (3.6) can be written as $2|\varphi_v - \varphi_d|\left(|x-\overline{x}_r| + |\theta_v - \overline{\theta}| + |\overline{x}_r - \overline{\theta}|\right) < \gamma$ which must be fulfilled for $\mathbf{z}_v = \mathbf{z}_d$ and $\mathbf{z}_v = \hat{\mathbf{z}}$. It is clear that in the former case the condition is trivially satisfied. For the later case, the region is restricted to $|\varphi - \varphi_d| \leq \overline{\varphi},\ \overline{\varphi} > 0$ and condition (3.6) is such that

$$|x-\overline{x}_r| + |\hat{\theta} - \overline{\theta}| \leq \frac{\gamma}{2\overline{\varphi}} - k. \tag{3.10}$$

By choosing the design functions R_v and $\Gamma(\mathbf{x})$ as above and under conditions (3.9) or (3.10), *semi-global* contraction of the virtual system is concluded. The next step, in the down-level of the design, is to realize the adaptive algorithm (3.3). For this purpose, the

[1]See the Notation section for definition of variables.

following PI form of adaptive algorithms is considered

$$\hat{\theta} = \hat{\theta}_I(t) + \hat{\theta}_P(x, \varphi), \tag{3.11}$$

and its time derivative along (3.2) is fitted with (3.3). We obtain that $\frac{\partial \hat{\theta}_P(x,\varphi)}{\partial \varphi} = \Gamma(x) = -\gamma$, and

$$\hat{\theta}_P(x, \varphi) = -\gamma\varphi, \tag{3.12}$$

$$\dot{\hat{\theta}}_I(t) = R(\varphi, \varphi_d, x, \hat{\theta}) - \gamma\lambda_\varphi \varphi. \tag{3.13}$$

The adaptive control scheme was applied to system and tested in simulation. The parameter was adjusted by (3.11)-(3.13). The initial conditions were chosen as $x_0 = 2$ and $\hat{\theta}_0 = 1$. The nominal parameter was selected as $\theta = 1.5$ and $\overline{\theta} = 2$ was considered. The reference was taken as $x_r(t) = \sin(t)$, then $\overline{x}_r = 1$. $R(\varphi, \varphi_d, x, \hat{\theta})$ was selected as in the aforementioned analysis: *i)* $\hat{R} = 0$ with $\lambda_\varphi = 4$, $\gamma = 2$, *ii)* $\hat{R} = -\varphi(1 + (x - \hat{\theta})^2)$ by considering $\varphi_d(0) = 0$ and $\overline{\varphi} = |\varphi(0)| = x_0$, for $\gamma = 15$. For the given selections, conditions (3.9) and (3.10) are fulfilled with $k = 1$. Tracking and estimated parameter errors are depicted in Figure 3.1.

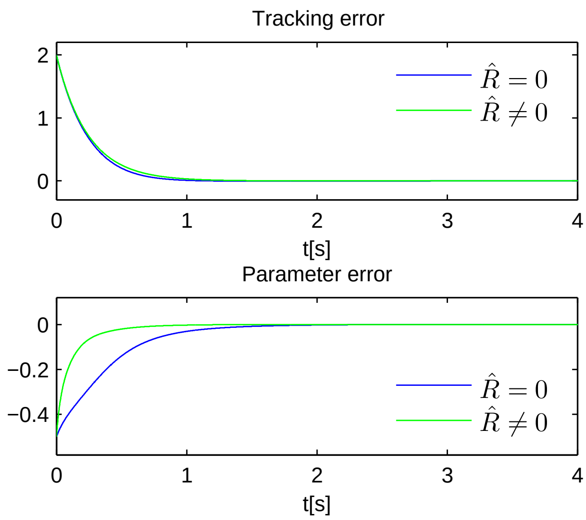

FIGURE 3.1: Tracking and parameter errors for the system (3.1) in closed-loop with the adaptive control scheme, the parameter θ is adjusted with (3.11)-(3.13). Responses for $\hat{R} = 0$ and $\hat{R} \neq 0$ are depicted in blue and green, respectively.

3.2 Problem statement

Consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}, \mathbf{u}, t) + \mathbf{g}(\mathbf{x}, \boldsymbol{\theta}, t), \;\; \mathbf{x}(t_0) = \mathbf{x}_0, \tag{3.14}$$

where $\mathbf{x} \in \mathbb{R}^n$ denotes the plant state assumed accessible for measurement and $\mathbf{u} \in \mathbb{R}^q$ stands for the control input. $\boldsymbol{\theta} \in \mathbb{R}^p$ is the unknown parameters vector which belongs to a compact set $\Omega_\theta \subset \mathbb{R}^p$. Functions $\mathbf{f}_p : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ are assumed to have all the continuity and differentiability requirements needed. In addition, it is supposed that all the solutions of (3.14) are well defined for all initial conditions and $\forall t \geq t_0$. The problem this chapter addresses consists in designing an adaptive control law

$$\begin{aligned} \mathbf{u} &= \mathbf{u}(\mathbf{x}, \hat{\boldsymbol{\theta}}, t), \\ \dot{\hat{\boldsymbol{\theta}}} &= \boldsymbol{\Psi}(\mathbf{x}, \hat{\boldsymbol{\theta}}, t), \end{aligned}$$

such that all trajectories of the closed-loop system remain bounded and the state converges to

$$\Omega_0 = \{\mathbf{x} \in \mathbb{R}^n | \varphi(\mathbf{x}(t), t) = 0\},$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$ is a $\mathcal{C}^1$ class *error* function which defines the target set. Furthermore, if possible, to find conditions to reconstruct the unknown parameters vector $\boldsymbol{\theta}$.

3.3 Adaptive controller design

The design of the adaptive control law starts by considering the time evolution of φ along the trajectories of (3.14)

$$\dot{\varphi} = \frac{\partial \varphi}{\partial \mathbf{x}} \, \mathbf{f}_p(\mathbf{x}, \mathbf{u}, t) + \frac{\partial \varphi}{\partial \mathbf{x}} \, \mathbf{g}(\mathbf{x}, \boldsymbol{\theta}, t) + \frac{\partial \varphi}{\partial t}. \tag{3.15}$$

The following assumptions are in order.

Assumption 3.1. *There exists an ideal control* $\mathbf{u}_i(\mathbf{x}, \boldsymbol{\theta}, t)$*, such that in closed-loop with (3.15) achieves the control objective according to the target dynamics*

$$\dot{\varphi} = \frac{\partial \varphi}{\partial \mathbf{x}} \, \mathbf{f}_p(\mathbf{x}, \mathbf{u}_i(\mathbf{x}, \boldsymbol{\theta}, t), t) + \frac{\partial \varphi}{\partial \mathbf{x}} \, \mathbf{g}(\mathbf{x}, \boldsymbol{\theta}, t) + \frac{\partial \varphi}{\partial t} = -\psi(\varphi).$$

△

Assumption 3.2. *$\psi : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ class contracting function, i.e. there exists $\lambda_\psi > 0$ such that $-\frac{\partial \psi(\varphi)}{\partial \varphi} \leq -\lambda_\psi$.* △

The (certainty-equivalence) adaptive control $\mathbf{u}_i(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$ in closed-loop with (3.15) gives

$$\dot{\varphi} = -\psi(\varphi) + f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t), \tag{3.16}$$

where the definition $\frac{\partial \varphi}{\partial \mathbf{x}} \mathbf{f}_p(\mathbf{x}, \mathbf{u}_i(\mathbf{x}, \boldsymbol{\theta}, t), t) \triangleq f(\mathbf{x}, \boldsymbol{\theta}, t)$ was taken, analogously for $f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$.

Assumption 3.3. *The first and the second partial derivatives of $f(\mathbf{x}, \boldsymbol{\theta}, t)$ with respect to $\boldsymbol{\theta}$ exist and are bounded in Ω_θ.* △

For the error dynamics (3.16), the following (non-implementable) adaptive algorithm is proposed

$$\dot{\hat{\boldsymbol{\theta}}} = \mathbf{R}(\varphi, \varphi_d, \mathbf{x}, \hat{\boldsymbol{\theta}}) + \boldsymbol{\Gamma}(\mathbf{x}) \left(f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t) \right), \tag{3.17}$$

where $\hat{\mathbf{R}}$ and $\boldsymbol{\Gamma}(\mathbf{x})$ are the degrees of freedom needed to satisfy the generalized contraction conditions.

Assumption 3.4. *Functions $\boldsymbol{\Gamma} : \mathbb{R}^n \to \mathbb{R}^p$ and $\mathbf{R} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ are $\mathcal{C}^1$ class such that $\frac{\partial \psi(\varphi_v)}{\partial \varphi_v} \bar{\lambda}(\mathbf{J}_{2_s}) \geq \sigma^2(\mathbf{G})$ for $\mathbf{z}_v$ evaluated in $\mathbf{z}_d$ and $\hat{\mathbf{z}}$, where matrices $\mathbf{J}_{2_s}$, $\mathbf{G}$ are given by*

$$\mathbf{J}_{2_s} = -\left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\theta}_v} + \boldsymbol{\Gamma}(\mathbf{x}) \frac{\partial f_v}{\partial \boldsymbol{\theta}_v} \right]_s, \tag{3.18}$$

$$\mathbf{G} = -\frac{1}{2} \left(\frac{\partial \mathbf{R}_v}{\partial \varphi_v} + \frac{\partial f_v^T}{\partial \boldsymbol{\theta}_v} \right). \tag{3.19}$$

△

3.3.1 Stability analysis

The main goal of the adaptive control law can be written in terms of desired system trajectories, namely

$$\dot{\varphi}_d = -\psi(\varphi_d), \;\; \dot{\boldsymbol{\theta}}_d = \mathbf{0}, \;\; \boldsymbol{\theta}_d(t_0) = \boldsymbol{\theta},$$

which, suggested by the structure of (3.16) and (3.17), can be written as

$$\begin{aligned} \dot{\varphi}_d &= -\psi(\varphi_d) + f(\mathbf{x}, \boldsymbol{\theta}_d, t) - f(\mathbf{x}, \boldsymbol{\theta}, t), && (3.20a)\\ \dot{\boldsymbol{\theta}}_d &= \mathbf{R}(\varphi_d, \varphi_d, \mathbf{x}, \boldsymbol{\theta}_d) + \boldsymbol{\Gamma}(\mathbf{x})\left(f(\mathbf{x}, \boldsymbol{\theta}_d, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)\right),\ \boldsymbol{\theta}_d(t_0) = \boldsymbol{\theta}, && (3.20b) \end{aligned}$$

where $\mathbf{R}(\varphi_d, \varphi_d, \mathbf{x}, \boldsymbol{\theta}_d) = 0$ is assumed. Based on this representation, the virtual system

$$\begin{aligned} \dot{\varphi}_v &= -\psi(\varphi_v) + f(\mathbf{x}, \boldsymbol{\theta}_v, t) - f(\mathbf{x}, \boldsymbol{\theta}, t), && (3.21a)\\ \dot{\boldsymbol{\theta}}_v &= \mathbf{R}(\varphi_v, \varphi_d, \mathbf{x}, \boldsymbol{\theta}_v) + \boldsymbol{\Gamma}(\mathbf{x})\left(f(\mathbf{x}, \boldsymbol{\theta}_v, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)\right), && (3.21b) \end{aligned}$$

is proposed. It can be readily verified that this virtual system has the trajectories of the actual system (3.16)-(3.17) and that of the desired system (3.20) as two particular solutions. Therefore the design objectives are accomplished if this virtual system is contracting under the approach of Lemma 2.2.

Theorem 3.1. *Consider the system (3.14) in closed loop with the control* $\mathbf{u}_i(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$ *and the adaptive algorithm (3.17). Suppose that Assumptions 3.1-3.4 are satisfied. Let* $\boldsymbol{\Gamma}(\mathbf{x})$ *and* $\mathbf{R}_v$ *be such that* $\forall t \geq t_0$, $\forall \mathbf{x} \in \mathcal{X}$,

$$\begin{aligned} \int_t^{t+T} \mathbf{J}_{2_s}(\tau) d\tau &\geq \lambda_\theta \mathbf{I}_{p\times p}, && (3.22)\\ \sigma^2\left(\int_t^{t+T} \mathbf{G}(\tau) d\tau\right) &< \lambda_\psi \lambda_\theta T, && (3.23) \end{aligned}$$

for some $\lambda_\theta > 0,\ T > 0$, *for* $\mathbf{z}_v = \mathbf{z}_d$ *and* $\mathbf{z}_v = \hat{\mathbf{z}}$, *where* $\mathbf{J}_{2_s}$ *and* $\mathbf{G}$ *are defined in (3.18) and (3.19), respectively. Then all the trajectories of (3.16) and (3.17) tend exponentially to those of (3.20), i.e.* $\lim_{t\to\infty} \varphi(t) = \varphi_d(t)$ *and* $\lim_{t\to\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. The Jacobian of the virtual system (3.21) leads to a matrix $-\mathbf{J}_s$ as in (2.7) with $J_{1_s} = \frac{\partial\psi(\varphi_v)}{\partial\varphi_v}$, $\mathbf{J}_{2_s}$ and $\mathbf{G}$ as in (3.18) and (3.19), respectively. $-\mathbf{J}_s$ is USPD by Assumptions 3.2 and 3.4. Hence the approach of Lemma 2.2 can be applied. By Assumption 3.2 and condition (3.22), $\int_t^{t+T} J_{1_s}(\tau)d\tau$ and $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau$ are UPD. Requirement (3.23) is equivalent to (2.8). The result follows by reversing (2.5) of Lemma 2.2 and by taking $\lambda \leq \underline{\lambda}\left(-\int_t^{t+T} \mathbf{J}_s(\tau)d\tau\right)$. □

By taking advantage of the choice of $\mathbf{R}_v$, it is possible to make the off-diagonal element $\mathbf{G}$ equal to zero. Therefore, $-\mathbf{J}_s$ can be written as a block diagonal matrix whose elements are J_{1_s} and $\mathbf{J}_{2_s}$. The analysis of tracking and parameters error is *decoupled* and requirements

(3.22)-(3.23) are reduced to check only the positivity property (3.22). These ideas are established in the following corollary.

Corollary 3.1. *Under Assumptions 3.1-3.3, let* $\hat{\mathbf{R}}$ *be given by*

$$\hat{\mathbf{R}} = -(\varphi - \varphi_d)\frac{\partial \hat{f}^T}{\partial \hat{\boldsymbol{\theta}}}. \tag{3.24}$$

Consider the matrix

$$\mathbf{J}_{2_s} = \left[(\varphi_v - \varphi_d)\frac{\partial^2 f_v}{\partial \boldsymbol{\theta}_v^2} - \boldsymbol{\Gamma}(\mathbf{x})\frac{\partial f_v}{\partial \boldsymbol{\theta}_v}\right]_s. \tag{3.25}$$

Let $\boldsymbol{\Gamma}(\mathbf{x})$ *be such that* $\mathbf{J}_{2_s}$ *is USPD and condition (3.22) is satisfied for* $\mathbf{z}_v = \mathbf{z}_d$ *and* $\mathbf{z}_v = \hat{\mathbf{z}}$, $\forall t \geq t_0$, $\forall \mathbf{x} \in \mathcal{X}$.

(P1) If $\lambda_\theta = 0$ *then all the trajectories of (3.16)-(3.17) are bounded and* $\lim_{t\to\infty} \varphi(t) = \varphi_d(t)$.

(P2) If $\lambda_\theta > 0$ *then all the trajectories of (3.16)-(3.17) tend exponentially to the trajectories of (3.20), i.e.* $\lim_{t\to\infty} \varphi(t) = \varphi_d(t)$ *and* $\lim_{t\to\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. Given the selection of $\hat{\mathbf{R}}$, the corresponding $\mathbf{R}_v$ is such that $\mathbf{G} = \mathbf{0}$ in (3.19). Hence, the diagonal elements of $-\mathbf{J}_s$ are $J_{1_s} = \frac{\partial \psi(\varphi_v)}{\partial \varphi_v}$ and $\mathbf{J}_{2_s}$ as in (3.25). By Assumption 3.2 and the properties of $\boldsymbol{\Gamma}(\mathbf{x})$, $-\mathbf{J}_s$ is USPD. By Assumption 3.2, we have $\int_t^{t+T} J_{1_s}(\tau)d\tau \geq \lambda_\psi T$. If (3.22), *i.e.* $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau \geq \lambda_\theta \mathbf{I}_{p\times p}$, is satisfied for $\lambda_\theta = 0$, $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau$ is positive semidefinite and all the trajectories started in $\mathcal{X} \times \Omega_\theta$ will remain therein for all $t \geq t_0$. Let $\mathcal{M} \triangleq \{(\varphi, \hat{\boldsymbol{\theta}}) | \varphi = 0, \hat{f} - f = 0, \ \forall t \geq t_0, \forall \mathbf{x} \in \mathcal{X}\} \subset \mathbb{R} \times \mathbb{R}^p$ and $\mathbf{V} = [1 \ 0 \ ... \ 0] \in \mathbb{R}^{1\times(p+1)}$. Notice that $\mathcal{M}$ is flow-invariant. It follows from the fact that $\mathbf{V} \int_t^{t+T} \mathbf{J}_s(\tau)d\tau \ \mathbf{V}^T = -\int_t^{t+T} J_{1_s}(\tau)d\tau \leq -\lambda_\psi T$ and Lemma 2.3 that all trajectories will converge to $\mathcal{M}$. This proves (P1). If $\lambda_\theta > 0$ then (P2) follows by reversing (2.5) of Lemma 2.2 and taking $\lambda \leq \min(\lambda_\psi T, \lambda_\theta)$. □

3.4 Implementation of the adaptive algorithm

Realization schemes will be searched through different PI based estimation vectors $\hat{\boldsymbol{\theta}}(t)$ proposed accordingly to the available signals. By adjusting the time derivative of $\hat{\boldsymbol{\theta}}(t)$ along (3.16) with (3.17), implementation expressions will be obtained. To this aim, let

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}_I(t) + \hat{\boldsymbol{\theta}}_P(\mathbf{x}, \varphi, t) + \boldsymbol{\rho}(\mathbf{x}, t), \tag{3.26}$$

be, where $\hat{\boldsymbol{\theta}}_I : \mathbb{R}_{\geq t_0} \to \mathbb{R}^p$ is a continuous and differentiable function, $\hat{\boldsymbol{\theta}}_P : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^p$ is a degree of freedom responsible for introducing interactions between tracking and parameter error dynamics, and $\boldsymbol{\rho} : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ is continuous and differentiable in its arguments designed to avoid $\dot{\mathbf{x}}$ coefficients. By comparing (3.17) and temporal evolution of (3.26) along (3.16) we have

$$\begin{aligned} \boldsymbol{\Gamma}(\mathbf{x}) &= \frac{\partial \hat{\boldsymbol{\theta}}_P(\mathbf{x}, \varphi, t)}{\partial \varphi}, & (3.27) \\ \dot{\hat{\boldsymbol{\theta}}}_I &= \mathbf{R}(\varphi, \varphi_d, \mathbf{x}, \hat{\boldsymbol{\theta}}) + \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \varphi}\psi(\varphi) - \frac{\partial \boldsymbol{\rho}}{\partial t} - \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial t}, & (3.28) \end{aligned}$$

and $\boldsymbol{\rho}(\mathbf{x}, t)$ is such that

$$\frac{\partial \boldsymbol{\rho}(\mathbf{x}, t)}{\partial \mathbf{x}} = -\frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \mathbf{x}}. \tag{3.29}$$

The key condition to exclude $\dot{\mathbf{x}}$ influence (3.29) will have solutions if and only if the Poincaré Lemma [6]

$$\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \mathbf{x}}\right) = \left(\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \mathbf{x}}\right)\right)^T, \tag{3.30}$$

is satisfied. If (3.30) does not hold or if signal $\dot{\mathbf{x}}$ is available, the following strategy is adopted. By setting $\boldsymbol{\rho}(\cdot)$ to $\mathbf{0}$ in (3.26), a term proportional to the partial derivative of $\hat{\boldsymbol{\theta}}_P(\mathbf{x}, \varphi, t)$ w.r.t. $\mathbf{x}$ arises in (3.17). This term can be canceled by means of modifying $\hat{\boldsymbol{\theta}}_I(t)$, namely

$$\dot{\hat{\boldsymbol{\theta}}}_I = \mathbf{R}(\varphi, \varphi_d, \mathbf{x}, \hat{\boldsymbol{\theta}}) + \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \varphi}\psi(\varphi) - \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \mathbf{x}}\dot{\mathbf{x}} - \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial t}. \tag{3.31}$$

Integral action (3.31) can be readily implemented if $\dot{\mathbf{x}}$ is available. Otherwise precise estimations of $\dot{\mathbf{x}}$ must be obtained from derivative observers (see *e.g.* [27], [28]). Instead of excluding $\dot{\mathbf{x}}$ theoretically, realization (3.31) includes its signal explicitly in the implementation which avoids complexities in the verification of (3.30). However, if $\dot{\mathbf{x}}$ is unknown, design becomes more intricate since it has to be considered in combination with an observation scheme.

3.5 Particular forms of parameterization

If nothing is said about the structure specifying the plant dynamics in (3.14), or equivalently $f(\mathbf{x}, \boldsymbol{\theta}, t)$ in (3.16), it is hard, if not impossible, to go further than conditions (3.22)-(3.23) or (3.22) with matrix (3.25) to analyze contraction of the overall system. In these conditions, the role of persistent excitation of closed-loop signals in the convergence of estimated parameters is unclear. To address this issue and considering the large class of practical problems of interest, particular structures describing the plant dynamics are considered in the following.

3.5.1 Nonlinear function of linear parameterization

Let the function $f(\mathbf{x}, \boldsymbol{\theta}, t)$ describe the error dynamics in (3.16) be of the form

$$f(\mathbf{x}, \boldsymbol{\theta}, t) \triangleq f_\alpha(\mathbf{x}, \boldsymbol{\alpha}^T(\mathbf{x})\boldsymbol{\theta}, t), \tag{3.32}$$

i.e. a nonlinear function of a linear parameterization, where $\boldsymbol{\alpha} : \mathbb{R}^n \to \mathbb{R}^p$ is the regressor, assumed to be bounded and continuous function of the state $\mathbf{x}$.

The structure of (3.32) leads to the following partial derivatives

$$\begin{aligned} \frac{\partial f_\alpha}{\partial \boldsymbol{\theta}_v} &= \left(\left. \frac{\partial f_\alpha(\mathbf{x}, \xi_v, t)}{\partial \xi_v} \right|_{\xi_v = \boldsymbol{\alpha}^T \boldsymbol{\theta}_v} \right) \boldsymbol{\alpha}^T(\mathbf{x}), \\ \frac{\partial^2 f_\alpha}{\partial \boldsymbol{\theta}_v^2} &= \left(\left. \frac{\partial^2 f_\alpha(\mathbf{x}, \xi_v, t)}{\partial \xi_v^2} \right|_{\xi_v = \boldsymbol{\alpha}^T \boldsymbol{\theta}_v} \right) \boldsymbol{\alpha}(\mathbf{x})\boldsymbol{\alpha}^T(\mathbf{x}). \end{aligned}$$

Consider $\hat{\mathbf{R}}$ as in (3.24) and the implementation condition (3.27) with

$$\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}(\mathbf{x}), \varphi) = -\bar{\boldsymbol{\Gamma}} \operatorname{sign} \left(\left. \frac{\partial f_\alpha(\mathbf{x}, \hat{\xi}, t)}{\partial \hat{\xi}} \right|_{\hat{\xi} = \boldsymbol{\alpha}^T \hat{\boldsymbol{\theta}}} \right) \boldsymbol{\alpha}(\mathbf{x})\varphi, \tag{3.33}$$

where $\bar{\boldsymbol{\Gamma}} = \bar{\boldsymbol{\Gamma}}^T \in \mathbb{R}^{p \times p}$ is a constant gain matrix. $\mathbf{J}_{2_s}$ takes the particular form

$$\mathbf{J}_{2_s} = \left[\mathbf{E}_\alpha \boldsymbol{\alpha}(\mathbf{x})\boldsymbol{\alpha}^T(\mathbf{x}) \right]_s, \tag{3.34}$$

where

$$\mathbf{E}_\alpha = \left. \left((\varphi_v - \varphi_d) \frac{\partial^2 f_{v_\alpha}}{\partial \xi_v^2} \mathbf{I}_{p \times p} + \left| \frac{\partial f_{v_\alpha}}{\partial \xi_v} \right| \bar{\boldsymbol{\Gamma}} \right) \right|_{\xi_v = \boldsymbol{\alpha}^T \boldsymbol{\theta}_v}. \tag{3.35}$$

In this case, due to the particular choices of $\hat{\mathbf{R}}$ and $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}(\mathbf{x}), \varphi)$, the design of the algorithm depends only on the nonlinear function of the linear parameterization. Here is when the condition of persistent excitation of the regressor naturally arises. The result is established in the following theorem.

Theorem 3.2. *Consider the class of systems given by (3.14) resulting in the tracking error dynamics (3.16) whose nonlinearity is given by (3.32), in closed-loop with the control*

$\mathbf{u}_i(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$ and the adaptive algorithm (3.26)-(3.29), where $\hat{\mathbf{R}}$ and $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}(\mathbf{x}), \varphi)$ are given in (3.24) and (3.33), respectively. Suppose that Assumptions 3.1-3.3 are valid. Assume that there exists a matrix gain $\bar{\boldsymbol{\Gamma}} = \bar{\boldsymbol{\Gamma}}^T \in \mathbb{R}^{p\times p}$ given in (3.35) such that $\forall t \geq t_0$, $\forall \mathbf{x} \in \mathcal{X}$

$$\mathbf{E}_\alpha \geq \lambda_E \mathbf{I}_{p\times p}, \tag{3.36}$$

for some $\lambda_E > 0$, and for $\mathbf{z}_v = \mathbf{z}_d$ and $\mathbf{z}_v = \hat{\mathbf{z}}$.

(P3) Then, all trajectories of (3.16)-(3.17) are bounded and $\lim_{t\to\infty} \varphi(t) = \varphi_d(t)$.

(P4) If, in addition, the regressor $\boldsymbol{\alpha}(\mathbf{x})$ is uniformly persistently exciting in the sense of Definition 2.2, then all the trajectories of (3.16)-(3.17) tend exponentially to those of (3.20), i.e. $\lim_{t\to\infty} \varphi(t) = \varphi_d(t)$ and $\lim_{t\to\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. Under the given selections of design functions, the diagonal blocks of matrix $-\mathbf{J}_s$ are $J_{1_s} = \frac{\partial\psi(\varphi_v)}{\partial\varphi_v}$ and $\mathbf{J}_{2_s}$ as in (3.34) which, by condition (3.36), is USPD. By this fact and Assumption 3.2, $-\mathbf{J}_s$ is USPD. If the regressor is not UPE then $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau \geq \lambda_\theta \mathbf{I}_{p\times p}$ with $\lambda_\theta = 0$, for any $T > 0$. Hence, all the trajectories started in $\mathcal{X} \times \Omega_\theta$ will remain therein for all $t \geq t_0$. Moreover, by (P1) Corollary 3.1, they will tend exponentially to the flow-invariant set $\mathcal{M} \triangleq \{(\varphi, \hat{\boldsymbol{\theta}})|\varphi = 0, \hat{f}_\alpha - f_\alpha = 0, \forall t \geq t_0, \forall \mathbf{x} \in \mathcal{X}\} \subset \mathbb{R}\times\mathbb{R}^p$. Then (P3) follows. Now consider the case when the regressor is UPE and take $T = \delta$. By Assumption 3.2, $\int_t^{t+T} J_{1_s}(\tau)d\tau \geq \lambda_\psi T$. By inequality (3.36) and considering that there exists $\epsilon_\alpha > 0$ given in Definition 2.2, the lower diagonal block is such that $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau \geq \lambda_E\epsilon_\alpha \mathbf{I}_{p\times p}$. The result follows by taking $\lambda_\theta \leq \lambda_E\epsilon_\alpha$, reversing (2.5) of Lemma 2.2 and selecting $\lambda \leq \min(\lambda_\psi T, \lambda_\theta)$. □

3.5.2 Linear parameterization

Consider the case when the function in (3.32) is linear in the parameterization, *i.e.*,

$$f_\alpha(\mathbf{x}, \boldsymbol{\alpha}^T(\mathbf{x})\boldsymbol{\theta}, t) = k\boldsymbol{\alpha}^T(\mathbf{x})\boldsymbol{\theta},$$

where $k \neq 0$ is known, $\boldsymbol{\alpha}(\mathbf{x}) = [\alpha_1(\mathbf{x})\ \alpha_2(\mathbf{x})\ \ldots\ \alpha_p(\mathbf{x})]^T \in \mathbb{R}^p$, and $\boldsymbol{\theta} = [\theta_1\ \theta_2\ \ldots\ \theta_p]^T \in \mathbb{R}^p$. Take the target dynamics as $\dot{\varphi}_d = -\lambda_\varphi\varphi_d$, $\lambda_\varphi > 0$, $\hat{\mathbf{R}}$ as in (3.24), $\boldsymbol{\rho}(\mathbf{x}, t)$ satisfying[2] (3.29), $\hat{\boldsymbol{\theta}}_P = -\bar{\boldsymbol{\Gamma}}\,\mathrm{sign}(k)\boldsymbol{\alpha}(\mathbf{x})\varphi$, and $\dot{\hat{\boldsymbol{\theta}}}_I$ as in (3.28). It is easy to verify that this adaptive

[2]The adaptive algorithm (3.31) can be equivalently used if (3.29) is not achieved.

law is equivalent to the composite adaptation scheme [29] (Section 9.2, p. 411) with an error dynamics

$$\dot{\tilde{\boldsymbol{\theta}}} = -|k|\bar{\boldsymbol{\Gamma}}\boldsymbol{\alpha}(\mathbf{x})\boldsymbol{\alpha}^T(\mathbf{x})\tilde{\boldsymbol{\theta}} - k\boldsymbol{\alpha}(\mathbf{x})\varphi.$$

This composite adaptation law was shown to have an improved convergence property [30].

3.6 Extension of the stability analysis for a generalized Jacobian

Conditions (3.22)-(3.23) in Theorem 1 may be stated in terms of a generalized Jacobian by introducing a transformation matrix $\mathbf{P}$ whose influence is established in Definition 1, see (2.4). Throughout this thesis, we will work with *constant*, block-diagonal transformation matrices $\mathbf{P} = \operatorname{diag}(p_\psi, \mathbf{P}_\theta)$, where $p_\psi > 0$ and $\mathbf{P}_\theta = \mathbf{P}_\theta^T \in \mathbb{R}^{p\times p}$ is nonsingular. Elements of matrix $-\mathbf{J}_s$ are modified as

$$\mathbf{J}_{2_m} \triangleq -\left[\mathbf{P}_\theta\left(\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\theta}_v} + \boldsymbol{\Gamma}(\mathbf{x})\frac{\partial f_v}{\partial \boldsymbol{\theta}_v}\right)\mathbf{P}_\theta^{-1}\right]_s, \tag{3.37}$$

$$\mathbf{G}_m = -\frac{1}{2}\left(\frac{1}{p_\psi}\mathbf{P}_\theta\frac{\partial \mathbf{R}_v}{\partial \varphi_v} + p_\psi\mathbf{P}_\theta^{-1}\frac{\partial f_v^T}{\partial \boldsymbol{\theta}_v}\right). \tag{3.38}$$

The contraction properties of the corresponding virtual system are stated in the following Theorem.

Theorem 3.3. *Under Assumptions 3.1-3.3, take the design function*

$$\hat{\mathbf{R}} = -(\varphi - \varphi_d)(p_\psi \mathbf{P}_\theta^{-1})^2\frac{\partial \hat{f}^T}{\partial \hat{\boldsymbol{\theta}}}. \tag{3.39}$$

Let $\boldsymbol{\Gamma}(\mathbf{x})$*,* $p_\psi > 0$*, and non singular* $\mathbf{P}_\theta = \mathbf{P}_\theta^T$ *be such that*

$$\mathbf{J}_{2_m} = \left[\mathbf{P}_\theta\left((p_\psi\mathbf{P}_\theta^{-1})^2\frac{\partial^2 f_v}{\partial \boldsymbol{\theta}_v^2}(\varphi_v - \varphi_d) - \boldsymbol{\Gamma}(\mathbf{x})\frac{\partial f_v}{\partial \boldsymbol{\theta}_v}\right)\mathbf{P}_\theta^{-1}\right]_s,$$

is USPD and

$$\int_t^{t+T} \mathbf{J}_{2_m}(\tau)d\tau \geq \lambda_m \mathbf{I}_{p\times p}, \tag{3.40}$$

for some $\lambda_m > 0$*,* $T > 0$*,* $\forall t \geq t_0$*,* $\forall \mathbf{x} \in \mathcal{X}$*, and for* $\mathbf{z}_v = \mathbf{z}_d$*,* $\mathbf{z}_v = \hat{\mathbf{z}}$*.*

(P5) *If* $\lambda_m = 0$ *then all the trajectories of* (3.16)-(3.17) *are bounded and* $\lim_{t\to\infty}\varphi(t) = \varphi_d(t)$.

(P6) *If* $\lambda_m > 0$ *then all the trajectories of* (3.16)-(3.17) *tend exponentially to the trajectories of* (3.20)*, i.e.* $\lim_{t\to\infty}\varphi(t) = \varphi_d(t)$ *and* $\lim_{t\to\infty}\hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. The proof follows the same lines as in Corollary 3.1 with the generalized Jacobian. □

Remark 3.1. In comparison with $\mathbf{J}_{2_s}$ in (3.25), the presence of the non-identity metric leads to an extra factor $(p_\psi \mathbf{P}_\theta^{-1})^2$ in the second partial derivative term, which gives more flexibilities to deal with convex/concave *parameters*. As the second partial derivative has fixed sign, the factor due to the metric allows to make this term sufficiently *large*, while $\boldsymbol{\Gamma}(\mathbf{x})$ is used to reduce the first partial derivative term. The same idea can be used to deal with convex/concave *parameterizations* as will be seen in the following. △

3.6.1 Generalization of the system parameterization: one linear-one nonlinear forms

In practice, the uncertain parameters may appear in both linear and nonlinear forms distributed in several nonlinearities. To deal with this situation, the tracking error dynamics (3.16) is extended to

$$\dot{\varphi} = -\psi(\varphi) + \boldsymbol{\alpha}_0^T(\mathbf{x})\tilde{\boldsymbol{\theta}}_0 + \sum_{j=1}^{N_p}(\hat{f}_j - f_j), \tag{3.41}$$

where $\tilde{\boldsymbol{\theta}}_0 = \hat{\boldsymbol{\theta}}_0 - \boldsymbol{\theta}_0 \in \mathbb{R}^{p_0}$, and $\hat{f}_j,\ f_j$ are defined in the Notation section. To clarify the notation, in this subsection we take $N_p = 1$ and the nonlinearity is

$$f(\mathbf{x}, \boldsymbol{\theta}, t) = \boldsymbol{\alpha}_0^T(\mathbf{x})\boldsymbol{\theta}_0 + f_1(\mathbf{x}, \boldsymbol{\alpha}_1^T(\mathbf{x})\boldsymbol{\theta}_1, t), \tag{3.42}$$

where $\boldsymbol{\theta}_i \in \mathbb{R}^{p_i}$ stands for the linear ($i = 0$) and the nonlinear ($i = 1$) parameters vectors. Let $\boldsymbol{\theta} = [\boldsymbol{\theta}_0^T\ \boldsymbol{\theta}_1^T]^T \in \mathbb{R}^p$, with $p_0 + p_1 = p$, be the compiled vector of parameters. Functions $\boldsymbol{\alpha}_i(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^{p_i},\ i = 0, 1$ are the regressors considered bounded functions of the state $\mathbf{x}$ and

$$\boldsymbol{\alpha}(\mathbf{x}) = [\boldsymbol{\alpha}_0^T(\mathbf{x})\ \boldsymbol{\alpha}_1^T(\mathbf{x})]^T \in \mathbb{R}^p. \tag{3.43}$$

In this case, matrix $\mathbf{P}_\theta$ is subdivided as $\mathbf{P}_\theta = \mathrm{diag}\,(\mathbf{P}_0, \mathbf{P}_1)$ where matrices $\mathbf{P}_i = \mathbf{P}_i^T \in \mathbb{R}^{p_i \times p_i},\ i = 0, 1$ are nonsingular. By following the steps of Theorem 3.2, the functions $\hat{\mathbf{R}}$

and $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}(\mathbf{x}),\varphi)$ are established and the analysis is carried out depending on the intrinsic properties of the system parameterization. Take $\hat{\mathbf{R}}$ as in (3.39) and define the gain matrix of (3.33) as $\bar{\boldsymbol{\Gamma}} = \text{diag}\,(\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1)$,where $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^T \in \mathbb{R}^{p_i \times p_i}$, $i = 0, 1$. Define the following matrix

$$\mathbf{E}_1 = \frac{\partial^2 f_{v_1}}{\partial \xi_{v_1}^2}(\varphi_v - \varphi_d)(p_\psi \mathbf{P}_1^{-1})^2 + \left|\frac{\partial f_{v_1}}{\partial \xi_{v_1}}\right| \boldsymbol{\Gamma}_1. \tag{3.44}$$

Matrix $\mathbf{J}_{2_m}$ of (3.40) is given by

$$\int_t^{t+T} \mathbf{J}_{2_m}(\tau) d\tau = \int_t^{t+T} \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{12}^T & \mathbf{F}_{22} \end{bmatrix} d\tau, \tag{3.45}$$

where

$$\mathbf{F}_{11} = \left[\mathbf{P}_0 \boldsymbol{\Gamma}_0 \boldsymbol{\alpha}_0 \boldsymbol{\alpha}_0^T \mathbf{P}_0^{-1}\right]_s, \tag{3.46}$$

$$\mathbf{F}_{22} = \left[\mathbf{P}_1 \mathbf{E}_1 \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1^T \mathbf{P}_1^{-1}\right]_s, \tag{3.47}$$

$$\mathbf{F}_{12} = \frac{1}{2}(\partial f_{v_1} \mathbf{P}_0 \boldsymbol{\Gamma}_0 \boldsymbol{\alpha}_0 \boldsymbol{\alpha}_1^T \mathbf{P}_1^{-1} + s_f \mathbf{P}_0^{-1} \boldsymbol{\alpha}_0 \boldsymbol{\alpha}_1^T \boldsymbol{\Gamma}_1 \mathbf{P}_1), \tag{3.48}$$

and $\partial f_{v_1} \triangleq \frac{\partial f_1(\mathbf{x}, \xi_{v_1}, t)}{\partial \xi_{v_1}}$, $s_f \triangleq \text{sign}(\partial f_{v_1})$. Stability in this section is established by analyzing the USPD condition for $\mathbf{J}_{2_m}$ and UPD of (3.45). A relationship with the persistent excitation condition for the complete regressor (3.43) will arise.

Theorem 3.4. *Consider the class of systems (3.14) which lead to error functions of the form (3.41), with* $N_p = 1$, *in closed loop with the control* $\mathbf{u}_i(\mathbf{x}, \hat{\boldsymbol{\theta}}, t)$ *and the adaptive algorithm (3.26)-(3.29) where* $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}(\mathbf{x}),\varphi)$ *and* $\hat{\mathbf{R}}$ *are given in (3.33) and (3.39), respectively. Assume that the compiled regressor (3.43) is persistently exciting in the sense of Definition 2.2 for some positive constants* δ, ϵ_α. *In addition, suppose that there exist* $p_\psi > 0$, $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^T \in \mathbb{R}^{p_i \times p_i}$, *and nonsingular matrices* $\mathbf{P}_i = \mathbf{P}_i^T \in \mathbb{R}^{p_i \times p_i}$, $i = 0, 1$, *such that* $\mathbf{E}_1 \geq \lambda_{E_1} \mathbf{I}_{p_1 \times p_1}$, *for some* $\lambda_{E_1} > 0$, *and the following requirements are simultaneously achieved* $\forall t \geq t_0$, $\forall \mathbf{x} \in \mathcal{X}$ *and for* $\mathbf{z}_v = \mathbf{z}_d$ *and* $\mathbf{z}_v = \hat{\mathbf{z}}$

$$\mathbf{D}_{11} = \int_t^{t+T} \left(\mathbf{F}_{11}(\tau) - \epsilon' \boldsymbol{\alpha}_0(\tau) \boldsymbol{\alpha}_0^T(\tau)\right) d\tau > d_0 \mathbf{I}_{p_0 \times p_0}, \tag{3.49}$$

$$\mathbf{D}_{22} = \int_t^{t+T} \left(\mathbf{F}_{22}(\tau) - \epsilon' \boldsymbol{\alpha}_1(\tau) \boldsymbol{\alpha}_1^T(\tau)\right) d\tau > d_1 \mathbf{I}_{p_1 \times p_1}, \tag{3.50}$$

$$\bar{\lambda}(\mathbf{D}_{11}) \bar{\lambda}(\mathbf{D}_{22}) > \sigma^2(\mathbf{D}_{12}), \tag{3.51}$$

where

$$\mathbf{D}_{12} = \int_t^{t+T} \left(\mathbf{F}_{12}(\tau) - \epsilon' \boldsymbol{\alpha}_0(\tau) \boldsymbol{\alpha}_1^T(\tau)\right) d\tau,$$

for some $\epsilon' > 0,\ T > 0,\ d_i > 0,\ i = 0, 1$. *Then all the trajectories of (3.41) and (3.17) tend exponentially to those of (3.20), i.e.* $\lim_{t\to\infty} \varphi(t) = \varphi_d(t)$ *and* $\lim_{t\to\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. From (3.41), (3.21a) can be reformulated as $\dot{\varphi}_v = -\psi(\varphi_v) + \boldsymbol{\alpha}_0^T(\mathbf{x})\tilde{\boldsymbol{\theta}}_{v0} + f_{v_1} - f_1$. Analogously, the virtual parameters error is $\dot{\boldsymbol{\theta}}_v = \mathbf{R}_v + \boldsymbol{\Gamma}(\mathbf{x})(\boldsymbol{\alpha}_0^T(\mathbf{x})\tilde{\boldsymbol{\theta}}_{v_0} + f_{v_1} - f_1)$. The symmetric generalized Jacobian $-\mathbf{J}_s$ of this new virtual system is block diagonal whose elements are $J_{1_m} = \frac{\partial\psi(\varphi_v)}{\partial\varphi_v}$ and $\mathbf{J}_{2_m}$ given by

$$\mathbf{J}_{2_m} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{12}^T & \mathbf{F}_{22} \end{bmatrix}.$$

It will be shown that $\mathbf{J}_{2_m}$ is USPD by demonstrating that there exists some $\beta_N > 0$ such that $\mathbf{M} = \mathbf{J}_{2_m} - \beta_N \boldsymbol{\alpha}\boldsymbol{\alpha}^T$ is USPD, where $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_0^T\ \boldsymbol{\alpha}_1^T]^T \in \mathbb{R}^p$ for any p. By proceeding by induction take $p = 2$ and $\alpha_i \in \mathbb{R},\ i = 0, 1$, *i.e.* scalar functions α_i. Define $\boldsymbol{\Gamma}_i = \gamma_i$, $\mathbf{P}_i = p_i > 0,\ i = 0, 1,\ \gamma' = \frac{1}{2}\left(\partial f_{v_1}\frac{p_0}{p_1}\gamma_0 + s_f\frac{p_1}{p_0}\gamma_1\right)$. In this case

$$\mathbf{M} = \begin{bmatrix} (\gamma_0 - \beta_N)\alpha_0^2 & (\gamma' - \beta_N)\alpha_0\alpha_1 \\ (\gamma' - \beta_N)\alpha_0\alpha_1 & (\lambda_{E_1} - \beta_N)\alpha_1^2 \end{bmatrix},$$

which is USPD if $0 < \beta_N < \gamma_0$ and $\beta_N \leq \frac{\gamma_0\lambda_{E_1} - \gamma'^2}{\gamma_0\lambda_{E_1} - 2\gamma'}$.

Let us suppose the result valid when $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_0^T\ \boldsymbol{\alpha}_1^T\ \dots\ \boldsymbol{\alpha}_{N_p}^T]^T$, $\sum_{i=0}^{N_p} p_i = p = k$. This means that there exists $\beta_{N_k} > 0$ such that $\mathbf{M}_k = \mathbf{J}_{2_m} - \beta_{N_k}\boldsymbol{\alpha}\boldsymbol{\alpha}^T$ is USPD, where $\mathbf{J}_{2_m}$ given in the general matrix (4.50) with $j = k$. The result must be shown for $p = k+1$. Without loss of generality take $\mathbf{P} = \mathbf{I}_{k+1\times k+1}$, $\boldsymbol{\Gamma}_i = \gamma_i \mathbf{I}_{p_i\times p_i},\ i = 0, \dots, k+1$ and one scalar additional term, *i.e.* $\alpha_{k+1} \in \mathbb{R}$. Matrix $\mathbf{M}_{k+1}$ can be defined under the definition (2.9) with $\mathbf{A} = \mathbf{M}_k$, $\mathbf{b} = \frac{1}{2}[\gamma_0'\boldsymbol{\alpha}_0^T\alpha_{k+1}\ \dots \gamma_k'\boldsymbol{\alpha}_k^T\alpha_{k+1}]^T$, where $\gamma_l' = \partial f_{v_l}\gamma_{k+1} + s_{f_{k+1}}\gamma_l - 2\beta_{N_{k+1}},\ \beta_{N_{k+1}} > 0,\ l = 0, ..., k$, and $d = (\lambda_{E_{k+1}} - \beta_{N_{k+1}})\alpha_{k+1}^2$. The conclusion follows if it is possible to find γ_i such that $\det(\mathbf{A}_0) \geq 0$. In summary, by Assumption 3.2 and the previous argument, matrix $-\mathbf{J}_s$ is USPD. On the other hand, by taking $T = \delta$ of Definition 2.2, and if requirements (3.49)-(3.51) are satisfied, then

$$\int_t^{t+T} \mathbf{J}_{2_m}(\tau)d\tau > \epsilon' \int_t^{t+T} \boldsymbol{\alpha}(\mathbf{x}(\tau))\boldsymbol{\alpha}^T(\mathbf{x}(\tau))d\tau \geq \epsilon_\alpha\epsilon'\mathbf{I}_{p\times p}. \tag{3.52}$$

The result follows by defining $\lambda_m \leq \epsilon_\alpha\epsilon'$, reversing (2.5) of Lemma 2.2 and taking $\lambda \leq \min(\lambda_\psi T, \lambda_m)$. □

In general, the metric selection is not obvious. In the following, a systematic approach for choosing the metric and a *sufficient* condition for positivity of (3.45) are obtained.

Let $\mathbf{A}$ and $\mathbf{B}$ be two positive definite matrices of the same dimensions. Then there exists some positive constant ϵ' such that $\mathbf{A}-\epsilon'\mathbf{B}$ is PD. By the previous fact, if $\mathbf{\Gamma}_i,\ i=0,1$ and $\mathbf{P}$ are selected such that (3.45) is positive definite, then such ϵ' which satisfies (3.49)-(3.51) exists. If linear regressor is of persistent excitation, $\int_t^{t+T}\boldsymbol{\alpha}_0(\mathbf{x}(\tau))\boldsymbol{\alpha}_0^T(\mathbf{x}(\tau))d\tau \geq \epsilon_0\mathbf{I}_{p_0\times p_0}$ for some $\epsilon_0>0$. By simple calculations it can be shown that

$$\int_t^{t+T}\mathbf{F}_{11}(\tau)d\tau \geq \epsilon_0\left[\mathbf{P}_0\mathbf{\Gamma}_0\mathbf{P}_0^{-1}\right]_s. \tag{3.53}$$

On the other hand, matrix $\mathbf{E}_1$ in (3.44) is symmetric because $\mathbf{P}_1$ (equivalently $\mathbf{P}_1^{-1}$) and $\mathbf{\Gamma}_1$ are symmetric. Assume that p_ψ, $\mathbf{P}_1$, and $\mathbf{\Gamma}_1$ can be chosen such that

$$\mathbf{E}_1 \geq \lambda_{E_1}\mathbf{I}_{p_1\times p_1}, \tag{3.54}$$

for some $\lambda_{E_1}>0$, uniformly in t and $\mathbf{x}$, for $\mathbf{z}_v=\mathbf{z}_d$ and $\mathbf{z}_v=\hat{\mathbf{z}}$.

By condition (3.54), matrix $\int_t^{t+T}\mathbf{F}_{22}(\tau)d\tau$ is bounded by

$$\int_t^{t+T}\mathbf{F}_{22}(\tau)d\tau \geq \lambda_{E_1}\left[\mathbf{P}_1\int_t^{t+T}\boldsymbol{\alpha}_1(\mathbf{x}(\tau))\boldsymbol{\alpha}_1^T(\mathbf{x}(\tau))d\tau\mathbf{P}_1^{-1}\right]_s.$$

If regressor $\boldsymbol{\alpha}_1(\mathbf{x})$ is persistently exciting then $\int_t^{t+T}\boldsymbol{\alpha}_1(\mathbf{x}(\tau))\boldsymbol{\alpha}_1^T(\mathbf{x}(\tau))d\tau \geq \epsilon_1\mathbf{I}_{p_1\times p_1}$, for some $\epsilon_1>0$. The previous calculations lead to

$$\int_t^{t+T}\mathbf{F}_{22}(\tau)d\tau \geq \lambda_{E_1}\epsilon_1\mathbf{I}_{p_1\times p_1}, \tag{3.55}$$

valid for any matrix $\mathbf{P}_1=\mathbf{P}_1^T$. By the use of the monotonicity result for symmetric matrices ([31], p. 469) in (3.53) and (3.55), and considering that the largest eigenvalues are positive constants, it is possible to write

$$\bar{\lambda}\left(\int_t^{t+T}\mathbf{F}_{11}(\tau)d\tau\right)\bar{\lambda}\left(\int_t^{t+T}\mathbf{F}_{22}(\tau)d\tau\right) \geq \lambda_{E_1}\epsilon_0\epsilon_1\bar{\lambda}\left(\left[\mathbf{P}_0\mathbf{\Gamma}_0\mathbf{P_0}^{-1}\right]_s\right). \tag{3.56}$$

On the other hand, by defining

$$\mathbf{\Lambda}_f \triangleq \int_t^{t+T} \partial f_{v_1} \boldsymbol{\alpha}_0(\mathbf{x}(\tau))\boldsymbol{\alpha}_1^T(\mathbf{x}(\tau))d\tau, \tag{3.57}$$

$$\mathbf{\Lambda}_\alpha \triangleq \int_t^{t+T} s_f \boldsymbol{\alpha}_0(\mathbf{x}(\tau))\boldsymbol{\alpha}_1^T(\mathbf{x}(\tau))d\tau, \tag{3.58}$$

the off-diagonal element can be written as

$$\int_t^{t+T} \mathbf{F}_{12}(\tau)d\tau = \frac{1}{2}\begin{bmatrix} \mathbf{P}_0\mathbf{\Gamma}_0 & \mathbf{P}_0^{-1} \end{bmatrix} \mathbf{\Lambda} \begin{bmatrix} \mathbf{P}_1^{-1} \\ \mathbf{\Gamma}_1\mathbf{P}_1 \end{bmatrix},$$

where $\mathbf{\Lambda} \triangleq \operatorname{diag}(\mathbf{\Lambda}_f, \mathbf{\Lambda}_\alpha)$. Due to the fact that for no necessarily square matrices of appropriate dimensions $\mathbf{A}$ and $\mathbf{B}$, the inequality $\sigma(\mathbf{AB}) \leq \sigma(\mathbf{A})\sigma(\mathbf{B})$ holds ([32], p. 178), by using it successively for a product of three matrices, and by squaring both sides of the inequality, we have

$$\sigma^2\left(\int_t^{t+T} \mathbf{F}_{12}(\tau)d\tau\right) \leq \frac{1}{4}\sigma^2\left(\begin{bmatrix} \mathbf{P}_0\mathbf{\Gamma}_0 & \mathbf{P}_0^{-1} \end{bmatrix}\right)\sigma^2(\mathbf{\Lambda})\,\sigma^2\left(\begin{bmatrix} \mathbf{P}_1^{-1} \\ \mathbf{\Gamma}_1\mathbf{P}_1 \end{bmatrix}\right). \tag{3.59}$$

By using (3.56) and (3.59) in condition (2.8), the complete matrix (3.45) will be positive definite if matrices $\mathbf{P}_i$ and $\mathbf{\Gamma}_i$ can be designed such that the sufficient condition

$$\sigma^2(\mathbf{\Lambda}) < \frac{4\lambda_{E_1}\epsilon_0\epsilon_1\bar{\lambda}\left(\left[\mathbf{P}_0\mathbf{\Gamma}_0\mathbf{P}_0^{-1}\right]_s\right)}{\sigma^2\left(\begin{bmatrix} \mathbf{P}_0\mathbf{\Gamma}_0 & \mathbf{P}_0^{-1} \end{bmatrix}\right)\sigma^2\left(\begin{bmatrix} \mathbf{P}_1^{-1} \\ \mathbf{\Gamma}_1\mathbf{P}_1 \end{bmatrix}\right)}, \tag{3.60}$$

is satisfied uniformly in t, $\forall \mathbf{x} \in \mathcal{X}$, and for $\mathbf{z}_v = \mathbf{z}_d$ and $\hat{\mathbf{z}}$.

From Definition 2.2, it is clear that if $\boldsymbol{\alpha}(\mathbf{x}) = [\boldsymbol{\alpha}_0^T(\mathbf{x})\ \boldsymbol{\alpha}_1^T(\mathbf{x})]^T$ is UPE then each regressor $\boldsymbol{\alpha}_i(\mathbf{x}, t)$ is UPE, but the converse is in general not true. That is, to produce a compiled UPE regressor it requires that not only each regressor is UPE separately, but also the correlation between them, involved through $\sigma^2(\mathbf{\Lambda})$, must satisfy (3.60). The contraction requirement (3.60) ensures parameter convergence hence, may be viewed as an extended UPE condition: it is fulfilled if each regressor $\boldsymbol{\alpha}_i(\mathbf{x}, t)$ is UPE and the correlation in terms of (3.57)-(3.58), is sufficiently small to satisfy it. This is important in practice to guide how to select the input signals in order to guarantee the parameter estimates convergence in some applications. To clarify this statement, the following example, an extension of that developed in [33], is analyzed[3]

[3]All related values for all implementations are given in Table 4.1.

3.6.1.1 Examples: regressors dependent on internal and external signals

The academic system is given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\lambda_g x_2 + \boldsymbol{\alpha}_0^T(t)\boldsymbol{\theta}_0 + \exp(-\boldsymbol{\alpha}_1^T(t)\boldsymbol{\theta}_1) + u, \ \lambda_g > 0, \end{aligned} \tag{3.61}$$

where $\mathbf{x} = [x_1 \ x_2]^T$ is the state, the scalar signal u is the control input, and $\boldsymbol{\alpha}_i(t) \in \mathbb{R}^2$, $\boldsymbol{\theta}_i \in \mathbb{R}^2$, are the linear ($i = 0$) and nonlinear ($i = 1$) regressors and parameters, respectively. To show the importance of the persistent excitation of $\boldsymbol{\alpha}(t) = [\boldsymbol{\alpha}_0^T(t) \ \boldsymbol{\alpha}_1^T(t)]^T$ and its relationship with nonlinear parameters convergence, the time-dependent regressors $\boldsymbol{\alpha}_i(t) = [\sin(\omega_{i_1} t) \ \ \cos(\omega_{i_2} t)]^T$, $i = 0, 1$ are taken.

The tracking error is $\varphi = x_2 + \lambda_\varphi x_1, \ \lambda_\varphi > 0$, and the adaptive control law is $u = -\lambda_t \varphi + x_2(\lambda_g - \lambda_\varphi) - \boldsymbol{\alpha}_0(t)^T \hat{\boldsymbol{\theta}}_0 - \exp(-\boldsymbol{\alpha}_1(t)^T \hat{\boldsymbol{\theta}}_1), \ \lambda_t > 0$. Tracking error dynamics takes the form (3.41), with $j = 1$, with differences on signs of the last three terms. In this example $\hat{\boldsymbol{\theta}}_I(t)$ is given by (3.31) (since we consider that $\dot{\mathbf{x}}$ is available), where $\hat{\mathbf{R}}$ as in (3.39) and $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}(t), \varphi)$ as in (3.33). All Assumptions 3.1-3.3 and definitions (3.44)-(3.48) are valid and uniform semi-positive definiteness of $\mathbf{J}_{2_m}$ is verified. In this case, regressors $\boldsymbol{\alpha}_i(t), \ i = 0, 1$ are persistently exciting separately and their corresponding levels of excitation, denoted by ϵ_i, are obtained off-line from numerical analysis. Condition (3.60) is verified as follows. Matrix (3.44) is $\mathbf{E}_1 = \exp(-\boldsymbol{\alpha}_1^T(t)\boldsymbol{\theta}_{v_1}) \left(\boldsymbol{\Gamma}_1 - (\varphi_v - \varphi_d)\left(p_\psi \mathbf{P}_1^{-1}\right)^2\right)$ which has to fulfill (3.54) for some $\lambda_{E_1} > 0$. As all the signals in the closed-loop system are bounded, there exists $k_0 > 0$ such that $\boldsymbol{\alpha}_1^T \hat{\boldsymbol{\theta}}_1 < k_0, \ \forall t \geq 0$. By inverting this inequality and applying exponential function in both sides, it is clear that $\exp(-\boldsymbol{\alpha}_1^T \hat{\boldsymbol{\theta}}_1) > \exp(-k_0) \triangleq k_1$. The previous bound is also valid if $\theta_{v_1} = \theta_{d_1}$. Assume that there exists $k_2 > 0$ such that $\boldsymbol{\Gamma}_1 - (\varphi_v - \varphi_d)\left(p_\psi \mathbf{P}_1^{-1}\right)^2 > k_2 \mathbf{I}_{2\times 2}$ for $\varphi_v = \varphi$ (fulfillment for the case $\varphi_v = \varphi_d$ is trivial). Set $\varphi_d(0) = 0$ and suppose that $|\varphi| < \overline{\varphi}$ for some $\overline{\varphi} > 0$. The previous inequality is fulfilled if

$$\left(p_\psi \mathbf{P}_1^{-1}\right)^2 \overline{\varphi} < \boldsymbol{\Gamma}_1. \tag{3.62}$$

Hence, $\mathbf{E}_1 > \lambda_{E_1} \mathbf{I}_{2\times 2}, \ \forall t \geq 0$, where $\lambda_{E_1} \leq k_1 k_2$ which, by construction, gives an estimate of λ_{E_1}. Gains $\boldsymbol{\Gamma}_i$ and metric values $\mathbf{P}_i$ were calculated such that the right hand side of (3.60), restricted to (3.62), was maximum. Finally, values of the linear and nonlinear frequencies $\omega_{i_1}, \ \omega_{i_1}, \ i = 0, 1$ were set in order to minimize the largest singular value of matrix $\boldsymbol{\Lambda}$, which is composed by (3.57)-(3.58). Such singular value was obtained on-line because the factor ∂f_{v_1} depends on internal simulation variables. As a verification of uniform positive definiteness of (3.45), the *difference* between product of diagonal largest

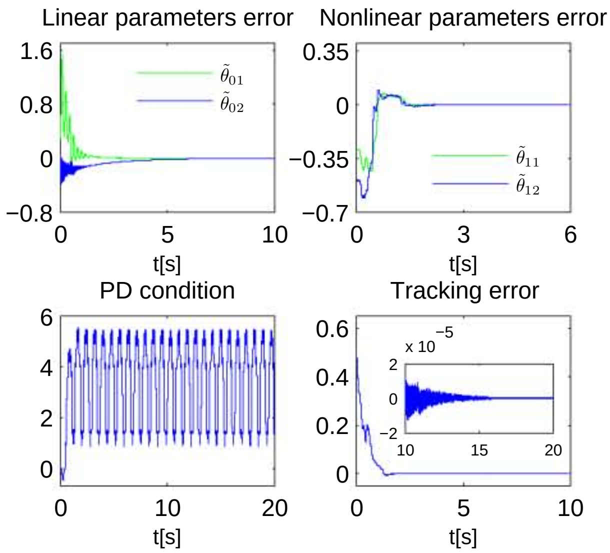

FIGURE 3.2: Linear and nonlinear parameters errors for system (3.61) under the adaptive control scheme. The difference between product of diagonal largest eigenvalues and off-diagonal squared largest singular value of matrix (3.45) was obtained from simulations and it is denoted by PD condition. Tracking error tends to zero within the first $10s$, the subplot depicts its behavior on the time interval $[10s, 20s]$.

eigenvalues and off-diagonal squared largest singular value, equivalent to condition (2.8), was plotted.

The responses are depicted in Figure 3.2. First row shows the linear and nonlinear parameters *errors*, while the second row shows the condition for uniform positive definiteness of (3.45) and tracking error.

3.6.1.2 Regressors dependent on the state

The case when regressors of (3.61) are dependent on the state is analyzed. The regressors are $\boldsymbol{\alpha}_0(\mathbf{x}) = [x_1 \ x_2]^T$ and $\boldsymbol{\alpha}_1(\mathbf{x}) = [x_2^2 \ x_2^3]^T$. The control objective here is to make the tracking error $\varphi = (x_{2_m} - x_2) + \lambda_\varphi(x_{1_m} - x_1) \to 0$, $\lambda_\varphi > 0$, asymptotically where $\mathbf{x}_m = [x_{1_m}, x_{2_m}]^T$ is the state of the second order oscillator

$$\dot{x}_{1_m} = x_{2_m}, \tag{3.63a}$$

$$\dot{x}_{2_m} = -\omega_n^2 x_{1_m} - 2\zeta\omega_n x_{2_m} + \omega_n^2 r(t), \tag{3.63b}$$

with ζ, ω_n positive constants, and $r(t)$ a smooth, bounded, input signal. By defining $\hat{F} = \boldsymbol{\alpha}_0^T(\mathbf{x})\hat{\boldsymbol{\theta}}_0 + \exp(-\boldsymbol{\alpha}_1^T(\mathbf{x})\hat{\boldsymbol{\theta}}_1)$, and $D(\mathbf{x}_m) = -\omega_n^2 x_{1_m} + x_{2_m}(\lambda_\varphi - 2\zeta\omega_n) + \omega_n^2 r(t)$, the adaptive control is given by $u_a(\mathbf{x}, \mathbf{x}_m, \hat{\boldsymbol{\theta}}) = \lambda_t\varphi + D(\mathbf{x}_m) + x_2(\lambda_g - \lambda_\varphi) - \hat{F}$, with $\lambda_t > 0$.

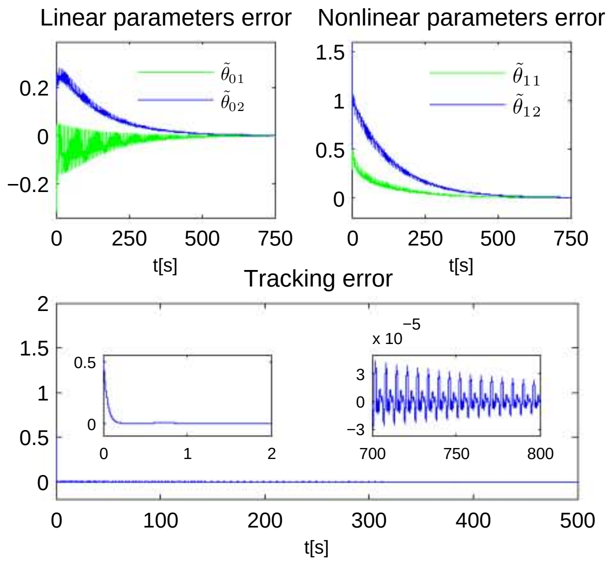

FIGURE 3.3: Linear and nonlinear parameter errors and tracking error of the system (3.61) with regressors dependent on the state $\mathbf{x}$ under the adaptive control scheme. Subplots represent the tracking error behavior in the time intervals $[0s, 2s]$ and $[700s, 800s]$.

The dynamics error has the form (3.41) and the same selections as the preceding example were considered for $\hat{\boldsymbol{\theta}}_I(t)$, $\hat{\mathbf{R}}$, $\hat{\boldsymbol{\theta}}_P(\alpha(\mathbf{x}), \varphi)$, adaptive gains, and metric parameters. In the previous case it was possible to select the frequency parameters in order to minimize singular values of (3.60) left hand side. In the present example, we do not have a quantified control over that singular values. However, it is possible to reach parametric convergence even in this situation because (3.60) is only a sufficient condition. The form of verifying positive definiteness as we did in the previous example is not conclusive either. This argument shows that more general procedures of gains and metric values selection are needed. Parametric and tracking errors for this example are shown in Figure 3.3.

3.7 Application examples: direct control of fermentation and friction systems

In this section, the developed methodology is applied to two systems with nonlinearly parameterized uncertainties. First example is a fermentation process model whose parameters are in the argument of a nonlinear function of linear parameterization (Theorem 3.2). Second example is a friction model which can be written in terms of the error equation (3.41) (Theorem 3.4). In both cases the control objective is to track the state of a reference model, hence controller and target dynamics involved in verification of Assumptions 3.1 and 3.2 are easily found. Comments on validation of Assumption 3.3 are given in each case. Control objectives are reached and parameters are reconstructed under the generalized PE condition[4].

3.7.1 Fermentation process model.

The generic model of a fermentation process can be written as [7]

$$\dot{x} \;=\; \frac{x^2}{\theta_1 + \theta_2 x} + u. \tag{3.64}$$

The state of the plant is $x \in \mathbb{R}_+$, $\forall t \geq 0$, and each component of $\boldsymbol{\theta} = [\theta_1 \ \theta_2]^T \in \mathbb{R}^2$ is restricted to belong to an interval where $\underline{\theta}_i$ and $\overline{\theta}_i$ are the lower and upper bounds, respectively, $i = 1, 2$. The control input is denoted by u with the objective to make the tracking error $\varphi(x, x_m) = x - x_m \to 0$ asymptotically according to the target dynamics $\dot{\varphi}_d = -\lambda_m \varphi_d$, $\lambda_m > 0$. x_m is the state of the reference model $\dot{x}_m = -\lambda_m x_m + b_m r(t)$, with $b_m > 0$ and $r(t)$ a smooth bounded reference trajectory. By defining $\boldsymbol{\alpha}(x) = [1 \ x]^T$ and $\boldsymbol{\theta} = [\theta_1 \ \theta_2]^T$, the adaptive controller $u = -\lambda_m x + b_m r(t) + \hat{f}_\alpha$, where $\hat{f}_\alpha = -\frac{x^2}{\boldsymbol{\alpha}^T(x)\hat{\boldsymbol{\theta}}}$, is such that (3.16) is obtained. Assumption 3.3 is satisfied because $\underline{\theta}_i > 0$. By (3.33), $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}(x), \varphi) = -\bar{\boldsymbol{\Gamma}}\boldsymbol{\alpha}(x)\varphi$, $\bar{\boldsymbol{\Gamma}} = \gamma \mathbf{I}_{2\times 2}$, $\gamma > 0$, and $\hat{\mathbf{R}}$ as in (3.24) was taken. Condition (3.30) for existence of function $\boldsymbol{\rho}(x, x_m(t))$ is satisfied, hence, by solving (3.29), we have

$$\boldsymbol{\rho}(x, x_m) = \gamma \begin{bmatrix} 0 \\ \frac{x^2}{2} - x x_m \end{bmatrix}. \tag{3.65}$$

Finally $\hat{\boldsymbol{\theta}}_I(t)$ as in (3.28) and $\hat{\boldsymbol{\theta}}$ as in (3.26) are taken. Condition (3.36) conduces to the inequality

$$\frac{x^2}{(\boldsymbol{\alpha}^T\boldsymbol{\theta}_v)^2}\left(\frac{2(\varphi_v - \varphi_d)}{\boldsymbol{\alpha}^T\boldsymbol{\theta}_v} + \gamma\right) \geq \lambda_E, \tag{3.66}$$

which has to be fulfilled for $\mathbf{z}_v$ evaluated in $\mathbf{z}_d$ and $\hat{\mathbf{z}}$. A complete verification of (3.66) is widely explained in [16] and [17].

Responses of estimated parameters, levels of excitation (denoted by PE condition), and tracking error are shown in Figure 3.4. The excitation level $d \triangleq \delta \int_t^{t+\delta} x(\tau)^2 d\tau - \left(\int_t^{t+\delta} x(\tau) d\tau\right)^2$ was numerically implemented for two different references $r(t)$. If $r(t) = 10 + 15\sin(2\pi t)$, the excitation level does not tend to zero and parametric convergence is reached (Figure 3.4, first column). In contrast, if reference is $r(t) = 2$, excitation level tends to zero and parameters do not converge to their nominal values (Figure 3.4, second column).

[4]Values of all the constants involved in this section are defined in Table 4.1.

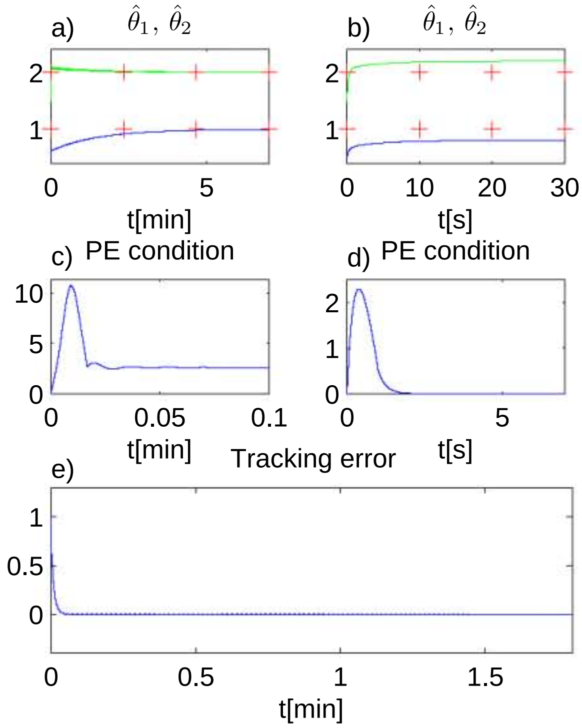

FIGURE 3.4: Estimated parameters with (a) and without (b) parametric convergence. Nominal values are denoted by '+'. Levels of excitation (denoted by PE condition) with (c) and without (d) persistent excitation. If persistent excitation is present, parametric convergence is reached, otherwise parameters do not converge to their nominal values. Tracking error is depicted in (e).

3.7.2 Friction model

In the range of low velocities, frictional force changes nonlinearly as it is proposed in the model of [34], which describes the force as $F = F_c \operatorname{sign}(\dot{x}) + (F_s - F_c)\operatorname{sign}(\dot{x})\exp(-\dot{x}/v_s)^2 + F_v\dot{x}$, where x is the angular position of the motor shaft, F_c represents the Coulomb friction, F_v stands for the viscous friction coefficient, and v_s is the Stribeck parameter. In order to have uncertainties of similar order of magnitude, we redefine the meter unit as $m' = 0.054m$ and rewrite the original parameters under this normalization.

The equation of motion can be written as $\ddot{x} = F + u$ where u is the control input (torque) to be determined. If $x \triangleq x_1$ and $\dot{x} \triangleq x_2$, the definitions $\boldsymbol{\alpha}_0(x_2) = [\operatorname{sign}(x_2)\ x_2]^T$, $\boldsymbol{\theta}_0 = [F_c\ F_v]^T$, $\boldsymbol{\alpha}_1(x_2) = [x_2^2\ 1]^T$, $\boldsymbol{\theta}_1 = [\eta\ \ln(\frac{1}{\sigma})]^T$ where $\eta = 1/v_s^2$, and $\sigma = F_s - F_c$, lead to the following model written in the appropriate form

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= \boldsymbol{\alpha}_0^T(x_2)\boldsymbol{\theta}_0 + \operatorname{sign}(x_2)\exp(-\boldsymbol{\alpha}_1^T(x_2)\boldsymbol{\theta}_1) + u.\end{aligned} \tag{3.67}$$

The control objective is to track the state of the reference model (3.63) under the target dynamics $\varphi_d = -\lambda_t\varphi_d$, $\lambda_t > 0$, where $\varphi = x_2 - x_{2_m} + \lambda_\varphi(x_1 - x_{1_m})$, $\lambda_\varphi > 0$. In this case, by defining $\hat{F} = \boldsymbol{\alpha}_0^T(x_2)\hat{\boldsymbol{\theta}}_0 + \text{sign}(x_2)\exp(-\boldsymbol{\alpha}_1^T(x_2)\hat{\boldsymbol{\theta}}_1)$ and $D(\mathbf{x}_m) = -\omega_n^2 x_{1_m} - 2\zeta\omega_n x_{2_m} + \omega_n^2 r(t)$, the expression of the adaptive controller is $u_a(\mathbf{x}, \mathbf{x_m}, \hat{\boldsymbol{\theta}}) = -\lambda_t\varphi - \hat{F} + D(\mathbf{x}_m) - \lambda_\varphi(x_2 - x_{2_m})$. Assumption 3.3 is clearly satisfied by the exponential function. Since condition for existence of $\boldsymbol{\rho}(\cdot)$ is not satisfied, $\hat{\boldsymbol{\theta}}_I(t)$ is taken as in (3.31) where $\hat{\mathbf{R}}$ is as in (3.39). $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}(\mathbf{x}), \varphi) = \bar{\boldsymbol{\Gamma}}\boldsymbol{\alpha}(\mathbf{x})\varphi$ is given by (3.33), where $\bar{\boldsymbol{\Gamma}} = \text{diag}(\boldsymbol{\Gamma}_0, -\,\text{sign}(x_2)\boldsymbol{\Gamma}_1)$. Initial conditions and gains are calculated as in the illustrative example (3.61). Some additional details can be seen in [35]. Results of simulation are shown in Figure 3.5 where linear and nonlinear parameters *errors* and tracking error are depicted. Response in the subplot corresponds to tracking error in the final time interval of simulation. One main feature of our development is pointed out: while the work done in [11] was focused in identifying the *complete* function F or the methodology in [36] allowed to identify the linear parameters in the model, our scheme is capable to reconstruct *individually* each nominal parameter of the plant. To the best of our knowledge, this is the first report of parametric convergence of uncertainties with these characteristics.

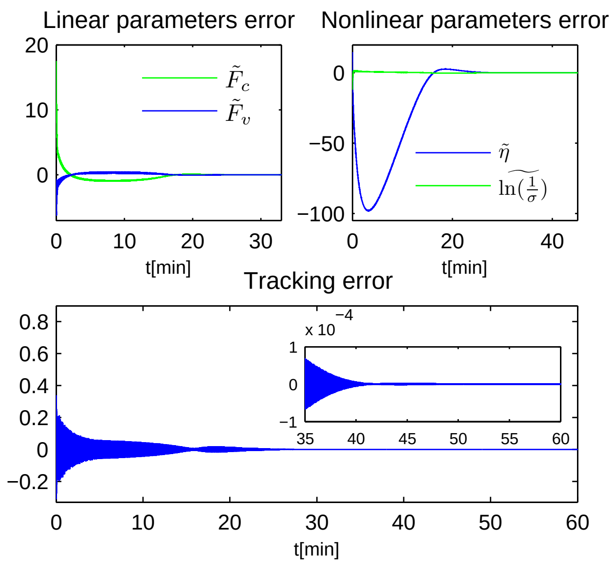

FIGURE 3.5: Linear and nonlinear parameters errors and tracking error for the system (3.67) under the adaptive control scheme. Subplot depicts the tracking error behavior in the time interval [35min, 60min].

	Regressors dependent on $\sin(\cdot)$ **and** $\cos(\cdot)$		
Plant	$\mathbf{\Gamma}$, $\mathbf{P}$, λ_t, λ_φ	$\hat{\boldsymbol{\theta}}(0)$	Additional par.
$\mathbf{x}_0 = [0\ 0.5]^T$ $\lambda_g = 5$ $\boldsymbol{\theta}_0 = [0.7\ 1.2]^T$ $\boldsymbol{\theta}_1 = [1.3\ 1.5]^T$	$\mathbf{\Gamma}_0 = \text{diag}(4,1)$ $\mathbf{\Gamma}_1 = \text{diag}(2.5,3.5)$ $\mathbf{P}_0 = \text{diag}(0.4,0.4)$ $\mathbf{P}_1 = \text{diag}(2,2)$ $p_\psi = 3$ $\lambda_t = 3$ $\lambda_\varphi = 3$	$\hat{\boldsymbol{\theta}}_0(0) = [1\ 1]^T$ $\hat{\boldsymbol{\theta}}_1(0) = [1\ 1]^T$	$[\omega_{0_1}\ \omega_{0_2}]^T = [35\ 126.6]^T$ $[\omega_{1_1}\ \omega_{1_2}]^T = [8\ 48]^T$ $\delta = 0.5$
	Regressors dependent on x_1 **and** x_2		
$\mathbf{x}_0 = [0\ 1]^T$ $\lambda_g = 5$ $\boldsymbol{\theta}_0 = [0.7\ 1.2]^T$ $\boldsymbol{\theta}_1 = [1.3\ 1.5]^T$	$\mathbf{\Gamma}_i$ as in previous case. $\mathbf{P}_0 = \text{diag}(3,5)$ $\mathbf{P}_1 = \text{diag}(2,5)$ $p_\psi = 3$ $\lambda_t = 20$ $\lambda_\varphi = 10$	$\hat{\boldsymbol{\theta}}_0(0) = [1\ 1]^T$ $\hat{\boldsymbol{\theta}}_1(0) = [1\ 1]^T$	$r(t) = 1 + \sin(t)$ $\omega_n = 5 \quad \zeta = 0.707$ $x_{1_{m0}} = 0 \quad x_{2_{m0}} = 1.5$
	Fermentation process model		
Plant par.	Model par.	$\hat{\boldsymbol{\theta}}(0)$	
$x_0 = 4$ $\boldsymbol{\theta} = [1\ 2]^T$	$x_m(0) = 3$ $b_m = 1$ $\lambda_m = 2$	$\hat{\boldsymbol{\theta}}(0) = [0.5\ 1.5]^T$	$r(t) = 10 + 15\sin(2\pi t)$ $r(t) = 2$ $\gamma = 1$
	Friction model		
$\mathbf{x}_0 = [0\ 1.3]^T$ $F_c = 1N$ $F_v = 0.4Ns/m$ $v_s = 0.018m/s$ $F_s = 1.5N$	$x_{1_m}(0) = 0$ $x_{2_m}(0) = 1$ $\omega_n = 0.5$ $\zeta = 0.70$	$\hat{\boldsymbol{\theta}}_0(0) = [1\ 1]^T$ $\hat{\boldsymbol{\theta}}_1(0) = [1\ 10]^T$	$F'_c = 18.52N'$ $F'_v = 0.4N's/m'$ $v'_s = \frac{1}{3}m'/s$ $F'_s = 27.77N'$ $\lambda_\varphi = 50$ $\lambda_t = 60$ $\gamma = \text{diag}(1,1)$ $r(t) = \sin(t) + \sin(4t + 1.5)$

Table 3.1: Simulation parameters for system (3.61), fermentation and friction models (3.64) and (3.67), respectively.

Chapter 4

Identification of systems with nonlinear parameterization

The direct adaptive control scheme developed on Chapter 3 was based on the analysis of a virtual system which involves generic interactions between tracking and parameters errors. Averaged conditions for contraction of such system led to closed-loop stability properties in which convexity, concavity, or monotonicity in the parameters or in parameterization were not assumed. When parameterizations were particularized, such conditions became more accurate and motivated relationships with well known results on parametric convergence (uniform persistent excitation).

In the present chapter, we formulate an identification algorithm for nonlinearly parameterized systems by focusing only on nonlinear functions of linear parameterizations. Although analysis in this chapter is parallel to that done in the previous one, further results are proposed. Given that parameterization is specified, the formulation facilitates to focus on the selection flexibility of $\mathbf{\Gamma}(\mathbf{x})$ which is related with implementation schemes through the estimation proportional term (see (3.27)). Such selection freedom will show that the realization procedures of Chapter 3 can be extended and an approach based on filtering, which makes use only of available signals, can be proposed. In contrast with the previous chapter, where the introduction of a non-conventional metric let to deal with systems with one linear-one nonlinear blocks of uncertainties, in this chapter parameterization extension consists in many nonlinear blocks of uncertainties which are identified through the same metric change.

4.1 Problem statement

Consider the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}, \mathbf{u}, t) + \mathbf{g}(\mathbf{x}, \boldsymbol{\alpha}^T(\mathbf{x}, t)\boldsymbol{\theta}), \ \ \mathbf{x}(0) = \mathbf{x}_0, \tag{4.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ denotes the plant state assumed accessible for measurement and $\mathbf{u} \in \mathbb{R}^q$ stands for a bounded input signal. $\boldsymbol{\theta} \in \mathbb{R}^p$ is the unknown parameters vector which belongs to a compact set $\Omega_\theta \subset \mathbb{R}^p$ and $\boldsymbol{\alpha}(\mathbf{x}, t)$ is detailed later in Assumption 4.2. Functions $\mathbf{f}_p : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are assumed to have all the continuity and differentiability requirements needed. In addition, it is supposed that all the solutions of (4.1) are well defined for all initial conditions and $\forall t \geq t_0$. The following Assumptions over (4.1) are made.

Assumption 4.1. *The system (4.1) is* input to state stable *(ISS).* △

Assumption 4.2. *Function $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^p$ is continuous, differentiable in its arguments and uniformly bounded $\forall t \geq t_0$, for all $\mathbf{x} \in \mathcal{X}$,* i.e. *$\|\boldsymbol{\alpha}(\mathbf{x}, t)\| \leq \bar{\alpha}$, for some $\bar{\alpha} > 0$.* △

The identification model of (4.1) is

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}_p(\hat{\mathbf{x}}, \mathbf{u}, t) + \mathbf{g}(\mathbf{x}, \boldsymbol{\alpha}^T(\mathbf{x}, t)\hat{\boldsymbol{\theta}}), \tag{4.2}$$

where $\hat{\mathbf{x}} \in \mathbb{R}^n$ is the estimated state and $\hat{\boldsymbol{\theta}} \in \mathbb{R}^p$ denote the estimated parameters vector. If $\mathbf{c} \in \mathbb{R}^n$ is a non-trivial constant vector, output signals of (4.1) and (4.2) can be defined as $y = \mathbf{c}^T\mathbf{x}$ and $\hat{y} = \mathbf{c}^T\hat{\mathbf{x}}$, respectively. Time evolution of estimation error $\varphi = \hat{y} - y$ is written under the following assumption.

Assumption 4.3. *There exists some $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^T\mathbf{f}_p(\hat{\mathbf{x}}, \mathbf{u}, t) - \mathbf{c}^T\mathbf{f}_p(\mathbf{x}, \mathbf{u}, t) = -\psi(\varphi)$ where $\psi : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ class contracting function, i.e. $-\frac{\partial \psi(\varphi)}{\partial \varphi} \leq -\lambda_\psi$ for some $\lambda_\psi > 0$.* △

By defining $\mathbf{c}^T\mathbf{g}(\mathbf{x}, \boldsymbol{\alpha}^T(\mathbf{x}, t)\boldsymbol{\theta}) = f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta})$, similarly for the estimated variables, estimation error dynamics is

$$\dot{\varphi} = -\psi(\varphi) + f(\mathbf{x}, \boldsymbol{\alpha}^T\hat{\boldsymbol{\theta}}) - f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta}). \tag{4.3}$$

Assumption 4.4. *The first and second partial derivatives of* $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *w.r.t* $\boldsymbol{\theta}$ *exist and are bounded* $\forall \boldsymbol{\theta} \in \Omega_\theta$*, in particular,* $\left\| \frac{\partial f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\| \leq \max_{\boldsymbol{\theta}\in\Omega_\theta} \left\| \frac{\partial f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\|$. △

The problem this Chapter is concerned is identification of the plant nonlinear parameters from available measurements. Identification, in terms of a desired estimation error dynamics, can be written as

$$\dot{\varphi}_d = -\psi(\varphi_d), \;\; \dot{\boldsymbol{\theta}}_d = \mathbf{0}, \;\; \boldsymbol{\theta}_d(0) = \boldsymbol{\theta}. \tag{4.4}$$

4.2 Identification algorithm

At the top level design, the non-implementable identification algorithm

$$\dot{\hat{\boldsymbol{\theta}}} = \mathbf{R}(\varphi, \varphi_d, \mathbf{x}, \hat{\boldsymbol{\theta}}) + \boldsymbol{\Gamma}(\mathbf{x}, t)(f(\mathbf{x}, \boldsymbol{\alpha}^T\hat{\boldsymbol{\theta}}) - f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta})), \tag{4.5}$$

is proposed, where $\mathbf{R} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ and $\boldsymbol{\Gamma} : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^p$ are degrees of freedom related with the realization of (4.5) through generalized contraction analysis.

4.2.1 Parametric convergence analysis

The identification objective, expressed as a dynamics in (4.4), is rewritten as

$$\dot{\varphi}_d = -\psi(\varphi_d) + f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta}_d) - f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta}), \tag{4.6a}$$

$$\dot{\boldsymbol{\theta}}_d = \mathbf{R}(\varphi_d, \varphi_d, \mathbf{x}, \boldsymbol{\theta}_d) + \boldsymbol{\Gamma}(\mathbf{x}, t)(f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta}_d) - f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta})), \tag{4.6b}$$

where $\boldsymbol{\theta}_d(0) = \boldsymbol{\theta}$, and $\mathbf{R}(\varphi_d, \varphi_d, \mathbf{x}, \boldsymbol{\theta}_d) = \mathbf{0} \in \mathbb{R}^p$. Actual system (4.3), (4.5) and desired system (4.6) are particular solutions of the virtual system

$$\dot{\varphi}_v = -\psi(\varphi_v) + f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta}_v) - f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta}), \tag{4.7a}$$

$$\dot{\boldsymbol{\theta}}_v = \mathbf{R}(\varphi_v, \varphi_d, \mathbf{x}, \boldsymbol{\theta}_v) + \boldsymbol{\Gamma}(\mathbf{x}, t)(f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta}_v) - f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta})). \tag{4.7b}$$

Given the specific parameterization, it is possible to write[1] $\frac{\partial f_v}{\partial \boldsymbol{\theta}_v} = \frac{\partial f_v}{\partial \xi_v}\boldsymbol{\alpha}^T$. Hence, Jacobian symmetric part of (4.7), $-\mathbf{J}_s$, is constituted by matrices (see (2.7))

$$
\begin{aligned}
J_{1_s} &= \frac{\partial \psi(\varphi_v)}{\partial \varphi_v}, && (4.8)\\
\mathbf{J}_{2_s} &= -\left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\theta}_v} + \frac{\partial f_v}{\partial \xi_v}\boldsymbol{\Gamma}(\mathbf{x},t)\boldsymbol{\alpha}^T\right]_s, && (4.9)\\
\mathbf{G} &= -\frac{1}{2}\left(\frac{\partial \mathbf{R}_v}{\partial \varphi_v} + \frac{\partial f_v}{\partial \xi_v}\boldsymbol{\alpha}\right). && (4.10)
\end{aligned}
$$

The generalized contraction of system (4.7) and the consequent identification of parameters conditions for plant (4.1), for a specific choice of $\hat{\mathbf{R}}$, are stated in the following theorem.

Theorem 4.1. *Consider the system (4.1) and its identification model (4.2) where* $\hat{\boldsymbol{\theta}}$ *is adjusted by (4.5). In addition to Assumptions 4.1-4.4, suppose that* $\hat{\mathbf{R}}$ *is given by*

$$\hat{\mathbf{R}} = -\frac{\partial \hat{f}}{\partial \hat{\xi}}(\varphi - \varphi_d)\boldsymbol{\alpha}(\mathbf{x},t). \tag{4.11}$$

Let $\boldsymbol{\Gamma}(\mathbf{x},t)$ *be such that* $\mathbf{J}_{2_s}$ *is USPD, where*

$$\mathbf{J}_{2_s} = \left[\frac{\partial^2 f_v}{\partial \xi_v^2}(\varphi_v - \varphi_d)\boldsymbol{\alpha}\boldsymbol{\alpha}^T - \frac{\partial f_v}{\partial \xi_v}\boldsymbol{\Gamma}(\mathbf{x},t)\boldsymbol{\alpha}^T\right]_s, \tag{4.12}$$

and such that, for some $T > 0$ *and* $\lambda_\theta > 0$,

$$\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau \geq \lambda_\theta \mathbf{I}_{p\times p}, \tag{4.13}$$

$\forall t \geq t_0$, $\forall \mathbf{x} \in \mathcal{X}$, *for* $\mathbf{z}_v = \mathbf{z}_d$ *and* $\mathbf{z}_v = \hat{\mathbf{z}}$. *Then, the actual system trajectories (4.3), (4.5) tend to the desired system trajectories (4.6),* i.e. $\lim_{t\to\infty}\varphi(t) = \varphi_d(t)$ *and* $\lim_{t\to\infty}\hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. By the selection (4.11), the corresponding $\mathbf{R}_v$ will make $\mathbf{G} = \mathbf{0} \in \mathbb{R}^p$ in (4.10). Then $-\mathbf{J}_s = \mathrm{diag}(J_{1_s}, \mathbf{J}_{2_s})$ where J_{1_s} given in (4.8) and $\mathbf{J}_{2_s}$ as in (4.12). By Assumption 4.3 and the given properties of $\boldsymbol{\Gamma}(\mathbf{x},t)$, $-\mathbf{J}_s$ is USPD and usage of Lemma 2.2 is justified. By Assumption 4.3 and condition (4.13), requirement (2.5) of Lemma 2.2 will be satisfied if $\lambda \leq \min(\lambda_\psi T, \lambda_\theta)$ and the conclusion follows. □

[1] See the Notation section for definition of variables.

Instead of considering general parameterizations, as was done in Corollary 1 of Chapter 3, Theorem 4.1 focuses only on parameters structures of the form (4.1). The main advantage of restricting to the underlined parameterizations arises when showing the flexibility in choosing $\mathbf{\Gamma}(\mathbf{x}, t)$. This claim is detailed as follows.

By means of a proper definition of $\mathbf{\Gamma}(\mathbf{x}, t)$, a term $\boldsymbol{\alpha}\boldsymbol{\alpha}^T$ can be factored and certain structure can be imposed in $\mathbf{J}_{2_s}$ (4.12). Such structure leads to the property of persistent excitation and a clear relationship between parametric convergence and averaged contraction analysis will arise. To formalize this affirmation, define the matrix

$$\mathbf{E}_\alpha = \frac{\partial^2 f_v}{\partial \xi_v^2}(\varphi_v - \varphi_d)\mathbf{I}_{p\times p} + \left|\frac{\partial f_v}{\partial \xi_v}\right| \bar{\mathbf{\Gamma}}, \tag{4.14}$$

where $\bar{\mathbf{\Gamma}} = \bar{\mathbf{\Gamma}}^T \in \mathbb{R}^{p\times p}$ is a constant gain matrix. The following corollary is established.

Corollary 4.1. *Under the same Assumptions of Theorem 4.1, consider* $\mathbf{\Gamma}(\mathbf{x}, t)$ *is given by*

$$\mathbf{\Gamma}(\mathbf{x}, t) = -\bar{\mathbf{\Gamma}}\,\mathrm{sign}\left(\frac{\partial \hat{f}}{\partial \hat{\xi}}\right)\boldsymbol{\alpha}(\mathbf{x}, t). \tag{4.15}$$

Let $\bar{\mathbf{\Gamma}} = \bar{\mathbf{\Gamma}}^T \in \mathbb{R}^{p\times p}$ *be such that*

$$\mathbf{E}_\alpha \geq \lambda_E \mathbf{I}_{p\times p}, \tag{4.16}$$

for some $\lambda_E > 0$, $\forall t \geq t_0$, *for all* $\mathbf{x} \in \mathcal{X}$, *and for* $\mathbf{z}_v = \mathbf{z}_d$ *and* $\mathbf{z}_v = \hat{\mathbf{z}}$. *Furthermore, assume that the regressor* $\boldsymbol{\alpha}(\mathbf{x}, t)$ *is uniformly persistently exciting in the sense of Definition 2.2, for some* ϵ_α, $\delta > 0$. *Then all the trajectories of (4.3), (4.5) tend exponentially to those of (4.6), i.e.* $\lim_{t\to\infty} \varphi(t) = \varphi_d(t)$ *and* $\lim_{t\to\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. By selection (4.15), $-\mathbf{J}_s = \mathrm{diag}(J_{1_s}, \mathbf{J}_{2_s})$ has elements given by J_{1_s} as in (4.8) and $\mathbf{J}_{2_s} = \left[\mathbf{E}_\alpha \boldsymbol{\alpha}\boldsymbol{\alpha}^T\right]_s$. Under condition (4.16), $\mathbf{J}_{2_s}$ is USPD which, together with Assumption 4.3, guarantee that $-\mathbf{J}_s$ is USPD. Given that there exist ϵ_α and δ for $\boldsymbol{\alpha}(\mathbf{x}, t)$ of uniform PE, take $T = \delta$. By (4.16), $\mathbf{J}_{2_s}$ satisfies $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau \geq \lambda_E \epsilon_\alpha \mathbf{I}_{p\times p}$. Then, (4.13) is fulfilled if $\lambda_\theta \leq \lambda_E \epsilon_\alpha$. By Assumption 4.3 and the previous considerations, condition (2.5) of Lemma 2.2 will be satisfied if $\lambda \leq \min(\lambda_\psi T, \lambda_\theta)$. □

The condition $\lambda_\theta = 0$ is not allowed in the studied cases of Corollary 4.1 since the algorithm would not be able to reach the identification objective. This is the main difference between previous Corollary and Theorem 2 of Chapter 3 where $\lambda_\theta = 0$ implies convergence to the invariant manifold where tracking error tends to zero but nominal parameters may not be reconstructed.

An alternative definition of $\mathbf{\Gamma}(\mathbf{x},t)$ can be established by imposing a dependence on a *filtered* version of the regressor $\boldsymbol{\alpha}(\mathbf{x},t)$, denoted in the large by $\boldsymbol{\alpha}_f(\mathbf{x},t)$. Filtered variable is such that

$$\dot{\boldsymbol{\alpha}}_f = \mathbf{Q}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_f), \tag{4.17}$$

where $\mathbf{Q} \in \mathbb{R}^{p\times p}$ is a PD constant matrix. By simplification purposes, take $\mathbf{Q} = q\mathbf{I}_{p\times p}$, $q > 0$. The following Lemma establishes, for fixed frequencies, an asymptotic convergence property between $\boldsymbol{\alpha}_f$ and $\boldsymbol{\alpha}$.

Lemma 4.2. *Let ω be a fixed value of the frequency and define $G(s) = \frac{s}{s+q}$. For any $\epsilon > 0$ there exists a sufficiently large $q > 0$ such that $\|G(s)\|_2 < \epsilon$. Moreover, for such q, the relationship $\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_f\| \leq \epsilon\|\boldsymbol{\alpha}\|$, $\forall \boldsymbol{\alpha} \in \mathbb{R}^p$ holds.*

Proof. By direct application of induced 2-norm definition we have $\|G(s)\|_2 = \frac{\omega}{\sqrt{\omega^2+q^2}}$. If q is chosen such that $\omega\sqrt{\frac{1}{\epsilon^2} - 1} < q$ the result in first part follows. To show the second part of the Lemma, from (4.17) we have $\boldsymbol{\alpha}_f = \frac{q}{s+q}\boldsymbol{\alpha}$, hence $\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_f\| = \|(1 - \frac{q}{s+q})\mathbf{I}_{p\times p}\boldsymbol{\alpha}\| \leq \|G(s)\|_2\|\boldsymbol{\alpha}\|$. By using the first part of the result, the conclusion follows. □

Remark 4.1. If $\epsilon \ll 1$, then $\omega\sqrt{\frac{1}{\epsilon^2} - 1} < q$ can be approximated by $\frac{\omega}{\epsilon} < q$. If q increases, ϵ must decrease in order to preserve the inequality. This means that, by increasing q value, the difference $\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_f\|$ tends to be smaller proportionally with ϵ. △

Stability for the case when $\mathbf{\Gamma}(\mathbf{x},t)$ depends on filtered signals is stated in the following corollary.

Corollary 4.2. *Under the same Assumptions of Theorem 4.1, define*

$$\mathbf{\Gamma}(\mathbf{x},t) = -\bar{\mathbf{\Gamma}}\,\mathrm{sign}\left(\frac{\partial \hat{f}}{\partial \hat{\xi}}\right)\boldsymbol{\alpha}_f(\mathbf{x},t). \tag{4.18}$$

Suppose that the regressor $\boldsymbol{\alpha}(\mathbf{x},t)$ is uniformly persistently exciting in the sense of Definition 2.2, for some ϵ_α, $\delta > 0$. Let $\bar{\mathbf{\Gamma}} = \bar{\mathbf{\Gamma}}^T \in \mathbb{R}^{p\times p}$ be such that bound $\mathbf{E}_\alpha \geq \lambda_E\mathbf{I}_{p\times p}$ holds, for some $\lambda_E > 0$, and let $q > 0$ of Lemma 4.2 be such that

$$\lambda_E\epsilon_\alpha - \bar{\lambda}(\bar{\mathbf{\Gamma}})\max\left|\frac{\partial f_v}{\partial \xi_v}\right|\epsilon\bar{\alpha}^2 T > \lambda_\theta, \tag{4.19}$$

for some $\epsilon > 0$, $T > 0$ and $\lambda_\theta > 0$, $\forall t \geq t_0$, for all $\mathbf{x} \in \mathcal{X}$ and for $\mathbf{z}_v = \mathbf{z}_d$ and $\mathbf{z}_v = \hat{\mathbf{z}}$. Then all the trajectories of (4.3), (4.5) tend exponentially to those of (4.6), i.e. $\lim_{t\to\infty}\varphi(t) = \varphi_d(t)$ and $\lim_{t\to\infty}\hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. By selection (4.18), elements of $-\mathbf{J}_s = \mathrm{diag}(J_{1_s}, \mathbf{J}_{2_s})$ are J_{1_s} as in (4.8) and

$$\mathbf{J}_{2_s} = \left[\mathbf{E}_\alpha \boldsymbol{\alpha}\boldsymbol{\alpha}^T + \bar{\boldsymbol{\Gamma}} \left|\frac{\partial f_v}{\partial \xi_v}\right| (\boldsymbol{\alpha}_f - \boldsymbol{\alpha})\boldsymbol{\alpha}^T\right]_s .$$

By Assumption 4.3, given that filtering is a linear operator, and by the structure of $\mathbf{J}_{2_s}$, matrix $-\mathbf{J}_s$ is USPD. By Assumptions 4.2 and 4.4, $\|\frac{\partial f_v}{\partial \boldsymbol{\theta}_v}\| \leq \left|\frac{\partial f_v}{\partial \xi_v}\right| \|\boldsymbol{\alpha}\| \leq \max\left|\frac{\partial f_v}{\partial \xi_v}\right| \bar{\alpha}$. By this fact, Lemma 4.2, Assumption 4.2, and by noting that $\bar{\boldsymbol{\Gamma}} \leq \bar{\lambda}(\bar{\boldsymbol{\Gamma}})\mathbf{I}_{p\times p}$ for any symmetric matrix $\bar{\boldsymbol{\Gamma}}$, then

$$\left[\bar{\boldsymbol{\Gamma}} \left|\frac{\partial f_v}{\partial \xi_v}\right| (\boldsymbol{\alpha} - \boldsymbol{\alpha}_f)\boldsymbol{\alpha}^T\right]_s \leq \bar{\lambda}(\bar{\boldsymbol{\Gamma}}) \max\left|\frac{\partial f_v}{\partial \xi_v}\right| \epsilon\bar{\alpha}^2 \mathbf{I}_{p\times p}.$$

Given that there exist ϵ_α, $\delta > 0$ for $\boldsymbol{\alpha}(\mathbf{x}, t)$ of uniform PE, take $T = \delta$. By this property, the previous calculations, and (4.16), $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau \geq (\lambda_E \epsilon_\alpha - \bar{\lambda}(\bar{\boldsymbol{\Gamma}}) \max\left|\frac{\partial f_v}{\partial \xi_v}\right| \epsilon\bar{\alpha}^2 T)\mathbf{I}_{p\times p}$. By (4.19) and Assumption 4.3, Lemma 2.2 will be satisfied if $\lambda \leq \min(\lambda_\psi T, \lambda_\theta)$. □

Remark 4.2. By Remark 4.1, second term of $\mathbf{J}_{2_s}$ can be made appropriately small through a proper selection of sufficiently large q. This point is shown in the example of Section 4.4.5. △

4.3 Identification algorithm implementation

Realization of the identification algorithm can be accomplished by choosing different forms of estimation schemes. In this section, equivalent implementation procedures to those developed in Chapter 3, for the particular parameterizations being considered, are explicitly calculated. It will be shown that, due to the selection flexibility of $\boldsymbol{\Gamma}(\mathbf{x}, t)$, an additional implementation procedure, based on a filtering approach, can be added to the list of realization schemes.

4.3.1 State time derivative based implementation

As a first case, a theoretical cancellation of $\dot{\mathbf{x}}$ factors will be carried out by means of an additional design function in the PI estimation. To begin with, $\hat{\boldsymbol{\theta}}(t)$ is defined as

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}_I(t) + \hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}, \varphi) + \boldsymbol{\rho}(\mathbf{x}, \hat{\mathbf{x}}), \tag{4.20}$$

where $\boldsymbol{\rho} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ is designed to avoid $\dot{\mathbf{x}}$ action through an appropriate algebraic condition. The objective of shaping the time derivative of (4.20) along (4.3) to obtain (4.5), is achieved if

$$\boldsymbol{\Gamma}(\mathbf{x}, t) = \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \varphi}, \tag{4.21}$$

$$\frac{\partial \boldsymbol{\rho}}{\partial \mathbf{x}} = -\frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \boldsymbol{\alpha}} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{x}}, \tag{4.22}$$

$$\dot{\hat{\boldsymbol{\theta}}}_I(t) = \hat{\mathbf{R}} + \psi(\varphi) \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \varphi} - \frac{\partial \boldsymbol{\rho}}{\partial \hat{\mathbf{x}}} \dot{\hat{\mathbf{x}}} - \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \boldsymbol{\alpha}} \frac{\partial \boldsymbol{\alpha}}{\partial t}. \tag{4.23}$$

By considering the definition of $\boldsymbol{\Gamma}(\mathbf{x}, t)$ in (4.15) and condition (4.21), implementation concludes by taking

$$\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}, \varphi) = -\bar{\boldsymbol{\Gamma}} \operatorname{sign} \left(\frac{\partial \hat{f}}{\partial \hat{\xi}} \right) \boldsymbol{\alpha} \varphi. \tag{4.24}$$

The key condition to exclude $\dot{\mathbf{x}}$ influence (4.22) will have solutions if and only if the corresponding Poincaré Lemma [6]

$$\frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \boldsymbol{\alpha}} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{x}} \right) = \left(\frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \boldsymbol{\alpha}} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{x}} \right) \right)^T, \tag{4.25}$$

holds. If signal $\dot{\mathbf{x}}$ is available, or observable, $\boldsymbol{\rho}(\mathbf{x}, \hat{\mathbf{x}}) = \mathbf{0}$ is set and $\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}_I(t) + \hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}, \varphi)$ whose time derivative along (4.3) will fit in (4.5) if (4.21) is satisfied, proportional term is as in (4.24), and

$$\dot{\hat{\boldsymbol{\theta}}}_I(t) = \hat{\mathbf{R}} + \psi(\varphi) \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \varphi} - \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \boldsymbol{\alpha}} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{x}} \dot{\mathbf{x}} - \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \boldsymbol{\alpha}} \frac{\partial \boldsymbol{\alpha}}{\partial t}. \tag{4.26}$$

From (4.26) it is clear that, when necessary time derivatives are available, integral action facilitates the implementation and complexities in verifying (4.25) are avoided. In Section 4.3.3, an example in which the required time derivative is known and implementable is developed.

4.3.2 Adaptive algorithm implementation by filtering approach

If the measurable signal $\boldsymbol{\alpha}_f(\mathbf{x}, t)$ is introduced to the estimation vector, we have

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}_I(t) + \hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}_f, \varphi). \tag{4.27}$$

Time derivative of estimation (4.27) along (4.3) is adjusted to (4.5) if

$$\dot{\hat{\boldsymbol{\theta}}}_I(t) = \hat{\mathbf{R}} + \psi(\varphi)\frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \varphi} - \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \boldsymbol{\alpha}_f}\mathbf{Q}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_f). \tag{4.28}$$

As $\boldsymbol{\Gamma}(\mathbf{x}, t) = \frac{\partial \hat{\boldsymbol{\theta}}_P}{\partial \varphi}$ is required, implementation in this section can be concluded by taking

$$\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}_f, \varphi) = -\bar{\boldsymbol{\Gamma}}\,\mathrm{sign}\left(\frac{\partial \hat{f}}{\partial \hat{\xi}}\right)\boldsymbol{\alpha}_f \varphi. \tag{4.29}$$

4.3.3 Example: identification of non convex, non concave, non monotone scalar parameter

The problem consists in identifying a neither convex, concave nor monotone scalar parameter. The system is given by (see [33])

$$\dot{x} = -\lambda_g x + \exp(-\sin(t)\theta) + u, \;\; \lambda_g > 0. \tag{4.30}$$

The state is $x \in \mathbb{R}$, and $u \in \mathbb{R}$ stands for a smooth, bounded input signal. Parameter θ belongs to the set $\Omega_\theta = \{\theta \in \mathbb{R} | \; \underline{\theta} \leq \theta \leq \overline{\theta}\}$ whose bounds are known. Jacobian of the open loop ($u = 0$) system is $J = -\lambda_g$ hence, all the solutions tend global and exponentially to the equilibrium trajectory $x_e(t) = \exp(\alpha(t)\theta)$, where $\alpha(t) = -\sin(t)$. If $\theta \in \Omega_\theta$, the equilibrium trajectory will remain limited within $\exp(-\overline{\theta}) \leq x_e(t) \leq \exp(\overline{\theta})$, $\forall t \geq 0$. The previous argument justify that the system satisfies Assumption 4.1. Assumptions 4.2-4.4 are readily verified. By following the approach of Corollary 4.1, $\hat{R}$ is taken such that $R_v = -(\varphi_v - \varphi_d)\alpha(t)\exp(\alpha(t)\theta_v)$ and $\Gamma(x, t) = \gamma \sin(t)$, $\gamma > 0$, results from (4.15). In this case $E_\alpha = \exp(-\sin(t)\theta_v)\,(\varphi_v - \varphi_d + \gamma)$. If $\hat{\theta}$ is initialized within the region Ω_θ, the bound $\exp(-\overline{\theta}) \leq \exp(-\sin(t)\theta_v)$ holds for $\theta_v = \hat{\theta}$. If $\theta_v = \theta_d$ the bound is trivially satisfied. On the other hand if estimation error is restricted to $|\varphi_v - \varphi_d| < \overline{\varphi} < \gamma$ for some $\overline{\varphi} > 0$ and for $\varphi_v = \varphi$ (if $\varphi_v = \varphi_d$, it holds trivially), then there exists some $\gamma_b > 0$ such that $\gamma_b \leq \varphi - \varphi_d + \gamma$, $\forall t \geq 0$. In summary, condition $E_\alpha \geq \lambda_E$ is satisfied if $\gamma_b \exp(-\overline{\theta}) \geq \lambda_E$. Implementation procedure results in $\hat{\theta}(t) = \hat{\theta}_I(t) + \hat{\theta}_P(\varphi, t)$ where

$$\hat{\theta}_P(\varphi, t) = \gamma \sin(t)\varphi, \tag{4.31}$$

$$\dot{\hat{\theta}}_I(t) = \hat{R} + \lambda_g \varphi \frac{\partial \hat{\theta}_P}{\partial \varphi} - \frac{\partial \hat{\theta}_P}{\partial t}. \tag{4.32}$$

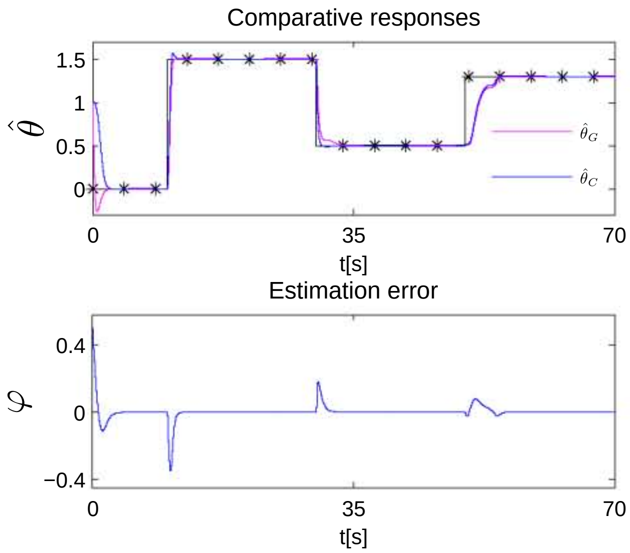

FIGURE 4.1: Estimation error and comparative responses of the parameter identified by (4.31)-(4.32) and the scheme of [33] denoted by $\hat{\theta}_C$ and $\hat{\theta}_G$, respectively. Piecewise constant nominal parameter is denoted by '*'.

As the regressor is a function dependent only on t, the third term of (4.32) is available and implementable. Hence it is not necessary to check condition of existence of solutions (4.25).

The identification scheme (4.31)-(4.32) was applied on the example and the following parameters were taken. Initial conditions: $x_0 = 4.5$, $\hat{x}_0 = 5$, and $\hat{\theta}_0 = 1$. $\lambda_g = 2$ and adaptive gain was $\gamma = 3$. For comparative purposes with the methodology reported in [33], our scheme is applied to estimate a piecewise constant parameter. The comparison is shown in Figure 4.1, top row. Estimation with methodology of [33] is denoted by $\hat{\theta}_G$ and $\hat{\theta}_C$ stands for the estimation response with scheme (4.31)-(4.32).

4.4 Extension of convergence analysis for generalized Jacobian

Introduction of a *constant* non-identity metric $\mathbf{P} = \text{diag}(p_\psi, \mathbf{P}_\theta)$, where $p_\psi > 0$ and $\mathbf{P}_\theta = \mathbf{P}_\theta^T \in \mathbb{R}^{p\times p}$ is non-singular, will lead to a methodology capable to deal with more general parameterizations. Since J_{1_s} is scalar, this element remains without changes and is given by (4.8). On the other hand, $\mathbf{J}_{2_s}$ and $\mathbf{G}$, defined in (4.9) and (4.10), respectively, are changed to

$$\mathbf{J}_{2_s} = -\left[\mathbf{P}_\theta\left(\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\theta}_v} + \frac{\partial f_v}{\partial \xi_v}\boldsymbol{\Gamma}(\mathbf{x},t)\boldsymbol{\alpha}^T\right)\mathbf{P}_\theta^{-1}\right]_s, \tag{4.33}$$

$$\mathbf{G} = -\frac{1}{2}\left(p_\psi \frac{\partial f_v}{\partial \xi_v}\mathbf{P}_\theta^{-1}\boldsymbol{\alpha} + \frac{1}{p_\psi}\mathbf{P}_\theta\frac{\partial \mathbf{R}_v}{\partial \varphi_v}\right). \tag{4.34}$$

In order to establish the main stability theorem for generalized Jacobian, define the matrix

$$\mathbf{E}_m = p_\psi^2 \frac{\partial^2 f_v}{\partial \xi_v^2}(\varphi_v - \varphi_d)\left(\mathbf{P}_\theta^{-1}\right)^2 + \left|\frac{\partial f_v}{\partial \xi_v}\right|\bar{\boldsymbol{\Gamma}}. \tag{4.35}$$

Theorem 4.3. *Consider the system (4.1) and its identification model (4.2) where $\hat{\boldsymbol{\theta}}$ is adjusted by the non-implementable algorithm (4.5). In addition to Assumptions 4.1-4.4, suppose $\boldsymbol{\Gamma}(\mathbf{x},t)$ is given by (4.15) and $\hat{\mathbf{R}}$ is taken as*

$$\hat{\mathbf{R}} = -p_\psi^2 \frac{\partial \hat{f}}{\partial \hat{\xi}}(\varphi - \varphi_d)\left(\mathbf{P}_\theta^{-1}\right)^2\boldsymbol{\alpha}. \tag{4.36}$$

Let $p_\psi > 0$, non singular $\mathbf{P}_\theta = \mathbf{P}_\theta^T \in \mathbb{R}^{p\times p}$, and $\bar{\boldsymbol{\Gamma}} = \bar{\boldsymbol{\Gamma}}^T \in \mathbb{R}^{p\times p}$ be such that $\mathbf{E}_m \geq \lambda_m \mathbf{I}_{p\times p}$, for some $\lambda_m > 0$, uniformly $\forall t \geq t_0$, for all $\mathbf{x} \in \mathcal{X}$, and for $\mathbf{z}_v = \mathbf{z}_d$ and $\mathbf{z}_v = \hat{\mathbf{z}}$. Moreover, assume that the regressor $\boldsymbol{\alpha}(\mathbf{x},t)$ is uniformly persistently exciting in the sense of Definition 2.2, for some ϵ_α, $\delta > 0$. Then all the trajectories of (4.3), (4.5) tend exponentially to those of (4.6), i.e. $\lim_{t\to\infty}\varphi(t) = \varphi_d(t)$ and $\lim_{t\to\infty}\hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. By selection (4.36), the corresponding $\mathbf{R}_v$ will make (4.34) equals $\mathbf{0} \in \mathbb{R}^p$, then $-\mathbf{J}_s = \text{diag}(J_{1_s}, \mathbf{J}_{2_s})$ where J_{1_s} is the same as in (4.8) and $\mathbf{J}_{2_s} = \left[\mathbf{P}_\theta \mathbf{E}_m \boldsymbol{\alpha}\boldsymbol{\alpha}^T \mathbf{P}_\theta^{-1}\right]_s$. By Assumption 4.3 and the given bound of $\mathbf{E}_m$, $-\mathbf{J}_s$ is USPD. Given that there exist ϵ_α, $\delta > 0$ from Definition 2.2, take $T = \delta$. By the given bound on $\mathbf{E}_m$, due to $\mathbf{P}_\theta$ is constant, and uniform PE of $\boldsymbol{\alpha}(\mathbf{x},t)$, the following is obtained $\int_t^{t+T}\mathbf{J}_{2_s}(\tau)d\tau \geq \lambda_m\epsilon_\alpha \mathbf{I}_{p\times p}$. Hence, by Assumption 4.3 and if $\lambda_\theta < \lambda_m\epsilon_\alpha$, Lemma 2.2 will be satisfied for $\lambda \leq \min(\lambda_\psi T, \lambda_\theta)$. □

4.4.1 Generalization of the system parameterization: one linear-one nonlinear in the parameters function

Nonlinearity $f(\mathbf{x}, \boldsymbol{\alpha}^T\boldsymbol{\theta})$ consists in one single nonlinear function of linear parameterization. Introduction of non-conventional metric allows to deal with more general parameterizations *e.g.* $f^\dagger(\mathbf{x}, \boldsymbol{\alpha}_0^T\boldsymbol{\theta}_0, \boldsymbol{\alpha}_1^T\boldsymbol{\theta}_1)$. With the purpose of clarifying such extension, the class of functions given by

$$f^\dagger(\mathbf{x}, \boldsymbol{\alpha}_0^T\boldsymbol{\theta}_0, \boldsymbol{\alpha}_1^T\boldsymbol{\theta}_1) = \boldsymbol{\alpha}_0^T\boldsymbol{\theta}_0 + \sum_{j=1}^{N_p} f_j(\mathbf{x}, \boldsymbol{\alpha}_j^T\boldsymbol{\theta}_j), \tag{4.37}$$

with $N_p = 1$ is considered. Parameters of (4.37) are arranged in a vector $\boldsymbol{\theta} = [\boldsymbol{\theta}_0^T\ \boldsymbol{\theta}_1^T]^T$, where $\boldsymbol{\theta}_i \in \mathbb{R}^{p_i}$, $i = 0, 1$, and $p_0 + p_1 = p$. The *compiled* regressor

$$\boldsymbol{\alpha}(\mathbf{x}, t) = [\boldsymbol{\alpha}_0^T(\mathbf{x}, t)\ \boldsymbol{\alpha}_1^T(\mathbf{x}, t)]^T, \tag{4.38}$$

is composed by bounded and continuous functions $\boldsymbol{\alpha}_i : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}^{p_i}$, $i = 0, 1$. Extension (4.37) implies that the partial derivative w.r.t. the parameter is

$$\frac{\partial f_v^\dagger}{\partial \boldsymbol{\theta}_v} = \left[\frac{\partial f_{v_0}}{\partial \xi_{v_0}}\boldsymbol{\alpha}_0^T\ \frac{\partial f_{v_1}}{\partial \xi_{v_1}}\boldsymbol{\alpha}_1^T\right], \tag{4.39}$$

where $f_{v_0} = \boldsymbol{\alpha}_0(\mathbf{x}, t)^T\boldsymbol{\theta}_{v_0}$, hence $\frac{\partial f_{v_0}}{\partial \xi_{v_0}} = 1$. By doing the same contraction analysis for nonlinearities (4.37) and by following the steps of Theorem 4.3, particular $\boldsymbol{\Gamma}(\mathbf{x}, t)$ and $\hat{\mathbf{R}}$, appropriately extended for the class of uncertainties being considered, will be selected with the objective of imposing certain structure on the resulting $\mathbf{J}_{2_s}$. Such matrix will be written in terms of a generalized metric $\mathbf{P} = \text{diag}(p_\psi, \mathbf{P}_\theta)$, with $p_\psi > 0$ and $\mathbf{P}_\theta = \text{diag}(\mathbf{P}_i)$ with nonsingular $\mathbf{P}_i = \mathbf{P}_i^T \in \mathbb{R}^{p_i \times p_i}$, $i = 0, 1$. Define the matrix

$$\mathbf{E}_1 = p_\psi^2(\varphi_v - \varphi_d)\left(\mathbf{P}_1^{-1}\right)^2\frac{\partial^2 f_{v_1}}{\partial \xi_{v_1}^2} + \boldsymbol{\Gamma}_1\left|\frac{\partial f_{v_1}}{\partial \xi_{v_1}}\right|, \tag{4.40}$$

where the adaptive gains of $\boldsymbol{\Gamma}(\mathbf{x}, t)$ are $\bar{\boldsymbol{\Gamma}} = \text{diag}(\boldsymbol{\Gamma}_i)$, with $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^T \in \mathbb{R}^{p_i \times p_i}$, $i = 0, 1$.

Structure searched in $\mathbf{J}_{2_s}$ and its relationship with the persistent excitation of compiled regressor (4.38) are emphasized if the calculation is stopped before taking the symmetric part. Under the appropriate choices of $\hat{\mathbf{R}}$ and $\boldsymbol{\Gamma}(\mathbf{x}, t)$, it will be shown that $\mathbf{J}_2 = -\mathbf{P}_\theta\left(\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\theta}_v} + \boldsymbol{\Gamma}(\mathbf{x}, t)\frac{\partial f_v^\dagger}{\partial \boldsymbol{\theta}_v}\right)\mathbf{P}_\theta^{-1}$ will result in

$$\mathbf{J}_2 = \begin{bmatrix} \mathbf{P}_0\boldsymbol{\Gamma}_0\boldsymbol{\alpha}_0\boldsymbol{\alpha}_0^T\mathbf{P}_0^{-1} & h_{01}\mathbf{P}_0\boldsymbol{\Gamma}_0\boldsymbol{\alpha}_0\boldsymbol{\alpha}_1^T\mathbf{P}_1^{-1} \\ h_{10}\mathbf{P}_1\boldsymbol{\Gamma}_1\boldsymbol{\alpha}_1\boldsymbol{\alpha}_0^T\mathbf{P}_0^{-1} & \mathbf{P}_1\mathbf{E}_1\boldsymbol{\alpha}_1\boldsymbol{\alpha}_1^T\mathbf{P}_1^{-1} \end{bmatrix}, \tag{4.41}$$

where $h_{ij} = \text{sign}\left(\frac{\partial f_{v_i}}{\partial \xi_{v_i}}\right)\frac{\partial f_{v_j}}{\partial \xi_{v_j}}$, $i, j \in \{0, 1\}$. The stability theorem for nonlinearities (4.37) is enunciated.

Theorem 4.4. *Consider the system (4.1) and its identification model (4.2) where* $\hat{\boldsymbol{\theta}}$ *is* *adjusted by (4.5) whose nonlinearities are of the form (4.37). In addition to Assumptions 4.1-4.4, suppose that* $\hat{\mathbf{R}}$ *and* $\boldsymbol{\Gamma}(\mathbf{x},t)$ *are taken as*

$$\hat{\mathbf{R}} = -p_\psi^2(\varphi-\varphi_d)\,\mathrm{col}\left[\frac{\partial \hat{f}_i}{\partial \hat{\xi}_i}(\mathbf{P}_i^{-1})^2\boldsymbol{\alpha}_i\right], \tag{4.42}$$

$$\boldsymbol{\Gamma}(\mathbf{x},t) = -\,\mathrm{diag}(\boldsymbol{\Gamma}_i)\,\mathrm{col}\left[\mathrm{sign}\left(\frac{\partial \hat{f}_i}{\partial \hat{\xi}_i}\right)\boldsymbol{\alpha}_i\right],\ i=0,1, \tag{4.43}$$

where $\hat{f}_0 = \boldsymbol{\alpha}_0^T\hat{\boldsymbol{\theta}}_0$ *hence* $\frac{\partial \hat{f}_0}{\partial \hat{\xi}_0} = 1$. *Let the elements of the metric* $p_\psi > 0$ *and* $\mathbf{P}_\theta = \mathrm{diag}(\mathbf{P}_i)$ *with nonsingular* $\mathbf{P}_i = \mathbf{P}_i^T \in \mathbb{R}^{p_i\times p_i}$, *and the adaptive gains* $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^T \in \mathbb{R}^{p_i\times p_i}$, $i=0,1$ *be such that* $\mathbf{E}_1 \geq \lambda_1\mathbf{I}_{p_1\times p_1}$ *for some* $\lambda_1 > 0$, *and*

$$\int_t^{t+T}\mathbf{J}_{2_s}(\tau)d\tau \geq \epsilon'\int_t^{t+T}\boldsymbol{\alpha}(\mathbf{x},\tau)\boldsymbol{\alpha}^T(\mathbf{x},\tau)d\tau, \tag{4.44}$$

where $\mathbf{J}_{2_s}$ *given by the symmetric part of (4.41) and* $\boldsymbol{\alpha}(\mathbf{x},t)$ *defined in (4.38), for some* $T > 0$ *and* $\epsilon' > 0$, *uniformly for all* $t \geq t_0$, $\forall \mathbf{x}\in\mathcal{X}$, *and for* $\mathbf{z}_v = \mathbf{z}_d$ *and* $\mathbf{z}_v = \hat{\mathbf{z}}$. *Furthermore, assume that* $\boldsymbol{\alpha}(\mathbf{x},t)$ *is uniformly persistently exciting in the sense of Definition 2.2, for some* ϵ_α, $\delta > 0$. *Then all the trajectories of (4.3), (4.5) tend exponentially to those of (4.6), whose nonlinearities given in (4.37), i.e.* $\lim_{t\to\infty}\varphi(t) = \varphi_d(t)$ *and* $\lim_{t\to\infty}\hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. Upper diagonal element of matrix $-\mathbf{J}_s = \mathrm{diag}(J_{1_s}, \mathbf{J}_{2_s})$ is, as in previous cases, given by (4.8). On the other hand, $\mathbf{J}_{2_s}$ is the symmetric part of (4.41). By Assumption 4.3, J_{1_s} is UPD and a demonstration that $\mathbf{J}_{2_s}$ is USPD is given in the proof of Theorem 3.4, Chapter 3. Hence $-\mathbf{J}_s$ is USPD. By uniform persistent excitation of $\boldsymbol{\alpha}(\mathbf{x},t)$ and by (4.44), it follows that $\int_t^{t+T}\mathbf{J}_{2_s}(\tau)d\tau \geq \epsilon'\epsilon_\alpha\mathbf{I}_{p\times p}$, where $T=\delta$. If $\lambda_\theta < \epsilon'\epsilon_\alpha$ and by Assumption 4.3, Lemma 2.2 will be fulfilled if $\lambda \leq \min(\lambda_\psi T, \lambda_\theta)$. □

As is shown by condition (4.44), linear and nonlinear parameters can be simultaneously identified if compiled regressor is of uniform persistent excitation. This point is illustrated in Section 4.4.4 where the problem of estimating the linear-nonlinear scalar parameters of a system by taking the state time derivative implementation (Section 4.3.1), is solved.

4.4.2 Approach based on filtering for extended parameterization

Filtered signals of the compiled regressor (4.38) are obtained in terms of its components, *i.e.* $\boldsymbol{\alpha}_f(\mathbf{x},t) = [\boldsymbol{\alpha}_{f_0}^T(\mathbf{x},t)\ \boldsymbol{\alpha}_{f_1}^T(\mathbf{x},t)]^T$, where $\boldsymbol{\alpha}_{f_i}(\mathbf{x},t) = \frac{q}{q+s}\boldsymbol{\alpha}_i(\mathbf{x},t)$, $q > 0$, $i = 0,1$. By following the filtering approach of Section 4.3.2 to identify uncertainties with the extended parameterization, a function $\boldsymbol{\Gamma}(\mathbf{x},t)$ depending on the compiled $\boldsymbol{\alpha}_f(\mathbf{x},t)$ will be proposed. The following Theorem changes the selection of $\boldsymbol{\Gamma}(\mathbf{x},t)$ given in Theorem 4.4 and establishes stability and convergence when the approach of filtering is examined.

Theorem 4.5. *Consider the system (4.1) and its identification model (4.2) where $\hat{\boldsymbol{\theta}}$ is adjusted by (4.5) whose nonlinearities are of the form (4.37). In addition to Assumptions 4.1-4.4, suppose that $\hat{\mathbf{R}}$ is given by (4.42) but $\boldsymbol{\Gamma}(\mathbf{x},t)$ is changed to*

$$\boldsymbol{\Gamma}_f(\mathbf{x},t) = -\operatorname{diag}(\boldsymbol{\Gamma}_i)\operatorname{col}\left[\operatorname{sign}\left(\frac{\partial \hat{f}_i}{\partial \hat{\xi}_i}\right)\boldsymbol{\alpha}_{f_i}\right],\ i = 0,1. \tag{4.45}$$

Assume that $\boldsymbol{\alpha}(\mathbf{x},t)$ in (4.38) is uniformly persistently exciting in the sense of Definition 2.2, for some ϵ_α, $\delta > 0$. Additionally, suppose that the elements of the metric $p_\psi > 0$ and $\mathbf{P}_\theta = \operatorname{diag}(\mathbf{P}_i)$ with nonsingular $\mathbf{P}_i = \mathbf{P}_i^T \in \mathbb{R}^{p_i\times p_i}$, and the adaptive gains $\bar{\boldsymbol{\Gamma}} = \operatorname{diag}(\boldsymbol{\Gamma}_i)$, $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^T \in \mathbb{R}^{p_i\times p_i}$, $i = 0,1$ are such that $\mathbf{E}_1 \geq \lambda_1\mathbf{I}_{p_1\times p_1}$, for some $\lambda_1 > 0$, and that (4.44) hold. Let $q > 0$ of Lemma 4.2 be such that for all $t \geq t_0$, $\forall\mathbf{x}\in\mathcal{X}$, and for $\mathbf{z}_v = \mathbf{z}_d$ and $\mathbf{z}_v = \hat{\mathbf{z}}$

$$\epsilon'\epsilon_\alpha - \bar{\lambda}\left(\bar{\boldsymbol{\Gamma}}\right)\epsilon\bar{\alpha}\max\left\|\frac{\partial f_v^\dagger}{\partial \boldsymbol{\theta}_v}\right\| T > \lambda_\theta, \tag{4.46}$$

is true for some $\epsilon > 0$, $T > 0$, and $\lambda_\theta > 0$.

Then all the trajectories of (4.3), (4.5) tend exponentially to those of (4.6), whose nonlinearities given in (4.37), i.e. $\lim_{t\to\infty}\varphi(t) = \varphi_d(t)$ and $\lim_{t\to\infty}\hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. Matrix $-\mathbf{J}_s = \operatorname{diag}(J_{1_s},\mathbf{J}_{2_s})$ is given by J_{1_s} as in (4.8) and, due to the selection of $\boldsymbol{\Gamma}_f(\mathbf{x},t)$, $\mathbf{J}_{2_s} = [\mathbf{J}_2 - \mathbf{J}_2^\star]_s$ where $\mathbf{J}_2$ defined in (4.41) and $\mathbf{J}_2^\star$ given by

$$\mathbf{J}_2^\star = \begin{bmatrix} \left|\frac{\partial f_{v0}}{\partial \xi_{v0}}\right|\mathbf{P}_0\boldsymbol{\Gamma}_0(\boldsymbol{\alpha}_0-\boldsymbol{\alpha}_{f_0})\boldsymbol{\alpha}_0^T\mathbf{P}_0^{-1} & h_{01}\mathbf{P}_0\boldsymbol{\Gamma}_0(\boldsymbol{\alpha}_0-\boldsymbol{\alpha}_{f_0})\boldsymbol{\alpha}_1^T\mathbf{P}_1^{-1} \\ h_{10}\mathbf{P}_1\boldsymbol{\Gamma}_1(\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_{f_1})\boldsymbol{\alpha}_0^T\mathbf{P}_0^{-1} & \left|\frac{\partial f_{v_1}}{\partial \xi_{v_1}}\right|\mathbf{P}_1\boldsymbol{\Gamma}_1(\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_{f_1})\boldsymbol{\alpha}_1^T\mathbf{P}_1^{-1}\end{bmatrix}. \tag{4.47}$$

Matrix (4.47) can be written as $\mathbf{J}_2^\star = \mathbf{P}_\theta \bar{\mathbf{\Gamma}} \,\mathrm{col}\left[\mathrm{sign}\left(\frac{\partial f_{v_i}}{\partial \xi_{v_i}}\right)(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{f_i})\right] \frac{\partial f_v^\dagger}{\partial \boldsymbol{\theta}_v} \mathbf{P}_\theta^{-1},\ i = 0,1$ with $\frac{\partial f_v^\dagger}{\partial \boldsymbol{\theta}_v}$ defined in (4.39). By Lemma 4.2 and Assumptions 4.2 and 4.4, the following bound holds $\mathbf{J}_2^\star \leq \bar{\lambda}(\bar{\mathbf{\Gamma}})\epsilon\bar{\alpha} \max \left\|\frac{\partial f_v^\dagger}{\partial \boldsymbol{\theta}_v}\right\|$. By uniform persistent excitation of compiled $\boldsymbol{\alpha}(\mathbf{x}, t)$ and given that (4.44) is true, we have $\int_t^{t+T} [\mathbf{J}_2(\tau) - \mathbf{J}_2^\star(\tau)]_s d\tau \geq (\epsilon'\epsilon_\alpha - \bar{\lambda}(\bar{\mathbf{\Gamma}})\epsilon\bar{\alpha} \max \left\|\frac{\partial f_v^\dagger}{\partial \boldsymbol{\theta}_v}\right\| T)\mathbf{I}_{p\times p}$. By Assumption 4.3 and (4.46), it is guaranteed that Lemma 2.2 is satisfied by taking $\lambda \leq \min(\lambda_\psi T, \lambda_\theta)$. □

The extension of the filtering approach for the generalized parameterization lead to $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}_f, \varphi)$ given by

$$\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}_f, \varphi) = -\,\mathrm{diag}(\mathbf{\Gamma}_i)\,\mathrm{col}\left[\mathrm{sign}\left(\frac{\partial \hat{f}_i}{\partial \hat{\xi}_i}\right)\boldsymbol{\alpha}_{f_i}\right]\varphi,\ i = 0, 1. \tag{4.48}$$

4.4.3 Generalization of the system parameterization: several nonlinear forms.

In the following, several nonlinear forms are considered in (4.37), *i.e.* $N_p > 1$. In this case, $\boldsymbol{\theta}_i \in \mathbb{R}^{p_i}$, and $\sum_{i=0}^{N_p} p_i = p$. Metric $\mathbf{P}_\theta$ is written as $\mathbf{P}_\theta = \mathrm{diag}(\mathbf{P}_i)$, where $\mathbf{P}_i = \mathbf{P}_i^T \in \mathbb{R}^{p_i \times p_i}$, $i = 0, \ldots, N_p$ are non-singular. The next assumption simplifies the partial derivatives with respect to the parameters vector.

Assumption 4.5. *The function* $f_k = f_k(\mathbf{x}, \boldsymbol{\alpha}_k^T(\mathbf{x}, t)\boldsymbol{\theta}_k)$ *depends only on vector* $\boldsymbol{\theta}_k$*, i.e.* $\frac{\partial f_k}{\partial \boldsymbol{\theta}_l} \neq \mathbf{0}_{1\times p_l}$ *if and only if* $k = l$*.* $k, l \in [1, \ldots, N_p]$*.* △

State the following matrix

$$\mathbf{E}_j = \frac{\partial^2 f_{v_j}}{\partial \xi_{v_j}^2}(\varphi_v - \varphi_d)(p_\psi \mathbf{P}_j^{-1})^2 + \left|\frac{\partial f_{v_j}}{\partial \xi_{v_j}}\right| \mathbf{\Gamma}_j,\ j = 1, \ldots, N_p. \tag{4.49}$$

Recall that $\frac{\partial f_{v0}}{\partial \xi_{v0}} = 1$ where $f_{v0} = \boldsymbol{\alpha}_0^T(\mathbf{x}, t)\boldsymbol{\theta}_{v0} = \xi_{v0}$ and $\mathrm{sign}\,(\partial f_{v0}/\partial \xi_{v0}) = 1$. Define the matrix $\mathbf{H}_{k,l} = h_{kl}\mathbf{P}_k\mathbf{\Gamma}_k\boldsymbol{\alpha}_k\boldsymbol{\alpha}_l^T\mathbf{P}_l^{-1}$, $h_{kl} = \mathrm{sign}\left(\frac{\partial f_{v_k}}{\partial \xi_{v_k}}\right)\frac{\partial f_{v_l}}{\partial \xi_{v_l}}$, $k, l \in [0, \ldots, N_p]$. The lower block of the diagonal matrix $-\mathbf{J}_s$ is

$$\mathbf{J}_{2_s} = \begin{bmatrix} [\mathbf{P}_0\mathbf{\Gamma}_0\boldsymbol{\alpha}_0\boldsymbol{\alpha}_0^T\mathbf{P}_0^{-1}]_s & \mathbf{G} \\ \mathbf{G}^T & [\mathbf{P}_j\mathbf{E}_j\boldsymbol{\alpha}_j\boldsymbol{\alpha}_j^T\mathbf{P}_j^{-1}]_s \end{bmatrix}, \tag{4.50}$$

where $j = 1, \ldots, N_p$, $\mathbf{G} = \mathbf{H}_{k,l} + \mathbf{H}_{l,k}^T$, and $(l+1)$ and $(k+1)$ indicates the row and column number respectively.

Theorem 4.6. *Consider the system (4.1) and its identification model (4.2) where $\hat{\boldsymbol{\theta}}$ is adjusted by (4.5) whose nonlinearities are of the form (4.37) with $N_p > 1$. In addition to Assumptions 4.1-4.4, suppose that $\hat{\mathbf{R}}$ is given by (4.42) and $\boldsymbol{\Gamma}(\mathbf{x}, t)$ is as in (4.43) with $i = 0, \ldots, N_p$. Assume that the compiled regressor $\boldsymbol{\alpha}(\mathbf{x}, t) = [\boldsymbol{\alpha}_0^T(\mathbf{x}, t) \ldots \boldsymbol{\alpha}_{N_p}^T(\mathbf{x}, t)]^T$ is persistently exciting i.e. it satisfies Definition 2.2 for some positive constants δ, ϵ_α. In addition, suppose that there exist $p_\psi > 0$, $\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^T$, and nonsingular $\mathbf{P}_i = \mathbf{P}_i^T \in \mathbb{R}^{p_i \times p_i}$, $i = 0, ..., N_p$ such that $\mathbf{J}_{2_s}$ in (4.50) is USPD and $\int_t^{t+T} \mathbf{J}_{2_s}(\tau) d\tau \geq \epsilon' \int_t^{t+T} \boldsymbol{\alpha}(\mathbf{x}(\tau), \tau) \boldsymbol{\alpha}^T(\mathbf{x}(\tau), \tau) d\tau$, for some $T > 0$, $\epsilon' > 0$, $\forall t \geq t_0$, $\forall \mathbf{x} \in \mathcal{X}$, and for $\mathbf{z}_v$ evaluated in $\mathbf{z}_d$ and $\hat{\mathbf{z}}$.*

Then all the trajectories of (4.3), (4.5) tend exponentially to those of (4.6), whose nonlinearities given in (4.37) with $N_p > 1$, i.e. $\lim_{t \to \infty} \varphi(t) = \varphi_d(t)$ and $\lim_{t \to \infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

Proof. A demonstration that (4.50) is USPD is given in the proof of Theorem 3.4, Chapter 3. The proof follows the same lines of Theorem 4.4. □

4.4.4 Example: identification of one linear-one nonlinear scalar parameters by taking $\dot{\mathbf{x}}$ known

The model

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\lambda_\psi x_2 - \boldsymbol{\alpha}_0^T(\mathbf{x}) \boldsymbol{\theta}_0 + \exp(-\boldsymbol{\alpha}_1^T(\mathbf{x}) \boldsymbol{\theta}_1) + u, \ \ \lambda_\psi > 0, \end{aligned} \tag{4.51}$$

is based on [33]. Scalar parameters and regressors are taken in first place, in particular $\alpha_0(\mathbf{x}) = x_1$, and $\alpha_1(\mathbf{x}) = x_2^2$. In addition to the fact that (4.51) is Lipschitz $\forall \mathbf{x} = [x_1 \; x_2]^T \in \mathbb{R}^2$, the open loop equilibrium point namely, $\mathbf{x}_e = (\frac{1}{\theta_1}, 0)$, is globally asymptotically stable, hence, Assumption 4.1 holds. Identification error is given by $\varphi = \hat{x}_2 - x_2$ where $\hat{x}_2$ comes from the identification model (4.2). The algorithm is implemented based on selections of Theorem 4.4, *i.e.* $\hat{\mathbf{R}}$ and $\boldsymbol{\Gamma}(\mathbf{x}, t)$ as in (4.42) and (4.43), respectively. Estimation vector is given by PI scheme (4.20) (with $\boldsymbol{\rho} = \mathbf{0}$). $\hat{\boldsymbol{\theta}}_I(t)$ is as in (4.26) and $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}, \varphi)$ is as in (4.24). By the fact that parameters are scalars we take $p_\psi > 0$, $\mathbf{P}_i = p_i > 0$, and $\boldsymbol{\Gamma}_i = \gamma_i > 0$, $i = 0, 1$. Factor $E_1 = \exp(-\alpha_1 \theta_{v_1}) \left[\frac{p_\psi^2}{p_1^2} (\varphi_v - \varphi_d) + \gamma_1 \right]$ is uniformly bounded below if there exists $\lambda_1 > 0$ such that $\frac{p_\psi^2}{p_1^2} (\varphi_v - \varphi_d) + \gamma_1 > \lambda_1$, $\forall t \geq 0$ and for $\varphi_v = \varphi$ (fulfillment for the case $\varphi_v = \varphi_d$ is trivial). Set $\varphi_d(0) = 0$ and suppose that $|\varphi| < \overline{\varphi}$ for

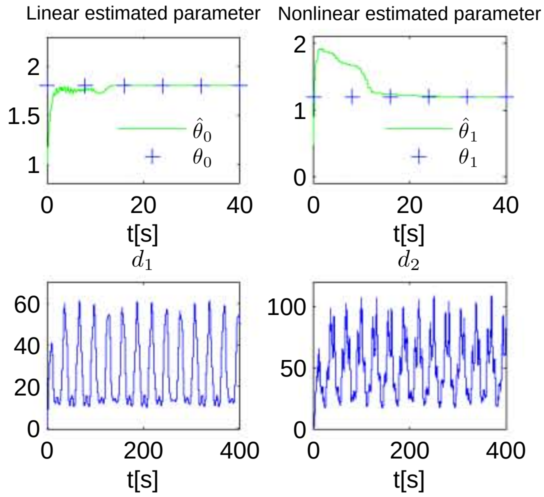

FIGURE 4.2: Linear and nonlinear estimated parameters given by the identification algorithm applied to the plant (4.51). Nominal parameters are denoted by '+'. The first and second leader principal minors of $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau$ are denoted by d_1 and d_2, respectively.

some $\overline{\varphi} > 0$. The previous inequality holds if γ_1, p_ψ, and p_1 are such that

$$\frac{p_\psi^2}{p_1^2}\,\overline{\varphi} < \gamma_1. \tag{4.52}$$

Initial conditions on the plant, identification model, and estimated parameters were chosen to met (4.52). All their values are given in Table 4.1. A *sufficient* condition for the existence of such ϵ' which satisfies (4.44) is that $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau$ is UPD (see the selection of the metric procedure given in Section 3.6.1). Adaptive gains and metric elements were chosen guided by this criterion subordinated to (4.52). Responses of estimated parameters and the two leading principal minors of $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau$ (denoted by d_1 and d_2) are depicted in Figure 4.2. Parametric convergence is shown in this example due to the persistent excitation of regressor $\boldsymbol{\alpha}(\mathbf{x}) = [x_1\ x_2^2]^T$ obtained through an appropriate input $u(t)$.

4.4.5 Example: identification of two linear-two nonlinear parameters by filtering approach

The model (4.51) is now extended to consider two linear-two nonlinear parameters. In this case $\boldsymbol{\alpha}_0(\mathbf{x}) = [x_1\ x_2]^T$ and $\boldsymbol{\alpha}_1(\mathbf{x}) = [x_2^2\ x_2^3]^T$. By an argument similar to that given in the previous example, Assumption 4.1 is fulfilled. In this case the term $\hat{\mathbf{R}}$ is as in (4.42) and $\boldsymbol{\Gamma}_f(\boldsymbol{\alpha}, \varphi)$ as in (4.45), which leads to the implementation function $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}_f, \varphi)$

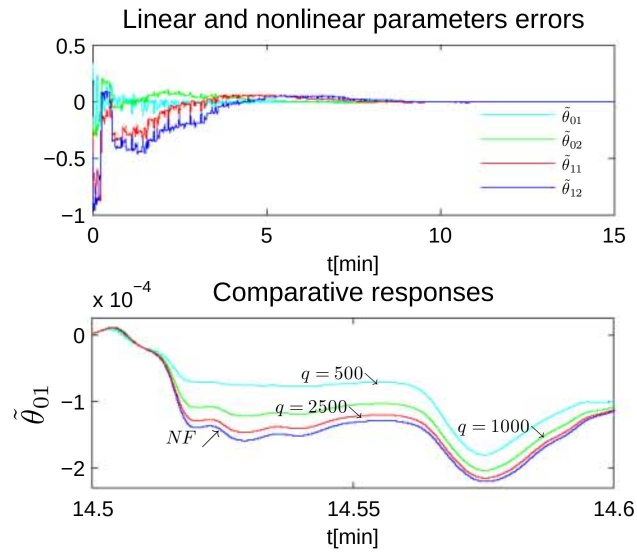

FIGURE 4.3: Linear and nonlinear parameters errors and comparative responses of the error $\tilde{\theta}_{01}$ for each increasing value of q in the time interval [14.5 min, 14.6 min]. The reference error is obtained when estimation considers $\dot{\mathbf{x}}$ available. This is denoted by *NF*.

given in (4.48). Estimation vector is given by (4.20) with $\boldsymbol{\rho} = \mathbf{0}$ and $\hat{\boldsymbol{\theta}}_I(t)$ is taken as in Section 4.3.2. In this case we have $\mathbf{E}_1 = \exp(-\boldsymbol{\alpha}_1^T\boldsymbol{\theta}_{v_1})\left(\boldsymbol{\Gamma}_1 - (\varphi_v - \varphi_d)(p_\psi \mathbf{P}_1^{-1})^2\right)$ which is bounded below by a simple extension of condition (4.52). Implementation values for this example are given in Table 4.1. In this case it was not possible to calculate analytically adaptive gains and metric elements which ensure that $\int_t^{t+T} \mathbf{J}_{2_s}(\tau)d\tau$ is UPD. However, it was possible to accomplish parametric convergence because that is only a sufficient condition. On the other hand, sufficiently large $q > 0$ values were chosen such that ϵ of Lemma 4.2 was sufficiently small. Parameters errors obtained through the identification algorithm implemented by filtering with $q = 2500$ are shown in Figure 4.3. The comparative responses between errors $\tilde{\theta}_{01} = \hat{\theta}_{01} - \theta_{01}$ obtained by filtering with increasing values of q are depicted in the time interval [14.5 min, 14.6 min], Figure 4.3. The reference error (denoted by NF) is obtained by considering the implementation with availability of $\dot{\mathbf{x}}$.

4.5 Application example: identification of electro hydraulic valve system (EHVS)

The electro hydraulic valvetrain system (EHVS) consists in two coupled hydraulic actuators which replace the conventional mechanical camshaft on engines. EHVS's have been recently studied as a flexible alternative to control fuel intake-waste exhaust timings for improving full load torque and to reduce exhaust emissions [37]. It has been shown that a nonlinear adaptive control improves performance over a non-adaptive scheme [38]. In the work of [39], a model-based on-line parameter identification scheme for EHVS based on a least squares approach was developed. Difficulties for identifying certain nonlinear parameter are overcome by taking a Taylor expansion about such value. In the following, we show that such approximation is not necessary in our methodology since the model can be written the one linear-one nonlinear blocks form. EHVS model consists in a fluid dynamics part

$$\begin{array}{ll} Q_1 = C_{d_1} w_1 x_{v_1}\,\mathrm{sign}\,(P_s - P_1)\sqrt{\frac{2|P_s-P_1|}{\rho}}, & V_1 = V_{0_1} + A_{p_1}x_1, \\ Q_2 = C_{d_2} w_2 x_{v_2}\,\mathrm{sign}\,(P_1 - P_r)\sqrt{\frac{2|P_1-P_r|}{\rho}}, & Q_1 - Q_2 = \dot{V}_1 + \frac{V_1}{\beta_e}\dot{P}_1, \end{array} \tag{4.53}$$

where $x_{v_i},\ i = 1,2$ denotes the discrete input signals of the solenoid valves. These are depicted in Figure 4.4, 'Input signal', and represent the available input to the designer. $Q_i,\ i = 1,2$ denotes flux, V_1 and V_{0_1} are volume and initial value of volume, while P_1 and P_r are the chamber pressure and reservoir pressure, respectively. β_e is a coefficient related with pressure and ρ is density. Constants C_{d_i} are the discharge coefficients for solenoid valves $i = 1,2$, and w_i are area gradients[2]. All the constants involved in this part of the model are assumed known and are defined in Table 4.1. The remaining parameters are also implicated in the mechanical part of the model which is given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ M_t\dot{x}_2 &= A_{p_1}P_1 - A_{p_2}P_s - F_0 - Bx_2 - RB(x_1,x_2), \end{aligned} \tag{4.54}$$

with M_t, x_1, x_2, A_{p_1}, and A_{p_2} the mass, position, velocity, areas of the top and of the bottom of the valve piston, respectively. P_s is the supply pressure. F_0 is the dead force on valve and B is the damping of valve piston. Finally, $RB(x_1,x_2)$ is a variable damping present only immediately before the valve closes. Model of $RB(x_1,x_2)$ is obtained under a phenomenological basis and is formulated as

$$RB(x_1,x_2) = \frac{D}{2}\frac{(x_2 - |x_2|)}{\left(D_b + x_1^{2k}\right)}, \tag{4.55}$$

where k, D, and D_b are dimensionless. Unknown parameters are B, D and D_b. By simple algebraical manipulations it is clear that $RB(x_1,x_2) = k_1\frac{x_2-|x_2|}{2}\frac{1}{\boldsymbol{\alpha}_1^T\boldsymbol{\theta}_1}$, where $\boldsymbol{\alpha}_1 = [1\ x_1^{2k}]^T$ and $\boldsymbol{\theta}_1 = [\frac{k_1D_b}{D}\ \frac{k_1}{D}]^T$ in which a scaling factor $k_1 > 0$ was introduced to obtain

[2]For more details on the EHVS model, see [39].

uncertainties of the same order of magnitude. By defining $\alpha_0 = \frac{x_2}{M_t}$, $\theta_0 = B$, $u = \frac{1}{M_t}(A_{p_1}P_1 - A_{p_2}P_s - F_0)$, and $\kappa(x_2) = k_1\frac{x_2-|x_2|}{2M_t}$, the system (4.54) can be written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\alpha_0\theta_0 - \kappa(x_2)\frac{1}{\boldsymbol{\alpha}_1^T\boldsymbol{\theta}_1} + u, \end{aligned} \tag{4.56}$$

which clearly shows the one linear-one nonlinear parameterization structure. Identification algorithm was implemented under the approach of Theorem 4.4 and Section 4.3.1 where measurements of all derivatives needed are assumed available. In this case $\mathbf{E}_1 = -\frac{\kappa(x_2)}{(\boldsymbol{\alpha}_1^T\boldsymbol{\theta}_{v_1})^2}\left(\frac{2p_\psi^2(\varphi_v-\varphi_d)}{\boldsymbol{\alpha}_1^T\boldsymbol{\theta}_{v_1}}(\mathbf{P_1}^{-1})^2 + \boldsymbol{\Gamma}_1\right)$. Note that, by definition, $\kappa(x_2) \leq 0$, $\forall x_2$. By limiting the estimation error in $|\varphi_v - \varphi_d| \leq \bar{\varphi}$, $\bar{\varphi} > 0$, for $\varphi_v = \varphi$ and $\varphi_v = \varphi_d$, considering that $0 \leq x_1 \leq \bar{x}_1$, for some $\bar{x}_1 > 0$, and assuming that each component of nonlinear parameters vector is bounded by $\underline{\theta}_{1i} \leq \hat{\theta}_{1i} \leq \bar{\theta}_{1i}$ for known $\underline{\theta}_{1i}$, $\bar{\theta}_{1i}$, $i = 1, 2$, then $\mathbf{E}_1 \geq \lambda_1\mathbf{I}_{2\times 2}$ for some $\lambda_1 \geq 0$ if

$$\frac{2p_\psi^2\bar{\varphi}}{\underline{\theta}_{11}}(\mathbf{P}_1^{-1})^2 < \boldsymbol{\Gamma}_1. \tag{4.57}$$

Initial conditions and all the parameters involved in implementation are given in Table 4.1. Input signals to the EHVS and responses of piston position x_1, piston velocity x_2, and parameters errors are depicted in Figure 4.4 in which parametric convergence, due to the persistent excitation of input, is shown.

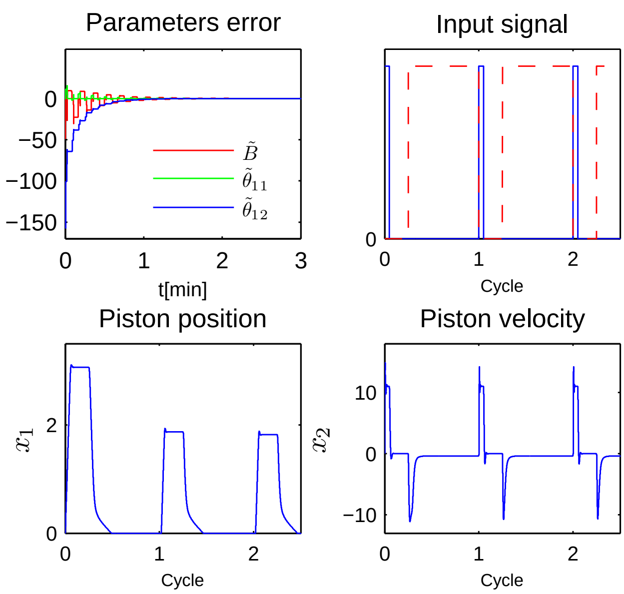

FIGURE 4.4: Parameters errors, input signal, piston position and piston velocity for the EHVS.

$\mathbf{x}_0$ and $\boldsymbol{\theta}$	$\bar{\mathbf{\Gamma}}$, $\mathbf{P}$, and λ_ψ	$\hat{\mathbf{x}}_0$ and $\hat{\boldsymbol{\theta}}_0$	Additional par.
Identification one linear-one nonlinear parameters by taking $\dot{\mathbf{x}}$ known			
$\mathbf{x}_0 = [0.5\ 1]^T$ $\boldsymbol{\theta} = [1.8\ 1.2]^T$	$\bar{\mathbf{\Gamma}} = \text{diag}(2, 5)$ $\mathbf{P} = \text{diag}(2.5, 5, 2)$ $\lambda_\psi = 5$	$\hat{\mathbf{x}}_0 = [0\ 1.5]^T$ $\hat{\boldsymbol{\theta}}_0 = [1\ 0.5]^T$	$u(t) = 5\sin(3\pi t) + 2\sin(t)$ $+3\cos(\pi t/15)$
Identification two linear-two nonlinear parameters by the approach of filtering			
$\mathbf{x}_0 = [0.5\ 1]^T$ $\boldsymbol{\theta} = [0.7\ 1.2\ 1.3\ 1.5]^T$	$\mathbf{\Gamma}_0 = \text{diag}(2, 2)$ $\mathbf{\Gamma}_1 = \text{diag}(5, 5)$ $\mathbf{P} = \mathbf{I}_{5\times 5}$ $\lambda_\psi = 8$	$\hat{\mathbf{x}}_0 = [0\ 1.5]^T$ $\hat{\boldsymbol{\theta}}_0 = [1\ 1\ 1\ 1]^T$	$q = 500, 1000, 2500$ $u(t) = 1 + 10\sin(t)$ $+\cos(3\pi t) + 0.5\cos(t + 10)$ $+5\sin(\pi t/10 + 2)$
Identification of the EHVS			
$\mathbf{x}_0 = [0\ 0]^T$ $P_{1_0} = 125$ $B = 50$ $D = 60$ $D_b = 0.01$	$\mathbf{\Gamma}_0 = \text{diag}(5, 5)$ $\mathbf{\Gamma}_1 = \text{diag}(7, 7)$ $\mathbf{P}_0 = \mathbf{I}_{2\times 2}$ $\mathbf{P}_1 = \text{diag}(3, 3.1)$ $p_\psi = 1$ $\lambda_\psi = 50$	$\hat{\mathbf{x}}_0 = [0\ 0.7]^T$ $\hat{\boldsymbol{\theta}}_0 = [1\ 0.9\ 9]^T$	$\underline{\theta}_{11} = 0.5$, $M_t = 1$ $A_{p_1} = 20$, $A_{p_2} = 10$ $P_s = 250$, $k = 2$ $k_1 = 10000$, $F_0 = 3$ $V_{0_1} = 5$, $P_r = 10$ $C_{d_i} = 10$, $w_i = 5, i = 1, 2$ $\rho = 10$, $\beta_e = 250.$

Table 4.1: Simulation parameters for identification of one linear-one nonlinear and two linear-two nonlinear generalized parameterizations and identification of EHV system.

Chapter 5

Indirect adaptive control

Nonlinearly parameterized systems identification formulated in Chapter 4 can be applied on the on-line calculation of an estimated plant model. From the input-output measurements of the actual and the estimated models, the on-line computation of specific controller parameters is possible. Such adaptive control strategy is the so called indirect control scheme which is developed in the present chapter for systems with nonlinear parameterizations.

It will be shown that closed-loop time evolution of the system with the adaptive controller leads to a tracking error composed by an exponentially stable term perturbed by a factor dependent on the mismatch between nominal and adjusted parameters. If the input signals are sufficiently exciting and the identification algorithm reaches its objective, estimated parameters tend to the true ones and the perturbation will tend exponentially to zero. Therefore, the perturbed tracking dynamics will not lose its convergence properties and control purposes will be achieved.

5.1 Problem statement

Indirect adaptive control consists of designing a suitable identification model which generates asymptotic estimates of the plant parameters. These parameters are substituted in the expression of a controller, under the certain-equivalence principle, whose objective is to track some desired signal while maintaining bounded all trajectories of the closed-loop system. Recall the nonlinear dynamics

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}, \mathbf{u}, t) + \mathbf{g}(\mathbf{x}, \boldsymbol{\theta}, t), \;\; \mathbf{x}(0) = \mathbf{x}_0, \tag{5.1}$$

under the same definitions of system (3.14). Let $e(\mathbf{x}, t)$ denote the tracking error function, where $e : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$ is continuous and differentiable in its arguments. The existence of an ideal controller and the main features of the target dynamics are stated in the following assumptions.

Assumption 5.1. *Suppose that there exists an ideal control* $\mathbf{u}_i(\mathbf{x}, \boldsymbol{\theta}, t)$ *such that the control objective (tracking or regulation) is accomplished under the target dynamics*

$$\frac{\partial e}{\partial \mathbf{x}} \left(\mathbf{f}_p(\mathbf{x}, \mathbf{u}_i(\mathbf{x}, \boldsymbol{\theta}, t), t) + \mathbf{g}(\mathbf{x}, \boldsymbol{\theta}, t) \right) + \frac{\partial e}{\partial t} = -\Phi(e). \tag{5.2}$$

$\triangle$

Assumption 5.2. *Function* $\Phi : \mathbb{R} \to \mathbb{R}$ *is a* $\mathcal{C}^1$ *class contracting function,* i.e. $-\frac{\partial \Phi(e)}{\partial e} \leq -\lambda_\Phi$ *for some* $\lambda_\Phi > 0$. $\triangle$

It is clear that Assumption 5.2 implies that

$$\dot{e}_d = -\Phi(e_d), \tag{5.3}$$

is a contracting dynamics whose particular solutions tend to $e(\mathbf{x}, t) = 0$ exponentially. By implementing the adaptive control law

$$\mathbf{u}_a = \mathbf{u}_i(\mathbf{x}, \hat{\boldsymbol{\theta}}, t), \tag{5.4}$$

and defining $\frac{\partial e}{\partial \mathbf{x}} \left(\mathbf{f}_p(\mathbf{x}, \mathbf{u}_i(\mathbf{x}, \boldsymbol{\theta}, t), t) \right) = f(\mathbf{x}, \boldsymbol{\theta}, t)$, similarly for the estimated parameters, dynamics (5.2) turns into

$$\dot{e} = -\Phi(e) + f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t). \tag{5.5}$$

To make use of the estimation algorithm developed in Chapter 4, let establish the following Assumption.

Assumption 5.3. *Assume that* $f(\cdot)$ *is uniformly Lipschitz in* $\boldsymbol{\theta} \in \Omega_\theta$ *for all* $t \geq t_0$ *and* $\forall \mathbf{x} \in \mathcal{X}$ *with Lipschitz constant* l_f. $\triangle$

5.2 Closed loop stability analysis

The following Lemma states the stability of the overall closed loop system.

Lemma 5.1. *Consider the plant (5.1) in closed loop with the estimator (4.3), (4.5), and the adaptive control (5.4). Suppose that conditions of Theorem 4.1 and Assumptions 5.1-5.3 hold. Then* $e(t) \to e_d(t)$ *exponentially.*

Proof. By following the analysis of perturbed systems given in [18], the distance $R(t) \triangleq \|e(t) - e_d(t)\|$ satisfies $\dot{R} + \lambda_\Phi R(t) \leq \|f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)\|$. In virtue of Assumption 5.3, $\|f(\mathbf{x}, \hat{\boldsymbol{\theta}}, t) - f(\mathbf{x}, \boldsymbol{\theta}, t)\| \leq l_f \|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\|$, $l_f > 0$. By Theorem 4.1, $\lim_{t\to\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$ exponentially, which means that $\|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\| \leq \|\hat{\boldsymbol{\theta}}(0) - \boldsymbol{\theta}\| \exp(-\beta t)$, $\beta > 0$, $\forall \hat{\boldsymbol{\theta}}(0) \in \Omega_\theta$. Then, the distance between trajectories $e(t)$ and $e_d(t)$ tends exponentially to zero and the conclusion follows. □

5.3 Indirect control of fermentation and friction systems

Identification of fermentation and friction systems is not directly applicable since ISS property is not verified. To overcome such obstacle, the indirect control scheme is used and plant parameters are estimated on-line with the aid of an appropriate control law. Model expressions and control objectives are those given in Section 3.7 but the additional implementation of the identification guidelines given in Chapter 4 are considered.

Fermentation process model is established in (3.64). The existence of the ideal control required by Assumption 5.1 is fulfilled since such controller was explicitly found (see Section 3.7) and led to the target dynamics $\dot{e} = -\lambda_m e$, $\lambda_m > 0$, $e = x - x_m$, which verifies the requirements of Assumption 5.2. Estimation starts by taking (4.2) and is built relying on the Corollary 4.2 and all its conditions. Realization comes from the framework explained in Section 4.3.2 with q sufficiently large ($q = 170$). Persistent excitation is crucial to reach the control goal so the signal $r(t)$ which gave parameter convergence was taken ($r(t) = 10 + 15\sin(2\pi t)$). Estimation expressions were given by $\hat{\boldsymbol{\theta}}(t)$ as in (4.27), $\dot{\hat{\boldsymbol{\theta}}}_I(t)$ as in (4.28) where $\hat{\mathbf{R}}$ as in (4.11) and $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}_f, \varphi)$ that given in (4.29). All the remaining values involved in the implementation are the same as those given in Table 4.1. Results for estimated parameters and tracking error are shown in Figure 5.1.

Friction model rewritten in the one linear-one nonlinear parameterization form is given in (3.67). This is taken as a practical example of indirect adaptive control for the extended parameterization. Ideal control was already set up (Section 3.7) resulting in a control target dynamics equal to $\varphi_d = -\lambda_t \varphi_d$, $\lambda_t > 0$, where $\varphi = x_2 - x_{2_m} + \lambda_\varphi(x_1 - x_{1_m})$, $\lambda_\varphi > 0$,

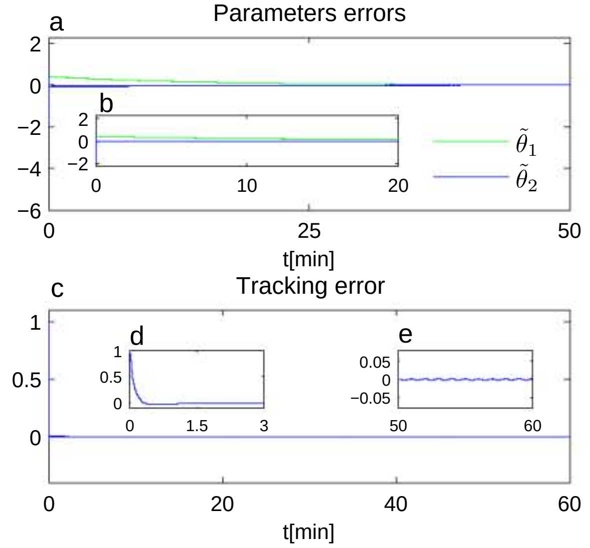

FIGURE 5.1: Global responses of parameters errors (a) and tracking error (c) for the fermentation plant under the indirect adaptive control. Subplot (b) shows parameters errors in the time interval $[0, 20]$min. Subplots (d) and (e) show the tracking error response in the time intervals $[0, 3]$min, $[50, 60]$ min, respectively.

$\mathbf{x}_m = [x_{1_m} \ x_{2_m}]^T$ is the state of the reference model (3.63), and $\mathbf{x} = [x_1 \ x_2]^T$ is the state of (3.67). Since desired controller exists and control target dynamics is contracting, Assumptions 5.1 and 5.2 are guaranteed.

Identification model (4.2) is applied to (3.67) and estimation is based on the filtering approach for extended parameterization justified in Theorem 4.5 and all its Assumptions, and considers general choices given in Section 4.4.2. Estimation expressions are given by $\hat{\boldsymbol{\theta}}(t)$ as in (4.27), $\dot{\hat{\boldsymbol{\theta}}}_I(t)$ as in (4.28) where $\hat{\mathbf{R}}$ as in (4.42) and $\hat{\boldsymbol{\theta}}_P(\boldsymbol{\alpha}_f, \varphi)$ that given in (4.48). Results for parameters and tracking errors are shown in Figure 5.2.

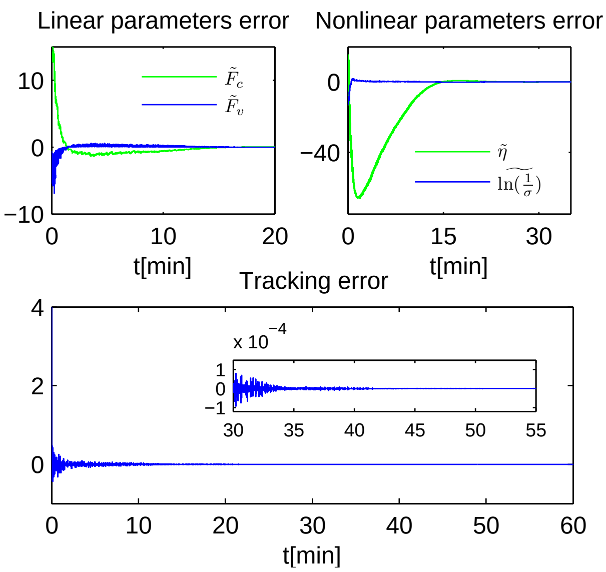

FIGURE 5.2: Linear and nonlinear parameters errors, and tracking error of the plant (3.67) in closed loop with the indirect adaptive control. Subplot shows the tracking error response in the time interval $[30, 55]$ min.

Chapter 6

Conclusions and future perspectives

Necessity of weaker conditions for contraction for analyzing stability and convergence in adaptive control problems is clear in the literature. Until now, parametric convergence conclusions have been obtained by mixing contraction tools of analysis and Lyapunov based techniques, such as the Barbalat Lemma. In this work, we proposed an extended condition for contraction which naturally associates the persistent excitation property with the contraction theory framework. Such condition, called averaged condition for contraction, was instrumental for studying the boundedness of closed loop trajectories on the three different areas of adaptive control: direct and indirect adaptive control, and identification. It allowed to design a top-down adaptive methodology which consisted in two main stages of design: at the top-level, stability and convergence were established by means of the averaged contraction analysis. At the down-level, diverse realization methods of the non-implementable adaptive law were suggested by means of the proportional-integral adaptive algorithms. Since these two stages are designed independently and PI is not the only methodology appropriate for the algorithm implementation, it can be concluded that the adaptive control scheme developed in this work is endowed with modularity properties which improved flexibility and transparency. A wide variety of examples were tested and performance was analyzed by means of numerical simulations. It is relevant to mention that, to the best of our knowledge, parameters given in two different blocks of parameterization, such as one linear-one nonlinear, had not been successfully identified. Our methodology reached such objective by a simple extension of the one block results.

Future perspectives are diverse and widely recognized. Many control problems of practical importance or identification of systems in interdisciplinary areas such as bio-chemical processes or biology and neuroscience, are written in terms of complex and nonlinearly parameterized dynamics. Adaptive control of such kind of systems is currently a challenging problem and an active area of research.

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