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GENERAL BOUNDARY QUANTUM FIELD THEORY IN ANTI DE SITTER SPACETIMES

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Introducción y Resumen

Esta tesis doctoral trata sobre Teoría de Campos Cuánticos en espaciotiempo curvo, en particular sobre la construcción de una matriz S para un campo Klein-Gordon real en un fondo Anti de Sitter (AdS). AdS es un espaciotiempo con curvatura negativa constante y la misma topología que el espaciotiempo de Minkowski. En el límite de curvatura muy pequeña, (una parte local de) AdS asintóticamente se torna en el espaciotiempo de Minkowski. Esto lo llamamos el límite plano. La construcción cada vez que aparece de la matriz S usa como ingrediente crucial estados cuánticos asintóticamente libres para tiempos largos. Sin embargo, en AdS estos estados no existen y por lo tanto la matriz S estandar no puede ser construida.

Resolvemos este problema aplicando la Formulación de Fronteras Generales (GBF) de la teoría cuántica. La GBF es un reciente marco axiomático que describe cómo reformular las teorías cuánticas, a la vez generalizando la formulación estandar y conservando sus resultados. En breve, la GBF asocia a cada hipersuperficie Σ del espaciotiempo su propio espacio $\mathcal{H}_\Sigma$ de estados cuánticos, el cual es un espacio de Hilbert. Un subconjunto particular de hipersuperficies son las fronteras $\partial\mathbb{M}$ de regiones $\mathbb{M}$ del espaciotiempo. Cada una de estas regiones tiene asociada su mapeo de amplitud (el cual depende de la teoría cuántica particular considerada), que asigna una amplitud a cada estado en la frontera. De estas amplitudes se derivan probabilidades consistentes con la regla de Born.

En el contexto estandar, la región que se considera es el producto cartesiano de todo el espacio (normalmente dado por $\mathbb{R}^d$) y un intervalo de tiempo. Esto lo llamamos una región tipo intervalo de tiempo. Su frontera consiste de dos superficies de tiempo constante, sobre cuales viven los estados entrantes y salientes. Lo más relevante para nosotros es la región tipo bastón, la cual consiste del producto cartesiano de todo el tiempo (normalmente dado por $\mathbb{R}_t$) y una bola sólida $\mathbb{B}^d$ en el espacio. La frontera del bastón es una sola hipersuperficie conexa, la cual llamamos hipercilindro (producto de todo el tiempo y una esfera $\mathbb{S}^{d-1}$ en el espacio). Cada estado cuántico sobre un hipercilindro entonces encodifica a la vez partículas entrantes y salientes. Las regiones tipo bastón son importantes, porque la métrica AdS causa que los estados en su frontera se vuelvan asintóticamente libres para radios grandes (de hecho lo mismo pasa en el espaciotiempo de Minkowski). Esto justifica la interpretación de sus amplitudes como matriz S.

Usamos el método de cuantización holomorfa. Sus estados son funcion(al)es sobre el espacio de soluciones clásicas de las ecuaciones de movimiento (en vez del espacio fase). Sus ingredientes más importantes son una estructura simpléctica ω y una estructura compleja J sobre tal espacio. Juntas, las dos definen un producto interno $\omega(\cdot, J\cdot)$. La estructura simpléctica está fijada por el Lagrangiano clásico, mientras que la estructura compleja no es fijada por la teoría clásica. La estructura compleja más bien es un ingrediente cuántico, y cada elección corresponde uno-a-uno a una elección del estado de vacío. Fijar la estructura compleja determina completamente el mapeo de amplitud.

Para regiones de intervalo de tiempo, hay una elección estandar de la estructura compleja, la cual determina las amplitudes de estas regiones. Para regiones tipo bastón no hay tal elección estandar, y por lo tanto necesitamos construirla según principios físicos. Primero, queremos que nuestra J induzca amplitudes que sean invariantes bajo las acciones de isometrías del espaciotiempo. Segundo, deseamos que J haga nuestras amplitudes del bastón coincidir con las amplitudes estándares de las regiones intervalo de tiempo. Esto lo llamamos equivalencia de amplitudes. Tercero, nos gustaría que el producto interno inducido $\omega(\cdot, J\cdot)$ sea positivo definido. Sin embargo, esto no es una condición necesaria, ya que un producto indefinido también se puede usar para una cuantización consistente (donde espacios de Krein generalizan los de Hilbert). Cuarto, para AdS podemos usar el límite plano como guía, requiriendo que en este límite las amplitudes de AdS reproduzcan las de Minkowski.

Para el espaciotiempo de Minkowski, una estructura compleja para regiones bastón que cumple con invariancia bajo isometrías, equivalencia de amplitudes y producto positivo definido ha sido encontrada ya hace un tiempo. Para AdS, aquí construimos estructuras complejas para regiones bastón según las propiedades mencionadas arriba. Encontramos, que no existe ninguna J que cumpla todos los requisitos. Hay una estructura compleja J^{iso} compatible con las isometrías de AdS que induce equivalencia de amplitud, pero induce un producto interno indefinido. Además, su límite plano solo recupera las amplitudes de Minkowski para un subconjunto discreto de las frecuencias/energías. Una segunda estructura compleja J^{pos} es compatible con las traslaciones de tiempo y las rotaciones espaciales de AdS (pero no con los boosts), también induce equivalencia de amplitudes, y nos da un producto positivo definido. Su límite plano reproduce las amplitudes de Minkowski para todas frecuencias. Los resultados de esta tesis estan publicados en [30] y en [31].

Summary

This PhD thesis mainly treats a particular aspect of Quantum Field Theory on curved spacetime: the construction of an S-matrix for a real Klein-Gordon field on an Anti de Sitter background (AdS). AdS is a spacetime with constant negative curvature and the same topology as Minkowski spacetime. In the limit of very small curvature, (a part of) AdS asymptotically turns into Minkowski spacetime. We call this the flat limit. The standard construction of an S-matrix uses asymptotically free quantum states at large times as a crucial ingredient. However, on AdS these states do not exist and hence the standard S-matrix cannot be constructed.

We solve this problem by applying the General Boundary Formulation (GBF) of Quantum Theory. The GBF is a recent axiomatic framework describing how quantum theories can be formulated, both generalizing the standard formulation and conserving its results. In short, the GBF associates to each hypersurface Σ on spacetime its own quantum state space $\mathcal{H}_\Sigma$, which is a Hilbert space. A particular subset of hypersurfaces are the boundaries $\partial\mathbb{M}$ of spacetime regions $\mathbb{M}$. Each of these regions has its associated amplitude map (which depends on the particular quantum theory under consideration), that assigns an amplitude to each boundary state. These amplitudes lead to probabilities consistent with Born's rule.

In the standard context, the spacetime region under consideration is the cartesian product of all of space (usually given by $\mathbb{R}^d$) and a time-interval. We call this the time-interval region. It's boundary consists of two equal-time hypersurfaces, on which the IN and OUT states are living. Most relevant for us is here the solid hypercylinder region (rod region for short), which is the cartesian product of all of time (usually $\mathbb{R}_t$) and a solid ball $\mathbb{B}^d$ in space. The rod's boundary is a single connected hypersurface, which we call hypercylinder (product of all of time and a sphere $\mathbb{S}^{d-1}$ in space). Each quantum state on a hypercylinder thus encodes both incoming and outgoing particles. Rod regions are important, because the Ads metric makes states on their boundary asymptotically free for large radius (actually the same happens on Minkowski spacetime). This justifies the interpretation of their amplitudes' limits as an S-matrix.

We use the method of Holomorphic Quantization. Its states are wave function(al)s on spaces of solutions of the classical equations of motion (instead of phase space). Its most important ingredients are a symplectic structure ω and a complex structure J on such a solution space. Together they define an inner product $\omega(\cdot, J\cdot)$. The symplectic structure is fixed by the classical Lagrangian, while the complex structure is not fixed by the classical theory. The complex structure is rather a quantum ingredient, and each choice of it corresponds one-to-one to a choice of the vacuum state. Fixing the complex structure completely determines the amplitude map.

For the time-interval regions, there is a distinguished standard choice for the complex structure, which determines the amplitudes on these regions. For rod regions, there is no such standard choice, and therefore we need to construct it according to physical principles. First, we wish our J to induce amplitudes which are invariant under the action of spacetime isometries. Second, we want J to make our rod amplitudes agree with the standard amplitudes of the time-interval regions. We call this requirement amplitude equivalence. Third, we would like the induced inner product $\omega(\cdot, J\cdot)$ to be positive-definite. However, this is not a necessary condition, since an indefinite product can also be used to construct a consistent quantization (wherein Krein spaces generalize the Hilbert spaces). And fourth, for AdS we can use the flat limit as a guideline by requiring that in this limit the AdS amplitudes reproduce the Minkowski amplitudes.

For Minkowski spacetime, a complex structure for rod regions fulfilling isometry invariance, amplitude equivalence and positive-definiteness has been found already some time ago. For AdS, here we construct complex structures for rod regions according to the above properties. We find that there is no J fulfilling all requirements. There is one complex structure J^{iso} compatible with all AdS isometries and inducing amplitude equivalence, but giving an indefinite inner product. Moreover, its flat limit recovers the Minkowski amplitudes only for a discrete subset of frequencies/energies. A second complex structure J^{pos} is compatible with AdS time-translations and spatial roations (but not with boosts), also inducing amplitude equivalence, and giving a positive-definite inner product. Its flat limit recovers the Minkowski amplitudes for all frequencies. The results of this thesis are published in [30] and [31].

Contents

Chapter 1

Introduction: General Boundary Formulation (GBF)

The goal of this chapter is to present the General Boundary Formulation (GBF) of Quantum Theory, show its relevance and give some context. This introduction is divided into two sections: Section 1.1 outlines the basic ideas of the GBF and how they arise naturally by requiring locality and operationalism. We also sketch the problem of the S-matrix for Anti de Sitter spacetimes (AdS), and how the GBF solves this problem. This application of the GBF is the main focus of the present thesis. After this overview, in Section 1.2 we proceed by describing the GBF in more detail. We start with how the GBF treats spacetime. Then we introduce the GBF's Core Axioms, which contain the main ideas about how Quantum Theory can be formulated in a general boundary way. This concerns mainly state spaces and amplitudes. After this, axioms for the vacuum state and the role of spacetime symmetries are considered. We complete the picture with the probability interpretation of the GBF amplitudes, how to include observables, and their expectation values. At the end of the section we describe how the GBF relates to Topological Quantum Field Theories. The version of the GBF we discuss here, is the one in which it has been developed originally and is now called *Amplitude Formalism*, since the fundamental objects therein are generalized complex transition amplitudes (and their extensions to observables). Recently, a new version of the GBF is being developed under the name of *Positive Formalism* [63], [65].

Apart from the introductory chapter, this thesis is divided into two main chapters: in Chapter 2 we treat classical field theory and how it enters the GBF. This provides many important ingredients for the quantization we apply in Chapter 3. Throughout the whole thesis, we have chosen to accomodate the more technical parts in the appendices in order to keep the main parts cohesive. In Section 2.1 we introduce the two main types of spacetime regions considered throughout this work. Then in Section 2.2 we review the classical data needed later for quantization. The most important ones are spaces L of classical solutions and symplectic structures ω on them. In order to give a more complete overview, in this section we also introduce complex structures J on these spaces, despite these not being a classical structure, but a quantum one. This allows us to also consider real $\mathrm{g}(\cdot,\cdot)$ and complex $\{\cdot,\cdot\}$ inner products on L, which arise from combining complex and symplectic structure. These structures are then studied in detail in Section 2.4 for general spacetimes. In Section 2.5 we review the classical Klein-Gordon theory on Minkowski spacetime, and in Section 2.6 for Anti de Sitter spacetime (AdS). For both spacetimes, we study the spaces of classical Klein-Gordon solutions, and the symplectic structures on these. We also calculate the action of isometries in the solution spaces, and further on the symplectic structures. We find that the symplectic structures on Minkowski spacetime are invariant under temporal and spatial translations and under spatial rotations - but not under boosts. The symplectic structures of AdS are invariant under all isometries of this spacetime. In Section 2.6.8 we set up a correspondence between classical Klein-Gordon solutions and boundary data on AdS, extending previous results of Warnick [73].

Chapter 3 follows the same pattern for the Quantum Theory. First, in Section 3.1, we review the method of Holomorphic Quantization, which fits naturally into the GBF framework. Then we clarify the relation between the amplitudes of the standard formulation and of the GBF, and how they give rise to S-matrices. Since the GBF produces amplitudes for various types of spacetime regions, we discuss how to compare those amplitudes to each other. This gives rise to what we call amplitude equivalence. Another important concept is the flat limit. Since QFT on Minkowski spacetime is well known, we can use its results as a reference. In a sense, for large curvature radius $R_{\text{AdS}} \to \infty$, AdS becomes asymptotically flat. (Other spacetimes become asymtotically flat when other parameters tend to zero, like e.g. the mass of a Schwarzschild black hole.) In this limit, we would like the AdS amplitudes to recover the corresponding Minkowski amplitudes. After these more general considerations, we review Klein-Gordon theory on Minkowski spacetime in Section 3.2. This collects the results which we later aim to recover in the flat limit. In Section 3.3 we then quantize Klein-Gordon theory on AdS. The crucial ingredient needed here is the complex structure. Since on AdS there is no standard complex structure, the main part of this section consists of constructing complex structures with as many nice properties as possible. We conclude with a summary of our results in Section 4.

1.1 An overview of the GBF

This section gives a first flavor of the General Boundary Formulation, before entering into a detailed presentation in Section 1.2. Section 1.1.1 introduces the main ideas of the GBF and Section 1.1.2 motivates them from locality and operationalism. Section 1.1.3 then shows how the application of the GBF resolves the S-matrix problem of Anti de Sitter spacetimes (AdS).

1.1.1 General Boundary Formulation (GBF)

The General Boundary Formulation (GBF) is a still rather young reformulation of Quantum Theory, which generalizes it while reproducing the results of its standard formulation. We emphasize that the GBF is *not* some particular quantum theory, but rather a specification how *any* quantum theory should be formulated.

Why is this reformulation necessary? On the fundamental level of Physics there are two highly successful theories: Quantum Mechanics (respectively Quantum Field Theory, QFT) and General Relativity (GR). The theory of General Relativity describes the dynamics of spacetime, the "stage" on which matter lives and moves. Quantum Field Theory describes the interactions of the matter fields. However, in their standard formulations these two theories are not compatible: the quantization of GR fails (due to an infinite number of counterterms). That is, using the standard formulations, it is impossible to combine GR and QFT into a unified theory which comprises the dynamics of both spacetime and quantum matter. (This unified theory is usually called Quantum Gravity.) One way of approaching this problem is thus to reformulate General Relativity and/or Quantum Theory, hoping that such reformulation may render their unification possible. One example for this is Loop Quantum Gravity (LQG), where GR is first rewritten (without changing it) using Ashtekar variables and then quantized, resulting in a discretized structure of spacetime.

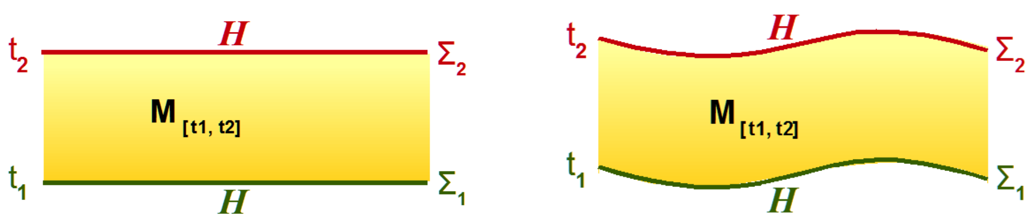
In contrast, the General Boundary Formulation reformulates Quantum Theory. One of its goals is doing this without referring to a (fixed) spacetime metric, hoping that through this the incorporation of GR into Quantum Theory might become possible one day. However, even without the perspective on dynamical gravity, the GBF provides many new insights for Quantum Theory in its own right. For example, the main topic of this thesis is (General Boundary) Quantum Field Theory on a (fixed) curved spacetime, which is Anti de Sitter (AdS) spacetime.

In order to become familiar with the GBF, let us now sketch its main features. (We do this thoroughly in Section 1.2.) To this end it is beneficial to briefly review the standard formulation of Quantum Theory. There, the physical system of interest is prepared at some initial time t_1 and the preparation is encoded by a normalized initial state $|\,\eta(t_1)\rangle$. After the system's components interacts among themselves for some time, a normalized final state $|\,\zeta(t_2)\rangle$ can then be observed at some final time $t_2 > t_1$. Both states live in the one and only state space $\mathcal{H}$ of the standard formulation, which is a Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$. Depending on the particular QFT under consideration, we can then calculate complex transition amplitudes (which we denote by ρ):

$$\rho(\eta(t_1), \zeta(t_2)) = \langle \zeta(t_2) \,|\, \mathcal{U}_{t_2,t_1} \,|\, \eta(t_1) \rangle\,, \tag{1.1}$$

wherein $\mathcal{U}_{t_2,t_1}$ is the unitary time-evolution operator of the theory. This operator can be described completely through its matrix elements, which are precisely the above transition amplitudes. The theoretical probability $P(\eta(t_1), \zeta(t_2)) = |\rho(\eta(t_1), \zeta(t_2))|^2$ of preparing $|\,\eta(t_1)\rangle$ and observing $|\,\zeta(t_2)\rangle$ is then obtained as the absolute value squared of the amplitude. This is known as the Born rule. For many initial and final states, the corresponding theoretical probabilities are then compared to the experimental probabilities that have actually been measured. This allows us to consider as falsified those theories, whose predictions do not match the experimental results, and continue only with those theories that do so.

How does the General Boundary Formulation generalize this? First, we note that the interaction/evolution of the system takes place within a region in spacetime. Usually this region is the time-interval $[t_1, t_2]$ times all of space, as drawn in Figure 1.2. Let us call this type of region a time-interval region $\mathbb{M}_{[t_1,t_2]}$. Second, the initial state $|\,\eta(t_1)\rangle$ is prepared on the early boundary of this spacetime region, which is the equal-time hypersurface Σ_1 given by the point in time t_1 times all of space. The final state $|\,\zeta(t_2)\rangle$ is observed on the late boundary Σ_2 of the region: t_2 times all of space.



(a) ... on Minkowski spacetime (b) ... on some curved spacetime

Figure 1.2: Time-interval region ...

While the standard formulation only considers time-interval regions (with equal-time hypersurfaces as their boundaries), the GBF generalizes the standard procedure to arbitrary regions (and their boundaries). The QFT under consideration is here applied on an arbitrary region $\mathbb{M}$ of spacetime. (For the rules about what qualifies as a region, see Section 1.2.1.) Quantum states then live on the boundary $\partial\mathbb{M}$ of the region. That is, each region's boundary has its own associated boundary state space $\mathcal{H}_{\partial\mathbb{M}}$. As an example, (leaving aside here details of orientation) the boundary of the time-interval region $\mathbb{M}_{[t_1,t_2]}$ consists of two components: $\partial\mathbb{M}_{[t_1,t_2]} = (\Sigma_1,\, \Sigma_2)$. The GBF then views the two states $\eta(t_1)$ and $\zeta(t_2)$ as one single boundary state $\xi_{\partial\mathbb{M}[t_1,t_2]} = (\eta(t_1),\, \zeta(t_2))$. The boundary state space is thus the tensor product of two copies of the usual state space (one for the intial and one for the final boundary component): $\mathcal{H}_{\partial\mathbb{M}[t_1,t_2]} = \mathcal{H} \otimes \mathcal{H}$. Since generic regions do not always have boundaries with two components, the boundary states may have an arbitrary number of components. The bra-ket notation is thus not very useful in the GBF. Instead, amplitudes are calculated by an amplitude map $\rho_{\mathbb{M}} : \ \mathcal{H}_{\partial\mathbb{M}} \ \to \ \mathbb{C}$. That is, each region $\mathbb{M}$ has its own associated amplitude map $\rho_{\mathbb{M}}$, which assigns amplitudes to its boundary states. The amplitude map $\rho_{\mathbb{M}}$ thus generalizes the time-evolution operator $\mathcal{U}_{t_2,t_1}$ of the standard formulation. Both depend on the quantum theory under consideration.

As a guideline for which regions are admissible in the GBF, we require that regions must make sense for scattering experiments to be conducted in them[1]. That is, they must have a boundary allowing for sending particles in, and also allowing for detecting particles coming out. Therefore,

[1]We thank Olivier Sarbach (IFM-UMNSH) for asking us to clarify this point.

what kinds of regions are admissible depends on the quantum theory under consideration and on the background spacetime. As an example, the region of Minkowski spacetime given only by the condition $t > t_1$ is not admissible, since we can only send particles in through the boundary Σ_1, but we have no way of detecting any outcoming particles on this boundary. By contrast, the time-interval region given by $t > t_1$ while $t < t_2$ allows for both of these actions and hence is admissible. Further, it is desirable to have sufficiently many regions to describe physics locally (compact regions) and to be able to glue them consitently.

Finally, the probability interpretation of the GBF allows us to relate theory with experiment. For the time-interval region, this interpretation reproduces the Born rule. However, for general regions it is not that simple. We review GBF probabilities in detail in Section 1.2.5. Observables and the corresponding expectation values also fit into the GBF, for the explicit treatment see Section 1.2.6. There, we also show how the probabilities and expectation values of the standard formulation are recovered from the General Boundary Formulation.

As a historical note, we mention that the idea of (generalized) transition amplitudes for regions of spacetime was conceived already in 1933 by Dirac in the famous paper [27], which later inspired Feynman to develop his path integral formulation. Therein, Dirac presents "generalized transformation functions" which essentially are the amplitudes of the GBF. He then formulates the gluing rule for regions in spacetime. For a more detailed discussion of his paper see Section 2.1 in [54].

The problem of time

In [51] the GBF is motivated from a viewpoint of nonrelativistic Quantum Mechanics, which is then extended towards Quantum Gravity. First, Schrödinger's Cat is revisited: Inside a box (thus hidden from our views) is a quantum system which produces some particular event with some probability, for example a radioactive isotope that has some probability for decaying within a certain amount of time. This quantum system is coupled to a detector for the particular event, which connects it to a macroscopic classical system inside the box; in the original thought experiment this is any device that kills the poor cat if a decay is detected. Therefore, the state of the macroscopic system depends on the state of the quantum system. The familiar experiment is thus divided into a quantum domain (the interior of the box) and a classical domain (its surroundings). Then, we cannot assume a definite classical evolution in the quantum domain, that is, inside the box (which is considered isolated from the classical world between preparation and observation). Repeating the experiment many times, the goal is to measure a probability $P(t)$ that the cat is still alive after a time t. If we consider space and time as quantum mechanical entities, then we cannot assume any definite passage of time inside the box. However, on the outside time remains classical. Then there are two ways of determining the time t: using a clock either outside or inside of the box. In both cases, a problem arises. Putting the clock outside of the box, then how can the quantum system know about the classical outside time t, if we really cannot assume any definite evolution of time inside? Putting the clock inside, we have no way of knowing when we shall open the box (e.g. if we are interested in the probability $P(t_0)$ for some particular time t_0). Putting the clock inside fundamentally changes the experiment and thus corresponds to a different measurement process, because the ability to predetermine the time t is a crucial part of the original experiment.

The way out of this dilemma is that the quantum system is actually not isolated from the classical world, but remains in contact with the surrounding classical spacetime through its boundary. The spacetime structure at this boundary is thus a boundary condition of the experiment, which we must regard as an integral part of the quantum mechanical measurement process. In particular, the spacetime structure on the boundary determines the time t on the classical clock, which can be calculated on paths on the boundary using the spacetime metric. For this to work, the boundary needs to be connected, that is, it must consist of only one connected component. This is called the principle of the integrity of the observer in [51].

In traditional approaches to Quantum Gravity, time-interval regions are used as sketched in Figure 1.2 (b). However, the boundary of a time-interval regions is not connected: it rather consists of two connected components which are spacelike hypersurfaces (e.g. Cauchy surfaces). Then, we can make sense of a time difference between two events on inital and final surface as follows: we need to assume inital and final spacetime metric to be described by quasi-classical quantum states, and further that from their peak metrics we can reconstruct the metric on the whole region. Then we can calculate how much time passes on some path from the initial to the final event. However, then there is no direct relation to our usual way of measuring time. This is called the problem of time. Such relation could be established by declaring some spacetime region as classical for placing a clock there, but then this region's boundary would connect the initial and final hypersurface.

This motivates why the GBF favours regions whose boundary is connected. The final goal is here to work with compact regions, since physical experiments are finite in space and time. A type of regions which are not compact, but already have a connected boundary, are the rod regions described in Section 2.1, which we shall use on Minkowski and Anti de Sitter spacetimes.

1.1.2 GBF from locality and operationalism

In the previous section we have seen the basic features of the General Boundary Formulation. In order to further motivate this method, we now sketch how it is induced naturally from general first principles as described in [64]. The two key principles considered therein are locality and operationalism.

In *classical physics* locality means that forces do not mysteriously act at a distance between some particle at one point of spacetime and another particle at some other point. Rather the forces are exerted by fields that fill out spacetime. By the term signal we shall denote such a force originating at one point and traveling to a second point. Locality then says that signals do not jump across distances on spacetime, but continuously propagate at finite speed. The locality principle is also respected in *quantum theory*, which describes particles and fields in a unified way. Thus, for both classical and quantum physics, locality means that interaction between particles and/or fields is only possible through their contiguous contact on spacetime.

One form in which locality appears is, that if we conduct an experiment in a laboratory, then its results should not depend on what happens outside the lab (except if the experiment is specifically designed to do so). That is, we divide the universe into two parts: one spacetime region, which is our laboratory for the current experiment, and the rest of the universe. Since locality excludes spooky actions at a distance, anything inside our lab/region can interact with the outside universe only by signals passing through (parts of) the region's boundary. While regions are topological spacetime submanifolds with the same dimension as spacetime itself, boundaries consist of one or more spacetime hypersurfaces of codimension one. (Let us recall that any boundary is a hypersurface, but not any hypersurface is a boundary. For example, a single equal-time hypersurface is not the boundary of an admissible region. However, the boundary of a time-interval region consists of two equal-time hypersurfaces, that is, of two hypersurfaces.) Locality thus directly requires a mathematical model of physics to use regions and boundaries.

Frecuently in physics the goal is to relate what happens in one experiment to what happens in an adjacent experiment, for example, when one experiment consists of various measurements which we can view as smaller experiments in their own right. Now the big experiment has its associated region, and the smaller experiments have their regions as well, which should be contained within the big region. According to locality, the interaction between these experiments can only occure by passing through the boundary parts shared by the corresponding smaller regions. Hence the composition of smaller experiments to a big one corresponds to gluing together the associated smaller regions along the shared parts of their boundaries, thereby forming a big region. This gluing is considered in detail in Section 1.2.1 and Axiom (T5b) of Section 1.2.2.

In classical physics the observed system and the observer are thaught of as independent and not influencing each other. However, Quantum Theory tells us that this is not the case: measurements *do affect* quantum systems. Operationalism now concludes that therefore Quantum Theory should not try to describe what quantum objects really are (that would be an ontologic approach), but rather describe what happens when they interact, for example, what we observe in quantum experiments.

The GBF incoporates operationalism by associating observables to each spacetime region[2]. This is natural, since physical experiments are extended in space and time. When regions are glued together to a bigger region for combining experiments, then there is also an induced gluing of the smaller regions' observables which results in observables of the bigger region. The GBF also includes a probability interpretation and expectation values for these observables. However, this is beyond the scope of this brief introduction, and therefore we get back to these points in Section 1.2.6.

In this and the previous section we have introduced the most important features of the General Boundary Formulation and sketched how they are induced by locality and operationalism. Before proceeding with a detailed description of the GBF in Section 1.2, we can already put to use our rough picture of the GBF in the following section for outlining how it can solve the S-matrix problem on Anti de Sitter spacetimes.

1.1.3 The S-matrix problem of Anti de Sitter spacetimes

As seen in Section 1.1.1, the General Boundary Formulation allows quantum states to live not only on equal-time hypersurfaces, but on general hypersurfaces. What benefits does this bring us? For instance, it can enable us to compute S-matrices for spacetimes where S-matrices cannot be constructed via the standard approach. This is precisely the case on Anti de Sitter spacetimes (AdS).

In standard QFT on Minkowski spacetime, the S-matrix is the limit $t \to \infty$ of the time-evolution operator $\mathcal{U}_{[-t,+t]}$. The S-matrix is completely determined by its matrix elements, which are the limit of the amplitudes (1.1):

$$\mathcal{S}_{\eta,\zeta} \;=\; \lim_{t\to\infty} \rho(\eta(-t),\zeta(+t)) \;=\; \lim_{t\to\infty} \langle \zeta(+t)\,|\,\mathcal{U}_{[-t,+t]}\,|\,\eta(-t)\rangle \;. \tag{1.3}$$

The particle interpretation of the initial and final states, (as incoming and outgoing particles with definite momenta) relies on states from the free theory. Hence we need initial and final state to be asymptotically free states. That is, there is no interaction between the particles at large times $|t| \to \infty$. The mathematical technique for switching off the interaction at large times is multiplying the interaction term by a bump function which is one at intermediate times, zero for large times, and varies smoothly and sufficiently slowly in between. If we assume the particle interaction to decrease with increasing distance, then switching the interaction off for large times is justified, if the particles are separated by large distances before and after their interaction. We assume that after the scattering the particles approximately move along timelike geodesics. In Minkowski spacetime, the timelike geodesics are straight lines which assure the large separation for large times. However, in a more general spacetime this is not necessarily so, since the shape of the geodesics depends on the geometry of the spacetime. For example, on AdS spacetime the timelike geodesics reconverge periodically, bringing the particles close together for possible interaction again and again. Therefore, on AdS there are no asymptotically free states for large times, which makes the S-matrix interpretation of amplitudes questionable. Let us have a closer look at AdS now.

The relevant geometric features of AdS are its constant negative curvature and its simple $\mathbb{R}^4$-topology (the same as 4-dimensional Minkowski spacetime). AdS has a timelike boundary at spatial infinity (not a lightlike boundary at null infinity as Minkowski). The negative curvature of AdS has a similar effect as a potential wall: it reflects particles back into the interior of the spacetime. This is sketched in Figure 1.4. Hence interacting particles after scattering are brought close to each other periodically in time, and therefore they never become asymptotically free. Without asymptoticically free states on equal-time hypersurfaces the standard S-matrix construction is not possible.

[2]This is done properly in the GBF version called Positive Formalism. In the Amplitude Formalism used in this thesis, there are subtleties that prevent us from considering n-point functions with $n \geq 2$ as proper expectation values.

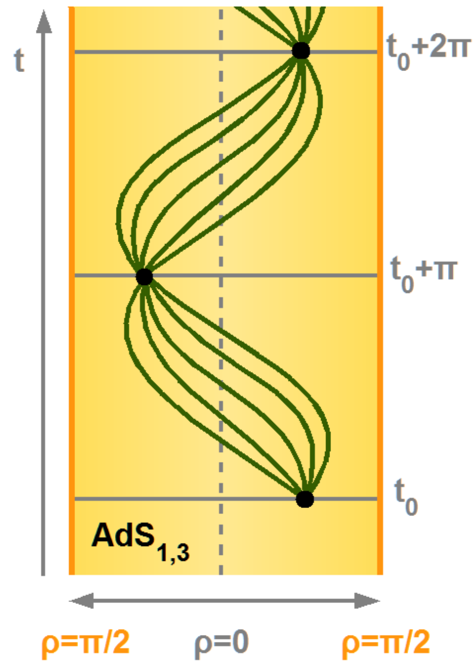


Figure 1.4: Conformal diagram of AdS: time extends infinitely above and below, while the radial coordinate ρ is compact with spatial infinity at $\rho = \frac{\pi}{2}$. Lightrays are at 45°. The dashed line is the time axis at $\rho = 0$. The orange lines represent the timelike boundary hypercylinder. The green lines are the timelike geodesics of free particles after an initial scattering at t_0. They reconverge periodically, making the particles interact again and again.

Fortunately there is another type of asympotically free states on AdS: since the metric becomes divergent for large radius (approaching the timelike boundary), it induces large distances for particles close to the boundary. Thus their interaction becomes negligible and the corresponding states are (radially) asymptotically free [37]. These states live on equal-radius hypersurfaces which we call hypercylinders $\mathbb{R}_t \times \mathbb{S}^2$: a sphere of fixed radius times all of time. Since the GBF formulates Quantum Theory for states on general hypersurfaces, it provides us the tools for a consistent construction of an S-matrix on AdS for these radially asymptotically free states. This is discussed in more depth in Section 3.1. Knowing now the basic ideas of the General Boundary Formulation and having sketched their application on AdS, we can move forward to a detailed axiomatic description of the GBF.

1.2 The GBF in detail

In the previous sections we have seen that the General Boundary Formulation considers geometric objects (spacetime regions and hypersurfaces) and associates to them algebraic objects (states, Hilbert spaces, amplitude maps). Now we will make these notions precise: Section 1.2.1 presents axioms for the geometric objects, which prepare for the GBF's core axioms in Section 1.2.2. These core axioms determine the association of algebraic objects to geometric objects (plus properties of these association laws). The algebraic objects are Hilbert spaces of states, and maps on and between these spaces. The geometric objects are topological manifolds, possibly but not necessarily with additional structure (such as a spacetime metric, as suggested by the name "geometric").

1.2.1 Geometric data: Regions and hypersurfaces

As a preparation for the following section, the axioms of the current section determine which geometric objects enter the GBF, closely following Section 2.1 in [59] and also Section 2.1 in [66]. The

GBF formalizes the structure of spacetime in a spacetime system, which consists of regions and hypersurfaces. First the spacetime dimension is fixed to $(d+1) \in \mathbb{N}^+$. Then the GBF takes as given a collection of $(d+1)$-dimensional oriented topological manifolds (possibly with boundary), and these manifolds we call regions. Further, there is another collection of oriented topological manifolds *without boundary* of dimension d, and these manifolds we call hypersurfaces. (A priori, boundaries are not required to meet differentiability conditions, although frequently they are taken to be piecewise smooth.) All manifolds in both collections may only have finitely many connected components. (The present condition that hypersurfaces have no boundaries is rather an intermediate step. In order to consistently treat compact regions and their gluings, it is necessary to allow hypersurfaces to have boundaries. These boundaries were introduced in [56] and are called corners therein.) We sometimes use the terms *admissible* manifold/region/ hypersurface, in order to stress that a given manifold is in one of those two collections. For a criterion on which regions are admissible, see Section 1.1.1. The GBF only considers collections satisfying the following requirements.

(GD1) Orientation:
If Σ is an admissible hypersurface, then the same manifold with opposite orientation (denoted by $\overline{\Sigma}$) is admissible as well.

(GD2) Components:
Any connected component of an admissible region (hypersurface) is itself an admissible region (hypersurface), that is, it is contained in the collection of regions (hypersurfaces).

(GD3) Unions:
Any disjoint union of finitely many admissible regions (hypersurfaces) is an admissible region (hypersurface).

(GD4) Boundaries:
The boundary of any admissible region is an admissible hypersurface.

The easiest way to think of a spacetime system is taking regions and hypersurfaces as submanifolds induced by a global spacetime manifold. In Section 6 of [55] this setting was introduced by the name of global background, which we review in Section 1.2.4. However, the regions of a spacetime system may also be viewed as independent pieces of spacetime which are not a priori embedded into any global manifold. It can be argued that it is actually more desirable not to assume any knowledge about spacetime outside of our laboratory or to even assume it to be some particular fixed global background.

Later on we will also need slice regions (called empty regions in earlier works). A slice region, denoted by $\hat{\Sigma}$, is associated to each hypersurface Σ. Topologically it is simply the hypersurface itself, but in the GBF we think of it as an "infinitesimally thin" region in a suitable sense. The boundary of a slice region is defined as the disjoint union of its hypersurface with an orientation-reversed copy of itself: $\partial\hat{\Sigma} := \Sigma \cup \overline{\Sigma}$. Forgetting orientation, for each hypersurface Σ there is exactly one slice region $\hat{\Sigma}$. When needed, we refer to regions that are not slice regions as regular regions.

Figure 1.5: Hypersurface Σ and associated slice region $\hat{\Sigma}$.

We recall that regions represent laboratories for conducting experiments. Since joining two labs results in a new (bigger) lab, the GBF also needs prescriptions for joining two regions. This is called gluing of regions in the GBF. In order to be most general, two different type of gluings must be covered by the GBF's notion of gluing. The first type (see for example Figure 1.7) is gluing *disjoint* regions together, along some parts of their respective boundaries. This represents bringing separated labs into contact. The second type (see for example Figure 1.6) is gluing some part of *one* region's boundary to another part of the *same* region's boundary. This represents changing the shape (connectedness, topology) of one single lab. Since in the first type (the union of) the disjoint smaller regions can be seen as one new bigger region, we can view the first type as a special case of the second type (since the glued boundary parts now belong to the same one big region, which is a union of small regions). Therefore, the most general way of formulating gluing is describing the second type. Suppose thus that we are given one region $\mathbb{M}_G$ (with G saying the region will be GLUED along Σ_G) with its boundary $\partial\mathbb{M}_G = \Sigma_N \cup \Sigma_G \cup \overline{\Sigma'_G}$ a disjoint union of some hypersurface Σ_N (possibly a union of hypersurfaces, or empty) with some hypersurface Σ_G (possibly a union of hypersurfaces), and a copy $\overline{\Sigma'_G}$ of Σ_G with opposite orientation. Then we may obtain a new manifold $\mathbb{M}_N$ (with N for NEW) with boundary $\partial\mathbb{M}_N = \Sigma_N$ by gluing $\mathbb{M}_G$ to itself along Σ_G and $\overline{\Sigma'_G}$, wherein gluing means identifying points of Σ_G with their copy on $\overline{\Sigma'_G}$.

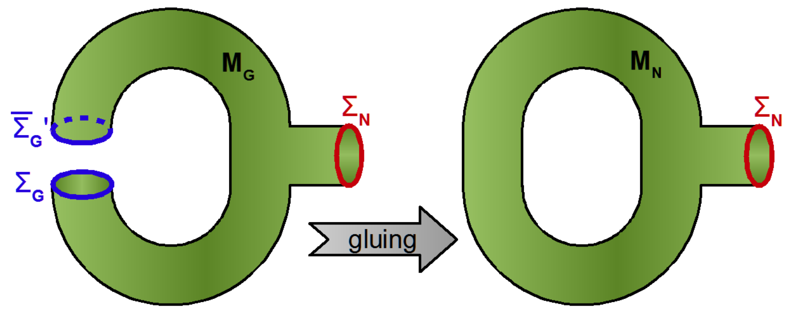


Figure 1.6: Gluing tube region along boundary components.

In Figure 1.6 we illustrate the gluing for a region consisting of one single connected component. The region is therein a two-dimensional "tube" whose boundary consists of three circles. Gluing along two of these circles results in a new region $\mathbb{M}_N$ whose boundary is the single circle $\partial\mathbb{M}_N = \Sigma_N$.

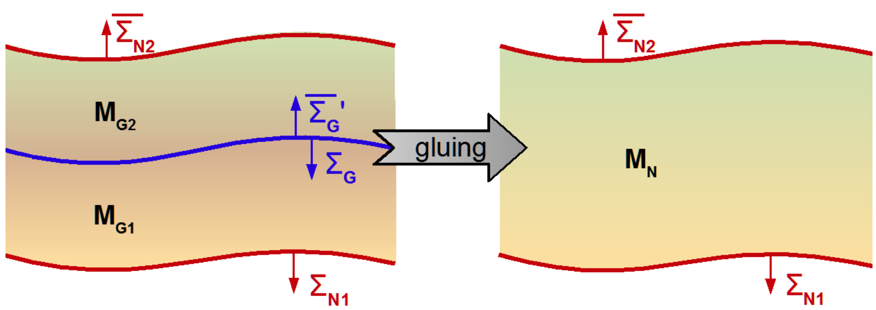


Figure 1.7: Gluing time interval regions along boundary components.

Figure 1.7 illustrates a simpler example: the gluing of two adjacent time interval regions. Here the original region $\mathbb{M}_G = \mathbb{M}_{G1} \cup \mathbb{M}_{G2}$ consists of two connected components which we view as disjoint (up to the shared boundary component). The boundary of $\mathbb{M}_G$ consists of four connected components: Σ_G, its copy $\overline{\Sigma}'_G$, and $\Sigma_N = \Sigma_{N1} \cup \overline{\Sigma}_{N2}$. Gluing along Σ_G results in the bigger time interval region $\mathbb{M}_N$ whose boundary consists of the two remaining components $\partial\mathbb{M}_N = \Sigma_N$. This shows why the above gluing procedure has been formulated for a single region: it enables the GBF to treat regions like the "tube" which consist of a single connected component, while applying as well for the gluing of disjoint regions (because according to Axiom (GD3) the union of these regions can be seen as a bigger region with various components).

If the manifold resulting from a gluing is inadmissible, then the gluing is not allowed. If there are different ways of making this identification, then we require the gluing to be unique. That is: we require the resulting manifolds to be indistinguishable in our setting. Different ways of identifying the gluing hypersurfaces might result in important differences depending on the theory one wants to model. This may be encoded through suitable additional structures on regions and hypersurfaces (for example differentiable structure and metric). If the manifolds carry such additional structure, then this must be taken into account in the gluing. This can make the gluing impossible or require it to be done in specific ways.

1.2.2 Core Axioms

After considering regions and hypersurfaces in the previous section, we now treat the core of the General Boundary Formulation. These core axioms determine the system of algebraic objects that become associated to the geometric ones. A first version of the core axioms was formulated in Section 2 of [55] and we follow the more recent version in Section 2.2 of [59] (which is identical to the one in Section 2.2 of [66]). In the axioms, by $\otimes$ we denote the tensor product of vector spaces, and use $\hat{\otimes}$ for the completed one of Hilbert spaces. As before, $\overline{\Sigma}$ denotes the hypersurface Σ with reversed orientation.

(T1) State spaces:
To each hypersurface Σ we associate a complex separable Hilbert space $\mathcal{H}_\Sigma$ and call it the state space of Σ. We denote its inner product by $\langle\, \cdot\, ,\, \cdot\, \rangle_\Sigma$.

(T1b) Orientation reversal and involution:
Associated to each hypersurface Σ is a conjugate linear isometric involution $\iota_\Sigma : \ \mathcal{H}_\Sigma \to \mathcal{H}_{\overline{\Sigma}}$. Linear isometric means preserving the norm: with $\eta_{\overline{\Sigma}} := \iota_\Sigma \eta_\Sigma$ we have

$$\|\, \eta_\Sigma \,\|_\Sigma = \|\, \eta_{\overline{\Sigma}} \,\|_{\overline{\Sigma}} \qquad\qquad \forall\ \eta_\Sigma \in \mathcal{H}_\Sigma. \tag{1.8}$$

It is an involution in the sense that $\iota_{\overline{\Sigma}} \circ \iota_\Sigma$ is the identity on $\mathcal{H}_\Sigma$. (Frequently we view $\mathcal{H}_\Sigma$ and $\mathcal{H}_{\overline{\Sigma}}$ as identified[3], and the involution simply as complex conjugation. That is: if $\xi_\Sigma \in \mathcal{H}_\Sigma$, then $\iota_\Sigma \xi_\Sigma = \overline{\xi_\Sigma}$.) Since the inner product is conjugate linear in the first argument and linear in the second, and the involution ι_Σ is conjugate linear while preserving norms, we have

$$\langle \eta_\Sigma, \zeta_\Sigma \rangle_\Sigma = \langle \zeta_{\overline{\Sigma}}, \eta_{\overline{\Sigma}} \rangle_{\overline{\Sigma}} \qquad\qquad \forall\ \eta_\Sigma, \zeta_\Sigma \in \mathcal{H}_\Sigma. \tag{1.9}$$

(T2) Unions of hypersurfaces:
We recall that, according to Axiom (GD3), a union of a finite number of hypersurfaces Σ_k is again a hypersurface Σ (with multiple components, $k = 1, ..., n$). Hence by Axiom (T1) this hypersurface has its associated state space $\mathcal{H}_\Sigma$. Now the current axiom says: if a hypersurface Σ decomposes into a disjoint union of hypersurfaces: $\Sigma = \Sigma_1 \cup ... \cup \Sigma_n$, then there is an isometric isomorphism of Hilbert spaces:

$$\tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma} : \ \ \mathcal{H}_{\Sigma_1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_{\Sigma_n} \ \to\ \mathcal{H}_\Sigma\ .$$

(The composition of the τ-maps of two consecutive hypersurface decompositions is identical to the τ-map of the resulting decomposition.) That is, for any states $\eta_{\Sigma_k}, \zeta_{\Sigma_k} \in \mathcal{H}_{\Sigma_k}$ with $k = 1, ..., n$ the inner product is compatible with hypersurface decomposition:

$$\Big\langle \tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma}(\eta_{\Sigma_1} \otimes \ldots \otimes \eta_{\Sigma_n}), \tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma}(\zeta_{\Sigma_1} \otimes \ldots \otimes \zeta_{\Sigma_n}) \Big\rangle_\Sigma = \Big\langle \eta_{\Sigma_1} \otimes \ldots \otimes \eta_{\Sigma_n}, \zeta_{\Sigma_1} \otimes \ldots \otimes \zeta_{\Sigma_n} \Big\rangle_{\mathcal{H}_{\Sigma_1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_{\Sigma_n}}$$
$$= \Big\langle \eta_{\Sigma_1},\ \zeta_{\Sigma_1} \Big\rangle_{\Sigma_1} \cdot\ \ldots\ \cdot \Big\langle \eta_{\Sigma_n},\ \zeta_{\Sigma_n} \Big\rangle_{\Sigma_n}$$

[3] Instad of identifying $\mathcal{H}_{\overline{\Sigma}}$ and $\mathcal{H}_\Sigma$, we could equivalently identify $\mathcal{H}_{\overline{\Sigma}}$ and $\mathcal{H}^{\mathrm{T}*}_{\overline{\Sigma}}$ (the topological dual, consisting of continuous linear maps from $\mathcal{H}_{\overline{\Sigma}}$ to $\mathbb{C}$. Then, a natural choice for the involution ι_Σ is the adjoint map, that is, $\iota_\Sigma \eta_\Sigma = {}^*\eta_\Sigma = \langle \eta_\Sigma,\ \cdot\, \rangle_\Sigma$. However, in our calculations we always use the former identification $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_\Sigma$ with ι_Σ given by complex conjugation, because this makes the equations simpler, see for example (3.1) which stems from [59], or the amplitude (24) in [53].

In order to keep equations more readable, frequently we do not explicitly write the τ-maps and instead assume their actions as understood. In other words: since the union's state space $\mathcal{H}_\Sigma$ is isometrically isomorphic to the tensor product of the components' state spaces $\mathcal{H}_{\Sigma_1} \hat{\otimes} \dots \hat{\otimes} \mathcal{H}_{\Sigma_n}$, we view them as identified as in $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \hat{\otimes} \dots \hat{\otimes} \mathcal{H}_{\Sigma_n}$, omitting the τ-map.

(T2b) The involution ι is compatible with the above hypersurface decomposition:

$$\tau_{\overline{\Sigma}_1,\dots,\overline{\Sigma}_n;\overline{\Sigma}} \circ (\iota_{\Sigma_1} \hat{\otimes} \dots \hat{\otimes} \iota_{\Sigma_n}) = \iota_\Sigma \circ \tau_{\Sigma_1,\dots,\Sigma_n;\Sigma} .$$

(T4) Amplitudes:
Associated with each region $\mathbb{M}$ is a linear amplitude map $\rho_\mathbb{M}$. It maps a dense subspace $\mathcal{H}^\circ_{\partial\mathbb{M}}$ of the region's boundary state space $\mathcal{H}_{\partial\mathbb{M}}$ to the complex numbers (the boundary $\partial\mathbb{M}$ carries the induced orientation and we call $\mathcal{H}^\circ_{\partial\mathbb{M}}$ the amplitude subspace of the state space $\mathcal{H}_{\partial\mathbb{M}}$):

$$\rho_\mathbb{M} : \mathcal{H}^\circ_{\partial\mathbb{M}} \to \mathbb{C}.$$

(T3x) Slice region's amplitude and inner product:
By definition, for any hypersurface Σ the boundary $\partial\hat{\Sigma}$ of the associated slice region $\hat{\Sigma}$ decomposes into the disjoint union $\partial\hat{\Sigma} = \Sigma \cup \overline{\Sigma}'$, where Σ' denotes a second copy of Σ. Then, the τ-image of (the tensor product of) the surfaces' state spaces is contained in the amplitude subspace of the slice region's boundary state space: $\tau_{\Sigma,\overline{\Sigma}';\partial\hat{\Sigma}}(\mathcal{H}_\Sigma \otimes \mathcal{H}_{\overline{\Sigma}'}) \subseteq \mathcal{H}^\circ_{\partial\hat{\Sigma}}$. Moreover, the slice region's amplitude map $\rho_{\hat{\Sigma}} \circ \tau_{\Sigma,\overline{\Sigma}';\partial\hat{\Sigma}}$ restricts to a bilinear pairing $(\,\cdot\,,\,\cdot\,)_\Sigma : \mathcal{H}_\Sigma \times \mathcal{H}_{\overline{\Sigma}'} \to \mathbb{C}$ such that the inner product is recovered by $\langle\,\cdot\,,\,\cdot\,\rangle_\Sigma = (\,\cdot\,, \iota_{\Sigma'}\,\cdot\,)_\Sigma$.

If we do not explicitly write the τ-maps, then the relation between amplitude and inner product writes as follows: let $\eta_\Sigma \in \mathcal{H}_\Sigma$ and $\zeta_{\Sigma\iota'_\Sigma} \in \mathcal{H}_{\Sigma\iota'_\Sigma}$ arbitrary states. Then, $(\eta_\Sigma \otimes \zeta_{\Sigma\iota'_\Sigma}) \in \mathcal{H}_{\partial\hat{\Sigma}}$ is a state on the slice region's boundary. Writing $\zeta_{\overline{\Sigma}'} = \iota_{\Sigma'}\zeta_{\Sigma'}$, the current axiom states that

$$\rho_{\hat{\Sigma}}(\eta_\Sigma \otimes \zeta_{\overline{\Sigma}'}) = (\eta_\Sigma, \zeta_{\overline{\Sigma}'})_\Sigma = \langle \eta_\Sigma, \zeta_{\Sigma'} \rangle_\Sigma. \tag{1.10}$$

(T5a) Product rule for amplitudes:
Let $\mathbb{M} = \mathbb{M}_1 \cup \mathbb{M}_2$ be the disjoint union of regions $\mathbb{M}_1$ and $\mathbb{M}_2$. (We may need to allow unions of regions that are disjoint only up to boundaries. However, we shall still call these regions "disjoint".) Then the boundary of the disjoint union is the disjoint union of the boundaries: $\partial\mathbb{M} = \partial\mathbb{M}_1 \cup \partial\mathbb{M}_2$. The τ-image of the amplitude subspaces of the constituent regions is contained in the amplitude subspace of their union: $\tau_{\partial\mathbb{M}_1,\partial\mathbb{M}_2;\partial\mathbb{M}}(\mathcal{H}^\circ_{\partial\mathbb{M}_1} \otimes \mathcal{H}^\circ_{\partial\mathbb{M}_2}) \subseteq \mathcal{H}^\circ_{\partial\mathbb{M}}$. Then, for all $\psi_1 \in \mathcal{H}^\circ_{\partial\mathbb{M}_1}$ and $\psi_2 \in \mathcal{H}^\circ_{\partial\mathbb{M}_2}$ the regions' amplitudes fulfill the product rule

$$\rho_\mathbb{M} \circ \tau_{\partial\mathbb{M}_1,\partial\mathbb{M}_2;\partial\mathbb{M}}(\psi_1 \otimes \psi_2) = \rho_{\mathbb{M}_1}(\psi_1) \cdot \rho_{\mathbb{M}_2}(\psi_2) . \tag{1.11}$$

(T5b) Gluing rule for amplitudes:
Let $\mathbb{M}_G$ be a region with its boundary decomposing as a disjoint union $\partial\mathbb{M}_G = \Sigma_N \cup \Sigma_G \cup \overline{\Sigma'_G}$, where Σ'_G is a copy of Σ_G (see discussion of gluing around Figures 1.6 and Figure 1.7). Σ_N is allowed to be empty or be some disjoint union of hypersurfaces. Let $\mathbb{M}_N$ denote the gluing of $\mathbb{M}_G$ with itself along Σ_G and $\overline{\Sigma'_G}$, and suppose that $\mathbb{M}_N$ is a region. Note $\partial\mathbb{M}_N = \Sigma_N$. Then, $\tau_{\Sigma_N,\Sigma_G,\overline{\Sigma'_G};\,\partial\mathbb{M}_G}(\psi_N \otimes \xi_G \otimes \iota_{\Sigma G}\xi_G) \in \mathcal{H}^\circ_{\partial\mathbb{M}_G}$ for all $\psi_N \in \mathcal{H}^\circ_{\partial\mathbb{M}_N}$ and $\xi_G \in \mathcal{H}_{\Sigma_G}$. Moreover, for any orthonormal basis $\{\xi^i_G\}_{i\in I}$ of $\mathcal{H}_{\Sigma_G}$ and for all $\psi_N \in \mathcal{H}^\circ_{\partial\mathbb{M}_N}$ the amplitudes obey the following gluing rule (wherein $c(\mathbb{M}_G, \Sigma_G, \overline{\Sigma'_G}) \in \mathbb{C}\setminus\{0\}$ is called the gluing anomaly factor and depends only on the geometric data):

$$\rho_{\mathbb{M}_N}(\psi_N) \cdot c(\mathbb{M}_G, \Sigma_G, \overline{\Sigma'_G}) = \sum_{i\in I} \rho_{\mathbb{M}_G} \circ \tau_{\Sigma_N,\Sigma_G,\overline{\Sigma'_G};\,\partial\mathbb{M}_G}(\psi_N \otimes \xi^i_G \otimes \iota_{\Sigma_G}\xi^i_G). \tag{1.12}$$

The relation (1.10) between amplitude and inner product is of course fulfilled in the standard formulation as well. Let $|\,\eta(t)\rangle$ an "initial" state and $|\,\zeta(t)\rangle$ a "final" state at the same time t. Then, (1.1) gives us the standard amplitude $\rho(\eta(t),\overline{\zeta(t)})$ for observing $|\,\zeta(t)\rangle$ "after" preparing $|\,\eta(t)\rangle$ as in $\rho(\eta(t),\overline{\zeta(t)}) = \langle\zeta(t)\,|\,\mathcal{U}_{t,t}\,|\,\eta(t)\rangle$, wherein $\mathcal{U}$ is the time evolution operator. (The complex conjugation is the involution ι of (T1b) which is applied because of the opposite orientation of the copy of Σ_t.) Since $\mathcal{U}_{t,t} = \mathbb{1}$, the standard "equal-time amplitude" reduces to the overlap $\rho(\eta(t),\overline{\zeta(t)}) = \langle\zeta(t)\,|\,\eta(t)\rangle$.

As introduced in [59], in Axiom (T4) the amplitude map $\rho_{\mathbb{M}}$ needs only be defined on a dense subspace $\mathcal{H}^{\circ}_{\partial\mathbb{M}}$ of the region $\mathbb{M}$'s boundary state space $\mathcal{H}_{\partial\mathbb{M}}$. This makes sense, because generically the amplitude map is not continuous and thus not bounded, since a linear map between two normed spaces is bounded if and only if it is continuous, see e.g. Theorem 1.32 in [71]. This unboundedness is due to the GBF's philosophy of considering states on the boundary as *one* state, even when the boundary consists of several disjoint components, see below.

The gluing rule in Axiom (T5b) is formulated for gluing a single region $\mathbb{M}_G$. This allows for gluing different parts of the boundary of one connected region. Moreover, it also allows for gluing boundary parts of various disconnected regions, because we can view the union of these regions as one new region according to Axiom (GD3), see Figures 1.6 and 1.7 with the related discussion.

Recovering the standard amplitudes

As already mentioned, the standard formulation always considers time-interval regions $\mathbb{M}_{[t_1,t_2]} = [t_1,t_2]\times\mathbb{R}^3$ in Minkowski spacetime. Let us orient all equal-time planes Σ_t backwards in time, and orient all boundaries outwards. Then, $\partial\mathbb{M}_{[t_1,t_2]} = \Sigma_{t_1}\cup\overline{\Sigma}_{t_2}$. Core Axioms (T2) and (T1b) allow us to identify the region's boundary state space with a tensor product: $\mathcal{H}_{\partial[t_1,t_2]} = \mathcal{H}_{\Sigma_{t1}}\hat{\otimes}\mathcal{H}^{\mathrm{T}*}_{\Sigma_{t2}}$. (For some Hilbert space $\mathcal{H}$, the Riesz representation theorem allows us to identify its complex conjugate $\overline{\mathcal{H}}$ with its topological dual $\mathcal{H}^{\mathrm{T}*}$.) The amplitude map $\rho_{[t_1,t_2]}$ can thus be seen as a bilinear map $\mathcal{H}_{\Sigma_{t1}}\hat{\otimes}\mathcal{H}^{\mathrm{T}*}_{\Sigma_{t2}} \to \mathbb{C}$, just like the inner product in bra-ket notation.

The standard formulation works with only one single state space H and dynamics is encoded by the time-evolution operator $\mathcal{U}_{t_2,t_1}:\ H\to H$. We can consider the three state spaces H, $\mathcal{H}_{\Sigma_{t1}}$ and $\mathcal{H}_{\Sigma_{t2}}$ as identified through the isometries of time-translations. The amplitude map $\rho_{[t_1,t_2]}$ and the time-evolution operator $\mathcal{U}_{t_2,t_1}$ are then related through

$$\rho_{[t_1,t_2]}(\eta_{\Sigma_{t1}}\otimes\overline{\zeta_{\Sigma_t 2}}) = \left\langle\zeta_{\Sigma_{t2}},\,\mathcal{U}_{t_2,t_1}\eta_{\Sigma_{t1}}\right\rangle_{\Sigma_{t2}} = {}_{t_2}\langle\zeta\,|\,\mathcal{U}_{t_2,t_1}\,|\,\eta\rangle_{t_1}\,. \tag{1.13}$$

In this context it is easy to show that the amplitude map is unbounded. Let $\{\xi_i\}_{i\in\mathbb{N}}$ an orthornormal basis of $\mathcal{H}_{\Sigma_{t1}}$, which by unitary evolution induces an orthonormal basis $\{\mathcal{U}_{t_2,t_1}\xi_i\}_{i\in\mathbb{N}}$ $\mathcal{H}_{\Sigma_{t2}}$. Then, we can construct the following countable set of normalized boundary states:

$$\psi_n := \tfrac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i\otimes\overline{\mathcal{U}_{t_2,t_1}\xi_i},$$

$$\|\,\psi_n\,\|^2 = \langle\psi_n,\,\psi_n\rangle = \tfrac{1}{n}\sum_{i,j=1}^{n}\langle\xi_i,\,\xi_j\rangle\cdot\langle\overline{\mathcal{U}_{t_2,t_1}\xi_i},\overline{\mathcal{U}_{t_2,t_1}\xi_j}\rangle = \tfrac{1}{n}\sum_{i,j=1}^{n}\delta_{i,j}\,\delta_{i,j} = 1.$$

For the amplitude of these states we get by linearity of ρ:

$$\rho_{[t_1,t_2]}(\psi_n) = \tfrac{1}{\sqrt{n}}\sum_{i=1}^{n}\rho_{[t_1,t_2]}(\xi_i\otimes\overline{\mathcal{U}_{t_2,t_1}\xi_i}) = \tfrac{1}{\sqrt{n}}\sum_{i=1}^{n}\langle\xi_i\,|\,\mathcal{U}^{\dagger}_{t_2,t_1}\mathcal{U}_{t_2,t_1}\,|\,\xi_i\rangle = \tfrac{1}{\sqrt{n}}\,n$$
$$= n^{1/2}.$$

Thus, if the Hilbert space is infinite-dimensional, then for any given positive number, we can construct a normalized boundary state ψ_n whose amplitude is greater than this number. That is, ρ is unbounded. The power of 1/2 arises because the time-interval region's boundary consists of 2 disconnected components, which makes the boundary Hilbert space the tensor product of the components' Hilbert spaces. In a sense, this makes the norm of the boundary Hilbert space the norm squared of one components' Hilbert space. In other words, in the bra-ket notation the normalizing factor $\frac{1}{\sqrt{n}}$ appears twice and thus cancels the n from the sum, giving the amplitude the value one. By contrast, in the ρ-notation the factor $\frac{1}{\sqrt{n}}$ appears only once and thus leaves $\sqrt{n}$ as GBF amplitude. However, this is not a problem at all, because the GBF's probability interpretation of the amplitudes assures that they induce probabilities with the usual properties, see Section 1.2.5.

1.2.3 Vacuum

As for the Core Axioms, we reproduce here the version of the Vacuum Axioms given in Section 2.3 of [59]. In standard QFT on Minkowski spacetime there is only one vacuum state and it is invariant under time-translations. In the GBF there is one vacuum state for each hypersurface. Nevertheless, these vacuum states are related through conditions induced by the axioms.

(V1) Vacuum state
For each hypersurface Σ there is a distinguished state $\psi_\Sigma^{\text{Vac}} \in \mathcal{H}_\Sigma$, called the vacuum state.

(V2) Opposite orientation and involution
The vacuum state is compatible with the isometric involution: for any hypersurface Σ the vacuum of its opposite orientation $\overline{\Sigma}$ can be obtained by involuting the original vacuum:

$$\psi_{\overline{\Sigma}}^{\text{Vac}} = \iota_\Sigma \, \psi_\Sigma^{\text{Vac}} .$$

(V3) Unions of hypersurfaces
The vacuum state is also compatible with decompositions. If a hypersurface Σ is the disjoint union of n hypersurfaces $\Sigma_1 \cup \ldots \cup \Sigma_n$, then the union's vacuum state is the τ-image of the components' vacuum states:

$$\psi_\Sigma^{\text{Vac}} = \tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma}(\psi_{\Sigma_1}^{\text{Vac}} \otimes \ldots \otimes \psi_{\Sigma_n}^{\text{Vac}}) .$$

(V5) Vacuum amplitude
The vacuum state of any region's boundary has unit amplitude, i.e.: for any region $\mathbb{M}$ we have

$$\rho_\mathbb{M}(\psi_{\partial\mathbb{M}}^{\text{Vac}}) = 1 .$$

The following consequences of these axioms are also discussed in [59]: a vacuum state is normalized and conserved under unitary evolution. (In Section 5 of [55] normalization was still a separate axiom (V4).) Unit normalization of the vacuum can be shown starting from the unit amplitude of a vacuum for the slice region $\hat{\Sigma}$ associated to a hypersurface Σ:

$$1 \overset{(V5)}{=} \rho_{\hat{\Sigma}}\big(\psi_{\partial\hat{\Sigma}}^{\text{Vac}}\big) \overset{(V3)}{=} \rho_{\hat{\Sigma}}\big(\tau_{\Sigma,\overline{\Sigma};\partial\hat{\Sigma}}(\psi_\Sigma^{\text{Vac}} \otimes \psi_{\overline{\Sigma}}^{\text{Vac}})\big) \overset{(T3x)}{=} \big\langle \psi_\Sigma^{\text{Vac}}, \, \psi_\Sigma^{\text{Vac}} \big\rangle_\Sigma = \big\| \, \psi_\Sigma^{\text{Vac}} \big\|^2 . \tag{1.14}$$

The axioms also imply that the vacuum is conserved under evolution in the following sense: assume first that we consider a region $\mathbb{M}$ with $\partial\mathbb{M} = \Sigma_1 \cup \overline{\Sigma}_2$ (both $\Sigma_{1,2}$ may again be unions of hypersurfaces). This assumption is completely natural, since "evolution" implies that we start from some hypersurface and evolute towards another. Next assume that evolution (e.g. in time) can be implemented via a unitary operator $\mathcal{U}_{21}^\mathbb{M} : \ \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$. We call it quantum evolution operator, and an explicit construction for it is given in Section 3.1.5. Unitarity means that the inner product is conserved:

$$\langle \eta_1, \zeta_1 \rangle_{\Sigma_1} = \langle \mathcal{U}_{21}^\mathbb{M} \eta_1, \mathcal{U}_{21}^\mathbb{M} \zeta_1 \rangle_{\Sigma_2} \qquad \forall \, \eta_1, \zeta_1 \in \mathcal{H}_{\Sigma_1}. \tag{1.15}$$

Moreover, let evolution be related to the amplitude map (as in standard QFT in Minkowski spacetime) for all $\eta_1 \in \mathcal{H}_{\Sigma_1}$ and $\zeta_2 \in \mathcal{H}_{\Sigma_2}$ via

$$\rho_\mathbb{M} \circ \tau_{\Sigma_1 \overline{\Sigma}_2;\partial\mathbb{M}}(\eta_1 \otimes \iota_{\Sigma_2} \zeta_2) = \langle \zeta_2, \mathcal{U}_{21}^\mathbb{M} \eta_1 \rangle_{\Sigma_2} . \tag{1.16}$$

Then evoluting one vacuum yields the other:

$$\mathcal{U}_{21}^{\mathbb{M}} \psi_{\Sigma_1}^{\text{Vac}} = \psi_{\Sigma_2}^{\text{Vac}} . \tag{1.17}$$

This can be seen by first noting that the evoluted vacuum is normalized due to unitarity:

$$1 = \langle \psi_{\Sigma_1}^{\text{Vac}}, \psi_{\Sigma_1}^{\text{Vac}} \rangle_{\Sigma_1} = \langle \mathcal{U}_{21}^{\mathbb{M}} \psi_{\Sigma_1}^{\text{Vac}}, \mathcal{U}_{21}^{\mathbb{M}} \psi_{\Sigma_1}^{\text{Vac}} \rangle_{\Sigma_2} . \tag{1.18}$$

Moreover, the inner product of the vacuum $\psi_{\Sigma_2}^{\text{Vac}}$ with the evoluted vacuum $\mathcal{U}_{21}^{\mathbb{M}} \psi_{\Sigma_1}^{\text{Vac}}$ also has value one:

$$1 \overset{(V5)}{=} \rho_{\mathbb{M}} \circ \tau_{\Sigma_1,\Sigma_2,\partial\mathbb{M}} \big(\psi_{\Sigma_1}^{\text{Vac}} \otimes \psi_{\overline{\Sigma}_2}^{\text{Vac}} \big) \overset{(1.16)}{=} \langle \psi_{\Sigma_2}^{\text{Vac}}, \mathcal{U}_{21}^{\mathbb{M}} \psi_{\Sigma_1}^{\text{Vac}} \rangle_{\Sigma_2} .$$

Since both $\psi_{\Sigma_2}^{\text{Vac}}$ and $\mathcal{U}_{21}^{\mathbb{M}} \psi_{\Sigma_1}^{\text{Vac}}$ are normalized, $\psi_{\Sigma_2}^{\text{Vac}}$ must be equal to $\mathcal{U}_{21}^{\mathbb{M}} \psi_{\Sigma_1}^{\text{Vac}}$. This "conservation" of the vacuum state under unitary evolution generalizes the invariance of the vacuum under time-translations.

1.2.4 Symmetries

The axioms about geometrical data in Section 1.2.1 are formulated using a minimal amount of ingredients and structures: regions are topological manifolds of fixed spacetime dimension $(d+1) \in \mathbb{N}^+$ and hypersurfaces are oriented topological manifolds of dimension d. Any additional structure (such as e.g. a metric structure, complex structure or volume form) is usually called *background*.

There are two ways of relating regions and hypersurfaces to backgrounds: First, regions and hypersurfaces can appear as submanifolds of a global spacetime manifold, which carries the background structure that is then inherited by the submanifolds. This is called a global background in Section 6.1 of [55]. For example, standard QFT assumes a global Minkowski background. Second, we can view regions and hypersurfaces as manifolds in their own right, each coming equipped with its background structure. This is called local background in [55]. Then, boundaries of a region must inherit the background of the region for consistency. Further, the gluing of regions must be done in a way respecting the background.

For our construction of an S-matrix in Anti de Sitter spacetime we work with a global background, determined by the AdS metric. However, for a description of QFT on general curved spacetime the use of local metric backgrounds appears more practical. Local backgrounds represent the principle of locality, implementing that events in a given region of spacetime should not depend on the background of another region, as discussed in Section 1.1.2.

Minkowski and AdS spacetimes are highly symmetric, that is, they have a high number of isometries. Isometries are the particular class of spacetime transformations which is characterized by leaving the metric background invariant. In general, a spacetime transformation is a bijective mapping from the spacetime manifold to itself. (The transformations considered may be either general or such that they leave background strucutres invariant, like the isometries for example.) Hence, spacetime transformations act on regions and hypersurfaces, and it is rather natural to suppose that these geometric transformations induce algebraic transformations on the associated state spaces and amplitude maps.

For global and local backgrounds the classes of spacetime transformations are different. For *global* backgrounds we consider global transformations of the whole spacetime, which then induce transformations of regions and hypersurfaces. For example, in standard QFT we work only with transformations leaving the global Minkowski background invariant. This group of isometries is called Poincaré group. For AdS we also consider only global isometries, and the group of isometries is SO(2,d). For *local* backgrounds we consider transformations of a region or hypersurface viewed as an independent manifold with background, that is, each region or hypersurface a priori is equipped with its own group of transformations. For QFT on general curved spacetime one would use general spacetime transformations (diffeomorphisms instead of isometries, since generically there are none). These diffeomorphisms then are local (transforming regions and hypersurfaces, and not the whole spacetime globally).

Since the transformation properties of state spaces and amplitude maps take a different form for global and local backgrounds, [55] treats them in separate sections. Because in this work we always use the global AdS background (and Minkowski for comparison), we only reproduce the symmetry axioms for global backgrounds.

Symmetries: global backgrounds

Let us denote the background spacetime by B and let K_{B} be a group of global spacetime transformations (the K stands for Killing, since frequently one deals with isometries). We require that this group maps regions to regions, and hypersurfaces to hypersurfaces. Let e the identity of K_{B} and $k \in K_{\mathrm{B}}$ arbitrary. Under the action of k, we denote the image of a hypersurface Σ by $k \triangleright \Sigma$ and the image of a region $\mathbb{M}$ by $k \triangleright \mathbb{M}$. The following global symmetry axioms are postulated in [55] (with minor adjustments to use the τ-maps of [59]). As mentioned therein, they are supposed to cover only the most simple situations and might require later modification.

(SG1) Induced action on state spaces
The action of K_{B} on any hypersurface Σ induces an action on the associated state space $\mathcal{H}_\Sigma$. That is, $k \in K_{\mathrm{B}}$ induces a linear isomorphism (which we also denote by k) between Hilbert spaces

$$\begin{aligned} k: \quad \mathcal{H}_\Sigma &\to \mathcal{H}_{k\triangleright\Sigma} \\ \psi_\Sigma &\mapsto k \triangleright \psi_\Sigma \, . \end{aligned}$$

It has the properties of a generalized action, i.e.:

$$\begin{aligned} k_1 \triangleright (k_2 \triangleright \psi_\Sigma) &= (k_1 k_2) \triangleright \psi_\Sigma && \forall\, k_{1,2} \in K_{\mathrm{B}} \\ e \triangleright \psi_\Sigma &= \psi_\Sigma && \forall\, \psi_\Sigma \in \mathcal{H}_\Sigma \, . \end{aligned}$$

Note that in spite of the suggestive notation this *is not* an action in the usual sense, because here a group element generally maps a state from one space to a state in a *different* state space. Nevertheless we use the word "action" for simplicity.

(SG2) Compatibility with involution map
The action of K_{B} on state spaces $\mathcal{H}_\Sigma$ is compatible with the involution maps ι_Σ. That is, for any $k \in K_{\mathrm{B}}$ and any hypersurface Σ we have

$$\iota_{k\triangleright\Sigma}\,(k \triangleright \psi_\Sigma) \;=\; k \triangleright (\iota_\Sigma \psi_\Sigma) \, .$$

(SG3) Unions of hypersurfaces
The action of K_{B} on state spaces is compatible with the decomposition of hypersurfaces into disconnected components. Suppose $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n$ is such a decomposition. The action of $k \in K_{\mathrm{B}}$ on this union is given by $k \triangleright \Sigma = (k \triangleright \Sigma_1) \cup \ldots \cup (k \triangleright \Sigma_n)$. For both unions we then have the τ-maps of core axiom (T2):

$$\begin{aligned} \tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma}: \quad & \mathcal{H}_{\Sigma_1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_{\Sigma_n} \to \mathcal{H}_\Sigma \\ \tau_{k\triangleright\Sigma_1,\ldots,k\triangleright\Sigma_n;\,k\triangleright\Sigma}: \quad & \mathcal{H}_{k\triangleright\Sigma_1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_{k\triangleright\Sigma_n} \to \mathcal{H}_{k\triangleright\Sigma} \, , \end{aligned}$$

and for any $k \in K_{\mathrm{B}}$ and any $\psi_{\Sigma_k} \in \mathcal{H}_{\Sigma_k}$ with $k = 1, \ldots, n$ we require

$$k \triangleright \tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma}\big(\psi_{\Sigma_1} \otimes \ldots \otimes \psi_{\Sigma_n}\big) \;=\; \tau_{k\triangleright\Sigma_1,\ldots,k\triangleright\Sigma_n;\,k\triangleright\Sigma}\big((k \triangleright \psi_{\Sigma_1}) \otimes \ldots \otimes (k \triangleright \psi_{\Sigma_n})\big) \, ,$$

that is: the following diagram commutes.

$$\begin{array}{ccc} \mathcal{H}_{\Sigma_1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_{\Sigma_n} & \xrightarrow{\;k\;} & \mathcal{H}_{k\triangleright\Sigma_1} \hat{\otimes} \ldots \hat{\otimes} \mathcal{H}_{k\triangleright\Sigma_n} \\ \Big\downarrow \tau_{\Sigma_1,\ldots,\Sigma_n;\,\Sigma} & & \Big\downarrow \tau_{k\triangleright\Sigma_1,\ldots,k\triangleright\Sigma_n;\,k\triangleright\Sigma} \\ \mathcal{H}_\Sigma & \xrightarrow{\;k\;} & \mathcal{H}_{k\triangleright\Sigma} \end{array}$$

(SG4) Compatibility with bilinear pairing and inner product

The action of K_{B} on state spaces is compatible with the inner product, and (T3x) together with (SG2) then imply compatibility with the bilinear pairing. That is, for any hypersurface Σ, any $k \in K_{\mathrm{B}}$ and any $\eta_\Sigma, \zeta_\Sigma \in \mathcal{H}_\Sigma$ with $\eta_{\overline{\Sigma}} := \iota_\Sigma \eta_\Sigma \in \mathcal{H}_{\overline{\Sigma}}$ we require unitarity:

$$\begin{aligned} & \left\langle k \triangleright \eta_\Sigma,\, k \triangleright \zeta_\Sigma \right\rangle_{k \triangleright \Sigma} \;=\; \left\langle \eta_\Sigma,\, \zeta_\Sigma \right\rangle_\Sigma \\ \overset{\text{(SG2)}}{\underset{\text{(T3x)}}{\Longleftrightarrow}} \quad & \left(k \triangleright \eta_\Sigma,\, k \triangleright \zeta_{\overline{\Sigma}} \right)_{k \triangleright \Sigma} \;=\; \left(\eta_\Sigma,\, \zeta_{\overline{\Sigma}} \right)_\Sigma \,. \end{aligned}$$

(SG5) Invariance of amplitudes

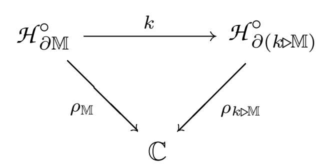
The action of K_{B} on regions leaves the amplitudes invariant. That is, for any region $\mathbb{M}$ the amplitude subspace $\mathcal{H}^\circ_{\partial\mathbb{M}}$ is preserved under the action of any $k \in K_{\mathrm{B}}$ as in

$$k \triangleright \mathcal{H}^\circ_{\partial\mathbb{M}} \;=\; \mathcal{H}^\circ_{\partial(k \triangleright \mathbb{M})} \;,$$

which is a short notation for $(k \triangleright \psi_{\partial\mathbb{M}}) \in \mathcal{H}^\circ_{\partial(k \triangleright \mathbb{M})}$ for all $\psi_{\partial\mathbb{M}} \in \mathcal{H}^\circ_{\partial\mathbb{M}}$. Further, for any vector $\psi_{\partial\mathbb{M}}$ of the amplitude subspace $\mathcal{H}^\circ_{\partial\mathbb{M}}$, and any $k \in K_{\mathrm{B}}$ we require

$$\rho_{k \triangleright \mathbb{M}} \left(k \triangleright \psi_{\partial\mathbb{M}} \right) \;=\; \rho_{\mathbb{M}} \left(\psi_{\partial\mathbb{M}} \right) \,,$$

that is: the following diagram commutes.



(SGV) Invariance of vacuum state

The vacuum state ψ^{Vac} is invariant under K_{B}, i.e., for all hypersurfaces Σ we require

$$k \triangleright \psi^{\mathrm{Vac}}_\Sigma \;=\; \psi^{\mathrm{Vac}}_{k \triangleright \Sigma} \qquad \forall\, k \in K_{\mathrm{B}} \,.$$

1.2.5 Probability interpretation of the GBF

In the previous sections we have introduced the axiomatic framework of the General Boundary Formulation. Now we turn to the question of how to extract measurable probabilities from the GBF. For the Amplitude Formalism of the GBF, this has been worked out in Section 4 of [55] and in [54], on which this section is based.

To ease us into the GBF's probability interpretation, we first review it in the standard formulation. For simplicity we assume that all state spaces are finite dimensional, avoiding difficulties of the infinite dimensional case (introduction of probability densities). We consider a time-interval of Minkowski spacetime $[t_1, t_2] \times \mathbb{R}^3$ which is bounded by two (outwards-oriented) equal-time hyperplanes $\Sigma_{1,2}$ at times $t_{1,2}$. Let $|\,\eta\rangle_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ be the normalized ket-state of a quantum system at time t_1 and ${}_{\overline{\Sigma}_2}\langle\,\zeta\,| \in \mathcal{H}^*_{\overline{\Sigma}_2}$ a normalized bra-state at time t_2. Usually one considers only one state space, since $\mathcal{H}_{\Sigma_1}$ and $\mathcal{H}_{\overline{\Sigma}_2}$ are canonically identified via time-translation invariance. However, here we distinguish them in order to prepare for the corresponding GBF expressions. The associated transition amplitude ρ is given by

$$\rho(\eta, \zeta) \;=\; {}_{\overline{\Sigma}_2}\langle\,\zeta\,|\,\mathcal{U}_{t_2,t_1}\,|\,\eta\rangle_{\Sigma_1} \tag{1.19}$$

where $\mathcal{U}_{t_2,t_1}: \quad \mathcal{H}_{\Sigma_1} \;\to\; \mathcal{H}_{\overline{\Sigma}_2}$ again is the time-evolution operator. The associated probability P then is the modulus square of the transition amplitude: $P(\eta,\zeta) \;=\; |\rho(\eta,\zeta)|^2$. The usual physical interpretation of P is being the probability of finding the normalized state $\zeta_{\overline{\Sigma}_2}$ at time t_2 given that the normalized state η_{Σ_1} has been prepared at time t_1. This means that P is actually a conditional probability. Such a probability usually depends on two types of data: *fixed data* (the condition) describing knowledge or preparation, and *open data* describing outcomes of observations/measurements which fix the answer to an open question. For two events A and B, the conditional probability $P(A|B)$ of finding A given B is defined as

$$P(A|B) := \frac{P(\, A \text{ AND } B\,)}{P(B)} \,.$$

Of course, this is only well defined if $P(B) \neq 0$. Indeed, the conditional probability $P(A|B)$ that A happens given that B happens becomes meaningless if B is impossible.

Example PS.1: In order to empasize that the probabilities are conditional, let us write the probability (1.19) as $P(\zeta_{\overline{\Sigma}_2}|\eta_{\Sigma_1})$ (read: the probability of observing $\zeta_{\overline{\Sigma}_2}$ conditional on the preparation of η_{Σ_1}):

$$P(\zeta_{\overline{\Sigma}_2}|\eta_{\Sigma_1}) \;=\; \big|_{\overline{\Sigma}_2}\langle\, \zeta \,|\, \mathcal{U}_{t_2,t_1} \,|\, \eta \rangle_{\Sigma_1}\big|^2. \tag{1.20}$$

A defining property of probabilities is that the cumulative probability of all exclusive possibilities is 1. Here this is implemented using the inner product which defines orthonormality and thus mutual exclusivity. Let $\{\xi_{\overline{\Sigma}_2,h}\}_{h\in H_2}$ be an orthonormal basis of $\mathcal{H}_{\overline{\Sigma}_2}$, representing a complete set of mutually exclusive measurement outcomes, then for any normalized η_{Σ_1} this implies

$$1 \;=\; \sum_{h\in H_2} P\big(\xi_{\overline{\Sigma}_2,h}\big|\eta_{\Sigma_1}\big) \;=\; \sum_{h\in H_2} \big|_{\overline{\Sigma}_2}\langle\, \xi_h \,|\, \mathcal{U}_{t_2,t_1} \,|\, \eta \rangle_{\Sigma_1}\big|^2 \,. \tag{1.21}$$

Example PS.2: As an extension of example PS.1 suppose now that we know a priori that only some specified measurement outcomes can occur. We could enforce this by selecting a specified subset of performed measurements in order to exclude the other outcomes. A way to formalize this is to say that the possible measurement outcomes form a closed subspace $\mathcal{P}_{\overline{\Sigma}_2}$ of $\mathcal{H}_{\overline{\Sigma}_2}$. We suppose now that the orthonormal basis (ONB) $\{\xi_{\overline{\Sigma}_2,h}\}_{h\in H_2}$ of $\mathcal{H}_{\overline{\Sigma}_2}$ restricts to an ONB $\{\xi_{\overline{\Sigma}_2,p}\}_{p\in P_2 \subseteq H_2}$ of $\mathcal{P}_{\overline{\Sigma}_2}$.

We now consider the probability of the outcome specified by a single state $\xi_{\overline{\Sigma}_2,k}$ with $k\in P_2$. The corresponding probability is conditional both on the prepared state being η_{Σ_1} and knowing that the outcome must lie in $\mathcal{P}_{\overline{\Sigma}_2}$. We denote this conditional probability by $P(\xi_{\overline{\Sigma}_2,k}|\eta_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$. In order to calculate it, we divide the conditional probability $P(\xi_{\overline{\Sigma}_2,k}|\eta_{\Sigma_1})$ by the conditional probability $P(\mathcal{P}_{\overline{\Sigma}_2}|\eta_{\Sigma_1})$ that the measurement's outcome lies in $\mathcal{P}_{\overline{\Sigma}_2}$ given the prepared state is η_{Σ_1}. The latter is simply

$$P\big(\mathcal{P}_{\overline{\Sigma}_2}\big|\eta_{\Sigma_1}\big) \;=\; \sum_{p\in P_2} P\big(\xi_{\overline{\Sigma}_2,p}\big|\eta_{\Sigma_1}\big) \;=\; \sum_{p\in P_2} \big|_{\overline{\Sigma}_2}\langle\, \xi_p \,|\, \mathcal{U}_{t_2,t_1} \,|\, \eta \rangle_{\Sigma_1}\big|^2 \,.$$

$$0 \;<\; P\big(\mathcal{P}_{\overline{\Sigma}_2}\big|\eta_{\Sigma_1}\big) \;\leq\; 1 \quad\Leftrightarrow\quad \varnothing \neq \mathcal{P}_{\overline{\Sigma}_2} \subseteq \mathcal{H}_{\overline{\Sigma}_2}$$

We suppose that this probability is not zero, which would imply the impossibility of obtaining any measurement outcome in $\mathcal{P}_{\overline{\Sigma}_2}$ and thus the meaninglessness of the quantity $P(\xi_k|\eta_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$. Then,

$$P\big(\xi_{\overline{\Sigma}_2,k}\big|\eta_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2}\big) \;=\; \frac{P\big(\xi_{\overline{\Sigma}_2,k}\big|\eta_{\Sigma_1}\big)}{P\big(\mathcal{P}_{\overline{\Sigma}_2}\big|\eta_{\Sigma_1}\big)} \;=\; \frac{\big|_{\overline{\Sigma}_2}\langle\, \xi_k \,|\, \mathcal{U}_{t_2,t_1} \,|\, \eta \rangle_{\Sigma_1}\big|^2}{\sum_{p\in P_2} \big|_{\overline{\Sigma}_2}\langle\, \xi_p \,|\, \mathcal{U}_{t_2,t_1} \,|\, \eta \rangle_{\Sigma_1}\big|^2} \,.$$

Example PS.3: We can further modify example PS.2 by considering as measurement outcome not just a single state, but a closed subspace $\mathcal{M}_{\overline{\Sigma}_2} \subseteq \mathcal{P}_{\overline{\Sigma}_2}$. We denote the corresponding conditional probability by $P(\mathcal{M}_{\overline{\Sigma}_2}|\eta_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$. It is the sum of the conditional probabilities $P(\xi_{\overline{\Sigma}_2,m}|\eta_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$

for an orthonormal basis $\{\xi_{\overline{\Sigma}_2,m}\}_{m\in M_2\subseteq P_2}$ of $\mathcal{M}_{\overline{\Sigma}_2}$ (to which again we suppose the ONB of $\mathcal{P}_{\overline{\Sigma}_2}$ to restrict):

$$P\big(\mathcal{M}_{\overline{\Sigma}_2}\big|\eta_{\Sigma_1},\mathcal{P}_{\overline{\Sigma}_2}\big) \;=\; \frac{\sum\limits_{m\in M_2}\big|\,{}_{\overline{\Sigma}_2}\langle\,\xi_m\,|\,\mathcal{U}_{t_2,t_1}\,|\,\eta\rangle_{\Sigma_1}\big|^2}{\sum\limits_{p\in P_2}\big|\,{}_{\overline{\Sigma}_2}\langle\,\xi_p\,|\,\mathcal{U}_{t_2,t_1}\,|\,\eta\rangle_{\Sigma_1}\big|^2}$$

$$0\;\le\;P\big(\mathcal{M}_{\overline{\Sigma}_2}\big|\eta_{\Sigma_1},\mathcal{P}_{\overline{\Sigma}_2}\big)\;\le\;1\quad\Leftrightarrow\quad\mathcal{M}_{\overline{\Sigma}_2}\subseteq\mathcal{P}_{\overline{\Sigma}_2}\subseteq\mathcal{H}_{\overline{\Sigma}_2}.$$

Example PS.4: A conceptually different extension of example PS.1 is the following: we suppose now that $\{\xi_{\Sigma_1,h}\}_{h\in H_1}$ is an orthonormal basis of $\mathcal{H}_{\Sigma_1}$. Then, for some $k\in H_1$

$$P\big(\xi_{\Sigma_1,k}\big|\zeta_{\overline{\Sigma}_2}\big)\;=\;\big|\,{}_{\overline{\Sigma}_2}\langle\,\zeta\,|\,\mathcal{U}_{t_2,t_1}\,|\,\xi_k\rangle_{\Sigma_1}\big|^2$$

is the conditional probability of the prepared state having been $\xi_{\Sigma_1,k}$ given that $\zeta_{\overline{\Sigma}_2}$ was measured. This somewhat counterintuitive interpretation may be understood as follows. Suppose we have prepared a large sample of measurements with random initial states $\xi_{\Sigma_1,h}$. We then measure whether the final state contains $\zeta_{\overline{\Sigma}_2}$ or not (the latter meaning that it is orthogonal to $\zeta_{\overline{\Sigma}_2}$). The probability distribution of the initial states $\xi_{\Sigma_1,h}$ in the subset of measurements resulting in $\zeta_{\overline{\Sigma}_2}$ is then given by $P(\xi_{\Sigma_1,h}|\zeta_{\overline{\Sigma}_2})$.

These four examples illustrate two points. First, the modulus square of a standard transition amplitude can be interpreted as a conditional probability in different ways. Second, the roles of different parts of the measurement process are not fixed (with respect to which is considered the conditional one and which the depending one). As shown by Example PS.4, the interpretation is not restricted to "final state conditional on initial state".

Probabilities: Amplitude map

In the General Boundary Formulation the dependence of probabilities on preparation data and observation data is preserved. The considerations of the previous section together with the GBF context lead us to the following probability interpretation, which was presented in [55]. The difference between standard and GBF approach is merely in direction: the standard approach usually starts with transition probabilities for single initial and final states, and can be generalized to cover preparation and measurement subspaces. The GBF starts with these subspaces from the outset, and the single states then appear as special cases.

Consider a process taking place in a spacetime region $\mathbb{M}$ with boundary $\partial\mathbb{M}$. Let $\mathcal{H}_{\partial\mathbb{M}}$ its boundary state space describing the given physical system in contact with preparation and measurement machinery. Then, both types of data are encoded through closed subspaces of $\mathcal{H}_{\partial\mathbb{M}}$: we suppose that a certain "prepared" knowledge about the process amounts to the specification of the closed preparation subspace $\mathcal{P}_{\partial\mathbb{M}}\subset\mathcal{H}_{\partial\mathbb{M}}$. We thus assume knowing a priori that the boundary state describing the measurement process lies in that subspace. We aim to find the probability whether the measurement's outcome lies in a closed measurement subspace $\mathcal{M}_{\partial\mathbb{M}}\subseteq\mathcal{P}_{\partial\mathbb{M}}\subset\mathcal{H}_{\partial\mathbb{M}}$. That is, we are interested in the conditional probability $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ of the measurement/observation process being described by the measurement subspace $\mathcal{M}_{\partial\mathbb{M}}$ given that its preparation is described by the preparation subspace $\mathcal{P}_{\partial\mathbb{M}}$. If $\mathcal{M}_{\partial\mathbb{M}}$ has dimension one, being spanned by one normalized state $\xi_{\partial\mathbb{M}}$, we also write $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})\;=\;P(\xi_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$.

We let again $\{\xi_h\}_{h\in H}$ an orthonormal basis of $\mathcal{H}_{\partial\mathbb{M}}$ which reduces to an ONB $\{\xi_p\}_{p\in P\subseteq H}$ of $\mathcal{P}_{\partial\mathbb{M}}$ and further to an ONB $\{\xi_m\}_{m\in M\subseteq P}$ of $\mathcal{M}_{\partial\mathbb{M}}$. Then a first way of expressing $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ is:

$$P\big(\mathcal{M}_{\partial\mathbb{M}}\big|\mathcal{P}_{\partial\mathbb{M}}\big)\;=\;\frac{\sum\limits_{m\in M}\big|\rho_{\mathbb{M}}(\xi_m)\big|^2}{\sum\limits_{p\in P}\big|\rho_{\mathbb{M}}(\xi_p)\big|^2}\,. \tag{1.22}$$

It turns out that in general it is not meaningful to interpret the numerator and denominator in (1.22) separately as probabilities. Further, again we assume that the denominator does not vanish. However, if $\mathcal{P}_{\partial\mathbb{M}}$ is such that this happens, then in (1.22) this would imply a vanishing probability for observing *anything* given the preparation $\mathcal{P}_{\partial\mathbb{M}}$. Thus the conditional probability would be physically meaningless. Moreover, because of $(\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{P}_{\partial\mathbb{M}})$ this implies that the numerator vanishes, too, making $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ undefined. Thus the knowledge encoded in such a $\mathcal{P}_{\partial\mathbb{M}}$ does not correspond to any physically allowed process. We now verify that $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ has all properties of a quantum mechanical probability.

- By construction $(\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{P}_{\partial\mathbb{M}})$ we have probabilities in the unit interval:

$$0 \leq P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) \leq 1 \,.$$

- For two mutually exclusive observations encoded by *orthogonal* subspaces $\mathcal{M}_{\partial\mathbb{M},1}$ and $\mathcal{M}_{\partial\mathbb{M},2}$, we have additive probabilities:

$$P\big(\mathcal{M}_{\partial\mathbb{M},1} \oplus \mathcal{M}_{\partial\mathbb{M},2} \,\big|\, \mathcal{P}_{\partial\mathbb{M}}\big) \;=\; P\big(\mathcal{M}_{\partial\mathbb{M},1}\big|\mathcal{P}_{\partial\mathbb{M}}\big) + P\big(\mathcal{M}_{\partial\mathbb{M},2}\big|\mathcal{P}_{\partial\mathbb{M}}\big)$$

  which can be quickly verified by inserting definition (1.22).

- For any (allowed) preparation $\mathcal{P}_{\partial\mathbb{M}}$ arbitrary (allowed) outcome has probability one, that is: the probability for $\mathcal{M}_{\partial\mathbb{M}} = \mathcal{P}_{\partial\mathbb{M}}$ equals unity.

$$1 \;=\; P\big(\mathcal{P}_{\partial\mathbb{M}}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \frac{\sum\limits_{m\in P}\big|\rho_{\mathbb{M}}(\xi_m)\big|^2}{\sum\limits_{p\in P}\big|\rho_{\mathbb{M}}(\xi_p)\big|^2} \qquad \forall\, \mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$$

- If we have $\mathcal{M}_{\partial\mathbb{M},2}$ implies $\mathcal{M}_{\partial\mathbb{M},1}$ implies $\mathcal{P}_{\partial\mathbb{M}}$, that is $\mathcal{M}_{\partial\mathbb{M},2} \subseteq \mathcal{M}_{\partial\mathbb{M},1} \subseteq \mathcal{P}_{\partial\mathbb{M}}$, then the following probability chain rule holds:

$$P\big(\mathcal{M}_{\partial\mathbb{M},2}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; P\big(\mathcal{M}_{\partial\mathbb{M},2}\big|\mathcal{M}_{\partial\mathbb{M},1}\big) \cdot P\big(\mathcal{M}_{\partial\mathbb{M},1}\big|\mathcal{P}_{\partial\mathbb{M}}\big).$$

  Again, this can quickly be checked by inserting (1.22).

Now let us rewrite Examples PS.1-4 of the standard formulation using the GBF's probability interpretation (1.22). In the standard formulation we always consider a time-interval region whose boundary consists of two (outwards oriented) disjoint components: $\partial\mathbb{M} \,=\, \Sigma_1 \cup \overline{\Sigma}_2$. Hence according to Core Axiom (T2) the state space factors into the tensor product $\mathcal{H}_{\partial\mathbb{M}} \,=\, \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}$. We recall the notation $\zeta_{\overline{\Sigma}_2} \,=\, \iota_{\Sigma_2}\zeta_{\Sigma_2}$.

Example PA.1 For the first example we select a normalized state $\eta_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ and write

$$\mathcal{P}_{\partial\mathbb{M}} \;=\; \text{"}\eta_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}}$$
$$\mathcal{P}_{\partial\mathbb{M}} \;:=\; \big\{\alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,|\, \exists\, \zeta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} \,=\, \eta_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2}\big\} \;\subset\; \mathcal{H}_{\partial\mathbb{M}}$$

Let again $\{\xi_{\overline{\Sigma}_2,h}\}_{h\in H_2}$ an orthonormal basis of $\mathcal{H}_{\overline{\Sigma}_2}$ and hence $\{\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,h}\}_{h\in H_2}$ is an orthonormal basis of $\mathcal{P}_{\partial\mathbb{M}}$. Then, the probability of observing the normalized state $\zeta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2}$, which corresponds to setting

$$\mathcal{M}_{\partial\mathbb{M}} \;=\; \eta_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2} \,,$$

subject to the preparation of $\eta_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ calculates to

$$P\big(\mathcal{M}_{\partial\mathbb{M}}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \frac{\big|\rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2})\big|^2}{\sum\limits_{p\in H_2}\big|\rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p})\big|^2} \;\overset{(1.26)}{=}\; \big|\rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2})\big|^2 \,. \tag{1.23}$$

Comparing the notation to the standard formulation as in (1.13), that is, recognizing

$$\rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2}) \;=\; {}_{\overline{\Sigma}_2}\langle \zeta \,|\, \mathcal{U}_{t_2,t_1} \,|\, \eta \rangle_{\Sigma_1}$$

shows that we recover the standard result $P(\zeta_{\overline{\Sigma}_2}|\eta_{\Sigma_1})$ of equation (1.20), as in (1.1).

Example PA.2 Similarly, the second example is recovered by setting

$$\begin{aligned}
\mathcal{P}_{\partial\mathbb{M}} &= \text{"}\eta_{\Sigma_1} \otimes \mathcal{P}_{\overline{\Sigma}_2}\text{"} \quad\subseteq\quad \text{"}\eta_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{P}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,|\, \exists \zeta_{\overline{\Sigma}_2} \in \mathcal{P}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} \,=\, \eta_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2} \right\} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &:= \eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,k} \qquad {}_{k \in P_2}
\end{aligned}$$

for an orthonormal basis $\{\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p}\}_{p \in P_2}$ of $\mathcal{P}_{\partial\mathbb{M}}$ with $(\varnothing \neq P_2 \subseteq H_2)$. Then we get agreement of the result for $P(\xi_{\overline{\Sigma}_2,k}|\eta_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$ in Example PS.2 with

$$P\big(\mathcal{M}_{\partial\mathbb{M}}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \frac{\big|\rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,k})\big|^2}{\sum\limits_{p \in P_2} \big|\rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p})\big|^2} \,.$$

Example PA.3 For the third example we keep $\mathcal{P}_{\partial\mathbb{M}}$ and its orthonormal basis and assume that it restricts to an ONB $\{\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,m}\}_{m \in M_2}$ of $\mathcal{M}_{\partial\mathbb{M}}$ with $(M_2 \subseteq P_2)$ and

$$\begin{aligned}
\mathcal{M}_{\partial\mathbb{M}} &= \text{"}\eta_{\Sigma_1} \otimes \mathcal{M}_{\overline{\Sigma}_2}\text{"} \quad\subseteq\quad \text{"}\eta_{\Sigma_1} \otimes \mathcal{P}_{\overline{\Sigma}_2}\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,|\, \exists \zeta_{\overline{\Sigma}_2} \in \mathcal{M}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} \,=\, \eta_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2} \right\} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \,.
\end{aligned}$$

Then we recover the $P(\mathcal{M}_{\overline{\Sigma}_2}|\eta_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$ of Example PS.3 via

$$P\big(\mathcal{M}_{\partial\mathbb{M}}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \frac{\sum\limits_{m \in M_2} \Big|\rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,m})\Big|^2}{\sum\limits_{p \in P_2} \Big|\rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p})\Big|^2} \,.$$

Example PA.4 For the fourth example we have

$$\begin{aligned}
\mathcal{P}_{\partial\mathbb{M}} &= \text{"}\mathcal{H}_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2}\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{P}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,|\, \exists \eta_{\Sigma_1} \in \mathcal{H}_{\Sigma_1} : \alpha_{\partial\mathbb{M}} \,=\, \eta_{\Sigma_1} \otimes \zeta_{\overline{\Sigma}_2} \right\} \;\subset\; \mathcal{H}_{\partial\mathbb{M}}
\end{aligned}$$

and denote by $\{\xi_{\Sigma_1,p} \otimes \zeta_{\overline{\Sigma}_2}\}_{p \in P_1}$ an orthonormal basis of $\mathcal{P}_{\partial\mathbb{M}}$. Then, the probability of "observing" (respectively "having observed" since $t_1 < t_2$) $\xi_{\Sigma_1,k} \in \mathcal{H}_{\Sigma_1}$, which corresponds to setting

$$\mathcal{M}_{\partial\mathbb{M}} \;=\; \xi_{\Sigma_1,k} \otimes \zeta_{\overline{\Sigma}_2} \,,$$

subject to the "retrospective preparation" of $\zeta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2}$ turns out as

$$P\big(\mathcal{M}_{\partial\mathbb{M}}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \frac{\big|\rho_{\mathbb{M}}(\xi_{\Sigma_1,k} \otimes \zeta_{\overline{\Sigma}_2})\big|^2}{\sum\limits_{p \in P} \big|\rho_{\mathbb{M}}(\xi_{\Sigma_1,p} \otimes \zeta_{\overline{\Sigma}_2})\big|^2} \;\overset{(1.26)}{=}\; \big|\rho_{\mathbb{M}}(\xi_{\Sigma_1,k} \otimes \zeta_{\overline{\Sigma}_2})\big|^2 \,, \tag{1.24}$$

which recovers the $P(\xi_{\Sigma_1,k}|\zeta_{\overline{\Sigma}_2})$ of Example PS.4. The important point here of course consists in "retrospectively preparing" the experimental setup at time $t_2 > t_1$ and "measuring" the initial state retrospectively. This illustrates the independence of the "preparation vs. observation" interpretation for certain data of the temporal sequence of the events described in the data. Thus preparation in a generalized sense can be expressed as "fixed (input and output) parts" of an experiment and observation as "open (input and output) parts becoming fixed" by this experiment. Here input (output) denotes anything flowing into (out of) the spacetime region in which the experiment is conducted.

Now let us justify why the denominator in (1.23) of Example PA.1 and in (1.24) of Example PA.4 is unity. Since we are in the standard setting of the time-interval region, the amplitude map $\rho_{\mathbb{M}}$ induces a linear, unitary map $\mathcal{U}_{21}^{\mathbb{M}}$ which fulfills

$$\mathcal{U}_{21}^{\mathbb{M}} : \; \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2} \qquad\qquad \rho_{\mathbb{M}}(\eta_1 \otimes \iota_2 \zeta_2) = \langle \zeta_2, \mathcal{U}_{21}^{\mathbb{M}} \eta_1 \rangle_{\Sigma_2} . \tag{1.25}$$

We call it quantum evolution map as above. In [55] this was still a separate axiom called (T4b), while in [59] it is treated as an induced property which holds under certain conditions (essentially that classical evolution in time is unitary). In the standard setting, these conditions are met, at least for Minkowski spacetime. In the standard language this map is just the time-evolution operator. In Section 3.1.5 we review an explicit construction of the map $\mathcal{U}_{21}^{\mathbb{M}}$ using the method of Holomorphic Quantization.

We now recall that in the first example PS.1 we choose η_{Σ_1} to be normalized and $\{\xi_{\overline{\Sigma}_2,p}\}_{p \in H_2}$ is an orthonormal basis of $\mathcal{H}_{\overline{\Sigma}_2}$. This then implies for these examples that

$$1 = \sum_{p \in P} \left| \rho_{\mathbb{M}}(\eta_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p}) \right|^2 = \sum_{p \in P} \left| \langle \xi_{\Sigma_2,p}, \mathcal{U}_{\mathbb{M}} \eta_{\Sigma_1} \rangle_{\Sigma_2} \right|^2 . \tag{1.26}$$

Either $\mathcal{U}_{\mathbb{M}}$ maps η_{Σ_1} directly to one vector of the orthonormal basis $\{\xi_{\Sigma_2,p}\}$ of $\mathcal{H}_{\Sigma_2}$ (which can always be arranged by choosing the ONB adequately) and hence the sum is over Kronecker deltas all vanishing but one. Or the same sum results by summing over all inner products (now each < 1) of $\mathcal{U}_{\mathbb{M}} \eta_{\Sigma_1}$ with the vectors $\{\xi_{\Sigma_2,p}\}$ of the ONB. By similar reasoning, the normalization factor in Example PA.4 equals unity, too (but not the ones PA.2+3, since therein not the whole Hilbert space $\mathcal{H}_{\overline{\Sigma}_2}$ is covered by $\mathcal{P}_{\partial\mathbb{M}}$).

Probabilities: Projectors

In the previous section the GBF probabilities are obtained from the amplitude maps via summing over an ONB basis of the preparation subspace respectively measurement subspace. There, the measurement subspace is required to be contained in the preparation subspace: $\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{P}_{\partial\mathbb{M}}$. Section 3 of [54] presents an equivalent formulation using projection operators. This formulation is then used in [60] to construct observables in the GBF context, see Section 1.2.6. We now consider the same setting as in the previous subsection: preparation is encoded in a closed preparation subspace $\mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$ and measurement outcome in a closed measurement subspace $\mathcal{M}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$. The second formula for the conditional probability $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ of the measurement being described by $\mathcal{M}_{\partial\mathbb{M}}$ given that its preparation is described by $\mathcal{P}_{\partial\mathbb{M}}$ is

$$P\big(\mathcal{M}_{\partial\mathbb{M}} \big| \mathcal{P}_{\partial\mathbb{M}}\big) = \frac{\| \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \|^2}{\| \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \|^2} . \tag{1.27}$$

Therein, $\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}$ and $\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}$ are the *orthogonal* projectors onto the respective subspaces and $\circ$ denotes the composition of maps. Orthogonality of a projector $\hat{\mathrm{P}}$ here refers to the inner product on $\mathcal{H}_{\partial\mathbb{M}}$, meaning that its range $\mathrm{R}(\hat{\mathrm{P}}) = \{\hat{\mathrm{P}}\psi | \psi \in \mathcal{H}_{\partial\mathbb{M}}\}$ and its null space $\mathrm{N}(\hat{\mathrm{P}}) = \{(\mathrm{Id} - \hat{\mathrm{P}})\psi | \psi \in \mathcal{H}_{\partial\mathbb{M}}\}$ are orthogonal subspaces of $\mathcal{H}_{\partial\mathbb{M}}$. A projector is orthogonal (with respect to some inner product) if and only if it is self-adjoint (for that inner product).

Hence in numerator and denominator of (1.27) we take the norm of linear maps $\mathcal{H}_{\partial\mathbb{M}} \to \mathbb{C}$, which thus are elements of the algebraic dual space $\mathcal{H}_{\partial\mathbb{M}}^{\mathrm{A}*}$. (The algebraic dual of a vector space consist of all linear maps on it, whereas the topological dual space of a topological vector space consists only of all *continuous* linear maps on it.) However, as discussed in Section 1.2.2, the amplitude map $\rho_{\mathbb{M}}$ is generically not bounded. Thus $\mathcal{P}_{\partial\mathbb{M}}$ must be "small enough" such that $\rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}$ becomes bounded, thus continuous, and hence an element not only of the algebraic dual $\mathcal{H}^{A*}_{\partial\mathbb{M}}$ of the state space, but also of its topological dual $\mathcal{H}^{T*}_{\partial\mathbb{M}} \subset \mathcal{H}^{A*}_{\partial\mathbb{M}}$. For $\mathcal{H}^{T*}_{\partial\mathbb{M}}$, the norm is defined as follows: let

$$\beta_{\partial\mathbb{M}} \in \mathcal{H}^{T*}_{\partial\mathbb{M}} : \quad \mathcal{H}_{\partial\mathbb{M}} \;\to\; \mathbb{C}$$

be a *bounded* linear map. Then by the Riesz representation theorem there exists ${}^*\beta_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}}$ such that

$$\beta_{\partial\mathbb{M}}(\eta_{\partial\mathbb{M}}) \;=\; \langle {}^*\beta_{\partial\mathbb{M}}, \eta_{\partial\mathbb{M}} \rangle_{\partial\mathbb{M}} \qquad \forall\, \eta_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}}$$

and we define

$$\| \beta_{\partial\mathbb{M}} \|_{\mathcal{H}^{T*}_{\partial\mathbb{M}}} := \| {}^*\beta_{\partial\mathbb{M}} \|_{\mathcal{H}_{\partial\mathbb{M}}} \;. \tag{1.28}$$

As in the previous section, in general it is not meaningful to interpret the numerator and denominator in (1.27) separately as probabilities. Again, we assume the denominator to be nonzero, the corresponding comments in the previous section apply here as well. Together thus the preparation subspace $\mathcal{P}_{\partial\mathbb{M}}$ must neither be "too small" such that $\rho_{\mathbb{M}} \circ \hat{P}_{\mathcal{P}_{\partial\mathbb{M}}}$ becomes zero, nor be "too large" such that $\rho_{\mathbb{M}} \circ \hat{P}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \hat{P}_{\mathcal{M}_{\partial\mathbb{M}}}$ becomes unbounded. Physically this says that $\mathcal{P}_{\partial\mathbb{M}}$ needs to be sufficiently restrictive but not too restrictive (causing an impossibility). This condition is satisfied in standard situations.

In the previous section the measurement subspace $\mathcal{M}_{\partial\mathbb{M}}$ is restricted to be a subspace of the preparation subspace $\mathcal{P}_{\partial\mathbb{M}}$. Here this restriction is not enforced, which represents more a formal than a physical difference. Making $\mathcal{M}_{\partial\mathbb{M}}$ a subspace of $\mathcal{P}_{\partial\mathbb{M}}$ just means that when we ask the question, then we already take into account the preparation knowledge that the answer must fall into a certain subspace. In particular, if $\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{P}_{\partial\mathbb{M}}$, then $\hat{P}_{\mathcal{P}_{\partial\mathbb{M}}}$ and $\hat{P}_{\mathcal{M}_{\partial\mathbb{M}}}$ commute, and then (1.27) can be written as [60]

$$P\big(\mathcal{M}_{\partial\mathbb{M}} \big| \mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \frac{\big\langle \rho_{\mathbb{M}} \circ \hat{P}_{\mathcal{P}_{\partial\mathbb{M}}},\; \rho_{\mathbb{M}} \circ \hat{P}_{\mathcal{M}_{\partial\mathbb{M}}} \big\rangle}{\big\| \rho_{\mathbb{M}} \circ \hat{P}_{\mathcal{P}_{\partial\mathbb{M}}} \big\|^2} \;. \tag{1.29}$$

For the calculations of the examples we mention a few technical properties. Projection operators are idempotent. Moreover, note that for any map $\beta_{\partial\mathbb{M}}$ as above and any orthogonal projector $\hat{P}_{\mathcal{S}_{\partial\mathbb{M}}}$ on a subspace $\mathcal{S}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$, since orthogonal projectors are hermitian, we have by the above definition:

$$\begin{aligned}
&(\beta_{\partial\mathbb{M}} \circ \hat{P}_{\mathcal{S}_{\partial\mathbb{M}}})\eta_{\partial\mathbb{M}} \;=\; \big\langle {}^*(\beta_{\partial\mathbb{M}} \circ \hat{P}_{\mathcal{S}_{\partial\mathbb{M}}}),\; \eta_{\partial\mathbb{M}} \big\rangle_{\partial\mathbb{M}} && \forall\, \eta_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \\
&(\beta_{\partial\mathbb{M}} \circ \hat{P}_{\mathcal{S}_{\partial\mathbb{M}}})\eta_{\partial\mathbb{M}} \;=\; \beta_{\partial\mathbb{M}}\,(\hat{P}_{\mathcal{S}_{\partial\mathbb{M}}}\eta_{\partial\mathbb{M}}) \;=\; \big\langle {}^*\beta_{\partial\mathbb{M}}\,,\; \hat{P}_{\mathcal{S}_{\partial\mathbb{M}}}\eta_{\partial\mathbb{M}} \big\rangle_{\partial\mathbb{M}} \;=\; \big\langle \hat{P}_{\mathcal{S}_{\partial\mathbb{M}}}\, {}^*\beta_{\partial\mathbb{M}},\; \eta_{\partial\mathbb{M}} \big\rangle_{\partial\mathbb{M}} \\
\Rightarrow \quad &{}^*(\beta_{\partial\mathbb{M}} \circ \hat{P}_{\mathcal{S}_{\partial\mathbb{M}}}) \;=\; \hat{P}_{\mathcal{S}_{\partial\mathbb{M}}}\, {}^*\beta_{\partial\mathbb{M}} \qquad \in \mathcal{H}_{\partial\mathbb{M}}.
\end{aligned} \tag{1.30}$$

In the same way one can show that for several projectors $\{\hat{P}_{\mathcal{S}_{\partial\mathbb{M},j}}\}_{j=1,\ldots,k}$ we have

$${}^*(\beta_{\partial\mathbb{M}} \circ \hat{P}_{\mathcal{S}_{\partial\mathbb{M},1}} \ldots \hat{P}_{\mathcal{S}_{\partial\mathbb{M},k}}) \;=\; \hat{P}_{\mathcal{S}_{\partial\mathbb{M},k}} \ldots \hat{P}_{\mathcal{S}_{\partial\mathbb{M},1}}\, {}^*\beta_{\partial\mathbb{M}}. \tag{1.31}$$

We continue by verifying again that $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ has the properties of a quantum mechanical probability.

- By construction we have probabilities in the unit interval:

$$0 \leq P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) \leq 1 \;.$$

- For two mutually exclusive observations encoded by *orthogonal* subspaces $\mathcal{M}_{\partial\mathbb{M},1}$ and $\mathcal{M}_{\partial\mathbb{M},2}$ we have additive probabilities:

$$\begin{gathered}
P\big(\mathcal{M}_{\partial\mathbb{M},1} \oplus \mathcal{M}_{\partial\mathbb{M},2} \,\big|\, \mathcal{P}_{\partial\mathbb{M}}\big) \;=\; P\big(\mathcal{M}_{\partial\mathbb{M},1} \big| \mathcal{P}_{\partial\mathbb{M}}\big) + P\big(\mathcal{M}_{\partial\mathbb{M},2} \big| \mathcal{P}_{\partial\mathbb{M}}\big) \\
\Updownarrow \\
\hat{P}_{\mathcal{M}_{\partial\mathbb{M},1}} \circ \hat{P}_{\mathcal{M}_{\partial\mathbb{M},2}}\, \xi_{\partial\mathbb{M}} \;=\; 0 \;=\; \hat{P}_{\mathcal{M}_{\partial\mathbb{M},2}} \circ \hat{P}_{\mathcal{M}_{\partial\mathbb{M},1}}\, \xi_{\partial\mathbb{M}} \qquad \forall\, \xi_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \;,
\end{gathered}$$

because of

$$P\big(\mathcal{M}_{\partial\mathbb{M},1} \oplus \mathcal{M}_{\partial\mathbb{M},2} \,\big|\, \mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \frac{\|\,\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}\oplus\mathcal{M}_{\partial\mathbb{M},2}}\,\|^2}{\|\,\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\,\|^2}$$

$$= \frac{\|\,\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ(\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}+\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}})\,\|^2}{\|\,\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\,\|^2}$$

$$= \frac{\big\langle {}^*(\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ(\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}+\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}})),\; {}^*(\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ(\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}+\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}}))\big\rangle}{\|\,\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\,\|^2}$$

$$= P\big(\mathcal{M}_{\partial\mathbb{M},1}\big|\mathcal{P}_{\partial\mathbb{M}}\big) + P\big(\mathcal{M}_{\partial\mathbb{M},2}\big|\mathcal{P}_{\partial\mathbb{M}}\big) + 2\,\mathbb{R}\mathrm{e}\;N \,/\, \|\,\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\,\|^2\,,$$

wherein

$$\begin{aligned} N \;&=\; \big\langle {}^*(\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}),\; {}^*(\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}})\big\rangle \\ &\overset{(1.30)}{=} \big\langle \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}{}^*\rho_{\mathbb{M}},\; \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}}{}^*\rho_{\mathbb{M}}\big\rangle \qquad \text{orthogonal projectors are hermitian} \\ &=\; \big\langle \underbrace{\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}{}^*\rho_{\mathbb{M}}}_{=0 \text{ if projectors commute}},\; {}^*\rho_{\mathbb{M}}\big\rangle \;=\; 0. \end{aligned}$$

- Arbitrary outcome has probability one for any (allowed) preparation subspace:

$$1 \;=\; P\big(\mathcal{H}_{\partial\mathbb{M}}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \frac{\|\,\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\circ\overbrace{\hat{\mathrm{P}}_{\mathcal{H}_{\partial\mathbb{M}}}}^{\mathrm{Id}_{\mathcal{H}_{\partial\mathbb{M}}}}\,\|^2}{\|\,\rho_{\mathbb{M}}\circ\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\,\|^2} \qquad \forall\,\mathcal{P}_{\partial\mathbb{M}} \,\subset\, \mathcal{H}_{\partial\mathbb{M}}$$

- If we have $\mathcal{M}_{\partial\mathbb{M},2}$ implies $\mathcal{M}_{\partial\mathbb{M},1}$ implies $\mathcal{P}_{\partial\mathbb{M}}$, that is $\mathcal{M}_{\partial\mathbb{M},2} \subseteq \mathcal{M}_{\partial\mathbb{M},1} \subseteq \mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$, then the following probability chain rule holds (it can be checked by inserting definition (1.27) into the chain rule and applying the premise $\hat{\mathrm{P}}_{\mathcal{P}}\circ\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}} = \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}$ etc.):

$$P\big(\mathcal{M}_{\partial\mathbb{M},2}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; P\big(\mathcal{M}_{\partial\mathbb{M},2}\big|\mathcal{M}_{\partial\mathbb{M},1}\big)\cdot P\big(\mathcal{M}_{\partial\mathbb{M},1}\big|\mathcal{P}_{\partial\mathbb{M}}\big)\,.$$

Now let us see again how the Examples PS.1-4 write using the projector method. This turns out quite similar to Examples PA.1-4, the difference being that now the measurement subspaces need no longer be contained in the preparation subspaces.

Example PP.1 For the first example, select two normalized states $\eta_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ and $\zeta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2}$ and set

$$\begin{aligned} \mathcal{P}_{\partial\mathbb{M}} \;&=\; \text{"}\,\eta_{\Sigma_1}\otimes\mathcal{H}_{\overline{\Sigma}_2}\,\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\ \mathcal{M}_{\partial\mathbb{M}} \;&=\; \text{"}\,\mathcal{H}_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\,\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\ \mathcal{P}_{\partial\mathbb{M}} \;&:=\; \big\{\alpha_{\partial\mathbb{M}}\in\mathcal{H}_{\partial\mathbb{M}} \,|\, \exists\,\beta_{\overline{\Sigma}_2}\in\mathcal{H}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} \;=\; \eta_{\Sigma_1}\otimes\beta_{\overline{\Sigma}_2}\big\} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\ \mathcal{M}_{\partial\mathbb{M}} \;&:=\; \big\{\alpha_{\partial\mathbb{M}}\in\mathcal{H}_{\partial\mathbb{M}} \,|\, \exists\,\beta_{\Sigma_1}\in\mathcal{H}_{\Sigma_1} : \alpha_{\partial\mathbb{M}} \;=\; \beta_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\big\} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \end{aligned}$$

Let us now denote by $\{\xi_{\Sigma_1,j}\otimes\xi_{\overline{\Sigma}_2,k}\}_{j,k=1,\dots,(\dim\mathcal{H}_{\Sigma_{1,2}})}$ an orthonormal basis of $\mathcal{H}_{\partial\mathbb{M}}$ with $\xi_{\overline{\Sigma}_2,k} = \iota_{\Sigma_2}\mathcal{U}_{\mathbb{M}}\xi_{\Sigma_1,k}$, and such that η_{Σ_1} is one of the $\xi_{\Sigma_1,j}$. Then, the probability of observing $\zeta_{\overline{\Sigma}_2}\in\mathcal{H}_{\overline{\Sigma}_2}$ subject to the preparation of $\eta_{\Sigma_1}\in\mathcal{H}_{\Sigma_1}$ turns out as in (1.25):

$$P\big(\mathcal{M}_{\partial\mathbb{M}}\big|\mathcal{P}_{\partial\mathbb{M}}\big) \;=\; \big|\big\langle\zeta_{\Sigma_2},\,\mathcal{U}_{\mathbb{M}}\eta_{\Sigma_2}\big\rangle_{\Sigma_2}\big|^2 \;=\; \big|{}_{\overline{\Sigma}_2}\big\langle\zeta\,\big|\,\mathcal{U}_{t_2,t_1}\,\big|\,\eta\big\rangle_{\Sigma_1}\big|^2\,,$$

which again recovers the standard result $P(\zeta_{\overline{\Sigma}_2}|\eta_{\Sigma_1})$ of equation (1.20). The calculation is done at the end of this section.

Example PP.2 For the second example we set

$$\begin{aligned}
\mathcal{P}_{\partial\mathbb{M}} &= \text{"}\,\eta_{\Sigma_1}\otimes\mathcal{P}_{\overline{\Sigma}_2}\text{"} \quad\subseteq\quad \text{"}\,\eta_{\Sigma_1}\otimes\mathcal{H}_{\overline{\Sigma}_2}\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &= \text{"}\,\mathcal{H}_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\text{"} \qquad \zeta_{\overline{\Sigma}_2}\in\mathcal{P}_{\overline{\Sigma}_2} \\
\mathcal{P}_{\partial\mathbb{M}} &:= \left\{\alpha_{\partial\mathbb{M}}\in\mathcal{H}_{\partial\mathbb{M}}\,|\,\exists\,\beta_{\overline{\Sigma}_2}\in\mathcal{P}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \eta_{\Sigma_1}\otimes\beta_{\overline{\Sigma}_2}\right\} \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &:= \left\{\alpha_{\partial\mathbb{M}}\in\mathcal{H}_{\partial\mathbb{M}}\,|\,\exists\,\beta_{\Sigma_1}\in\mathcal{H}_{\Sigma_1} : \alpha_{\partial\mathbb{M}} = \beta_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\right\} \subset \mathcal{H}_{\partial\mathbb{M}}
\end{aligned}$$

Example PP.3 For the third example we keep $\mathcal{P}_{\partial\mathbb{M}}$ and set

$$\begin{aligned}
\mathcal{M}_{\partial\mathbb{M}} &= \text{"}\,\mathcal{H}_{\Sigma_1}\otimes\mathcal{M}_{\overline{\Sigma}_2}\text{"} \quad\subseteq\quad \text{"}\,\mathcal{H}_{\Sigma_1}\otimes\mathcal{P}_{\overline{\Sigma}_2}\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &:= \left\{\alpha_{\partial\mathbb{M}}\in\mathcal{H}_{\partial\mathbb{M}}\,|\,\exists\,\beta_{\Sigma_1}\in\mathcal{H}_{\Sigma_1},\,\omega_{\overline{\Sigma}_2}\in\mathcal{P}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \beta_{\Sigma_1}\otimes\omega_{\overline{\Sigma}_2}\right\} \subset \mathcal{H}_{\partial\mathbb{M}}\,.
\end{aligned}$$

Example PP.4 For the fourth example we set:

$$\begin{aligned}
\mathcal{P}_{\partial\mathbb{M}} &= \text{"}\,\mathcal{H}_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &= \text{"}\,\eta_{\Sigma_1}\otimes\mathcal{H}_{\overline{\Sigma}_2}\text{"} \;\subset\; \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{P}_{\partial\mathbb{M}} &:= \left\{\alpha_{\partial\mathbb{M}}\in\mathcal{H}_{\Sigma}\,|\,\exists\,\beta_{\Sigma_1}\in\mathcal{H}_{\Sigma_1} : \alpha_{\partial\mathbb{M}} = \beta_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\right\} \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &:= \left\{\alpha_{\partial\mathbb{M}}\in\mathcal{H}_{\partial\mathbb{M}}\,|\,\exists\,\beta_{\overline{\Sigma}_2}\in\mathcal{H}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \eta_{\Sigma_1}\otimes\beta_{\overline{\Sigma}_2}\right\} \subset \mathcal{H}_{\partial\mathbb{M}}.
\end{aligned}$$

This reproduces the correct probabilities also for the case of retrospective measurement as the reader may verify following the steps of the first example. In the definition of probabilities via projectors this is not unexpected, since here with respect to example PP.1 we have only interchanged $\mathcal{P}_{\partial\mathbb{M}}$ with $\mathcal{M}_{\partial\mathbb{M}}$.

For example (PP.1) the calculation goes as follows. The two relevant projectors are:

$$\begin{aligned}
\hat{\mathrm{P}}_{\mathcal{P}_\Sigma} &= \eta_{\Sigma_1}\langle\eta_{\Sigma_1},\,\cdot\,\rangle_{\Sigma_1}\otimes\mathrm{Id}_{\mathcal{H}_{\overline{\Sigma}2}} \\
&= \eta_{\Sigma_1}\langle\eta_{\Sigma_1},\,\cdot\,\rangle_{\Sigma_1}\otimes\sum_{k=1}^{\dim\,\mathcal{H}_{\overline{\Sigma}2}}\xi_{\overline{\Sigma}_2,k}\langle\xi_{\overline{\Sigma}_2,k},\,\cdot\,\rangle_{\Sigma_2} \qquad (1.32) \\
\hat{\mathrm{P}}_{\mathcal{M}_\Sigma} &= \mathrm{Id}_{\mathcal{H}_{\Sigma1}}\otimes\zeta_{\overline{\Sigma}_2}\langle\zeta_{\overline{\Sigma}_2},\,\cdot\,\rangle_{\overline{\Sigma}_2} \\
&= \sum_{k=1}^{\dim\,\mathcal{H}_{\Sigma1}}\xi_{\Sigma_1,k}\langle\xi_{\Sigma_1,k},\,\cdot\,\rangle_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\langle\zeta_{\overline{\Sigma}_2},\,\cdot\,\rangle_{\overline{\Sigma}_2} \qquad (1.33)
\end{aligned}$$

The probability via projectors is then:

$$P(\mathcal{M}_\Sigma|\mathcal{P}_\Sigma) = \frac{\|\,\rho_\mathbb{M}\circ\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\circ\hat{\mathrm{P}}_{\mathcal{M}_\Sigma}\,\|^2_{\mathcal{H}^{\mathrm{T}*}_{\partial\mathbb{M}}}}{\|\,\rho_\mathbb{M}\circ\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\,\|^2_{\mathcal{H}^{\mathrm{T}*}_{\partial\mathbb{M}}}} = \frac{\left\langle {}^*(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\hat{\mathrm{P}}_{\mathcal{M}_\Sigma}),\;\; {}^*(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\hat{\mathrm{P}}_{\mathcal{M}_\Sigma})\right\rangle_{\mathcal{H}_{\partial\mathbb{M}}}}{\left\langle {}^*(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}),\;\; {}^*(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma})\right\rangle_{\mathcal{H}_{\partial\mathbb{M}}}} \qquad (1.34)$$

First let us consider the denominator of (1.34):

$$\begin{aligned}
\left\langle {}^*(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}),\;\; {}^*(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma})\right\rangle_{\mathcal{H}_{\partial\mathbb{M}}} &= \sum_{i,j}\left\langle {}^*(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}),\; \xi^i_{\Sigma_1}\otimes\xi^j_{\overline{\Sigma}_2}\right\rangle_{\mathcal{H}_{\partial\mathbb{M}}}\cdot\left\langle \xi^i_{\Sigma_1}\otimes\xi^j_{\overline{\Sigma}_2},\;\; {}^*(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma})\right\rangle_{\mathcal{H}_{\partial\mathbb{M}}} \\
&= \sum_{i,j}\left|\big(\rho_\mathbb{M}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\big)\big(\xi^i_{\Sigma_1}\otimes\xi^j_{\overline{\Sigma}_2}\big)\right|^2 = \sum_{i,j}\left|\rho_\mathbb{M}\big(\eta_{\Sigma_1}\langle\eta_{\Sigma_1},\,\xi^i_{\Sigma_1}\rangle_{\Sigma_1}\otimes\xi^j_{\overline{\Sigma}_2}\big)\right|^2 \\
&= \sum_j\left|\rho_\mathbb{M}\big(\eta_{\Sigma_1}\otimes\xi^j_{\overline{\Sigma}_2}\big)\right|^2 = 1.
\end{aligned}$$

The last equality is as in (1.26). For the numerator of (1.34) we obtain:

$$\begin{aligned}\Big\langle {}^*(\rho_{\mathbb{M}}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\hat{\mathrm{P}}_{\mathcal{M}_\Sigma}), {}^*(\rho_{\mathbb{M}}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\hat{\mathrm{P}}_{\mathcal{M}_\Sigma})\Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}} &= \sum_{i,j}\Big\langle {}^*(\rho_{\mathbb{M}}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\hat{\mathrm{P}}_{\mathcal{M}_\Sigma}), \xi^i_{\Sigma_1}\otimes\xi^j_{\overline{\Sigma}_2}\Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}}\cdot\Big\langle \xi^i_{\Sigma_1}\otimes\xi^j_{\overline{\Sigma}_2}, {}^*(\rho_{\mathbb{M}}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\hat{\mathrm{P}}_{\mathcal{M}_\Sigma})\Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}}\\
&= \sum_{i,j}\Big|\big(\rho_{\mathbb{M}}\hat{\mathrm{P}}_{\mathcal{P}_\Sigma}\hat{\mathrm{P}}_{\mathcal{M}_\Sigma}\big)(\xi^i_{\Sigma_1}\otimes\xi^j_{\overline{\Sigma}_2})\Big|^2\\
&= \sum_{i,j}\Big|\rho_{\mathbb{M}}\big(\eta_{\Sigma_1}\langle\eta_{\Sigma_1},\xi^i_{\Sigma_1}\rangle_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\langle\zeta_{\overline{\Sigma}_2},\xi^j_{\overline{\Sigma}_2}\rangle_{\overline{\Sigma}_2}\big)\Big|^2\\
&= \big|\rho_{\mathbb{M}}\big(\eta_{\Sigma_1}\otimes\zeta_{\overline{\Sigma}_2}\big)\big|^2 = \big|\langle\zeta_{\overline{\Sigma}_2},\mathcal{U}_{\mathbb{M}}\eta_{\Sigma_1}\rangle_{\Sigma_2}\big|^2.\end{aligned}$$

1.2.6 Observables

While the previous section deals with (generalized) amplitudes, here we consider observables in the GBF context. Operators have been mentioned first in Section 10 of [55] and then have been treated extensively in [60] on which this section is based. In standard quantum theory, self-adjoint operators (on the one and only Hilbert space of quantum states in the standard formulation) are the mathematical object representing observables. The algebra of these operators and commutation relations therein are important structures in quantization. Correspondence principles then link classical Poisson brackets to quantum commutators.

This correspondence is rooted in nonrelativistic Quantum Mechanics, where operators represent measurements that can be applied at any time. Further, the operational meaning of the product of two operators is the temporal composition of the corresponding measurements. In particular, $\hat{A}\hat{B}$ means that we first apply $\hat{B}$ followed by $\hat{A}$. This temporal ordering is crucial, since usually $\hat{A}\hat{B}\neq\hat{B}\hat{A}$, that is: reversing the sequence of measurements can change the result.

In QFT this is different, since operators are localized by a label specifying the spacetime point of their corresponding measurement as in $\hat{\phi}(t,\underline{x})$. Hence the only operationally meaningful way to combine operators is their time-ordered product as in $\mathbf{T}\hat{\phi}(t,\underline{x})\hat{\phi}(t',\underline{x}')$. A key property of this product is that it is commutative. Usually in QFT first a nonabelian algebra of field operators is constructed, starting with equal-time commutation relations. Because the property of having equal time is not invariant under Poincaré transformations, these commutators are then extended to field operators with differing times. Since two measurements with spacelike separation cannot influence each other in relativity, reversing their temporal sequence may not produce a different result. Hence we require for spacelike separations that $\hat{A}(t,\underline{x})\hat{B}(t',\underline{x}') = \hat{B}(t',\underline{x}')\hat{A}(t,\underline{x})$, that is: the commutator vanishes. This condition is Poincaré invariant and ensures that the time-ordered product is independent of the inertial frame we have chosen.

The commutative time-ordered product is usually considered to be derived from the unordered, noncommutative product. However, only the time-ordered product has a direct operational meaning: the amplitudes and S-matrix of QFT are defined using only the time-ordered product. And further, the nonordered product can be recovered from the time-ordered one [57]:

$$[\hat{A}(t,\underline{x}),\,\hat{B}(t,\underline{x}')] = \lim_{\epsilon\to+0}\mathbf{T}\big(\hat{A}(t{+}\epsilon,\underline{x})\hat{B}(t{-}\epsilon,\underline{x}') - \hat{A}(t{-}\epsilon,\underline{x})\hat{B}(t{+}\epsilon,\underline{x})\big).$$

Hence in the special relativistic setting of standard QFT the time-ordered operator product can be considered more fundamental than the unordered product. This is a hint to construct observables using the time-ordered product instead of the unordered. Further, progressing to a General Relativity context the spacetime metric becomes a dynamical object (as e.g. in Einstein's General Relativity). Therefore, we do not dispose of a metric allowing us to formulate the previous condition that commutators must vanish for spacelike separations. Hence setting up a nonabelian algebra of observables becomes even less convincing here.

[60] presents a formulation of quantum observables which takes the above considerations into account and naturally fits into the GBF. Here we only review the version of these axioms given in Section 4.2 of [66]. For the implementation of these axioms in Holomorphic Quantization we

refer to Section 3.1.7. The first axiom (O1) merely states that *for each region* there are observable maps, as does Core Axiom (T4) for the amplitude map. This represents the fact that physical measurements have finite extension in space and time, compared to theoretical measurements at some point in spacetime. The observable maps of a region arise as modifications of its amplitude map. This explains why these maps have very similar properties. However, which of the linear maps $\mathcal{H}^{\circ}_{\partial\mathbb{M}} \to \mathbb{C}$ can actually be considered observables in general depends on which particular quantum theory we consider.

The most important operation for observables is composition, which generalizes the temporal composition in standard Quantum Theory discussed above. This is exactly analoguous to the composition of amplitudes for unions and gluings of regions covered by Core Axioms (T5a+b). The corresponding observable axioms (O2a+b) are already written omitting the τ-maps of these core axioms[4]. By closedness under composition we mean the content of Axioms (O2a+b), that is, that the composition of two observables on their respective regions must be an observable on the composed region, and that the gluing of a region maps observables of the original region to observables of the glued region.

(O1) Observable maps
To each spacetime region $\mathbb{M}$ is associated a real vector space $\mathcal{O}_{\mathbb{M}}$ of linear maps $\mathcal{H}^{\circ}_{\partial\mathbb{M}} \to \mathbb{C}$ on its amplitude subspace. These maps we call observable maps, and the amplitude map is one of them: $\rho_{\mathbb{M}} \in \mathcal{O}_{\partial\mathbb{M}}$.

(O2a) Product rule for observables
As in Core Axiom (T5a), let $\mathbb{M} = \mathbb{M}_1 \cup \mathbb{M}_2$ be the disjoint union of regions $\mathbb{M}_1$ and $\mathbb{M}_2$. Then, there is an injective bilinear map $\diamond : \quad \mathcal{O}_{\mathbb{M}1} \times \mathcal{O}_{\mathbb{M}2} \hookrightarrow \mathcal{O}_{\mathbb{M}}$ such that for any $O_1 \in \mathcal{O}_{\mathbb{M}1}$ and $O_2 \in \mathcal{O}_{\mathbb{M}2}$ and for all $\psi_1 \in \mathcal{H}^{\circ}_{\partial\mathbb{M}1}$ and $\psi_2 \in \mathcal{H}^{\circ}_{\partial\mathbb{M}2}$ the following product rule is satisfied:

$$\big(O_1 \diamond O_2\big)(\psi_1 \otimes \psi_2) \;=\; O_1(\psi_1) \cdot O_2(\psi_2)\,. \tag{1.35}$$

We require the diamond product to be associative in the obvious way. It is defined such that equation (1.11) of Core Axiom (T5a) can be rewritten as $\rho_{\mathbb{M}} = \rho_{\mathbb{M}_1} \diamond \rho_{\mathbb{M}_2}$.

(O2b) Gluing rule for observables:
As in Core Axiom (T5b), let $\mathbb{M}_G$ be a region with its boundary decomposing as a disjoint union $\partial\mathbb{M}_G = \Sigma_N \cup \Sigma_G \cup \overline{\Sigma'_G}$, where Σ'_G is a copy of Σ_G. Let $\mathbb{M}_N$ denote the gluing of $\mathbb{M}_G$ with itself along Σ_G and $\overline{\Sigma'_G}$, and suppose that $\mathbb{M}_N$ is a region. Note $\partial\mathbb{M}_N = \Sigma_N$. Then, associated to the gluing surface Σ_G is a linear map $\diamond_{\Sigma_G} : \quad \mathcal{O}_{\mathbb{M}_G} \to \mathcal{O}_{\mathbb{M}_N}$ such that for any orthonormal basis $\{\xi^i_G\}_{i\in I}$ of $\mathcal{H}_{\Sigma_G}$, for any $O_G \in \mathcal{O}_{\mathbb{M}_G}$ and for all $\psi_N \in \mathcal{H}^{\circ}_{\partial\mathbb{M}_N}$ the observables obey the following gluing rule (wherein $c(\mathbb{M}_G, \Sigma_G, \overline{\Sigma'_G}) \in \mathbb{C}\setminus\{0\}$ is again the gluing anomaly factor of Core Axiom (T5b)):

$$\big(\diamond_{\Sigma_G} O_G\big)(\psi_N) \,\cdot\, c(\mathbb{M}_G, \Sigma_G, \overline{\Sigma'_G}) \;=\; \sum_{i\in I} O_G(\psi_N \otimes \xi^i_G \otimes \iota_{\Sigma_G} \xi^i_G). \tag{1.36}$$

We require this gluing map of observables to commute with itself and with the diamond product of (O2a) in the obvious way. It is defined such that equation (1.12) in Core Axiom (T5b) rewrites as $\rho_{\mathbb{M}_N} = \diamond_{\Sigma_G} \rho_{\mathbb{M}_G}$.

For each observable $O \in \mathcal{O}_{\mathbb{M}}$ of a region $\mathbb{M}$, its expectation value can be computed. As in the standard formulation, this expectation value depends on the preparation of the system, which is represented

[4]Since they are more general, we could use the Observable Axioms in order to reduce the Core Axioms. That is, (O1) would replace (T4), (O2a) would replace (T5a), and (O2b) would replace (T5b). While possible, this would make it somewhat more difficult to understand the Core Axioms from the outset. Further, it is useful to start by establishing an amplitude map first, and consider the implementation of observables only after this. The reason for this is, that there may be different ways for implementing observables. Each of these ways must give the amplitude map in case of the trivial observable. The established amplitude map thus serves as a guideline.

by a closed preparation subspace of the boundary state space $\mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$, see Section 1.2.5. Then, the expectation value of O for the preparation $\mathcal{P}_{\partial\mathbb{M}}$ is defined as[5]

$$\langle O\rangle_{\mathcal{P}_{\partial\mathbb{M}}} := \frac{\langle \rho_{\mathbb{M}}\circ\hat{P}_{\mathcal{P}_{\partial\mathbb{M}}},\, O\rangle}{\|\, \rho_{\mathbb{M}}\circ\hat{P}_{\mathcal{P}_{\partial\mathbb{M}}}\,\|^2}\,. \tag{1.37}$$

This expectation value is linear in the observable. Further, the probabilities of Section 1.2.5 arise for the special case of setting $O = \rho_{\mathbb{M}}\circ\hat{P}_{\mathcal{M}_{\partial\mathbb{M}}}$ (wherein $\mathcal{M}_{\partial\mathbb{M}}$ is the measurement subspace and $\hat{P}$ are the orthogonal projectors). Then, the expectation value (1.37) turns into the probability (1.29).

Recovering the standard observables and expectation values

In the standard formulation, observables are associated to instants of time, that is, equal-time hypersurfaces. We treat these hypersurfaces as "infinitesimally thin" regions, which we call slice regions as in Core Axiom (T3x). Let thus $\Sigma_t = \{t\}\times\mathbb{R}^3$ a backward-oriented equal-time plane in Minkowski spacetime and $\hat{\Sigma}_t$ its associated slice region. Then, by definition $\partial\hat{\Sigma}_t = \Sigma_t \cup \overline{\Sigma_t}$, and thus $\mathcal{H}_{\partial\hat{\Sigma}_t} = \mathcal{H}_{\Sigma_t}\hat{\otimes}\mathcal{H}^{\mathrm{T}*}_{\Sigma_t}$. Let again H the single state space of the standard formulation which we identify with $\mathcal{H}_{\Sigma_t}$, and let $\mathcal{H}^{\circ}_{\partial\hat{\Sigma}_t}$ the amplitude subspace of the slice region. A GBF observable map $O_t \in \mathcal{O}_{\hat{\Sigma}_t} : \mathcal{H}^{\circ}_{\partial\hat{\Sigma}_t} \to \mathbb{C}$ then relates to a standard operator $\hat{O}_t : H \to H$ in a way analogous to the relation (1.23) between the GBF amplitude map and the time-evolution operator $\mathcal{U}_{t_2,t_1}$:

$$\rho_{[t_1,t_2]}(\eta_{\Sigma_{t1}}\otimes\overline{\zeta_{\Sigma_{t2}}}) = \langle \zeta_{\Sigma_{t2}},\, \mathcal{U}_{t_2,t_1}\eta_{\Sigma_{t1}}\rangle_{\Sigma_{t2}} = {}_{t_2}\langle\,\zeta\,|\,\mathcal{U}_{t_2,t_1}\,|\,\eta\rangle_{t_1}\,.$$

We thus require for all $\eta_t, \zeta_t \in \mathcal{H}_{\Sigma_t} = H$ that

$$O_t(\eta_t\otimes\overline{\zeta_t}) = \langle \zeta_t,\, \hat{O}_t\eta_t\rangle_{\Sigma_t} = {}_t\langle\,\zeta\,|\,\hat{O}_t\,|\,\eta\rangle_t\,. \tag{1.38}$$

If we glue together two copies of the region $\hat{\Sigma}_t$, then we obtain again the slice region $\hat{\Sigma}_t$. Then, Axiom (O2b) induces a composition of GBF observable maps, which via (1.38) recovers the composition of the corresponding standard operators. Let thus $\hat{\Sigma}_1$ the "initial" copy of $\hat{\Sigma}_t$ and $\hat{\Sigma}_2$ the "final" copy as in Figure 1.39 (for better clarity we draw the slice regions as extended in time).

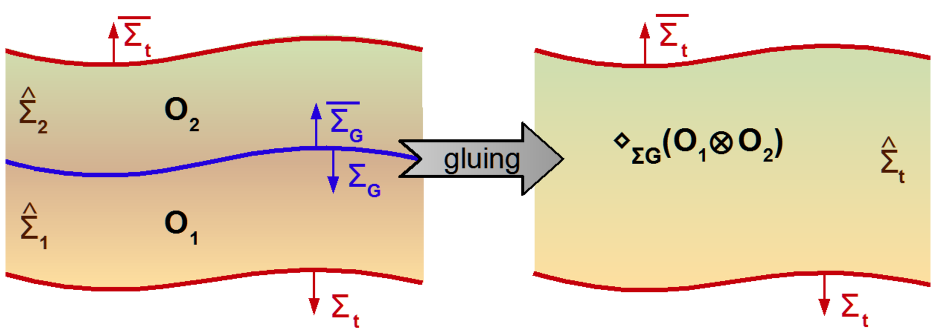


Figure 1.39: Gluing slice regions (here: infinitesimal time-interval).

For convenience let $\partial\hat{\Sigma}_1 = \Sigma_t \cup \overline{\Sigma}_G$ and $\partial\hat{\Sigma}_2 = \Sigma_G \cup \overline{\Sigma}_t$. The gluing is along Σ_G. Using the notation of (O2b), then $\mathbb{M}_G = \hat{\Sigma}_1 \cup \hat{\Sigma}_2$ and $\Sigma_N = \Sigma_t \cup \overline{\Sigma}_t$. The gluing results in $\mathbb{M}_N = \hat{\Sigma}_t$ with our notation chosen such that $\partial\hat{\Sigma}_t = \Sigma_t \cup \overline{\Sigma}_t$ as before. We consider two observables $O_1 \in \mathcal{O}_{\hat{\Sigma}_1}$ and $O_2 \in \mathcal{O}_{\hat{\Sigma}_2}$, thus in the notation of (O2b) we then have $O_G = O_1 \otimes O_2$. Further, the boundary state is here

[5] The interpretation of this formula as expectation value of a measurement only holds strictly, if the observable O is a 1-point function. The same actually occurs in the standard formulation of Quantum Theory. This issue will be treated with due care in the Positive Formalism.

$\psi_N = (\eta_t, \overline{\zeta_t})$. Then, with $\{\xi_j\}_{j\in J}$ an orthonormal basis of $\mathcal{H}_{\Sigma_G}$ and using (1.38), Axiom (O2b) results in

$$\begin{aligned}
\big(\diamond_{\Sigma_G} O_G\big)(\psi_N)\,\cdot\, c(\mathbb{M}_G, \Sigma_G, \overline{\Sigma'_G}) \;&=\; \big(\diamond_{\Sigma_G}(O_1\otimes O_2)\big)(\eta_t, \overline{\zeta_t})\,\cdot\, c(\mathbb{M}_G, \Sigma_G, \overline{\Sigma'_G}) \\
&= \sum_{j\in J}(O_1\otimes O_2)(\eta_t, \overline{\xi_j}, \xi_j, \overline{\zeta_t}) \;=\; \sum_{j\in J} O_1(\eta_t, \overline{\xi_j})\cdot O_2(\xi_j, \overline{\zeta_t}) \\
&= \sum_{j\in J}\langle \xi_j, O_1\eta_t\rangle_{\Sigma_t}\cdot\langle\zeta_t, O_2\xi_j\rangle_{\Sigma_t} \;=\; \sum_{j\in J} {}_t\langle\,\xi_j\,|\,O_1\,|\,\eta\rangle_t\cdot{}_t\langle\,\zeta\,|\,O_2\,|\,\xi_j\rangle_t \\
&= \sum_{j\in J}\overline{{}_t\langle\,\eta\,|\,O_1^\dagger\,|\,\xi_j\rangle_t}\cdot\overline{{}_t\langle\xi_j\,|\,O_2^\dagger\,|\,\zeta\rangle_t} \;=\; \overline{{}_t\langle\,\eta\,|\,O_1^\dagger O_2^\dagger\,|\,\zeta\rangle_t} \\
&= {}_t\langle\,\zeta\,|\,O_2O_1\,|\,\eta\rangle_t\,.
\end{aligned}$$

Thus the gluing of Axiom (O2b) recovers the usual operator product. Having treated operators, let us now turn to expectation values and evaluate (1.37) for the standard setting. A normalized state $\eta_t \in \mathcal{H}_{\Sigma_t} = H$ encodes the preparation of our system. As discussed in Example (PP.1) of Section 1.2.5, the GBF expresses this through the preparation subspace

$$\begin{aligned}
\mathcal{P}_{\partial\hat\Sigma_t} \;&=\; \text{"}\,\eta_t\otimes\mathcal{H}_{\overline{\Sigma}_t}\text{"} \;\subset\; \mathcal{H}_\Sigma \\
\mathcal{P}_{\partial\hat\Sigma_t} \;&:=\; \big\{\alpha_{\partial\hat\Sigma_t}\in\mathcal{H}_{\partial\hat\Sigma_t}\,\big|\,\exists\,\beta_{\overline{t}}\in\mathcal{H}_{\overline{\Sigma}_t} : \alpha_\Sigma \;=\; \eta_t\otimes\beta_{\overline{t}}\big\} \;\subset\; \mathcal{H}_{\partial\hat\Sigma_t}\,.
\end{aligned}$$

Next, Core Axiom (T3x) lets us write the amplitude $\rho_{\hat\Sigma_t}$ as the inner product on $\mathcal{H}_{\Sigma_t} = H$:

$$\rho_{\hat\Sigma_t}(\eta_t\otimes\zeta_{\overline{t}}) \;=\; \langle\zeta_t, \eta_t\rangle_{\Sigma_t} \;=\; {}_t\langle\,\zeta\,|\,\eta\rangle_t.$$

Thus, with $\hat P_\eta$ the orthogonal projector in $\mathcal{H}_{\Sigma_t} = H$ onto the subspace spanned by η_t, we get

$$\big(\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}}\big)(\xi_t\otimes\zeta_{\overline{t}}) \;=\; \langle\zeta_t, \hat P_\eta\xi_t\rangle_{\Sigma_t} \;=\; {}_t\langle\,\zeta\,|\,\hat P_\eta\,|\,\xi\rangle_t\,.$$

With this we can evaluate the denominator of (1.37): let $\{\xi_j\}_{j\in J}$ an orthonormal basis of $\mathcal{H}_{\Sigma_t} = H$, which we choose such that $\xi_1 = \eta_1$. Then, with $\delta_{i,j}$ denoting the Kronecker delta, we can write:

$$\begin{aligned}
\|\,\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}}\,\|^2 \;&=\; \Big\langle\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}},\ \rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}}\Big\rangle_{\mathcal{H}^{\mathrm{T}*}_{\partial\mathbb{M}}} \;=\; \Big\langle\,{}^*(\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}}),\ \ {}^*(\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}})\Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}} \\
&= \sum_{i,j\in J}\Big\langle\,{}^*(\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}}),\ \xi_i\otimes\overline{\xi_j}\Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}}\cdot\Big\langle\xi_i\otimes\overline{\xi_j},\ \ {}^*(\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}})\Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}} \\
&= \sum_{i,j\in J}\Big|\Big\langle\,{}^*(\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}}),\ \xi_i\otimes\overline{\xi_j}\Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}}\Big|^2 \;=\; \sum_{i,j\in J}\big|\rho_{\hat\Sigma_t}\circ\hat P_{\mathcal{P}}(\xi_i\otimes\overline{\xi_j})\big|^2 \\
&= \sum_{i,j\in J}\big|\langle\xi_j, \hat P_\eta\xi_i\rangle_{\Sigma_t}\big|^2 \;=\; \sum_{i,j\in J}\big|\delta_{1,i}\langle\xi_j, \xi_i\rangle_{\Sigma_t}\big|^2 \;=\; \sum_{i,j\in J}\big|\delta_{1,i}\delta_{i,j}\big|^2 \\
&= 1\,.
\end{aligned}$$

Next, we can write the numerator of (1.37) as

$$\begin{aligned}
\Big\langle \rho_{\hat{\Sigma}_t} \circ \hat{P}_{\mathcal{P}},\ O_t \Big\rangle_{\mathcal{H}^{\mathrm{T}*}_{\partial\mathbb{M}}} &= \Big\langle {}^*O_t,\ {}^*(\rho_{\hat{\Sigma}_t} \circ \hat{P}_{\mathcal{P}}) \Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}} = \sum_{i,j\in J} \Big\langle {}^*O_t,\ \xi_i \otimes \overline{\xi_j} \Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}} \cdot \Big\langle \xi_i \otimes \overline{\xi_j},\ {}^*(\rho_{\hat{\Sigma}_t} \circ \hat{P}_{\mathcal{P}}) \Big\rangle_{\mathcal{H}_{\partial\mathbb{M}}} \\
&= \sum_{i,j\,\in\,J} \overline{\big(\rho_{\hat{\Sigma}_t} \circ \hat{P}_{\mathcal{P}}\big)(\xi_i \otimes \overline{\xi_j})} \cdot O_t(\xi_i \otimes \overline{\xi_j}) \\
&= \sum_{i,j\,\in\,J} \langle \hat{P}_\eta \xi_i,\, \xi_j \rangle_{\Sigma_t} \cdot \langle \xi_j,\, \hat{O}_t \xi_i \rangle_{\Sigma_t} = \sum_{i,j\,\in\,J} \delta_{1,i} \langle \xi_i,\, \xi_j \rangle_{\Sigma_t} \cdot \langle \xi_j,\, \hat{O}_t \xi_i \rangle_{\Sigma_t} \\
&= \sum_{j\,\in\,J} \langle \eta_t,\, \xi_j \rangle_{\Sigma_t} \cdot \langle \xi_j,\, \hat{O}_t \eta_t \rangle_{\Sigma_t} = \langle \eta_t,\, \hat{O}_t \eta_t \rangle_{\Sigma_t} \\
&= {}_t\langle\, \eta \,|\, \hat{O}_t \,|\, \eta \rangle_t\,,
\end{aligned}$$

which recovers the standard expectation value.

1.2.7 The GBF and Topological Quantum Field Theory (TQFT)

As mentioned frequently throughout [59], [55], [56] and Oeckl's related articles, the General Boundary Formulation has been strongly influenced by Topological Quantum Field Theory (TQFT). Rather than being some particular kind of Quantum Field Theory, TQFT is an area of Mathematics which applies various QFTs as a tool to study the geometry of low-dimensional manifolds (that is, of spacetime dimension less or equal to 4). In Algebraic Topology, TQFT lead to the discovery of new invariants of knots and 3-manifolds. It is also applied in mathematical physics, however only to toy models without a physical interpretation of the amplitudes. In the following we briefly sketch the TQFT framework in order to show the similarities with the GBF.

An axiomatic framework for TQFTs is given by Atiyah in [5] as follows: A TQFT in dimension d defined over a base ring[6] R consists of the data (A, B) and axioms $(1-3)$. Therein, the map Z determines the TQFT.

(A) To each oriented, closed, smooth d-dimensional manifold Σ is associated a *finitely* generated R-module[7] denoted by $Z(\Sigma)$.

(B) To each oriented, smooth $(d+1)$-dimensional manifold $\mathbb{M}$ with boundary $\partial\mathbb{M}$ is associated an element $Z(\mathbb{M}) \in Z(\partial\mathbb{M})$.

(1) Z is functorial w.r.t. orientation preserving diffeomorphisms of Σ and $\mathbb{M}$.

(2) Z is involutory: $Z(\overline{\Sigma}) = (Z(\Sigma))^*$, wherein $\overline{\Sigma}$ is Σ with reversed orientation and $*$ denotes the dual module.

(3) Z is multiplicative.

---

[6] We recall that in short a ring is a set equipped with addition (commutative, associative, 0-element, inverse for each element) and multiplication (associative, 1-element, distributive w.r.t. addition) defined between its elements. Rings neither require that each element (except zero) must have its inverse under multiplication, nor that the multiplication be commutative. Rings which do fulfill these two extra requirements are called fields. The standard example of a ring is the set of integer numbers, and examples of fields are the sets of real and complex numbers.

[7] Further, if R is a ring, then a (left) module over the base ring R (also written as R-module) in short is an abelian group M (whose group operation we denote by the same symbol $+$ that we use for addition within the ring R) which is equipped with an operation called scalar multiplication: $R \times M \to M$ fulfilling the following conditions for all $r, r_{1,2} \in R$, $m, m_{1,2} \in M$. Distributive: $r(m_1 + m_2) = rm_1 + rm_2$ and $(r_1 + r_2)m = r_1 m + r_2 m$. Group action (compatible multiplications): $r_2(r_1 m) = (r_2 r_1)m$. Identity: $1m = m$ where $1 \in R$.

Vector spaces are defined by the same axioms as modules, the only difference being that the base ring R is required to be a field. Hence for fields F the terms F-module and F-vectorspace coincide, and finitely generated coincides with finite-dimensional.

How do these TQFT axioms relate to those of the GBF? In the General Boundary Formulation the base ring R is the field $\mathbb{C}$ and $Z(\Sigma)$ is the state space $\mathcal{H}_\Sigma$. As the notation suggests, the d-manifolds are the hypersurfaces of the GBF and the $(d+1)$-manifolds are the regions. A crucial difference is that in TQFT this space is required by Axiom (A) to be of finite dimension, while the state spaces of the GBF are infinite-dimensional [56] as needed for realistic QFT. The analogue of the distinguished state $Z(\mathbb{M}) \in Z(\partial\mathbb{M}) = \mathcal{H}_{\partial\mathbb{M}}$ in the GBF is the amplitude map ${}^*\rho_{\mathbb{M}}$.

The functorial Axiom (1) says that any orientation-preserving diffeomorphism $k_1 : \Sigma \to \Sigma_1$ must induce an isomorphism between state spaces $Z(k_1) : Z(\Sigma) \to Z(\Sigma_1)$. Here, Σ_1 is allowed to be Σ itself, . For a second such diffeomorphism $k_2 : \Sigma_1 \to \Sigma_2$ the map Z must respect compositions: $Z(k_2 \circ k_1) = Z(k_2) \circ Z(k_1)$. Further, any orientation-preserving diffeomorphism $K_1 : \mathbb{M} \to \mathbb{M}_1$ with $K(\partial\mathbb{M}) = \partial\mathbb{M}_1$ restricts to a $k_1 : \partial\mathbb{M} \to \partial\mathbb{M}_1$ as above, which must fulfill $Z(k_1) : Z(\mathbb{M}) \to Z(\mathbb{M}_1)$. In the GBF, these properties correspond to the Symmetry Axioms (SG1) and (SG5) of Section 1.2.4.

The involution property of Axiom (2) is contained in the GBF's Core Axiom (T1b) in Section 1.2.2: let $\mathcal{H}_\Sigma$ the state space of Σ. Then we can associate to $\overline{\Sigma}$ the dual of this state space: $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_\Sigma^*$. The multiplicative Axiom (3) requires two properties, called (3a) and (3b) in [5]. The part (3a) says that for disjoint unions $\Sigma = \Sigma_1 \cup \Sigma_2$ we have $Z(\Sigma) = Z(\Sigma_1) \otimes Z(\Sigma_2)$, as expressed by Core Axiom (T2). Second, Axiom (3b) requires a gluing property for the amplitude $Z(\mathbb{M})$, which the GBF incoporates in Core Axiom (T5b). [5] contains further axioms concerning subtle points. We only mention the non-triviality axioms

(4a) $Z(\varnothing) = R$ for empty d-manifolds $\Sigma = \varnothing$.

(4b) $Z(\varnothing) = 1$ for empty $(d+1)$-manifolds $\mathbb{M} = \varnothing$.

The GBF contains (4a) in the form of $\mathcal{H}_\varnothing = \mathbb{C}$, see footnote 2 in Section 2.2 of [59]. Axiom (4b) in the GBF would mean that the amplitude of an empty region is unity. However, the GBF considers regions without boundary (like the empty region) as not admissible. Anyway, for regions without boundary by (4a) we have $\mathcal{H}_\varnothing = \mathbb{C}$, that is, the whole state space consists of multiples of the vacuum state $\psi_\varnothing^{\text{Vac}} = 1$. Hence in that case Vacuum Axiom (V5) would ensure that $\rho_\varnothing = 1$.

Chapter 2

Classical Theory

This chapter serves mainly as a preparation for the quantization process of the next chapter. It is structured as follows. Since classical theories can be considered on different regions in spacetime, in Section 2.1 we define three types of regions which are naturally relevant on Minkowski, AdS, and many other spacetimes. Then we set up in axiomatic form, in which way the GBF expects the classical theory to be formulated, concerning spaces of solutions and structures thereon. In Section 2.4 we give explicit expressions for these structures for three levels of generality: a real, linear field theory (without gauge symmetries) with no metric background assumed, then we specialise to such a theory with a simple Lagrangian (quadratic in fields and derivatives), and finally Klein-Gordon theory on a spacetime with metric. We shall focus in particular on the symplectic structure, and on which (sub)spaces of solutions it vanishes. A second point we emphasize are the actions of the generators of spacetime symmetries on the spaces of solutions, and on their symplectic structures.

In Section 2.5, we review the above structures on the three regions within Minkowski spacetime for real Klein-Gordon theory, which will serve as a guiding reference for our calculations on AdS. We give a concise review of AdS geometry in Section 2.6.2. Next, in Section 2.6.3 we list the bounded Klein-Gordon solutions on three types of regions on AdS: time-interval regions, and tube and rod hypercylinder regions. We study their properties, in particular their radial behaviour. we list solutions and the associated symplectic structure for AdS rod regions in Section 2.6.4, and for time-interval regions in Section 2.6.5. Then we calculate the actions of the AdS isometry group on the solution spaces of these regions in Section 2.6.6. These actions are applied in Section 2.6.7 for showing the invariance of the symplectic structures under the isometries. We complete the classical picture by establishing one-to-one correspondences between initial data on hypersurfaces and bounded solutions for all three AdS regions in Section 2.6.8.

We often refer by e.g. AS[4.2.42] to formulas from the Handbook [1] of Abramowitz and Stegun, and by e.g. DLMF[4.2.42] to its online reincarnation, the Digital Library of Mathematical Functions [49]. Most of our results in this section (except for some results on Klein-Gordon theory in Minkowski spacetime) are published in [30].

2.1 Types of regions

Three types of spacetime regions will be of main interest for us. We shall always orient the boundary of a region such that its normal vector points outwards of the region.

The first type we call interval region. It arises whenever we can foliate our $(d+1)$-dimensional spacetime into leaves Σ_τ with topology $\mathbb{R}^d$, with τ the foliation parameter. Let thus spacetime be foliated as $\mathbb{R}_\tau \times \Sigma_\tau$. An interval $\mathbb{M}_{[\tau_1,\tau_2]}$ of such a spacetime is then given by $[\tau_1, \tau_2] \times \Sigma_\tau$. In general, τ can be any type of coordinate; it is not necessarily a time variable, despite this being the case which is used most frequently. If τ is a time variable[1], then the region is called time-interval region. This is the region used in the standard formulation, as pointed out various times in the introductory chapter. The boundary $\partial\mathbb{M}_{[\tau_1,\tau_2]}$ of an interval region is the disjoint union $\Sigma_{\tau_1}\sqcup\overline{\Sigma}_{\tau_2}$, and thus consists of two simply-connected components. Here, we have canonically oriented Σ_τ to point in negative τ-direction, and the bar denotes reversed orientation, in order to make $\overline{\Sigma}_{\tau_2}$ point outwards. The standard example is given by a time-interval $[t_1,t_2]\times\mathbb{R}^3$ of Minkowski spacetime. Another example in Minkowski spacetime is the spatial interval in x^1-direction $\mathbb{R}_t\times[x^1_1,x^1_2]\times\mathbb{R}^2$ ($\mathbb{R}_t$ is the whole line of time and $\mathbb{R}^2$ is the (x^2,x^3)-plane).

The second type of region is called solid hypercylinder (or rod region for short). It arises whenever we can foliate our $(d{+}1)$-dimensional spacetime into leaves Σ_t with topology $\mathbb{R}^d$, with here the time t being the foliation parameter. Let thus spacetime be $\mathbb{R}_t\times\Sigma_t$. We can then introduce some spherical coordinates on Σ_t with some radial coordinate, say r. A rod region $\mathbb{M}_{r_0}$ of such a spacetime is then given by $\mathbb{R}_t\times[0,r_0]\times\mathbb{S}^{d-1} = \mathbb{R}_t\times\mathbb{B}^{d-1}_{r_0}$ with r_0 being the radius of the rod. An example is directly given by a rod region in Minkowski spacetime using the standard spherical coordinates for $\mathbb{R}^3$. By hypercylinder Σ_r we denote the hypersurface $\mathbb{R}_t\times\mathbb{S}^{d-1}_r$ with orientation in negative r-direction (inwards orientation). The boundary $\partial\mathbb{M}_{r_0}$ of a rod region is then the hypercylinder $\overline{\Sigma}_{r_0} = \mathbb{R}_t\times\mathbb{S}^{d-1}_{r_0}$, that is, with outwards orientation. We note that this boundary consists of only one simply-connected component.

The third type of region is called pierced hypercylinder (or tube region for short). It arises in the same foliated spacetimes as its solid variant and is a larger rod with a concentrical smaller rod cut out of it. A tube region $\mathbb{M}_{[r_1,r_2]}$ is thus given by $\mathbb{R}_t\times[r_1,r_2]\times\mathbb{S}^{d-1}$ with r_1 the inner and r_2 the outer radius of the tube. An example is again given by a tube region in Minkowski spacetime. The boundary of the tube region is the disjoint union $\Sigma_{r_1}\sqcup\overline{\Sigma}_{r_2}$ with the same orientation conventions as above. In many calculations the tube region can be treated like the time-interval region. Therefore, we can also choose the radial coordinate r as our foliation parameter instead of the time t. Our leaves are then the hypercylinders Σ_r filling the whole tube region. These leaves do not have the topology of $\mathbb{R}^d$, and thus the tube is not an interval region. However, for many calculations this topological difference turns out to be irrelevant.

2.2 Classical data

Before considering classical Klein-Gordon theory on Minkowski and AdS spacetimes, we review here the axioms introduced in [59], about the form of a linear classical field theory. They are necessary for applying the Holomorphic Quantization method in the third chapter.

(C1) Classical solutions near hypersurfaces
Associated to each hypersurface Σ is a complex separable Hilbert space L_Σ with complex inner product denoted by $\{\cdot,\cdot\}_\Sigma$. We think of L_Σ as the space of real solutions which are well defined in the neighborhood of Σ. We also define the real g-product $\mathrm{g}_\Sigma(\cdot,\cdot) := \mathbb{R}\mathrm{e}\ \{\cdot,\cdot\}_\Sigma$ and symplectic form $\omega_\Sigma(\cdot,\cdot) := \frac{1}{2}\mathbb{I}\mathrm{m}\ \{\cdot,\cdot\}_\Sigma$. The Hilbert space L_Σ is complex in the sense that it is equipped with a complex structure $J_\Sigma:\ \mathrm{L}_\Sigma\to\mathrm{L}_\Sigma$.

(C2) Orientation reversal
Associated to each hypersurface Σ there is a conjugate-linear involution $\mathrm{L}_\Sigma\to\mathrm{L}_{\overline{\Sigma}}$, under which the inner product is complex conjugated. We will not write this map explicitly, but rather think of L_Σ as identified with $\mathrm{L}_{\overline{\Sigma}}$. Then, $\{\eta,\zeta\}_{\overline{\Sigma}} = \overline{\{\eta,\zeta\}_\Sigma}$ for all $\eta,\zeta\in\mathrm{L}_\Sigma$.

(C3) Unions of hypersurfaces
Suppose the hypersurface Σ decomposes into a disjoint union of hypersurfaces $\Sigma = \Sigma_1\cup\ldots\cup\Sigma_n$. Then, there is an isometric isomorphism of complex Hilbert spaces $\mathrm{L}_{\Sigma_1}\oplus\ldots\oplus\mathrm{L}_{\Sigma_n}\to\mathrm{L}_\Sigma$. Again, we will not write this map explicitly but rather think of it as an identification.

[1] That is, a time variable in the usual sense: τ is a function on spacetime whose gradient vector is everywhere timelike.

(C4) Solutions on regions
Associated to each region $\mathbb{M}$ there is a real vector space $\mathrm{L}_{\mathbb{M}}$, which we think of as space of real solutions on the region $\mathbb{M}$.

(C5) Restriction to boundary neighborhood
Associated to each region $\mathbb{M}$ there is a linear map of real vector spaces $\mathrm{r}_{\mathbb{M}} : \mathrm{L}_{\mathbb{M}} \to \mathrm{L}_{\partial\mathbb{M}}$. The image $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ is a real Hilbert space which is a closed subspace of $\mathrm{L}_{\partial\mathbb{M}}$. (The label $^{\mathrm{int}}$ means interior, since the solutions $\mathrm{L}_{\mathbb{M}}$ are those of $\mathrm{L}_{\partial\mathbb{M}}$ which can be continuously extended to the whole interior of $\mathbb{M}$.) Furthermore $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ is Lagrangian with respect to the symplectic form $\omega_{\partial\mathbb{M}}$. That is, for any $\mu \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ we have $0 = \omega_{\partial\mathbb{M}}(\mu, \lambda)$ for all $\lambda \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ (isotropic), plus if for some $\mu \in \mathrm{L}_{\partial\mathbb{M}}$ we have $0 = \omega_{\partial\mathbb{M}}(\mu, \lambda)$ for all $\lambda \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$, then $\mu \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ (coisotropic).

(C6) Unions of regions
Let $\mathbb{M} = \mathbb{M}_1 \cup \mathbb{M}_2$ the disjoint union of regions $\mathbb{M}_1$ and $\mathbb{M}_2$. Then, $\mathrm{L}_{\mathbb{M}} = \mathrm{L}_{\mathbb{M}_1} \oplus \mathrm{L}_{\mathbb{M}_2}$ and $\mathrm{r}_{\mathbb{M}} = (\mathrm{r}_{\mathbb{M}_1}, \mathrm{r}_{\mathbb{M}_2})$.

(C7) Gluing
As in Core Axiom (T5b), let $\mathbb{M}_G$ be a region with its boundary decomposing as a disjoint union $\partial\mathbb{M}_G = \Sigma_N \cup \Sigma_G \cup \overline{\Sigma'_G}$, where Σ'_G is a copy of Σ_G. Σ_N is allowed to be empty or be some disjoint union of hypersurfaces. Let $\mathbb{M}_N$ denote the gluing of $\mathbb{M}_G$ with itself along Σ_G and $\overline{\Sigma'_G}$, and suppose that $\mathbb{M}_N$ is a region. Note $\partial\mathbb{M}_N = \Sigma_N$, and that any solution on $\mathbb{M}_N$ restricts to a solution on $\mathbb{M}_G$, but not vice versa. Then, there is an injective linear map $\tilde{\mathrm{r}}_{\mathbb{M}_G;\Sigma_G,\overline{\Sigma'_G}} : \mathrm{L}_{\mathbb{M}_N} \hookrightarrow \mathrm{L}_{\mathbb{M}_G}$ such that (we use $\tilde{\mathrm{r}}$ as a shorthand)

$$\mathrm{L}_{\mathbb{M}_N} \xhookrightarrow{\tilde{\mathrm{r}}} \mathrm{L}_{\mathbb{M}_G} \rightrightarrows \mathrm{L}_{\Sigma_G} \tag{2.1}$$

is a (difference) exact sequence, meaning that each of the two arrows on the right hand side yields the same result. The upper one of these arrow represents the composition of the map $\mathrm{r}_{\mathbb{M}_G} : \mathrm{L}_{\mathbb{M}_G} \to \mathrm{L}_{\partial\mathbb{M}_G}$ with the orthogonal projection of $\mathrm{L}_{\partial\mathbb{M}_G}$ to L_{Σ_G}, while the lower arrow represents the composition of $\mathrm{r}_{\mathbb{M}_G}$ with the projection to $\mathrm{L}_{\overline{\Sigma'}_G}$. On the glued new region $\mathbb{M}_N$ the hypersurfaces Σ_G and $\overline{\Sigma}'_G$ are identified by the gluing, and hence solutions in $\mathrm{L}_{\mathbb{M}_N}$ must coincide on these surfaces. Therefore, those solutions on the original region $\mathbb{M}_G$ which stem from solutions on the glued region $\mathbb{M}_N$, must coincide on these two surfaces also when they are not glued.

Moreover, the following diagram commutes, wherein the bottom arrow is the orthogonal projection from $\mathrm{L}_{\partial\mathbb{M}_G} = \mathrm{L}_{\Sigma_N \cup \Sigma_G \cup \overline{\Sigma' G}}$ onto $\mathrm{L}_{\partial\mathbb{M}_N} = \mathrm{L}_{\Sigma_N}$.

$$\begin{array}{ccc} \mathrm{L}_{\mathbb{M}_N} & \xhookrightarrow{\quad\tilde{\mathrm{r}}\quad} & \mathrm{L}_{\mathbb{M}_G} \\ \downarrow{\scriptstyle \mathrm{r}_{\mathbb{M}_N}} & & \downarrow{\scriptstyle \mathrm{r}_{\mathbb{M}_G}} \\ \mathrm{L}_{\partial\mathbb{M}_N} = \mathrm{L}_{\Sigma_N} & \xleftarrow[\text{orth.proj.}]{} & \mathrm{L}_{\partial\mathbb{M}_G} \end{array} \tag{2.2}$$

We note that L_Σ being a Hilbert space includes that the complex inner product $\{\cdot, \cdot\}_\Sigma$ is positive-definite. Being the real part of this complex inner product, the g-product $\mathrm{g}_\Sigma(\cdot, \cdot)$ is a real, positive-definite symmetric bilinear form, which makes L_Σ into a real Hilbert space as well. ω_Σ is a real, antisymmetric, nondegenerate bilinear form making L_Σ into a symplectic vector space. The real and complex inner products g_Σ and $\{\cdot, \cdot\}_\Sigma$ are the structures that we will use in Holomorphic Quantization. Further, under orientation reversal we have

$$\mathrm{g}_{\overline{\Sigma}} = \mathrm{g}_\Sigma \qquad J_{\overline{\Sigma}} = -J_\Sigma \qquad \omega_{\overline{\Sigma}} = -\omega_\Sigma. \tag{2.3}$$

Moreover, for all $\eta, \zeta \in \mathrm{L}_\Sigma$ the following relations hold:

$$\{\eta,\,\zeta\}_\Sigma = \mathrm{g}_\Sigma(\eta,\zeta) + 2\mathrm{i}\,\omega_\Sigma(\eta,\zeta) \qquad \text{from Axiom (C1)} \tag{2.4}$$

$$\{\eta,\,\zeta\}_{\overline{\Sigma}} = \overline{\{\eta,\,\zeta\}_\Sigma} \qquad \text{from Axiom (C2)} \tag{2.5}$$

$$\omega_\Sigma(\eta,\zeta) = \omega_\Sigma(J_\Sigma\eta, J_\Sigma\zeta) \qquad J_\Sigma \text{ compatible with } \omega_\Sigma \tag{2.6}$$

$$\mathrm{g}_\Sigma(\eta,\zeta) = 2\omega_\Sigma(\eta, J_\Sigma\zeta) \tag{2.7}$$

$$\Rightarrow\ \mathrm{g}_\Sigma(\eta,\zeta) = \mathrm{g}_\Sigma(J_\Sigma\eta, J_\Sigma\zeta) \qquad J_\Sigma \text{ compatible with } \mathrm{g}_\Sigma \tag{2.8}$$

In order to determine the complex linearity properties of the complex inner product $\{\cdot,\,\cdot\}_\Sigma$, we recall that L_Σ is a complex Hilbert space due the complex structure J_Σ. Using the above properties, we can quickly check that in this sense the complex inner product is conjugate-linear in the first argument, and linear in the second:

$$\begin{aligned} \big\{(x{+}yJ_\Sigma)\eta,\,\zeta\big\}_\Sigma &= (x{-}\mathrm{i}y)\cdot\big\{\eta,\,\zeta\big\}_\Sigma \qquad &\forall\,\eta,\zeta\in\mathrm{L}_\Sigma \\ \big\{\eta,\,(x{+}yJ_\Sigma)\zeta\big\}_\Sigma &= (x{+}\mathrm{i}y)\cdot\big\{\eta,\,\zeta\big\}_\Sigma \qquad &\forall\,x,y\in\mathbb{R}. \end{aligned} \tag{2.9}$$

The following special cases then show that J_Σ indeed represents the multiplication with i in L_Σ:

$$\begin{aligned} \big\{J_\Sigma\eta,\,\zeta\big\}_\Sigma &= -\mathrm{i}\cdot\big\{\eta,\,\zeta\big\}_\Sigma \qquad &\forall\,\eta,\zeta\in\mathrm{L}_\Sigma \\ \big\{\eta,\,J_\Sigma\zeta\big\}_\Sigma &= +\mathrm{i}\cdot\big\{\eta,\,\zeta\big\}_\Sigma \qquad &\forall\,x,y\in\mathbb{R}. \end{aligned} \tag{2.10}$$

Lemma 2.11 For any region $\mathbb{M}$, the space $\mathrm{L}_{\partial\mathbb{M}}$ of solutions near the boundary (understood as a real Hilbert space) decomposes into the orthogonal direct sum $\mathrm{L}_{\partial\mathbb{M}} = \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}} \oplus J_{\partial\mathbb{M}}\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$. This is Lemma 4.1 from [59]. It allows us to view $\mathrm{L}_{\partial\mathbb{M}}$ as the complexification $(\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}})^{\mathbb{C}}$ of the original Hilbert space $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}} = \mathrm{r}_\mathbb{M}\mathrm{L}_\mathbb{M}$. In Section 2.3 we show how this decompostion looks in detail for time-interval (and tube) regions and for rod regions, and in Section 3.1 we describe how it is applied in the method of Holomorphic Quantization.

The symplectic structure ω_Σ is directly induced by the Lagrangian of the classical theory, and hence is a purely classical structure. By contrast, the complex structure J_Σ is not fixed by the classical theory. Rather, any choice of complex structure J_Σ corresponds one-to-one to a choice of vacuum state $\psi_\Sigma^{\mathrm{Vac}}$ as proven in Section of 3 [61]. Thus, the complex structure is of quantum nature. Nevertheless we introduce it already in this classical chapter, since it determines the structure of spaces of classical solutions and is necessary to relate the forms ω_Σ, g_Σ and $\{\cdot,\,\cdot\}_\Sigma$.

For general hypersurfaces Σ, there is no classical criterion that distinguishes any particular choice of complex structure J_Σ. However, in [4] it is shown how the classical energy of the field determines the quantum object J_Σ, which yields a unique complex structure, but only for Cauchy hypersurfaces Σ. For stationary spacetimes with timelike Killing vector field t this unique complex structure J is singled out by an energy condition (together with requiring g_Σ to be positive-definite). In short, this conditions says that the classical energy

$$E = \int_\Sigma \mathrm{d}S^a\ t^b\, T_{ab}(\phi)$$

must be reproduced by the inner product ($\mathcal{L}$ denotes the Lie derivative)

$$\{\phi,\,-J\mathcal{L}_t\phi\}_\Sigma\,.$$

This inner product actually represents the inner product of the one-particle quantum state associated to ϕ, wherein the operator $-J\mathcal{L}_t$ is the Hamiltonian. Hence the energy condition demands that the energy of the quantum state associated to ϕ be equal to the classical energy of ϕ. The quantum energy depends on the choice of J, while the classical energy is independent of it. This fixes J. The resulting complex structure is then

$$J = -\mathcal{L}_t/\sqrt{-\mathcal{L}_t^2}\,, \tag{2.12}$$

wherein $-\mathcal{L}_t^2$ is a positive, hermitian operator.

Units

Before studying the symplectic structure ω, the real g-product, and the complex inner product $\{\cdot,\cdot\}$ in more detail, let us comment on their units. In general, throughout this thesis we use natural units $c = \hbar = 1$ for simplicity of notation. This enables us to treat all three structures as mapping two classical solutions η, ζ to a complex number. However, since these structures and exponentials of them will be used later in quantization, it is useful to know their dimensions in SI units. This will make it clear, where Planck's quantum $\hbar$ appears in the quantization process.

First we need to find the SI units of the scalar field. They depend on the dimension of spacetime, which is $(d+1)$ in our notation. We consider a Klein-Gordon field, whose Lagrangian density in SI units writes as $\mathcal{L}(x) = \frac{1}{2}\sqrt{|g|}\,\big(g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - (\frac{mc}{\hbar})^2\phi^2\big)$. The action $S(\phi) = \int \mathrm{d}^{d+1}x\,\mathcal{L}(x)$ has the same dimension as $\hbar$, that is: $\mathrm{kg\,m^2/s}$. With the spacetime metric g dimensionless and all x^μ of dimension length, the Klein-Gordon field ϕ then has SI units $(\mathrm{m}^{3-d}\mathrm{kg/s})^{1/2}$.

The symplectic structure writes essentially as (see Section 2.4.2) $\omega(\eta,\zeta) = \int \mathrm{d}^d x\,(\eta\partial_x\zeta - \zeta\partial_x\eta)$. Using the SI units of the classical solutions, the symplectic structure has the SI units of an action, that is: $\mathrm{kg\,m^2/s}$. Since the complex structure J maps solutions to solutions, it is dimensionless in any system of units. Therefore, the real g-product $\mathrm{g}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ and the complex inner product $\{\cdot,\cdot\} = \mathrm{g}(\cdot,\cdot) + 2\mathrm{i}\omega(\cdot,\cdot)$ in SI units have the dimension of an action as well.

Therefore, when using natural units we have to keep in mind that the three structures implicitly appear with a factor of $\hbar^{-1}$ attached. Sometimes this factor is written explicitly, e.g. in Equation (14) of [25].

2.3 Classical solutions near boundaries

In the previous section we encountered on one hand classical solutions (which are well-defined and bounded) on whole regions of spacetime, and on the other hand classical solutions (which are well-defined and bounded only) near the boundaries of these regions. While it is intuitively clear that considering solutions on regions makes sense, this is less the case for considering solutions only near boundaries. Therefore, in this section we argue why it is both natural and relevant to include the latter. Often, the solutions near boundaries which we consider are solutions of the *free* classical theory. Therefore, we shall consider two simple examples of free solutions near boundaries.

Time-interval regions

The first example occurs for interval regions (and also for tube regions), see Section 2.1. We thus let $\mathbb{M}_{[t_1,t_2]} = [t_1,t_2]\times\mathbb{R}^d$ a time-interval region for simplicity. Its boundary consists of the two disjoint components Σ_1 and $\overline{\Sigma}_2$ (the bar denotes orientation reversal, since we orient boundaries to point outwards of the enclosed region), see Figure 2.13.

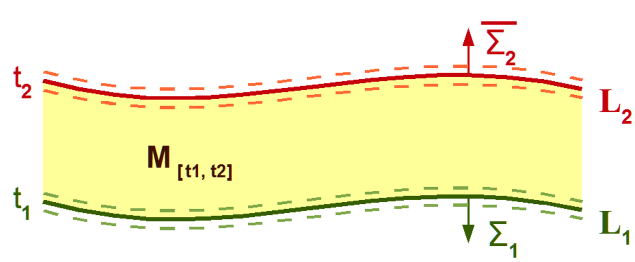


Figure 2.13: Time-interval region and neighborhood of its boundary.

A free solution ξ_{12} near the boundary here consists of two components: $\xi_{12} = (\eta_1, \zeta_2)$, wherein $\eta_1 \in \mathrm{L}_1$ is a free classical solution near Σ_1, and $\zeta_2 \in \mathrm{L}_2$ near $\overline{\Sigma}_2$, These two free solutions are independent of each other: if we extend η_1 continuously towards $\overline{\Sigma}_2$, then generically it will be different from ζ_2, as sketched in Figure 2.14.

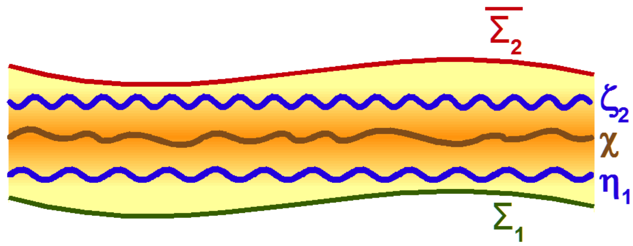


Figure 2.14: Free solutions η_1, ζ_2 and intermediate solution χ.

This kind of boundary solutions occurs naturally, if we have an interaction term or potential term in our classical Lagrangian, which only contributes for some intermediate times (darker orange part of region) while it vanishes for times near t_1 and t_2 (pale yellow part of region). Then, we start at t_1 with some initial free solution η_1, which evolutes classically in time. However, due to the interaction, at intermediate times this evolution does not yield free solutions, but other solutions χ (for example, solutions of the classical equations of motion with some source term). Only after the interaction vanishes again, the evolution returns a free solution ζ_2. However, this final free solution generically will be different from the initial one. That is, restricting to a neighborhood of the boundary, we there have two different free solutions, which is precisely the situation described above. This is the classical analogue of quantized theories, where quantum states are considered instead of classical solutions.

Let us now consider the decomposition of Lemma 2.11 for the time-interval region. (This holds as well for a tube region $\mathbb{M}_{[r_1,r_2]} = \mathbb{R}_t \times [r_1, r_2] \times \mathbb{S}^{d-1}$, we just need to interchange time t and radius r.) J_1 is the complex structure associated to Σ_1 with backwards orientation (in negative t-direction). We assume here that the complex structure is independent of the foliation parameter t. Then, since both boundary components have opposite orientation with respect to t, we have $J_2 = -J_1$. A solution $\xi_{12} \in \mathrm{L}_{\partial\mathbb{M}_{[t_1,t_2]}}$ near the boundary consists of two components here, such that the decomposition now writes as

$$\xi_{12} := \begin{pmatrix}\eta_1\\ \zeta_2\end{pmatrix} = \xi_{12}^{\mathrm{R}} + J_{\partial\mathbb{M}_{[t_1,t_2]}}\xi_{12}^{\mathrm{I}} = \begin{pmatrix}\xi_{12}^{\mathrm{R}}\\ \xi_{12}^{\mathrm{R}}\end{pmatrix} + \begin{pmatrix}J_1\xi_{12}^{\mathrm{I}}\\ J_2\xi_{12}^{\mathrm{I}}\end{pmatrix} = \begin{pmatrix}\xi_{12}^{\mathrm{R}}\\ \xi_{12}^{\mathrm{R}}\end{pmatrix} + \begin{pmatrix}J_1\xi_{12}^{\mathrm{I}}\\ -J_1\xi_{12}^{\mathrm{I}}\end{pmatrix}, \tag{2.15}$$

wherein $\xi_{12}^{\mathrm{R}}, \xi_{12}^{\mathrm{I}} \in \mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}$, whereas $J_{\partial\mathbb{M}_{[t_1,t_2]}}\xi_{12}^{\mathrm{I}} \notin \mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}$. This is solved by

$$\xi_{12}^{\mathrm{R}} = \tfrac{1}{2}(\eta_1 + \zeta_2) \qquad J_1\xi_{12}^{\mathrm{I}} = \tfrac{1}{2}(\eta_1 - \zeta_2) \qquad \xi_{12}^{\mathrm{I}} = \tfrac{1}{2}(-J_1\eta_1 + J_1\zeta_2). \tag{2.16}$$

Let us remark that the pair $(\xi_{12}^{\mathrm{R}}, \xi_{12}^{\mathrm{R}})$, with the first part seen as solution near Σ_1 and the second near Σ_2, is a solution near $\partial\mathbb{M}_{[t_1,t_2]}$ that extends continuously to the whole region $\mathbb{M}_{[t_1,t_2]}$, since both solutions coincide. By contrast, $J_{\partial\mathbb{M}_{[t_1,t_2]}}\xi_{12}^{\mathrm{I}} = (J_1\xi_{12}^{\mathrm{I}}, -J_1\xi_{12}^{\mathrm{I}})$ cannot be extended continuously to the whole time-interval region, because the first solution is the negative of the second. For this reason ξ_{12}^{R} is called a classically allowed solution near $\partial\mathbb{M}$, while $J_{\partial\mathbb{M}_{[t_1,t_2]}}\xi_{12}^{\mathrm{I}}$ is called classically forbidden in Section 4.3 of [59].

Rod regions

The second example occurs for rod regions $\mathbb{M}_0 = [0, r_0] \times \mathbb{B}_{r_0}^3$, whose boundary consists of one single component $\overline{\Sigma}_0$. Therefore, a boundary solution ξ_0 here consists of one component only, see Figure 2.17 (wherein the dark dashed line indicates that the rod region actually extends over all of time).

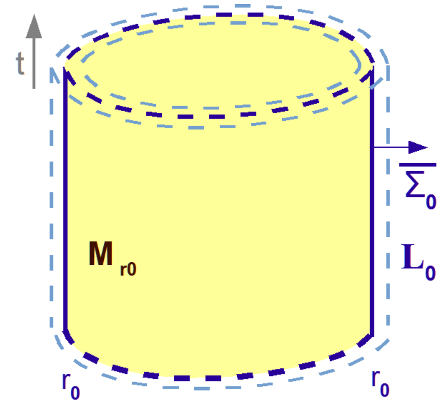


Figure 2.17: Rod region and neighborhood of its boundary.

However, typically the boundary solution is a sum of two types of modes (see Section 2.5.1 and Section 2.6.3): the first type of modes is regular in the whole ball of radius r_0, while the second type diverges at some points, for example at the origin $r = 0$. Then, the content of regular modes of ξ_0 is a boundary solution that can be extended continuously over the whole rod region, whereas the content of divergent modes is a boundary solution that cannot be extended in this way.

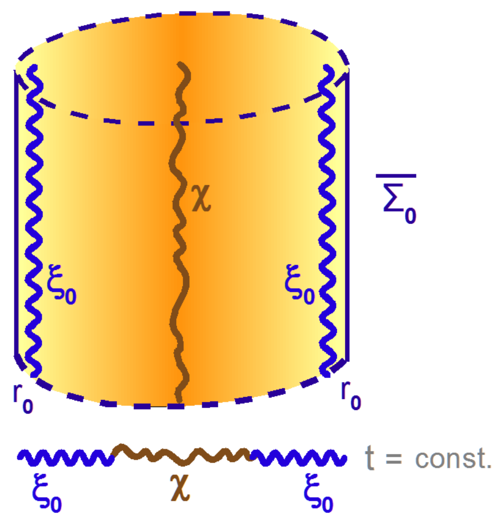


Figure 2.18: Rod region and neighborhood of its boundary.

As sketched in Figure 2.18, in analogy to the interval regions, boundary solutions like ξ_0 occur naturally, if we have an interaction or source that has some compact support inside $\mathbb{B}^3_{r_0}$ (darker orange part) and vanishes near the boundary (pale yellow part). (That is, for time-intervals we need the interaction to have compact support in *time*, whereas for rod regions we need it to have compact support in *space*.) Then, the solution χ on the interaction's support is not free, whereas the solution ξ_0 near the boundary is free. On the boundary of the interaction's support, the interacting solution transits smoothly into the free solution (sketched below the rod region for some constant time). However, this works only if the free solution consists of both regular and divergent modes (we can think of the regular modes as matching the values of the solutions, and of the divergent modes as matching the derivatives). Therefore we need to include the divergent modes when considering solutions near the rod's boundary.

The decomposition of Lemma 2.11 looks different for a rod region (compared to the time-interval region). We recall that (like for J_2 above), due to the inwards oriented hypercylinder Σ_0 and the outwards oriented boundary $\partial\mathbb{M}_0$, we here have $J_{\partial\mathbb{M}_0} = -J_0$. Thus the decomposition for a solution ξ_0 near the only boundary component Σ_0 of a rod region becomes

$$\xi_0 \;=\; \xi_0^{\mathrm{R}} - J_0\xi_0^{\mathrm{I}}, \tag{2.19}$$

wherein again $\xi_0^{\mathrm{R}}, \xi_0^{\mathrm{I}} \in \mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}$, whereas $J_0\xi_0^{\mathrm{I}} \notin \mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}$. That is, ξ_0^{R} is the classically allowed part of ξ_0 that can be extended continuously into the whole interior of $\mathbb{M}$, while $J_0\xi_0^{\mathrm{I}}$ is the classically forbidden part that cannot be extended in this way. For example, ξ_0^{R} can be a sum of modes that are regular on the whole region (like the Bessel modes of Klein-Gordon theory on Minkowski spacetime, see Section 2.5.1), while $J_0\xi_0^{\mathrm{I}}$ can be a sum of modes that are not regular on all of $\mathbb{M}$ (like the Neumann modes, which become singular on the Minkowski time axis where $r = 0$). Applying J_0 then turns the regionally well defined parts of solutions into parts that are not regionally well defined, and vice versa (for example, interchanges Bessel modes and Neumann modes). Without using information about the modes (which ones are regular on the rod's interior and which are not), it is not possible here to give an explicit construction of ξ_0^{R} and ξ_0^{I} (as in (2.16) for the time-interval region). However, once the modes are known, the principle of construction arises rather naturally through interchanging regionally regular and nonregular modes.

2.4 Structures on spaces of classical solutions

In this section we consider various structures on the space L_Σ of classical solutions in a neighborhood of a hypersurface Σ. The highest level of generality used here will be the one presented in Section 3.1 of [59], where no spacetime metric g is needed, and a real, linear field theory is considered. The linear field theory of most interest for us is real Klein-Gordon theory (which does need a metric), and therefore we also include the less general forms the equations take on for that particular example. By $\mathrm{L}_\mathbb{M}$ we denote the space of solutions on a region $\mathbb{M}$ of $(d{+}1)$-dimensional spacetime, and by $\mathrm{L}_{\partial\mathbb{M}}$ the space of solutions near its boundary, which is a hypersurface. (We shall write labels like $\mathbb{M}$ as subscripts or superscripts according to convenience, without changing the meaning.)

In particular we shall study the action $S_\mathbb{M}$, the symplectic potential θ_Σ, the symplectic structure ω_Σ, the complex structure J_Σ, a real inner product g_Σ and a complex inner product $\{\,\cdot\,,\,\cdot\,\}_\Sigma$. We shall call g_Σ real g-product and denote it by an upright letter g, in order to distinguish it from the spacetime metric g, even though the meaning should always be clear from the context. The complexified version of the space of real solutions on $\mathbb{M}$ will be denoted by $\mathrm{L}_\mathbb{M}^\mathbb{C}$. Since $\mathrm{L}_\mathbb{M}$ is a subspace of $\mathrm{L}_\mathbb{M}^\mathbb{C}$, naturally all relations valid for the latter hold for the first, too. We emphasize that we are *not* studying a complex linear theory here, but merely consider the various structures on the complexified space of real solutions. Further, for linear theories we can identify the solution spaces with their tangent spaces.

As discussed above in Section 2.2, the symplectic structure ω_Σ is a purely classical object, while the complex structure J_Σ relates to the quantum theory. Hence the real and complex inner products g_Σ and $\{\cdot,\cdot\}_\Sigma$ induced by the complex structure must be considered quantum objects as well. We choose to already introduce them in this classical chapter on a simple level, in order to give an overview of all related structures together. The symplectic structure ω_Σ is then considered in detail in this classical chapter, while the complex structure J_Σ and the other quantum objects are studied in greater detail in the following quantum chapter.

2.4.1 Action S

The action is a function(al) (that is, a 0-form) on the space $\mathrm{K}_\mathbb{M}$ of field configurations on the region $\mathbb{M}$ (which contains as a subset the space $\mathrm{L}_\mathbb{M}$ of classical solutions):

$$S_\mathbb{M}:\;\; \mathrm{K}_\mathbb{M} \;\to\; \mathbb{R} \qquad\qquad S_\mathbb{M}:\;\; \mathrm{K}_\mathbb{M}^\mathbb{C} \;\to\; \mathbb{C} \qquad\qquad \phi \;\mapsto\; S_\mathbb{M}(\phi)\;.$$

The action is determined by the Lagrangian density form Λ, a $(d{+}1)$-form on $\mathbb{M}$ determined in turn by the Lagrange density $\mathcal{L}$:

$$\Lambda \;=\; \Lambda(\varphi,\partial\varphi,x) \;=\; \mathcal{L}(\varphi,\partial\varphi,x)\,\mathrm{d}x^0\wedge\ldots\wedge\mathrm{d}x^d\;,$$

via integrating it over the spacetime region $\mathbb{M}$

$$S_{\mathbb{M}}(\phi) \;=\; \int\limits_{x\in\mathbb{M}}\!\Lambda(\phi(x),\partial\phi(x),x) \;=\; \int\limits_{\mathbb{M}}\mathrm{d}x^0\ldots\mathrm{d}x^d\;\mathcal{L}(\phi(x),\partial\phi(x),x)\;.$$

For generality, we denote here by $\phi = (\phi^1,\ldots,\phi^n)$ an n-tuple of fields, hence $\mathrm{K}_{\mathbb{M}}$ is actually the space of n-tuples of fields on $\mathbb{M}$. Volume weight factors (like $\sqrt{|g|}$ for metric manifolds) are contained in the Lagrange density $\mathcal{L}$ in our notation, and we assume the coordinate system chosen such that $\mathrm{d}x^0\wedge\ldots\wedge\mathrm{d}x^d$ corresponds to positive orientation. The differential of the action is a 1-form on the tangent bundle of $\mathrm{K}_{\mathbb{M}}$:

$$\begin{aligned}
\mathrm{d}S_{\mathbb{M}}:\;\; \mathrm{TK}_{\mathbb{M}} &\;\to\; \mathbb{R} &\qquad \mathrm{d}S^{\mathbb{M}}_{\phi}:\;\; (\mathrm{TK}_{\mathbb{M}})_{\phi} &\;\to\; \mathbb{R}\\
\mathrm{d}S_{\mathbb{M}}:\;\; \mathrm{TK}^{\mathbb{C}}_{\mathbb{M}} &\;\to\; \mathbb{C} &\qquad \mathrm{d}S^{\mathbb{M}}_{\phi}:\;\; (\mathrm{TK}^{\mathbb{C}}_{\mathbb{M}})_{\phi} &\;\to\; \mathbb{C}\\
(\phi,\eta) &\;\mapsto\; \mathrm{d}S^{\mathbb{M}}_{\phi}(\eta) & \eta &\;\mapsto\; \mathrm{d}S^{\mathbb{M}}_{\phi}(\eta)\,.
\end{aligned}$$

Here, η is an element of $(\mathrm{TK}_{\mathbb{M}})_{\phi}$, the tangent space of $\mathrm{K}_{\mathbb{M}}$ at ϕ. The differential of the action is defined in the usual way as

$$\begin{aligned}
\mathrm{d}S^{\mathbb{M}}_{\phi}(\eta) \;=\; \eta\, S^{\mathbb{M}}(\phi) \;&=\; \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} S^{\mathbb{M}}(\phi+\lambda\eta) \;=\; \int\limits_{\mathbb{M}}\left(\eta^a\,\frac{\delta\Lambda}{\delta\varphi^a}\bigg|_{\phi} + (\partial_\mu\eta^a)\,\frac{\delta\Lambda}{\delta(\partial_\mu\varphi^a)}\bigg|_{\phi}\right)\\
&=\; \int\limits_{\mathbb{M}}\left(\eta^a\,\frac{\delta\Lambda}{\delta\varphi^a}\bigg|_{\phi} - \eta^a\partial_\mu\,\frac{\delta\Lambda}{\delta(\partial_\mu\varphi^a)}\bigg|_{\phi} + \underbrace{\partial_\mu\left\{\eta^a\,\frac{\delta\Lambda}{\delta(\partial_\mu\varphi^a)}\bigg|_{\phi}\right\}}_{I_1}\right).
\end{aligned}\tag{2.20}$$

This can be transformed further because the $(d{+}1)$-form I_1 is exact. We have

$$I_1 \;=\; (\partial_\mu i^\mu)\,\mathrm{d}x^0\wedge\ldots\wedge\mathrm{d}x^d \qquad\qquad i^\mu \;=\; \eta^a\,\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi^a)}\bigg|_{\phi}\;,$$

and can define a $d-$form $\tilde{I}_1$ whose exterior derivative yields $\mathrm{d}\tilde{I}_1 = I_1$:

$$\begin{aligned}
\tilde{I}_1 \;&=\; \partial_\mu \lrcorner\, i^\mu\,\mathrm{d}x^0\wedge\ldots\wedge\mathrm{d}x^d\\
&=\; \sum_{\mu=0}^{d} i^\mu(-1)^\mu\,\mathrm{d}x^0\wedge\ldots\widehat{\mathrm{d}x^\mu}\ldots\wedge\mathrm{d}x^d\;,\\
\mathrm{d}\tilde{I}_1 \;&=\; \sum_{\mu=0}^{d}(\partial_\nu i^\mu)(-1)^\mu\mathrm{d}x^\nu\wedge\mathrm{d}x^0\wedge\ldots\widehat{\mathrm{d}x^\mu}\ldots\wedge\mathrm{d}x^d\\
&=\; (\partial_\mu i^\mu)\,\mathrm{d}x^0\wedge\ldots\wedge\mathrm{d}x^d\;,
\end{aligned}$$

wherein $\lrcorner$ denotes vector insertion into (contraction with) a form, and the wide $\widehat{\text{hat}}$ over a basic 1-form denotes its omission within the wedge product. Therefore we can employ Stoke's theorem to transform the integral over I_1, resulting in Equation (5) of [59]:

$$\mathrm{d}S^{\mathbb{M}}_{\phi}(\eta) \;=\; \int\limits_{\mathbb{M}}\!\eta^a\Bigg\{\underbrace{\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu\,\frac{\delta\Lambda}{\delta(\partial_\mu\varphi^a)}}_{\text{Euler-Lagrange derivative}}\Bigg\}_{\phi} \;+\; \int\limits_{\partial\mathbb{M}}\!\partial_\mu\lrcorner\,\eta^a\,\frac{\delta\Lambda}{\delta(\partial_\mu\varphi^a)}\bigg|_{\phi}\,.\tag{2.21}$$

The differential of the action thus is a sum of two terms: the integral of the Euler-Lagrange derivative over the interior of $\mathbb{M}$ (in short: bulk term), and the integral of $\tilde{I}_1$ over the boundary of $\mathbb{M}$ (boundary term). On $\mathrm{L}^{\mathbb{C}}_{\mathbb{M}}$ the bulk terms vanishes, and the differential of the action is a pure boundary term:

$$\mathrm{d}S^{\partial\mathbb{M}}_{\phi}(\eta) \;=\; \int\limits_{\partial\mathbb{M}} \partial_\mu \lrcorner\, \eta^a \, \frac{\delta\Lambda}{\delta(\partial_\mu\varphi^a)}\Bigg|_{\phi} \tag{2.22}$$

$$= \sum_{\mu=0}^{d} \int\limits_{\partial\mathbb{M}} (-1)^\mu \, \mathrm{d}x^0 \wedge ...\widehat{\mathrm{d}x^\mu}... \wedge \mathrm{d}x^d \; \eta^a \, \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi^a)}\Bigg|_{\phi}$$

$$= \operatorname{sign}(\partial\mathbb{M}) \int\limits_{\partial\mathbb{M}} \mathrm{d}x^0...\widehat{\mathrm{d}x^\mu}...\mathrm{d}x^d \; \eta^a \, \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi^a)}\Bigg|_{\phi} \tag{2.23}$$

The sign results from the following reasoning. The orientation of $\mathrm{d}x^0 \wedge ... \wedge \mathrm{d}x^d$ is positive, hence so is the one of $(-1)^\mu \mathrm{d}x^\mu \wedge \mathrm{d}x^0 \wedge ...\widehat{\mathrm{d}x^\mu}... \wedge \mathrm{d}x^d$ for each $\mu \in \{0, ..., d\}$. Deleting the first 1-form from a wedge product does not change the orientation. Hence $(-1)^\mu \mathrm{d}x^0 \wedge ...\widehat{\mathrm{d}x^\mu}... \wedge \mathrm{d}x^d$ has the positive orientation induced on $\partial\mathbb{M}$ by $\mathrm{d}x^0 \wedge ... \wedge \mathrm{d}x^d$. What remains to take into account is the orientation we choose for (possibly disjoint parts of) $\partial\mathbb{M}$ as pointing *outwards* of the enclosed region $\mathbb{M}$. To this end, we define

$$\operatorname{sign}(\partial\mathbb{M}) \;:=\; \begin{cases} +1 & \text{agreeing chosen and induced orientation of } \partial\mathbb{M} \\ -1 & \text{opposing chosen and induced orientation of } \partial\mathbb{M} \end{cases} .$$

We thus note, that first both the action S and its differential $\mathrm{d}S$ refer to solutions *on entire regions* of spacetime. Then, with the action's bulk term vanishing, the differential of the action can be calculated as an integral over the boundary of the region. For solutions $\phi \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ near the region's boundary which are induced by solutions on the region's interior, $\mathrm{d}S$ as a 1-form is indeed the differential of the 0-form S. However, for boundary solutions which are not induced by interior solutions, this ceases to be the case, and $\mathrm{d}S$ for those solutions is just some 1-form which is not exact. This difference will be important later, because it leads to Lagrangian subspaces of $\mathrm{L}_{\partial\mathbb{M}}$.

Further, the 1-form $\mathrm{d}S$ in its boundary form (2.23) can be defined for any oriented d-surface Σ in spacetime, by simply using the same integral for solutions in $\mathrm{L}^{\mathbb{C}}_{\Sigma}$. In this case, $\mathrm{d}S$ is not exact (except when Σ happens to be a boundary, then it becomes exact for interior solutions as before). For $\phi \in \mathrm{L}^{\mathbb{C}}_{\Sigma}$ and $\eta \in (\mathrm{T}\,\mathrm{L}^{\mathbb{C}}_{\Sigma})_\phi$ this means

$$\mathrm{d}S^{\Sigma}_{\phi}(\eta) \;=\; \int\limits_{\Sigma} \partial_\mu \lrcorner\, \eta^a \, \frac{\delta\Lambda}{\delta(\partial_\mu\varphi^a)}\Bigg|_{\phi} \tag{2.24}$$

$$= \operatorname{sign}(\Sigma) \int\limits_{\Sigma} \mathrm{d}x^0...\widehat{\mathrm{d}x^\mu}...\mathrm{d}x^d \; \eta^a \, \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi^a)}\Bigg|_{\phi} , \tag{2.25}$$

$$\operatorname{sign}(\Sigma) \;:=\; \begin{cases} +1 & \text{agreeing chosen and induced orientation of } \Sigma \\ -1 & \text{opposing chosen and induced orientation of } \Sigma \end{cases} . \tag{2.26}$$

As a simple example, we can consider an action that is quadratic both in fields an field derivatives:

$$\Lambda(\varphi, \partial\varphi, x) \;=\; \underbrace{\tfrac{1}{2}\Big\{ c^{\mu\nu}_{ab}\,(\partial_\mu\varphi^a)\,(\partial_\nu\varphi^b) - m_{ab}\varphi^a\varphi^b \Big\}}_{\mathcal{L}} \, \mathrm{d}x^0 \wedge ... \wedge \mathrm{d}x^d \, . \tag{2.27}$$

In order to get real actions for real fields, the coefficients must fulfill both $\overline{c^{\mu\nu}_{ab}} = c^{\nu\mu}_{ba}$ and $\overline{m_{ab}} = m_{ba}$. Free Klein-Gordon theory is obtained via setting $c^{\mu\nu}_{ab} = \operatorname{sign}(g_{00})\sqrt{|g|}\,g^{\mu\nu}\delta_{ab}$ with $m_{ab} = m^2\sqrt{|g|}\,\delta_{ab}$, where x^0 is the time coordinate and the sign of g_{00} renders the action independent of the metric's overall sign. We will denote the free action by S_0. Then, for the Lagrangian (2.27), the action's differential (2.23) becomes

$$\mathrm{d}S_\phi^{\partial\mathbb{M}}(\eta) \;=\; \mathrm{sign}\,(\partial\mathbb{M}) \int\limits_{\partial\mathbb{M}} \mathrm{d}x^0 ... \widehat{\mathrm{d}x^\mu} ... \mathrm{d}x^d \; c_{ab}^{\mu\nu}\, \eta^a\, (\partial_\nu \phi^b) \qquad \forall\, \phi, \eta \in \mathrm{L}_{\partial\mathbb{M}}, \tag{2.28}$$

being consistent with Equation (13) in [53]. Replacing $\partial\mathbb{M}$ by Σ as above, $\mathrm{d}S$ can again be defined for any d-surface Σ. For a free Klein-Gordon field on a spacetime region $\mathbb{M}$ with block-diagonal metric w.r.t. some coordinate τ (i.e.: $g_{\tau,\alpha} = 0$ for all $x^\alpha \neq \tau$), let Σ_τ a d-surface of constant τ. Then, the action's differential becomes (no Einstein summation)

$$(\mathrm{d}S_0^{\Sigma_\tau})_\phi(\eta) \;=\; \mathrm{sign}\,(\Sigma_\tau)\mathrm{sign}\,(g_{00}) \int\limits_{\Sigma_\tau} \mathrm{d}x^0 ... \widehat{\mathrm{d}x^\tau} ... \mathrm{d}x^d \, \sqrt{|g|}\; g^{\tau\tau}\; \eta\, (\partial_\tau \phi) \tag{2.29}$$

$$= \; \mathrm{sign}\,(g_{00} g^{\tau\tau}) \int\limits_{\Sigma_\tau} \mathrm{d}x^0 ... \widehat{\mathrm{d}x^\tau} ... \mathrm{d}x^d \, \sqrt{|\tilde g|}\; \eta\, (\partial_n \phi) \; . \tag{2.30}$$

Here, $\tilde g$ is the metric induced on Σ_τ, $\partial_n \;=\; \mathrm{sign}\,(\Sigma_\tau) \sqrt{|g^{\tau\tau}|}\,\partial_\tau \;=\; n_{\Sigma_\tau}$ is the normal derivative on Σ_τ, with n_{Σ_τ} the unit normal on Σ_τ pointing in the direction of its orientation, and

$$\mathrm{sign}\,(\Sigma_\tau) \;:=\; \begin{cases} +1 & \Sigma_\tau \text{ oriented in positive } \tau \text{ direction} \\ -1 & \Sigma_\tau \text{ oriented in negative } \tau \text{ direction} \end{cases} \; . \tag{2.31}$$

The Euler-Lagrange equation in (2.21) for the free Klein-Gordon field becomes the well known free (=homogeneous) Klein-Gordon equation

$$\left(\sigma_{00}\Box + m^2\right) \phi(x) \;=\; 0 \; . \tag{2.32}$$

Here, the factor $\sigma_{00} \;=\; \mathrm{sign}\, g_{00}$ gives us independence of the overall sign of the spacetime metric, which is physically irrelevant. For the Klein-Gordon theory with a source field $\mu(x)$, as in Section 3.1.2, the Lagrangian density has an additional coupling term:

$$\Lambda(\varphi, \partial\varphi, x) \;=\; \sqrt{|g|}\left(\tfrac{1}{2}\sigma_{00} g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - \;\tfrac{1}{2}\, m^2\varphi^2 + \mu\varphi\right) \mathrm{d}x^0 \wedge ... \wedge \mathrm{d}x^d \; . \tag{2.33}$$

The Euler-Lagrange equation for this Λ is the (inhomogeneous) Klein-Gordon equation with source term $\mu(x)$:

$$\left(\sigma_{00}\Box + m^2\right) \phi(x) \;=\; \mu(x) \; . \tag{2.34}$$

2.4.2 Symplectic potential θ and symplectic structure ω

We begin this section recalling some subspaces of symplectic vector spaces for later application. Following [76], for a vector space V (real or complex, of finite or infinite dimension) with symplectic form ω, the symplectic complement $\mathrm{S}^{\perp_\omega}$ for any subspace S of V is defined as

$$\mathrm{S}^{\perp_\omega} \;:=\; \left\{ v \in \mathrm{V} \;\middle|\; \omega(v, s) = 0 \quad \forall\, s \in \mathrm{S} \right\} \; . \tag{2.35}$$

The symplectic complement has the following properties (the last only for finite-dimensional V):

$$\mathrm{S}_2^{\perp_\omega} \subseteq \mathrm{S}_1^{\perp_\omega} \qquad \forall\, \mathrm{S}_1 \subseteq \mathrm{S}_2 \subseteq V \tag{2.36}$$

$$(\mathrm{S}^{\perp_\omega})^{\perp_\omega} \;=\; \mathrm{S} \qquad \forall\, \mathrm{S} \subseteq \mathrm{V} \tag{2.37}$$

$$(\mathrm{S}_1 \oplus \mathrm{S}_2)^{\perp_\omega} \;=\; \mathrm{S}_1^{\perp_\omega} \cap \mathrm{S}_2^{\perp_\omega} \qquad \forall\, \mathrm{S}_1, \mathrm{S}_2 \subseteq V \tag{2.38}$$

$$(\mathrm{S}_1 \cap \mathrm{S}_2)^{\perp_\omega} \;=\; \mathrm{S}_1^{\perp_\omega} \oplus \mathrm{S}_2^{\perp_\omega} \qquad \forall\, \mathrm{S}_1, \mathrm{S}_2 \subseteq V \tag{2.39}$$

$$\dim \mathrm{S} + \dim \mathrm{S}^{\perp_\omega} \;=\; \dim \mathrm{V} \qquad \forall\, \mathrm{S} \subseteq \mathrm{V} \; . \tag{2.40}$$

Therein, by $(\mathrm{S}_1 \oplus \mathrm{S}_2)$ we denote the subspace of V spanned by S_1 and S_2. Further, a subspace $\mathrm{S} \subseteq \mathrm{V}$ is called

$$\text{ISOTROPIC subspace if:} \qquad \mathrm{S} \subseteq \mathrm{S}^{\perp\omega} \quad \Leftrightarrow \quad 0 = \omega(s_1, s_2) \quad \forall\, s_1, s_2 \in \mathrm{S} \tag{2.41}$$

$$\text{COISOTROPIC subspace if:} \qquad \mathrm{S}^{\perp\omega} \subseteq \mathrm{S} \quad \Leftrightarrow \quad 0 = \omega(\tilde{s}_1, \tilde{s}_2) \quad \forall\, \tilde{s}_1, \tilde{s}_2 \in \mathrm{S}^{\perp\omega} \tag{2.42}$$

$$\text{LAGRANGIAN subspace if:} \qquad \mathrm{S}^{\perp\omega} = \mathrm{S} \quad \Leftrightarrow \quad \text{S is isotropic and coisotropic} \tag{2.43}$$

$$\text{SYMPLECTIC subspace if:} \qquad \mathrm{S}^{\perp\omega} \cap \mathrm{S} = \{0\} \quad \Leftrightarrow \quad \nexists\, (\tilde{s} \neq 0) \in \mathrm{S}: \quad 0 = \omega(\tilde{s}, s)\ \forall\, s \in S. \tag{2.44}$$

($\Leftarrow$ in (2.42) follows from (2.37).) Thus if S is isotropic, its symplectic complement is coisotropic, and conversely. Because of the dimension sum, any isotropic (or cosiotropic) subspace of dimension $\frac{1}{2}\dim V$ must be Lagrangian. An arbitrary subspace needs neither be (co)isotropic nor symplectic, therefore these categories are not exhaustive. A lemma in [76] assures that every finite-dimensional symplectic vector space (V, ω) has even dimension and contains a Lagrangian subspace.

After these definitions of special subspaces, let us return to the symplectic potential and structure themselves. The relation between the action's differential and the symplectic potential on $\mathrm{L}_\Sigma^{\mathbb{C}}$ is:

$$\theta_\phi^\Sigma(\eta) \;=\; \sigma_\theta\, \mathrm{d}S_\phi^\Sigma(\eta)\;. \tag{2.45}$$

Therein, σ_θ is a pure sign, which is chosen to be $\sigma_\theta = -1$ in Equation (3) in [58] in order to achieve agreement between holomorphic and path-integral quantization. We shall follow this choice throughout all our calculations. The symplectic structure is the exterior derivative of the symplectic potential:

$$\omega_\phi^\Sigma(\eta, \zeta) \;=\; \mathrm{d}\theta_\phi^\Sigma(\eta, \zeta)\;. \tag{2.46}$$

We recall that (in spite of the suggesting notation) $\mathrm{d}S_\Sigma$ itself is a 1-form on $\mathrm{L}_\Sigma^{\mathbb{C}}$, *but not* the exterior derivative of a 0-form (function) on $\mathrm{L}_\Sigma^{\mathbb{C}}$, see above. In particular, only if the hypersurface Σ is a boundary $\partial\mathbb{M}$ of some region $\mathbb{M}$, then $\mathrm{d}S_\Sigma$ is exact on $\mathrm{L}_{\partial\mathbb{M}}^{\text{int}}$. Hence the 1-form θ_Σ is generically not exact, except on $\mathrm{L}_{\partial\mathbb{M}}^{\text{int}}$ if Σ is $\mathbb{M}$'s boundary. Therefore, the symplectic structure ω_Σ generically *is not* the second exterior derivative of a 0-form, and hence does not vanish identically. Only if Σ is a boundary, then ω_Σ is the second exterior derivative of the 0-form S for interior solutions $\phi \in \mathrm{L}_{\partial\mathbb{M}}^{\text{int}}$. Hence $\omega_{\partial\mathbb{M}}$ vanishes identically on $\mathrm{L}_{\partial\mathbb{M}}^{\text{int}}$, that is, $\mathrm{L}_{\partial\mathbb{M}}^{\text{int}}$ is an isotropic subspace of $\mathrm{L}_{\partial\mathbb{M}}$. (In any case, θ_Σ is always a 1-form on L_Σ and thus $\mathrm{d}\omega = \mathrm{dd}\theta$ always vanishes identically.)

Let us now consider the subject of vanishing symplectic structures with regard to different types of regions. Recalling that the time-interval is a neighborhood of an equal-time hyperplane Σ_t, and the tube a neighborhood of an equal-ρ hypercylinder Σ_ρ, each of these hypersurfaces has an associated symplectic structure via (2.46).

Hypercylinders Σ_ρ are the boundaries of rod regions $\mathbb{M}_\rho$, and hence the symplectic structure (2.195) on an AdS hypercylinder vanishes on the space of solutions (2.191) on rod regions. The same happens in Minkowski spacetime for the symplectic structure (2.116) with solutions (2.112) on Minkowski rod regions.

By contrast, for regions $\mathbb{M}_{12}$ bounded by two hypersurfaces Σ_1 and Σ_2, the symplectic structure is not identically zero for each hypersurface. For example, a single equal-time hypersurface is not the boundary of a region, and hence the symplectic structures (2.209) and (2.96) do not vanish identically for solutions on such a region (these solutions correspond by evolution to solutions in the neighborhood of Σ_t). In the same way, a single hypercylinder Σ_ρ is not the boundary of a tube region, and hence the symplectic structures (2.195) and (2.116) do not vanish identically for solutions on such a region (these solutions correspond by evolution to solutions in the neighborhood of Σ_ρ). We recall that the space L_{Σ_ρ} of solutions near Σ_ρ is twice as large as the space $\mathrm{L}_\rho^{\text{rod}}$ of solutions on the whole rod region, which explains why ω_{Σ_ρ} can vanish for the latter but not the former.)

What is more, since the symplectic structure on the boundary $\omega_{\partial\mathbb{M}_{12}} = \omega_{\Sigma_2} - \omega_{\Sigma_1}$ is the second differential of the action, it vanishes on $\mathrm{L}_\mathbb{M}$. This implies, that on $\mathrm{L}_\mathbb{M}$ sthe symplectic structure is actually independent of the hypersurface: $\omega_{\Sigma_2} = \omega_{\Sigma_1}$. For Minkowski and AdS spacetimes, our explicit expressions (2.116), (2.96), (2.195) and (2.209) show that the symplectic structures associated to Σ_t and Σ_ρ are indeed independent of their associated hypersurface: $\omega_t(\eta,\zeta)$ and $\omega_\rho(\eta,\zeta)$ depend only on the two solutions η and ζ. This holds despite neither of our regions being compact, nor our solutions being restriced to compact support.

In order to evaluate ω we can use e.g. (A.1.16) from [76], wherein the fields are considered as vectors from the tangent space of L_Σ at ϕ, thus acting like a derivative on functions, giving:

$$\omega^\Sigma_\phi(\eta,\zeta) \;=\; \mathrm{d}\theta^\Sigma_\phi(\eta,\zeta) \;=\; \tfrac{1}{2}\,\eta\,\theta^\Sigma_\phi(\zeta) - \tfrac{1}{2}\,\zeta\,\theta^\Sigma_\phi(\eta) - \tfrac{1}{2}\,\theta^\Sigma_\phi([\eta,\zeta])\,. \tag{2.47}$$

The Lie bracket of vectors is defined in the usual way for elements η,ζ of the solution tangent space: $[\eta,\zeta] \;=\; \eta\,\zeta \;-\; \zeta\,\eta$. Combining the formula above with (2.24) reproduces equation (4) in [58]:

$$\begin{aligned}\omega^\Sigma_\phi(\eta,\zeta) \;=\; -\frac{1}{2}\int\limits_\Sigma\Big\{&\big(\eta^b\zeta^a - \eta^a\zeta^b\big)\,\partial_\mu \lrcorner\, \frac{\delta^2\Lambda}{\delta(\partial_\mu\varphi^a)\delta\varphi^b}\Bigg|_\phi \\ &+ \big(\zeta^a\partial_\nu\eta^b - \eta^a\partial_\nu\zeta^b\big)\,\partial_\mu \lrcorner\, \frac{\delta^2\Lambda}{\delta(\partial_\mu\varphi^a)\delta(\partial_\nu\varphi^b)}\Bigg|_\phi\Big\}\,.\end{aligned} \tag{2.48}$$

Thus $\omega^\Sigma_\phi(\eta,\zeta)$ is real only for real fields ϕ, η and ζ, and antisymmetric under exchange of their arguments:

$$\omega^\Sigma_\phi(\eta,\zeta) \;=\; -\omega^\Sigma_\phi(\zeta,\eta) \qquad \begin{array}{l}\forall\,\phi\in\mathrm{L}^{\mathbb{C}}_\Sigma\\ \forall\,\eta,\zeta\in(\mathrm{TL}^{\mathbb{C}}_\Sigma)_\phi\end{array}. \tag{2.49}$$

Evaluating symplectic potential and structure for the quadratic action in example (2.27) yields the relations

$$\theta^\Sigma_\phi(\eta) \;=\; -\mathrm{sign}\,(\Sigma)\int\limits_\Sigma \mathrm{d}x^0 ... \widehat{\mathrm{d}x^\mu} ... \mathrm{d}x^d\; c^{\mu\nu}_{ab}\,\eta^a\big(\partial_\nu\phi^b\big)\,, \tag{2.50}$$

$$\omega^\Sigma_\phi(\eta,\zeta) \;=\; -\tfrac{1}{2}\mathrm{sign}\,(\Sigma)\int\limits_\Sigma \mathrm{d}x^0 ... \widehat{\mathrm{d}x^\mu} ... \mathrm{d}x^d\; c^{\mu\nu}_{ab}\,\Big(\zeta^a\partial_\nu\eta^b - \eta^a\partial_\nu\zeta^b\Big)\,. \tag{2.51}$$

These provide another useful relation (valid for this example only):

$$\omega^\Sigma_\phi(\eta,\zeta) \;=\; \tfrac{1}{2}\,\big(\theta^\Sigma_\eta(\zeta) \;\; - \theta^\Sigma_\zeta(\eta)\big) \qquad \begin{array}{l}\forall\,\phi\in\mathrm{L}^{\mathbb{C}}_\Sigma\\ \forall\,\eta,\zeta\in(\mathrm{TL}^{\mathbb{C}}_\Sigma)_\phi\end{array}, \tag{2.52}$$

hence the symplectic structure for the case of a quadratic action is independent of the base point ϕ. Integration of d-forms over oriented d-surfaces induces that reversing the orientation of such surface $\Sigma \;\to\; \overline{\Sigma}$ results in a minus factor for the integral. Hence

$$\omega^{\overline{\Sigma}}_\phi(\eta,\zeta) \;=\; -\omega^\Sigma_\phi(\eta,\zeta) \qquad \begin{array}{l}\forall\,\phi\in\mathrm{L}^{\mathbb{C}}_\Sigma\\ \forall\,\eta,\zeta\in(\mathrm{TL}^{\mathbb{C}}_\Sigma)_\phi\end{array}. \tag{2.53}$$

For free Klein-Gordon theory symplectic potential and structure reproduce (7.2.6) in [76] by Woodhouse:

$$\theta^\Sigma_\phi(\eta) \;=\; -\mathrm{sign}\,(\Sigma)\mathrm{sign}\,(g_{00})\int\limits_\Sigma \mathrm{d}x^0 ... \widehat{\mathrm{d}x^\mu} ... \mathrm{d}x^d\; \sqrt{|g|}\,g^{\mu\nu}\,\eta\big(\partial_\nu\phi\big) \tag{2.54}$$

$$\omega^\Sigma_\phi(\eta,\zeta) \;=\; -\tfrac{1}{2}\mathrm{sign}\,(\Sigma)\mathrm{sign}\,(g_{00})\int\limits_\Sigma \mathrm{d}x^0 ... \widehat{\mathrm{d}x^\mu} ... \mathrm{d}x^d\; \sqrt{|g|}\,g^{\mu\nu}\,\big(\zeta\partial_\nu\eta - \eta\partial_\nu\zeta\big)\,. \tag{2.55}$$

2.4.3 Isometry invariance of symplectic structures

We show now the well known fact, that if two generators of isometries leave a symplectic structure invariant, then their Lie bracket does so as well. This will be needed in Sections 2.5.5 and 2.6.7, wherein we study explicitly only the actions of one boost (the simplest one) and can deduce the invariance of the remaining boosts from the fact that they arise as commutators of the simplest boost with spatial rotations. The action of an (infinitesimal or finite) isometry k on the symplectic structures ω fulfills

$$\begin{aligned} \big(k\triangleright\omega\big)(k\triangleright\eta, k\triangleright\zeta) &= \omega(\eta,\zeta) \\ \Rightarrow \qquad \big(k\triangleright\omega\big)(\eta,\zeta) &= \omega(k^{-1}\triangleright\eta, k^{-1}\triangleright\zeta)\ . \end{aligned} \tag{2.56}$$

Thus the action of an infinitesimal isometry $(1+\varepsilon K)$ generated by a differential operator K up to linear order in ε is given by

$$\big((1+\varepsilon K)\triangleright\omega\big)(\eta,\zeta) \;=\; \omega\big(\eta,\zeta\big) - \varepsilon\omega\big(K\triangleright\eta,\zeta\big) - \varepsilon\omega\big(\eta,K\triangleright\zeta\big)\ . \tag{2.57}$$

Or shorter, the action of a differential operator K on the symplectic structure is

$$\big(K\triangleright\omega\big)(\eta,\zeta) \;=\; -\omega\big(K\triangleright\eta,\zeta\big) - \omega\big(\eta,K\triangleright\zeta\big)\ . \tag{2.58}$$

That is, the symplectic structure is invariant under an infinitesimal isometry $(1+\varepsilon K)$ precisely if

$$0 \;=\; \big(K\triangleright\omega\big)(\eta,\zeta) \;=\; -\omega\big(K\triangleright\eta,\zeta\big) - \omega\big(\eta,K\triangleright\zeta\big)\ . \tag{2.59}$$

Therefore, the action of the commutator of two differential operators is determined already by the action of each of them:

$$\begin{aligned} \Big([K,L]\triangleright\omega\Big)(\eta,\zeta) &= -\omega\big(KL\triangleright\eta,\zeta\big) - \omega\big(\eta,KL\triangleright\zeta\big) + \omega\big(LK\triangleright\eta,\zeta\big) + \omega\big(\eta,LK\triangleright\zeta\big) \\ &\quad + \omega\big(K\triangleright\eta,L\triangleright\zeta\big) - \omega\big(K\triangleright\eta,L\triangleright\zeta\big) + \omega\big(L\triangleright\eta,K\triangleright\zeta\big) - \omega\big(L\triangleright\eta,K\triangleright\zeta\big) \\ &= -\Big(L\triangleright\omega\Big)(K\triangleright\eta,\zeta) - \Big(L\triangleright\omega\Big)(\eta,K\triangleright\zeta) + \Big(K\triangleright\omega\Big)(L\triangleright\eta,\zeta) + \Big(K\triangleright\omega\Big)(\eta,L\triangleright\zeta) \\ &= \Big(K\triangleright(L\triangleright\omega)\Big)(\eta,\zeta) - \Big(L\triangleright(K\triangleright\omega)\Big)(\eta,\zeta)\ . \end{aligned} \tag{2.60}$$

If now each of the differential operators leaves ω invariant as in

$$\begin{aligned} \big((1+\varepsilon K)\triangleright\omega\big)(\eta,\zeta) &= \omega(\eta,\zeta) \qquad \Longrightarrow \qquad \big(K\triangleright\omega\big)(\eta,\zeta) = 0 \\ \big((1+\varepsilon L)\triangleright\omega\big)(\eta,\zeta) &= \omega(\eta,\zeta) \qquad \Longrightarrow \qquad \big(L\triangleright\omega\big)(\eta,\zeta) = 0\ , \end{aligned}$$

then (2.60) implies that so does their commutator:

$$\begin{aligned} \big(K\triangleright\omega\big) &= 0 \\ \big(L\triangleright\omega\big) &= 0 \end{aligned} \qquad \Longrightarrow \qquad \big([K,L]\triangleright\omega\big) = 0. \tag{2.61}$$

2.4.4 Complex structure J

A complex structure on the real vector space $(\mathrm{TL}_\Sigma)_\phi$ is a linear operator, which acts on the vectors like the imaginary unit i on the complex numbers, and is compatible with the symplectic structure ω:

$$\begin{aligned} J^\Sigma_\phi : \ (\mathrm{TL}_\Sigma)_\phi &\to (\mathrm{TL}_\Sigma)_\phi \\ \big(J^\Sigma_\phi\big)^2 &= -\mathrm{Id} \\ \omega^\Sigma_\phi(J^\Sigma_\phi\eta, J^\Sigma_\phi\zeta) &= +\omega^\Sigma_\phi(\eta,\zeta) \qquad \forall\,\eta,\zeta\in(\mathrm{TL}_\Sigma)_\phi\ . \end{aligned}$$

Since compatibility with the symplectic structure and negative unit square are the only properties of a complex structure, there can exist various complex structures. If J^Σ_ϕ is a complex structure, then $-J^\Sigma_\phi$ is also a complex structure. For the complexified solution space $\mathrm{L}^{\mathbb{C}}_\Sigma$ we extend J^Σ_ϕ linearly to commute with the imaginary unit:

$$J^\Sigma_\phi\,(\mathrm{i}\eta) \;=\; \mathrm{i}J^\Sigma_\phi\,\eta \qquad\Longrightarrow\qquad J^\Sigma_\phi\overline{\eta} \;=\; \overline{J^\Sigma_\phi\eta} \qquad\qquad \forall\,\eta\in(\mathrm{TL}^{\mathbb{C}}_\Sigma)_\phi\;.$$

Thus we obtain

$$\begin{aligned} J^\Sigma_\phi :\;\; (\mathrm{TL}^{\mathbb{C}}_\Sigma)_\phi \;&\rightarrow\; (\mathrm{TL}^{\mathbb{C}}_\Sigma)_\phi \\ \big(J^\Sigma_\phi\big)^2 \;&=\; -\mathrm{Id} \\ \omega^\Sigma_\phi(J^\Sigma_\phi\eta, J^\Sigma_\phi\zeta) \;&=\; +\omega^\Sigma_\phi(\eta,\zeta) \qquad\qquad \forall\,\eta,\zeta\in(\mathrm{TL}^{\mathbb{C}}_\Sigma)_\phi\;. \end{aligned}$$

2.4.5 Commutation of complex structures with isometries

For AdS and Minkowski spacetime, we will look for complex structures J on the respective spaces of Klein-Gordon solutions whose actions on the solutions commute with the actions of the isometries on these solutions. That is, let K_{AB} the generator of any isometry, and let $\phi(x)$ any Klein-Gordon solution on some region $\mathbb{M}$. Then we require our complex structure to fulfill

$$\big(\,J\,(K_{AB}\triangleright\phi)\big)(x) \;\stackrel{!}{=}\; \big(K_{AB}\triangleright(J\phi)\big)(x) \qquad\qquad \forall\,\phi\in\mathrm{L}_{\mathbb{M}}\;. \tag{2.62}$$

This is what we mean by letting the commutator vanish:

$$\big[\,J\,,\;K_{AB}\big]\triangleright\phi(x) \;=\; 0\;. \tag{2.63}$$

Of the boosts, we only need to study one boost (the simplest one). This is sufficient, since the other boost's generators arise as Lie brackets of the simplest boost with rotation generators. It is quickly shown that if two operators K and L each commute with J, then their Lie bracket $[K,\,L]$ commutes with J as well:

$$\begin{aligned} \big[J,\;[K,\,L]\big] \;&=\; J\,(KL-LK) - (KL-LK)\,J & 0 \;&=\; [J,\,K] \\ &=\; J\,(KL-LK) - (KJL-LJK) & 0 \;&=\; [J,\,L] \\ &=\; J\,(KL-LK) - J\,(KL-LK) \\ &=\; 0\;. \end{aligned}$$

2.4.6 Real inner product g and complex inner product $\{\cdot,\,\cdot\}$

As described in Section 4.1 of [59], the symplectic structure ω^Σ_ϕ and the complex structure J^Σ_ϕ give rise to the real g-product and inner product through the definitions

$$\mathrm{g}^\Sigma_\phi(\eta,\zeta) \;:=\; 2\omega^\Sigma_\phi(\eta, J^\Sigma_\phi\zeta) \tag{2.64}$$

$$\{\eta,\,\zeta\}^\Sigma_\phi \;:=\; \mathrm{g}^\Sigma_\phi(\eta,\zeta) + 2\mathrm{i}\,\omega^\Sigma_\phi(\eta,\zeta)\;. \tag{2.65}$$

Requiring the real g-product to be independent of orientation and the inner product to be complex conjugated under orientation reversal, makes it necessary to associate a complex structure J^Σ_ϕ to each hypersurface and fix the relative sign to be

$$J^{\overline{\Sigma}}_\phi \;=\; -J^\Sigma_\phi\;. \tag{2.66}$$

Since the symplectic structure ω is real on TL_Σ, so is the real g-product g, while on $\mathrm{TL}^{\mathbb{C}}_\Sigma$ both are complex. The inner product $\{\cdot,\,\cdot\}$ by construction is complex for both TL_Σ and $\mathrm{TL}^{\mathbb{C}}_\Sigma$. For $\{\cdot,\,\cdot\}$ on TL_Σ the real g-product g provides the real part and the symplectic structure ω (half of) its imaginary part. The real g-product and inner product directly inherit their compatibility with the complex structure from the symplectic structure:

$$\mathrm{g}_\phi^\Sigma(\eta,\zeta) \;=\; \mathrm{g}_\phi^\Sigma\big(J_\phi^\Sigma\eta, J_\phi^\Sigma\zeta\big) \qquad \begin{array}{l}\forall\,\phi\in \mathrm{L}_\Sigma^{\mathbb{C}}\\ \forall\,\eta,\zeta\in(\mathrm{TL}_\Sigma^{\mathbb{C}})_\phi\end{array}, \tag{2.67}$$

$$\{\eta,\,\zeta\}_\phi^\Sigma \;=\; \Big\{J_\phi^\Sigma\eta,\, J_\phi^\Sigma\zeta\Big\}_\phi^\Sigma\,. \qquad \begin{array}{l}\forall\,\phi\in \mathrm{L}_\Sigma^{\mathbb{C}}\\ \forall\,\eta,\zeta\in(\mathrm{TL}_\Sigma^{\mathbb{C}})_\phi\end{array}. \tag{2.68}$$

Using the compatibility of ω and J together with the antisymmetry of ω shows that the real g-product is symmetric under exchange of its arguments while the inner product is conjugate symmetric (but only for real fields):

$$\mathrm{g}_\phi^\Sigma(\eta,\zeta) \;=\; +\mathrm{g}_\phi^\Sigma(\zeta,\eta) \qquad \begin{array}{l}\forall\,\phi\in \mathrm{L}_\Sigma^{\mathbb{C}}\\ \forall\,\eta,\zeta\in(\mathrm{TL}_\Sigma^{\mathbb{C}})_\phi\end{array}, \tag{2.69}$$

$$\{\eta,\,\zeta\}_\phi^\Sigma \;=\; \overline{\{\zeta,\,\eta\}_\phi^\Sigma}\,. \qquad \begin{array}{l}\forall\,\phi\in \mathrm{L}_\Sigma\\ \forall\,\eta,\zeta\in(\mathrm{TL}_\Sigma)_\phi\end{array}. \tag{2.70}$$

Moreover, the choice (2.66) gives us

$$\mathrm{g}_\phi^{\overline{\Sigma}}(\eta,\zeta) \;=\; +\mathrm{g}_\phi^\Sigma(\eta,\zeta) \qquad \begin{array}{l}\forall\,\phi\in \mathrm{L}_\Sigma^{\mathbb{C}}\\ \forall\,\eta,\zeta\in(\mathrm{TL}_\Sigma^{\mathbb{C}})_\phi\end{array} \tag{2.71}$$

$$\{\eta,\,\zeta\}_\phi^{\overline{\Sigma}} \;=\; \{\zeta,\,\eta\}_\phi^\Sigma \qquad \begin{array}{l}\forall\,\phi\in \mathrm{L}_\Sigma^{\mathbb{C}}\\ \forall\,\eta,\zeta\in(\mathrm{TL}_\Sigma^{\mathbb{C}})_\phi\end{array}. \tag{2.72}$$

The following property is often useful in calculations:

$$\big\{\eta,\,J_\phi^\Sigma\zeta\big\}_\phi^\Sigma \;=\; -\big\{J_\phi^\Sigma\eta,\,\zeta\big\}_\phi^\Sigma \;=\; \mathrm{i}\,\{\eta,\,\zeta\}_\phi^\Sigma \qquad \begin{array}{l}\forall\,\phi\in \mathrm{L}_\Sigma^{\mathbb{C}}\\ \forall\,\eta,\zeta\in(\mathrm{TL}_\Sigma^{\mathbb{C}})_\phi\end{array}. \tag{2.73}$$

Polarization projectors $\mathbf{P}^\pm$

The complex structure induces two polarization projectors on the complexified tangent space:

$$\mathrm{P}_{\Sigma,\phi}^\pm :\;\; (\mathrm{TL}_\Sigma^{\mathbb{C}})_\phi \;\to\; (\mathrm{TL}_\Sigma^{\mathbb{C}})_\phi$$

$$\mathrm{P}_{\Sigma,\phi}^\pm \;:=\; \tfrac{1}{2}\big(\mathrm{Id}\mp \mathrm{i}J_\phi^\Sigma\big) \tag{2.74}$$

$$\big(\mathrm{P}_{\Sigma,\phi}^\pm\big)^2 \;=\; \mathrm{P}_{\Sigma,\phi}^\pm \tag{2.75}$$

$$\mathrm{P}_{\Sigma,\phi}^+ + \mathrm{P}_{\Sigma,\phi}^- \;=\; \mathrm{Id} \tag{2.76}$$

$$\mathrm{P}_{\Sigma,\phi}^\pm\,\mathrm{P}_{\Sigma,\phi}^\mp \;=\; 0\,,$$

and the choice (2.66) causes

$$\mathrm{P}_{\overline{\Sigma},\phi}^\pm \;=\; \mathrm{P}_{\Sigma,\phi}^\mp\,. \tag{2.77}$$

The product and commutation relations with the complex structure are

$$J_\phi^\Sigma\mathrm{P}_{\Sigma,\phi}^\pm \;=\; \pm\mathrm{i}\,\mathrm{P}_{\Sigma,\phi}^\pm \qquad \mathrm{P}_{\Sigma,\phi}^\pm J_\phi^\Sigma \;=\; \mp\mathrm{i}\,\mathrm{P}_{\Sigma,\phi}^\pm \qquad \Longrightarrow \qquad \big[J_\phi^\Sigma,\,\mathrm{P}_{\Sigma,\phi}^\pm\big] \;=\; \pm 2\mathrm{i}\,\mathrm{P}_{\Sigma,\phi}^\pm\,. \tag{2.78}$$

All following relations in this section hold for $\mathrm{L}_\Sigma^{\mathbb{C}}$. First, it is easy to show that the symplectic structure of *equally* polarized fields vanishes by just plugging in the definitions:

$$\omega_\phi^\Sigma\big(\mathrm{P}_{\Sigma,\phi}^\pm\,\eta,\,\mathrm{P}_{\Sigma,\phi}^\pm\,\zeta\big)=0 \quad\Longrightarrow\quad \mathrm{g}_\phi^\Sigma\big(\mathrm{P}_{\Sigma,\phi}^\pm\,\eta,\,\mathrm{P}_{\Sigma,\phi}^\pm\,\zeta\big)=0 \quad\Longrightarrow\quad \Big\{\mathrm{P}_{\Sigma,\phi}^\pm\,\eta,\,\mathrm{P}_{\Sigma,\phi}^\pm\,\zeta\Big\}_\phi^\Sigma=0\,, \tag{2.79}$$

which leads to sums over opposed polarizations:

$$\begin{aligned}\omega_\phi^\Sigma(\eta,\,\zeta) \;&=\; \omega_\phi^\Sigma\big(\mathrm{P}_{\Sigma,\phi}^+\,\eta,\,\mathrm{P}_{\Sigma,\phi}^-\,\zeta\big) + \omega_\phi^\Sigma\big(\mathrm{P}_{\Sigma,\phi}^-\,\eta,\,\mathrm{P}_{\Sigma,\phi}^+\,\zeta\big)\\ \mathrm{g}_\phi^\Sigma(\eta,\,\zeta) \;&=\; \mathrm{g}_\phi^\Sigma\big(\mathrm{P}_{\Sigma,\phi}^+\,\eta,\,\mathrm{P}_{\Sigma,\phi}^-\,\zeta\big) + \mathrm{g}_\phi^\Sigma\big(\mathrm{P}_{\Sigma,\phi}^-\,\eta,\,\mathrm{P}_{\Sigma,\phi}^+\,\zeta\big)\,.\end{aligned} \tag{2.80}$$

Further, we can quickly check in the same way that

$$\begin{aligned} \omega^\Sigma_\phi\big(\mathrm{P}^\mp_{\Sigma,\phi}\,\eta,\,\mathrm{P}^\pm_{\Sigma,\phi}\,\zeta\big) &= \tfrac{1}{2}\,\omega^\Sigma_\phi\big(\eta,\,\zeta\big) \pm \tfrac{\mathrm{i}}{2}\,\omega^\Sigma_\phi\big(J^\Sigma_\phi\,\eta,\,\zeta\big) \\ \Longrightarrow \quad \mathrm{g}^\Sigma_\phi\big(\mathrm{P}^\mp_{\Sigma,\phi}\,\eta,\,\mathrm{P}^\pm_{\Sigma,\phi}\,\zeta\big) &= \pm\mathrm{i}\omega^\Sigma_\phi\big(\eta,\,\zeta\big) + \omega^\Sigma_\phi\big(\eta,\,J^\Sigma_\phi\,\zeta\big) \end{aligned} \tag{2.81}$$

which makes the inner product take into account only the negatively polarized part of its first argument and the positively polarized one of the second:

$$0 = \Big\{\mathrm{P}^+_{\Sigma,\phi}\,\eta,\,\mathrm{P}^-_{\Sigma,\phi}\,\zeta\Big\}^\Sigma_\phi \qquad \Longrightarrow \quad \{\eta,\,\zeta\}^\Sigma_\phi = \Big\{\mathrm{P}^-_{\Sigma,\phi}\,\eta,\,\mathrm{P}^+_{\Sigma,\phi}\,\zeta\Big\}^\Sigma_\phi\,. \tag{2.82}$$

2.5 Minkowski Spacetime: Classical Klein-Gordon Theory

2.5.1 Radial behaviour of the Minkowski Klein-Gordon solutions

In this section we have a closer look at the behaviour of the radial solutions in Minkowski spacetime, see e.g. Section A.1 of [22]. Separation of variables in spherical coordinates reduces the Klein-Gordon equation to a differential equation for the radial dependence of the modes, namely the spherical Bessel differential equation. We call its solutions "radial solutions" and reserve the term "solutions" for full solutions of the Klein-Gordon equation. By m we denote the Klein-Gordon mass of the field, and by E the energy/frequency of a mode, see for example (2.92).

For $E^2 > m^2$, the radial solutions are simply the spherical Bessel functions $j_l(p^{\mathbb{R}}_E r)$ and spherical Neumann functions $n_l(p^{\mathbb{R}}_E r)$. (The latter are also called spherical Bessel functions of the second kind.) Note that they are real here, since $p^{\mathbb{R}}_E := \sqrt{|E^2 - m^2|}$ is real by its definition, and r is real, too. Their Wronskian is positive for our arguments:

$$\big(j_l \overleftrightarrow{\partial}_r n_l\big)(p^{\mathbb{R}}_E r) = \frac{1}{p^{\mathbb{R}}_E\, r^2}\,.$$

For $E^2 > m^2$, the following complex radial solutions are also used very frequently: the spherical Hankel functions of the first and second kind

$$h^{(1)}_l(p^{\mathbb{R}}_E r) = j_l(p^{\mathbb{R}}_E r) + \mathrm{i}\, n_l(p^{\mathbb{R}}_E r), \qquad h^{(2)}_l(p^{\mathbb{R}}_E r) = j_l(p^{\mathbb{R}}_E r) - \mathrm{i}\, n_l(p^{\mathbb{R}}_E r). \tag{2.83}$$

(Both are also called spherical Bessel functions of the third kind.) Note that the alternative notation h_l for $h^{(1)}_l$ and $\overline{h_l}$ for $h^{(2)}_l$ may become misleading, since $\overline{h_l}(z) = j_l(z) - \mathrm{i} n_l(z)$ is equal to $\overline{h_l(z)} = \overline{j_l(z)} - \mathrm{i}\overline{n_l(z)}$ *only for real* arguments z. Their Wronskian is negative-imaginary for our arguments:

$$\big(h^{(1)}_l \overleftrightarrow{\partial}_r h^{(2)}_l\big)(p^{\mathbb{R}}_E r) = \frac{-2\mathrm{i}}{p^{\mathbb{R}}_E\, r^2}\,.$$

For $E^2 < m^2$, radial solutions are given by the modified spherical Bessel functions $\mathrm{i}^{-l} j_l(\mathrm{i}p^{\mathbb{R}}_E r)$ and the modified spherical Neumann functions $\mathrm{i}^{l+1} n_l(\mathrm{i}p^{\mathbb{R}}_E r)$. (The former are also called modified Bessel functions of the first kind, and the latter modified Bessel functions of the second kind.) Note that the factors in front are chosen such that both functions become real for imaginary arguments, and such that their Wronskian becomes positive for our arguments:

$$\big(\mathrm{i}^{-l} j_l \overleftrightarrow{\partial}_r \mathrm{i}^{l+1} n_l\big)(\mathrm{i}p^{\mathbb{R}}_E r) = \frac{1}{p^{\mathbb{R}}_E\, r^2}\,.$$

For $E^2 < m^2$, another pair of radial solutions is given by the modified spherical Hankel functions of the first kind

$$\begin{aligned} k^{(1)}_l(p^{\mathbb{R}}_E r) &= (-1)^l\,\mathrm{i}^{-l} j_l(\mathrm{i}p^{\mathbb{R}}_E r) + \mathrm{i}^{l+1} n_l(\mathrm{i}p^{\mathbb{R}}_E r) \\ &= \mathrm{i}^l j_l(\mathrm{i}p^{\mathbb{R}}_E r) + \mathrm{i}^{l+1} n_l(\mathrm{i}p^{\mathbb{R}}_E r) \end{aligned} \tag{2.84}$$

and the modified spherical Hankel functions of the second kind

$$\begin{aligned} k_l^{(2)}(p_E^{\mathbb{R}}r) &= -\mathrm{i}^{-l} j_l(\mathrm{i}p_E^{\mathbb{R}}r) - (-1)^{l+1}\mathrm{i}^{l+1} n_l(\mathrm{i}p_E^{\mathbb{R}}r) \\ &= -\mathrm{i}^{-l} j_l(\mathrm{i}p_E^{\mathbb{R}}r) - \mathrm{i}^{-(l+1)} n_l(\mathrm{i}p_E^{\mathbb{R}}r). \end{aligned} \tag{2.85}$$

(Both are also called modified Bessel functions of the third kind.) Note that the factors are chosen such that both functions become real for imaginary arguments, and such that their Wronskian becomes positive for our arguments:

$$\left(k_l^{(1)} \overleftrightarrow{\partial}_r k_l^{(2)}\right)(\mathrm{i}p_E^{\mathbb{R}}r) = \frac{2}{p_E^{\mathbb{R}}\, r^2}\,.$$

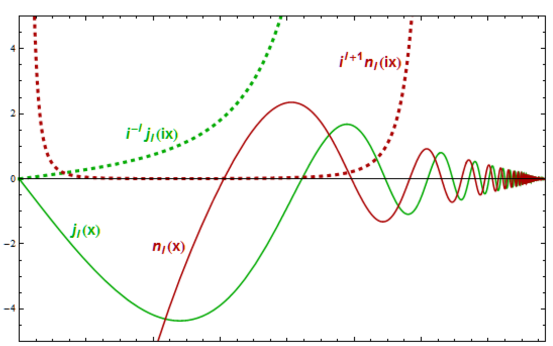
We can recover the modified Bessel and Neumann functions from the modified Hankel functions through

$$\begin{aligned} \mathrm{i}^{-l} j_l(\mathrm{i}p_E^{\mathbb{R}}r) &= \tfrac{1}{2}\Big((-1)^l k_l^{(1)}(p_E^{\mathbb{R}}r) - k_l^{(2)}(p_E^{\mathbb{R}}r)\Big) \\ \mathrm{i}^{l+1} n_l(\mathrm{i}p_E^{\mathbb{R}}r) &= \tfrac{1}{2}\Big(k_l^{(1)}(p_E^{\mathbb{R}}r) + (-1)^l k_l^{(2)}(p_E^{\mathbb{R}}r)\Big). \end{aligned}$$

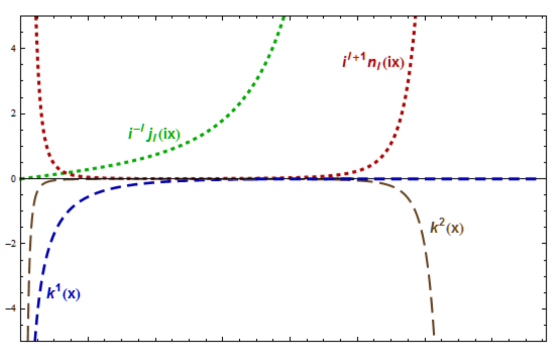
Our functions j_l, n_l, $h_l^{(1)}$, $h_l^{(2)}$, $i^{-l} j_l$ are the functions j_l, y_l, $\mathrm{h}_l^{(1)}$, $\mathrm{h}_l^{(2)}$, $\mathrm{i}_l^{(1)}$ as found in DLMF [10.47]. In contrast to these, the rest of the our functions (are closely related to but) do not coincide with functions in the DLMF. For us it is necessary to choose our functions as above, in order to avoid Wronskians whose signs depend on l as e.g. in $(-1)^l$.

Now let us consider the radial behaviour of the functions. On the time axis $r = 0$ the spherical Bessel functions $j_l(p_E^{\mathbb{R}}r)$ and the modified spherical Bessel functions $\mathrm{i}^{-l} j_l(p_E^{\mathbb{R}}r)$ are regular. (In particular, those with $l = 0$ have value one, and those with $l \geq 1$ have value zero on the time axis.) All the other functions behave like $r^{-(l+1)}$ near the time axis.

According to DLMF [10.52], near spatial infinity $r \to \infty$ the spherical Bessel functions $j_l(p_E^{\mathbb{R}}r)$, the spherical Neumann functions $n_l(p_E^{\mathbb{R}}r)$ and the spherical Hankel functions $h_l^{(1)}(p_E^{\mathbb{R}}r)$ and $h_l^{(2)}(p_E^{\mathbb{R}}r)$ decay essentially with $1/r$. The modified spherical Bessel and Neumann functions $\mathrm{i}^{-l} j_l(\mathrm{i}p_E^{\mathbb{R}}r)$ and $\mathrm{i}^{l+1} n_l(\mathrm{i}p_E^{\mathbb{R}}r)$ and the modified spherical Hankel functions of the second kind $k_l^{(2)}(p_E^{\mathbb{R}}r)$ diverge like $(p_E^{\mathbb{R}}r)^{-1}\exp(p_E^{\mathbb{R}}r)$ for $r \to \infty$, whereas the modified spherical Hankel functions of the first kind $k_l^{(1)}(p_E^{\mathbb{R}}r)$ decay like $(p_E^{\mathbb{R}}r)^{-1}\exp(-p_E^{\mathbb{R}}r)$. The behaviour of the radial functions for the generic cases is sketched in Figure 2.86.



(a) Unmodified Bessel functions for $E^2 > m^2$ and modified Bessel functions for $E^2 < m^2$



(b) Modified Bessel and Hankel functions for $E^2 < m^2$

Figure 2.86: Typical behaviour

Therein, we have $x \in [0, \frac{\pi}{2})$ with $p_E^{\mathbb{R}}r = \frac{1}{4}\tan x$ while for simplicity we let $p_E^{\mathbb{R}} = 1$. Thus $x = 0$ corresponds to $r = 0$ and $x \to \frac{\pi}{2}$ to $r \to \infty$. For better visibility we scaled some of the functions. On the left hand side, the continuous green line plots the spherical Bessel function $-10\, j_1(p_E^{\mathbb{R}}r)$, the continuous red line plots the spherical Neumann function $10\, n_1(p_E^{\mathbb{R}} r)$, the dashed green line plots the modified spherical Bessel function $\mathrm{i}^1 j_1(\mathrm{i} p_E^{\mathbb{R}} r)$, and the dashed red line plots the modified spherical Neumann function $-0.01\, \mathrm{i}^{2+1} n_2(\mathrm{i} p_E^{\mathbb{R}} r)$. On the right hand side, the dashed green and red lines are the same as on the left, the dashed blue line plots the modified spherical Hankel function $k_0^{(1)}(p_E^{\mathbb{R}} r)$, and the dashed brown line plots the modified spherical Hankel function $0.001\, k_2^{(2)}(p_E^{\mathbb{R}} r)$.

2.5.2 Time-interval regions: Solutions and structures

Time-interval regions $\mathbb{M}_{[t_1,t_2]} = [t_1, t_2] \times \mathbb{R}^3$ are foliated by equal-time hypersurfaces Σ_t which we orient all pastwards (backwards in time). Here, the spaces $\mathrm{L}_{[t_1,t_2]}$ of solutions on the time-interval region and L_{Σ_1} of solutions in a neighborhood of Σ_1 coincide, because any solution near Σ_1 evolutes to cover the whole region. Hence the space $\mathrm{L}_{\partial[t_1,t_2]}$ of solutions near the boundary $\partial\mathbb{M}_{[t_1,t_2]} = \Sigma_1 \cup \overline{\Sigma}_2$ is twice $\mathrm{L}_{\mathbb{M}}$. This is easiest seen as follows: let $\eta \in \mathrm{L}_{\Sigma_1}$ a solution near Σ_1 and $\zeta \in \mathrm{L}_{\overline{\Sigma_2}}$ a solution near Σ_2. Thus the pair $(\eta, \zeta) \in \mathrm{L}_{\partial[t_1,t_2]}$ is a solution near the boundary $\partial\mathbb{M}_{[t_1,t_2]} = \Sigma_1 \cup \overline{\Sigma}_2$. However, ζ is not required to be the solution obtained by evoluting η towards Σ_2, rather ζ is independent of η. That is, $\mathrm{L}_{\partial[t_1,t_2]} = \mathrm{L}_{\Sigma_1} \times \mathrm{L}_{\Sigma_2}$.

Hence for time-interval regions we need to consider solutions well-defined near equal-time planes Σ_t. From the preceeding section we know that the only modes well defined on the whole equal-time plane are the spherical Bessel modes with $E^2 > m^2$. We use the notation of Section 5.3 of [59] for spherical coordinates, that is: $\Omega = (\theta, \varphi)$ and $\mathrm{d}^2\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$. The metric is $\mathrm{diag}\,(+1, -1, -r^2, -r^2 \sin^2\theta)$. For the angular momentum numbers l and m_l, the label l only serves to distinguish m_l from the mass m. The momentum space needed to assure consistency is $l \in \mathbb{N}_0$ with $m_l \in \{-l, \ldots, +l\}$, and $p > 0$ such that $E_p := \sqrt{p^2 + m^2} > m$. The reason for this is the orthogonality relation DLMF [1.17.14] which requires only positive momenta in order to give a delta function:

$$\int_0^\infty \mathrm{d}r\; r^2\; 2p\,(2\pi)^{-1/2}\, j_l(pr)\; 2p'\,(2\pi)^{-1/2}\, j_l(p'r) \;=\; \delta(p-p') \qquad p, p' > 0 \quad l > -3/2 \tag{2.87}$$

$$\int_0^\infty \mathrm{d}p\; p^2\; 2r\,(2\pi)^{-1/2}\, j_l(pr)\; 2r'\,(2\pi)^{-1/2}\, j_l(pr') \;=\; \delta(r-r') \qquad p, p' > 0 \quad l > -3/2\,. \tag{2.88}$$

Hence we encounter only propagating waves. The radial part of the spherical Bessel modes is real. For a real solution we thus get the expansion

$$\begin{aligned} \phi(t, r, \Omega) \;&=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 2p\,(2\pi)^{-1/2}\, j_l(pr) \Big\{ \phi_{plm_l}\, \mathrm{e}^{-\mathrm{i}E_p t}\, Y_l^{m_l}(\Omega) + \text{ c.c. } \Big\} \\ &=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} \Big\{ \phi_{plm_l} \mu^{(j)}_{plm_l}(t,r,\Omega) + \overline{\phi_{plm_l}}\; \overline{\mu^{(j)}_{plm_l}(t,r,\Omega)} \Big\}\,, \end{aligned} \tag{2.89}$$

and for a complex solution

$$\phi(t, r, \Omega) \;=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 2p\,(2\pi)^{-1/2}\, j_l(pr) \Big\{ \phi^+_{plm_l}\, \mathrm{e}^{-\mathrm{i}E_p t}\, Y_l^{m_l}(\Omega) + \overline{\phi^-_{plm_l}}\, \mathrm{e}^{\mathrm{i}E_p t}\, \overline{Y_l^{m_l}(\Omega)} \Big\} \tag{2.90}$$

$$=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} \Big\{ \phi^+_{plm_l} \mu^{(j)}_{plm_l}(t,r,\Omega) + \overline{\phi^-_{plm_l}}\; \overline{\mu^{(j)}_{plm_l}(t,r,\Omega)} \Big\}\,. \tag{2.91}$$

The condition that $p > 0$ is incorporated for the real case through $\phi_{p=0,l,m_l} = 0$ and in the complex case through $\phi^\pm_{p=0,l,m_l} = 0$. We use the following definition for the $\mu^{(j)}$-modes:

$$\mu^{(j)}_{plm_l}(t, r, \Omega) \;:=\; \frac{2p}{\sqrt{2\pi}}\, \mathrm{e}^{-\mathrm{i}E_p t} Y_l^{m_l}(\Omega)\, j_l(pr) \qquad\qquad E_p \;:=\; \sqrt{p^2 + m^2}\,. \tag{2.92}$$

Note that we could equivalently write the integration over positive p as an integration over $E > m$:

$$\int_0^\infty \mathrm{d}p\; p\; f(p) \;=\; \int_m^\infty \mathrm{d}E\; E\; f(p_E)\;.$$

Knowing the expansions of solutions permits us to evaluate the structures discussed in Section 2.4. Our equal-time planes are oriented pastwards, hence $\operatorname{sign}\Sigma_t = -1$. With this, (2.54) gives us the symplectic potential

$$\theta_\phi^{\Sigma_t}(\eta) \;=\; +\int_{\Sigma_t} \mathrm{d}r\, \mathrm{d}^2\Omega\; r^2\; \eta\,(\partial_t\phi) \tag{2.93}$$

$$= +\mathrm{i}\int_0^\infty \mathrm{d}p \sum_{l,m_l} E_p \Big\{ -\phi^+_{p,l,-m_l}\,\eta^+_{plm_l}\,\mathrm{e}^{-2\mathrm{i}tE_p} + \overline{\phi^-_{plm_l}}\,\eta^+_{plm_l} \tag{2.94}$$

$$+\,\overline{\phi^-_{p,l,-m_l}}\;\overline{\eta^-_{plm_l}}\,\mathrm{e}^{+2\mathrm{i}tE_p} - \phi^+_{plm_l}\,\overline{\eta^-_{plm_l}} \Big\}\;.$$

The symplectic structure then results from (2.55) or (2.52):

$$\omega_t\big(\eta,\,\zeta\big) \;=\; -\frac{1}{2}\int_{\Sigma_t} \mathrm{d}r\, \mathrm{d}^2\Omega\; r^2\; \big(\eta\,\partial_t\zeta - \zeta\,\partial_t\eta\big) \tag{2.95}$$

$$= +\mathrm{i}\int_0^\infty \mathrm{d}p \sum_{l,m_l} E_p \Big\{ \overline{\eta^-_{plm_l}}\,\zeta^+_{plm_l} - \eta^+_{plm_l}\,\overline{\zeta^-_{plm_l}} \Big\}\;. \tag{2.96}$$

As for cartesian coordinates, in spherical coordinates the positive and negative frequency modes also form Lagrangian subspaces of the complexified space of Klein-Gordon solutions in a neighborhood of the equal-time plane Σ_t, see (1.2.3) in [76]. The full space of Klein-Gordon solutions on this neighborhood is the direct sum of these subspaces. The standard complex structure for equal-time planes is the same as for cartesian coordinates and acts in the momentum representation as

$$\big(J_{\Sigma_t}\phi\big)^\pm_{plm_l} \;=\; -\mathrm{i}\,\phi^\pm_{plm_l}\;. \tag{2.97}$$

Next, the real g-product follows from (2.64)

$$\mathrm{g}_t\big(\eta,\,\zeta\big) \;=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 2E_p \Big\{ \overline{\eta^-_{plm_l}}\,\zeta^+_{plm_l} + \eta^+_{plm_l}\,\overline{\zeta^-_{plm_l}} \Big\}\;, \tag{2.98}$$

and the inner product from (2.65)

$$\{\eta,\,\zeta\}_{\Sigma_t} \;=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 4E_p\; \eta^+_{plm_l}\,\overline{\zeta^-_{plm_l}}\;. \tag{2.99}$$

The momentum space representation of polarized solutions can be easily calculated using the projectors' definitions (2.74), resulting in

$$\begin{aligned} \big(\hat{\mathrm{P}}^+_{\Sigma_t}\,\phi\big)^+_{plm_l} &= 0 & \qquad \big(\hat{\mathrm{P}}^+_{\Sigma_t}\,\phi\big)^-_{plm_l} &= \phi^-_{plm_l} \\ \big(\hat{\mathrm{P}}^-_{\Sigma_t}\,\phi\big)^+_{plm_l} &= \phi^+_{plm_l} & \qquad \big(\hat{\mathrm{P}}^-_{\Sigma_t}\,\phi\big)^-_{plm_l} &= 0\;. \end{aligned} \tag{2.100}$$

2.5.3 Rod and tube regions: Solutions and structures

On a rod region $\mathbb{M}_{r_0} = \mathbb{R}_t \times \mathbb{B}^3_{r_0}$ we can only allow solutions well defined for all $r \leq r_0$, in particular on the time axis $r = 0$. That is, there we have the spherical Bessel modes with $E^2 > m^2$ and the modified spherical Bessel modes with $E^2 < m^2$. We have to exclude the Neumann modes since they diverge on the time axis $r = 0$. For the tube region $\mathbb{M}_{[r_1,r_2]} = \mathbb{R}_t \times [r_1, r_2] \times \mathbb{S}^2$ we can allow the Neumann modes, since the time axis is not part of tube regions. Hence on tube regions the spherical Bessel and Neumann modes with $E^2 > m^2$ and also their modified versions with $E^2 < m^2$ are all well defined. Counting modes, we can already see that the space $\mathrm{L}^{\mathrm{tub}}_{[r_1,r_2]}$ of solutions on a tube region is twice as big as the space $\mathrm{L}^{\mathrm{rod}}_{r_0}$ of solutions on a rod region.

For the S-matrices we aim to construct, the relevant region is the rod region, since it covers the whole spacetime for $r \to \infty$. The boundaries of rod regions are the hypercylinders Σ_r, which we orient all inwards (backwards in radius). Hence we need to consider also the space of solutions L_{Σ_r} of solutions in the neighborhood of a hypercylinder. We can view these neighborhoods as tube regions and hence the spaces L_{Σ_r} and $\mathrm{L}^{\mathrm{tub}}_{[r_1,r_2]}$ coincide, that is, they consist of the spherical Bessel and Neumann modes with $E^2 > m^2$ together with their modified versions with $E^2 < m^2$. As for the time-interval regions, also for the rod regions the space of solutions near the boundary is twice as large as the space of solutions in the interior: the (spherical and modified) Bessel modes can be continued smoothly from the boundary to cover the whole interior, while the Neumann modes cannot.

The metric is again $\mathrm{diag}(+1, -1, -r^2, -r^2 \sin^2 \theta)$. For an equal-$r$ hypercylinder Σ_r, the momentum space needed to assure consistency is $E \in \mathbb{R}$ and $l \in \mathbb{N}_0$ with $m_l \in \{-l, \ldots, +l\}$ because an orthonormal basis on the hypercylinder is given by $\mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)$. Since we have to include energies $|E| < m$, we encounter evanescent modes. With $D = E^2 - m^2$ and $p^{\mathbb{R}}_E := \sqrt{|E^2 - m^2|}$, we define the function $\check{h}$ (which is denoted by d in Section 5.3 of [59]) as

$$\check{h}_{El}(r) := \check{\jmath}_{El}(r) + \mathrm{i}\, \check{n}_{El}(r) \tag{2.101}$$

$$\check{\jmath}_{El}(r) = \begin{cases} j_l(p^{\mathbb{R}}_E r) & D \geq 0 \\ \mathrm{i}^{-l}\, j_l(\mathrm{i}p^{\mathbb{R}}_E r) & D < 0 \end{cases} \qquad \check{n}_{El}(r) = \begin{cases} n_l(p^{\mathbb{R}}_E r) & D \geq 0 \\ \mathrm{i}^{l+1}\, n_l(\mathrm{i}p^{\mathbb{R}}_E r) & D < 0 \end{cases} . \tag{2.102}$$

Note that both $\mathrm{i}^l\, j_l(\mathrm{i}p^{\mathbb{R}}_E r)$ and $\mathrm{i}^{l+1}\, n_l(\mathrm{i}p^{\mathbb{R}}_E r)$ are always real for real $p^{\mathbb{R}}_E r$. We thus reproduce the expansion given in Section 5.3 of [59] for a real solution on a tube region

$$\phi(t, r, \Omega) = \int \mathrm{d}E \sum_{l,m_l} \frac{p^{\mathbb{R}}_E}{4\pi} \left\{ \phi_{Elm_l}\, \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{h}_{El}(r) + \text{ c.c. } \right\} \tag{2.103}$$

$$= \int \mathrm{d}E \sum_{l,m_l} \frac{p^{\mathbb{R}}_E}{4\pi} \left\{ \phi^a_{Elm_l} \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{\jmath}_{El}(r) + \phi^b_{Elm_l} \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{n}_{El}(r) \right\} , \tag{2.104}$$

where the two momentum representations relate to each other through

$$\begin{aligned} \phi^a_{Elm_l} &= \phi_{Elm_l} + \overline{\phi_{-E,l,-m_l}} & \qquad \phi_{Elm_l} &= \frac{1}{2} \left(\phi^a_{Elm_l} - \mathrm{i}\phi^b_{Elm_l} \right) \\ \phi^b_{Elm_l} &= \mathrm{i}\phi_{Elm_l} - \mathrm{i}\overline{\phi_{-E,l,-m_l}} & \qquad \overline{\phi_{Elm_l}} &= \frac{1}{2} \left(\phi^a_{-E,l,-m_l} + \mathrm{i}\phi^b_{-E,l,-m_l} \right) . \end{aligned} \tag{2.105}$$

This fulfills $\overline{\phi^{a,b}_{Elm_l}} = \phi^{a,b}_{-E,l,-m_l}$. For a complex solution on a tube region we have the expansion

$$\phi(t,r,\Omega) = \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{4\pi} \left\{ \phi^+_{Elm_l}\, \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{h}_{El}(r) + \overline{\phi^-_{Elm_l}}\, \mathrm{e}^{+\mathrm{i}Et} \overline{Y_l^{m_l}(\Omega)}\, \overline{\check{h}_{El}(r)} \right\} \tag{2.106}$$

$$= \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{4\pi} \left\{ \phi^a_{Elm_l} \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{\jmath}_{El}(r) + \phi^b_{Elm_l} \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{n}_{El}(r) \right\} \tag{2.107}$$

$$= \int \mathrm{d}E \sum_{l,m_l} \left\{ \phi^a_{Elm_l} \mu^{(a)}_{Elm_l}(t,r,\Omega) + \phi^b_{Elm_l} \mu^{(b)}_{Elm_l}(t,r,\Omega) \right\} , \tag{2.108}$$

wherein we use the modes

$$\begin{aligned} \mu^{(a)}_{Elm_l}(t,r,\Omega) &:= \frac{p_E^{\mathbb{R}}}{4\pi}\, \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{\jmath}_{El}(r) \\ \mu^{(b)}_{Elm_l}(t,r,\Omega) &:= \frac{p_E^{\mathbb{R}}}{4\pi}\, \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{n}_{El}(r) . \end{aligned} \tag{2.109}$$

Now the momentum representations relate via

$$\begin{aligned} \phi^a_{Elm_l} &= \phi^+_{Elm_l} + \overline{\phi^-_{-E,l,-m_l}} & \qquad \phi^+_{Elm_l} &= \tfrac{1}{2}\, \phi^a_{Elm_l} - \tfrac{\mathrm{i}}{2}\, \phi^b_{Elm_l} \\ \phi^b_{Elm_l} &= \mathrm{i}\phi^+_{Elm_l} - \mathrm{i}\, \overline{\phi^-_{-E,l,-m_l}} & \qquad \overline{\phi^-_{-E,l,-m_l}} &= \tfrac{1}{2}\, \phi^a_{Elm_l} + \tfrac{\mathrm{i}}{2}\, \phi^b_{Elm_l} , \end{aligned} \tag{2.110}$$

which in matrix form writes

$$\begin{pmatrix} \phi^a_{Elm_l} \\ \phi^b_{Elm_l} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \phi^+_{Elm_l} \\ \overline{\phi^-_{-E,l,-m_l}} \end{pmatrix} \qquad \begin{pmatrix} \phi^+_{Elm_l} \\ \overline{\phi^-_{-E,l,-m_l}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} \begin{pmatrix} \phi^a_{Elm_l} \\ \phi^b_{Elm_l} \end{pmatrix}, \tag{2.111}$$

and turns into (2.105) by setting $\phi^+_{Elm_l} = \phi^-_{Elm_l}$. As for the equal-time plane, for $E^2 > m^2$ we can transform integration over positive p into integration over E and vice versa:

$$\begin{aligned} \int_m^\infty \mathrm{d}E\; E\; f(E) &= \int_0^\infty \mathrm{d}p\; p\; f(+E_p) & \qquad p &:= +\sqrt{|E^2 - m^2|} = +\sqrt{E^2 - m^2} \\ \int_{-\infty}^{-m} \mathrm{d}E\; E\; f(E) &= -\int_0^\infty \mathrm{d}p\; p\; f(-E_p) & \qquad E_p &:= +\sqrt{m^2 + p^2} . \end{aligned}$$

For $E^2 < m^2$ we get in turn

$$\begin{aligned} \int_0^m \mathrm{d}E\; E\; f(E) &= \int_0^m \mathrm{d}p\; p\; f(+E_p) & \qquad p &:= +\sqrt{|E^2 - m^2|} = +\sqrt{m^2 - E^2} \\ \int_{-m}^{0} \mathrm{d}E\; E\; f(E) &= -\int_0^m \mathrm{d}p\; p\; f(-E_p) & \qquad E_p &:= +\sqrt{m^2 - p^2} . \end{aligned}$$

For solutions on rod regions we only dispose of the Bessel modes and thus the expansion for complex solution becomes

$$\phi(t,r,\Omega) = \int \mathrm{d}E \sum_{l,m_l} \phi^a_{Elm_l} \mu^{(a)}_{Elm_l}(t,r,\Omega) . \tag{2.112}$$

Knowing the solutions' expansions, we can again evaluate the various structures on the space of solutions. Our hypercylinders are oriented inwards, hence $\operatorname{sign}\Sigma_r = -1$. The symplectic potential from (2.54) now takes the form

$$\theta_{\phi}^{\Sigma_r}(\eta) = -r^2 \int_{\Sigma_r} \mathrm{d}t\mathrm{d}^2\Omega\, \eta\, (\partial_r \phi) \tag{2.113}$$

$$= -\int \mathrm{d}E \sum_{l,m_l} \frac{(r p_E^{\mathbb{R}})^2}{8\pi} \Big\{ \phi^+_{-E,l,-m_l}\, \eta^+_{Elm_l}\, \check{h}_{El}(r)\, \partial_r\, \check{h}_{El}(r) + \overline{\phi^-_{Elm_l}}\, \eta^+_{Elm_l}\, \check{h}_{El}(r)\, \partial_r\, \overline{\check{h}_{El}(r)} \tag{2.114}$$

$$+ \overline{\phi^-_{-E,l,-m_l}}\, \overline{\eta^-_{Elm_l}}\, \overline{\check{h}_{El}(r)}\, \partial_r\, \overline{\check{h}_{El}(r)} + \phi^+_{Elm_l}\, \overline{\eta^-_{Elm_l}}\, \overline{\check{h}_{El}(r)}\, \partial_r\, \check{h}_{El}(r) \Big\} .$$

Then, (2.55) for the symplectic structure becomes

$$\omega_r(\eta, \zeta) = \frac{r^2}{2} \int_{\Sigma_r} \mathrm{d}t\mathrm{d}^2\Omega\, \big(\eta\, \partial_r \zeta - \zeta\, \partial_r \eta\big) \tag{2.115}$$

$$= \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{16\pi} \Big\{ \eta^a_{Elm_l}\, \zeta^b_{-E,l,-m_l} - \eta^b_{Elm_l}\, \zeta^a_{-E,l,-m_l} \Big\} \tag{2.116}$$

$$= \mathrm{i} \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{8\pi} \Big\{ \overline{\eta^-_{Elm_l}}\, \zeta^+_{Elm_l} - \eta^+_{Elm_l}\, \overline{\zeta^-_{Elm_l}} \Big\} . \tag{2.117}$$

The positive and negative frequency modes form Lagrangian subspaces of the complexified space of Klein-Gordon solutions in a neighborhood of the equal-r hypercylinder Σ_r, see (1.2.3) in [76]. The Bessel and Neumann modes form Lagrangian subspaces both on the real and complexified version of the solutions space. In both mode decompositions, the full space of Klein-Gordon solutions on this neighborhood is the direct sum of the two Lagrangian subspaces. One choice for a complex structure is given in equation (81) of [59] (we call it the positive complex structure because it induces a real g-product which is positive-definite for real solutions):

$$\big(J^{\mathrm{pos}}_{\Sigma_r}\, \phi\big)^{\pm}_{Elm_l} = -\mathrm{i}\, \phi^{\pm}_{Elm_l} , \tag{2.118}$$

which is equivalent to

$$\begin{pmatrix} \big(J^{\mathrm{pos}}_{\Sigma_r} \phi\big)^a_{Elm_l} \\ \big(J^{\mathrm{pos}}_{\Sigma_r} \phi\big)^b_{Elm_l} \end{pmatrix} = \begin{pmatrix} -\phi^b_{Elm_l} \\ +\phi^a_{Elm_l} \end{pmatrix} . \tag{2.119}$$

With that, the real g-product becomes

$$\mathrm{g}^{\mathrm{pos}}_{\Sigma_r}(\eta, \zeta) = \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{8\pi} \Big\{ \eta^a_{Elm_l}\, \zeta^a_{-E,l,-m_l} + \eta^b_{Elm_l}\, \zeta^b_{-E,l,-m_l} \Big\} \tag{2.120}$$

$$= \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{4\pi} \Big\{ \overline{\eta^-_{Elm_l}}\, \zeta^+_{Elm_l} + \eta^+_{Elm_l}\, \overline{\zeta^-_{Elm_l}} \Big\} , \tag{2.121}$$

and the inner product results as

$$\{\eta, \zeta\}^{\mathrm{pos}}_{\Sigma_r} = \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{8\pi} \Big\{ \eta^a_{Elm_l}\, \zeta^a_{-E,l,-m_l} + \eta^b_{Elm_l}\, \zeta^b_{-E,l,-m_l} + \mathrm{i}\eta^a_{Elm_l}\, \zeta^b_{-E,l,-m_l} - \mathrm{i}\eta^b_{Elm_l}\, \zeta^a_{-E,l,-m_l} \Big\}$$

$$= \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{2\pi}\, \eta^+_{Elm_l}\, \overline{\zeta^-_{Elm_l}} . \tag{2.122}$$

The momentum space representation of the polarized solutions with $J^{\text{pos}}_{\Sigma_r}$ becomes

$$\begin{aligned} \big(\hat{\mathrm{P}}^{+}_{\Sigma_r}\,\phi\big)^{+}_{Elm_l} &= 0 & \big(\hat{\mathrm{P}}^{+}_{\Sigma_r}\,\phi\big)^{-}_{Elm_l} &= \phi^{-}_{Elm_l} \\ \big(\hat{\mathrm{P}}^{-}_{\Sigma_r}\,\phi\big)^{+}_{Elm_l} &= \phi^{+}_{Elm_l} & \big(\hat{\mathrm{P}}^{-}_{\Sigma_r}\,\phi\big)^{-}_{Elm_l} &= 0\,. \end{aligned} \tag{2.123}$$

The results of this section agree with those of Section 5.3 in [59]. Note that in the symplectic potential there is still a case distinction according to the sign of $D = E^2 - m^2$, while in the symplectic structure there appears no such case distinction. This happens because in the symplectic structure the radial functions enter only via their Wronskian, which is the same for $D > 0$ and $D < 0$.

2.5.4 Isometries in Minkowski spacetime

The isometries on Minkowski spacetime are the three spatial translations in directions $x^{1,2,3}$, the time translation, the spatial rotations and the boosts. Together they form the Poincaré group. The standard S-matrix is invariant under the action of these isometries, and therefore we want to show the same for the GBF amplitudes (whose asymptotical limits can be regarded as S-matrices on Minkowski spacetime). However, here we only consider boosts and spatial translations, because for spatial rotations and time translations the actions and invariance/commutation for Minkowski are easily found and completely analogous to Anti de Sitter, for which we give the explicit expressions in Section 2.6.6. The spatial translations in x^3-direction are generated by

$$T_3 \,=\, \partial_{x^3} \,=\, \frac{\partial r}{\partial x^3}\,\partial_r + \frac{\partial \cos\theta}{\partial x^3}\,\partial_{\cos\theta} \,=\, \cos\theta\,\partial_r + \frac{(1-\cos^2\theta)}{r}\,\partial_{\cos\theta}, \tag{2.124}$$

and the boosts in the (t, x^3)-plane are generated by

$$K_{03} \,=\, -t\,\partial_{x^3} - x^3\,\partial_t \,=\, -t\cos\theta\,\partial_r - \frac{t}{r}\,(1-\cos^2\theta)\partial_{\cos\theta} - r\cos\theta\,\partial_t. \tag{2.125}$$

The quantization we shall use is called Holomorphic Quantization and is described in Section 3.1. In order to have amplitudes invariant under the actions of isometries it turns out that we need two properties to hold see Section 3.1.8. First, that the symplectic structure on a region's boundary be invariant under these actions, and second, that the complex structure commutes with these actions. For the symplectic structures on equal-time planes and hypercylinders we already dispose of the expressions (2.96) and (2.116) which use the momentum representation of the solutions. As in e.g. (2.119) and (2.97), we also express the action of the complex structures in the momentum representation. Hence for showing invariance/commutation, we also need the actions of the isometries in the momentum representation. That is, the derivatives with respect to spacetime coordinates in the above expressions for T_3 and K_{03} need to be rewritten as operators in momentum space. For the time-interval region we calculate these in Appendix B.2.2, and for rod and tube regions in Appendix B.2.1.

Minkowski time-interval region

Using expansion (2.91)

$$\phi(t,r,\Omega) \,=\, \int_0^\infty \mathrm{d}p \sum_{l,m_l} \Big\{ \phi^{+}_{plm_l}\,\mu^{(j)}_{plm_l}(t,r,\Omega) + \overline{\phi^{-}_{plm_l}}\;\overline{\mu^{(j)}_{plm_l}(t,r,\Omega)} \Big\}, \tag{2.126}$$

we find the following actions in the momentum representation:

$$\begin{aligned} \big(T_3 \triangleright \phi\big)^{+}_{plm_l} &= +p\,\chi^{(2)}_{-}(l,m_l)\,\phi^{+}_{p,l-1,m_l} - p\,\chi^{(2)}_{+}(l,m_l)\,\phi^{+}_{p,l+1,m_l} \\ \overline{\big(T_3 \triangleright \phi\big)^{-}_{plm_l}} &= +p\,\chi^{(2)}_{-}(l,m_l)\,\overline{\phi^{-}_{p,l-1,m_l}} - p\,\chi^{(2)}_{+}(l,m_l)\,\overline{\phi^{-}_{p,l+1,m_l}} \end{aligned} \tag{2.127}$$

and

$$\left(K_{03}\triangleright\phi\right)^{+}_{plm_l} = +\mathrm{i}\chi^{(2)}_{-}(l,m_l)\left(\partial_p E_p\phi^{+}_{p,l-1,m_l}\right) - \mathrm{i}(l)\,\tfrac{E_p}{p}\,\chi^{(2)}_{-}(l,m_l)\,\phi^{+}_{p,l-1,m_l} - \mathrm{i}\chi^{(2)}_{+}(l,m_l)\left(\partial_p E_p\phi^{+}_{p,l+1,m_l}\right) - \mathrm{i}(l+1)\,\tfrac{E_p}{p}\,\chi^{(2)}_{+}(l,m_l)\,\phi^{+}_{p,l+1,m_l} \tag{2.128}$$

$$\overline{\left(K_{03}\triangleright\phi\right)^{-}_{plm_l}} = -\mathrm{i}\chi^{(2)}_{-}(l,m_l)\left(\partial_p E_p\overline{\phi^{-}_{p,l-1,m_l}}\right) + \mathrm{i}(l)\,\tfrac{E_p}{p}\,\chi^{(2)}_{-}(l,m_l)\,\overline{\phi^{-}_{p,l-1,m_l}} + \mathrm{i}\chi^{(2)}_{+}(l,m_l)\left(\partial_p E_p\overline{\phi^{-}_{p,l+1,m_l}}\right) + \mathrm{i}(l+1)\,\tfrac{E_p}{p}\,\chi^{(2)}_{+}(l,m_l)\,\overline{\phi^{-}_{p,l+1,m_l}}. \tag{2.129}$$

Minkowski rod and tube regions

Near a hypercylinder, we expand a solution as in (2.108):

$$\phi(t,r,\Omega) \;=\; \int\!\mathrm{d}E\sum_{l,m_l}\Big\{\phi^{a}_{Elm_l}\mu^{(a)}_{Elm_l}(t,r,\Omega) + \phi^{b}_{Elm_l}\mu^{(b)}_{Elm_l}(t,r,\Omega)\Big\}\,. \tag{2.130}$$

With the χ-factors (A.17), the spatial translation T_3 then acts in the momentum representation for $E^2 \gtrless m^2$ as in (B.39):

$$\begin{aligned}(T_3\triangleright\phi)^{a}_{Elm_l} &= \pm p^{\mathbb{R}}_{E}\,\chi^{(2)}_{-}(l,m_l)\,\phi^{a}_{E,l-1,m_l} - p^{\mathbb{R}}_{E}\,\chi^{(2)}_{+}(l,m_l)\,\phi^{a}_{E,l+1,m_l}\\ (T_3\triangleright\phi)^{b}_{Elm_l} &= +p^{\mathbb{R}}_{E}\,\chi^{(2)}_{-}(l,m_l)\,\phi^{b}_{E,l-1,m_l} \mp p^{\mathbb{R}}_{E}\,\chi^{(2)}_{+}(l,m_l)\,\phi^{b}_{E,l+1,m_l}.\end{aligned} \tag{2.131}$$

The boost K_{03} for $E^2 \gtrless m^2$ results in the action (B.54):

$$(K_{03}\triangleright\phi)^{a}_{Elm_l} = \pm\chi^{(2)}_{-}(l,m_l)\big(\mathrm{i}p^{\mathbb{R}}_{E}\partial_E\phi^{a}_{E,l-1,m_l}\big) - \chi^{(2)}_{+}(l,m_l)\big(\mathrm{i}p^{\mathbb{R}}_{E}\partial_E\phi^{a}_{E,l+1,m_l}\big) - \mathrm{i}(E/p^{\mathbb{R}}_{E})(l-1)\chi^{(2)}_{-}(l,m_l)\,\phi^{a}_{E,l-1,m_l} \mp \mathrm{i}(E/p^{\mathbb{R}}_{E})(l+2)\chi^{(2)}_{+}(l,m_l)\,\phi^{a}_{E,l+1,m_l} \tag{2.132}$$

$$(K_{03}\triangleright\phi)^{b}_{Elm_l} = +\chi^{(2)}_{-}(l,m_l)\big(\mathrm{i}p^{\mathbb{R}}_{E}\partial_E\phi^{b}_{E,l-1,m_l}\big) \mp \chi^{(2)}_{+}(l,m_l)\big(\mathrm{i}p^{\mathbb{R}}_{E}\partial_E\phi^{b}_{E,l+1,m_l}\big) \mp \mathrm{i}(E/p^{\mathbb{R}}_{E})(l-1)\chi^{(2)}_{-}(l,m_l)\,\phi^{b}_{E,l-1,m_l} - \mathrm{i}(E/p^{\mathbb{R}}_{E})(l+2)\chi^{(2)}_{+}(l,m_l)\,\phi^{b}_{E,l+1,m_l}. \tag{2.133}$$

For both T_3 and K_{03}, the signs of the corresponding terms in the actions on a and b-modes coincide for $E^2 > m^2$ and are opposite for $E^2 < m^2$.

2.5.5 Invariance of symplectic structures under isometries

In this section we make use of the isometries' actions in the momentum representation in order to verify the invariance under them of the symplectic structures on the equal-time plane and on the hypercylinder.

Prelude: Boosts in cartesian coordinates

Showing the boost invariance of the symplectic structures ω_t on the equal-time plane and ω_r on the hypercylinder is somewhat unwieldy in spherical coordinates. In order to outline the method we use in a setting which allows for clearer equations, we here briefly treat the symplectic structure on the equal-time plane Σ_t in cartesian coordinates. There, solutions are just superpositions of planes waves, which greatly simplifies the calculations. As given in Section 5.1 of [59], the expansion of a solution $\phi(t,\underline{x})$ near Σ_t writes as

$$\phi(t,\underline{x}) \;=\; \int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{(2\pi)^3\,2E_{\underline{k}}}\left(\phi^{+}_{\underline{k}}\,\mathrm{e}^{-\mathrm{i}(Et-\underline{k}\,\underline{x})} + \overline{\phi^{-}_{\underline{k}}}\,\mathrm{e}^{+\mathrm{i}(Et-\underline{k}\,\underline{x})}\right), \tag{2.134}$$

and the symplectic structure becomes

$$\omega_t(\eta,\zeta) \;=\; \tfrac{\mathrm{i}}{2}\int_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{(2\pi)^3\,2E_{\underline{k}}}\left(\overline{\eta^{-}_{\underline{k}}}\,\zeta^{+}_{\underline{k}} - \eta^{+}_{\underline{k}}\,\overline{\zeta^{-}_{\underline{k}}}\right). \tag{2.135}$$

We note, that both the solution's expansion and the symplectic structure converge to finite values, if the momentum representation $\phi^{\pm}_{\underline{k}}$ vanishes for $|\underline{k}| \to \infty$. Let us now consider the action of the boost generator $K_{03} = -t\partial_3 - x^3\partial_t$, which calculates straightforwardly to

$$K_{03} \triangleright \mathrm{e}^{\mp\mathrm{i}(Et-\underline{k}\,\underline{x})} := -K_{03}\mathrm{e}^{\mp\mathrm{i}(Et-\underline{k}\,\underline{x})} = \mp\mathrm{i}\big(-tk_3 + x^3 E_{\underline{k}}\big)\,\mathrm{e}^{\mp\mathrm{i}(Et-\underline{k}\,\underline{x})}. \tag{2.136}$$

It is straightforward to verify that the right hand side is again a solution of the homogeneous Klein-Gordon equation $(-\partial_t^2+\underline{\partial}^2-m^2)\,\phi = 0$. We have calculated the action (2.136) using the configuration space representation $-t\partial_3 - x^3\partial_t$ of the boost generator K_{03}. Since both the solution's expansion (2.134) and the symplectic structure (2.135) use the momentum space, we need to find a momentum space representation of K_{03}, which reproduces the action (2.136). An intuitive first guess would be that the momentum representation of K_{03} reflects the configuration one through a form like $K_{03} = -E\partial_{k^3} - k^3\partial_E$. However, since E depends on $\underline{k}$, it turns out that the form of the boost generator actually is simpler. As can easily be verified, the action (2.136) is recovered by setting

$$K_{03} \triangleright \mathrm{e}^{\mp\mathrm{i}(Et-\underline{k}\,\underline{x})} := -E_{\underline{k}}\partial_{k^3}\,\mathrm{e}^{\mp\mathrm{i}(Et-\underline{k}\,\underline{x})}. \tag{2.137}$$

That is, with respect to the plane waves the boost generator's momentum representation writes as $K_{03} = E_{\underline{k}}\partial_{k^3}$. We thus have obtained

$$\begin{aligned}(K_{03} \triangleright \phi)(t,\underline{x}) &= \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{(2\pi)^3\,2E_{\underline{k}}} \Big(\phi^{+}_{\underline{k}}\,(-K_{03})\mathrm{e}^{-\mathrm{i}(Et-\underline{k}\,\underline{x})} + \overline{\phi^{-}_{\underline{k}}}\,(-K_{03})\mathrm{e}^{+\mathrm{i}(Et-\underline{k}\,\underline{x})}\Big) \\ &= \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{(2\pi)^3\,2E_{\underline{k}}} \Big(\phi^{+}_{\underline{k}}\,(-E_{\underline{k}}\partial_{k^3})\mathrm{e}^{-\mathrm{i}(Et-\underline{k}\,\underline{x})} + \overline{\phi^{-}_{\underline{k}}}\,(-E_{\underline{k}}\partial_{k^3})\mathrm{e}^{+\mathrm{i}(Et-\underline{k}\,\underline{x})}\Big).\end{aligned} \tag{2.138}$$

In order to obtain the momentum representation $(K_{03} \triangleright \phi)^{\pm}_{\underline{k}}$, we now need to let $E_{\underline{k}}\partial_{k^3}$ act to the left. This can be done by noting that it is an antisymmetric operator with respect to the integral we use here:

$$\begin{aligned}\int\limits_{\mathbb{R}} \frac{\mathrm{d}k}{(2\pi)^3\,2E_{\underline{k}}}\, f(k^3)\,(-E_{\underline{k}}\partial_{k^3})\,g(k^3) &= \int\limits_{\mathbb{R}} \frac{\mathrm{d}k}{(2\pi)^3\,2}\, f(k^3)\,(-\partial_{k^3})\,g(k^3) \\ &= (2\pi)^{-3}\tfrac{1}{2}\,fg|^{+\infty}_{-\infty} + \int\limits_{\mathbb{R}} \frac{\mathrm{d}k}{(2\pi)^3\,2}\, g(k^3)\,(\partial_{k^3})\,f(k^3) \\ &= +\int\limits_{\mathbb{R}} \frac{\mathrm{d}k}{(2\pi)^3\,2E_{\underline{k}}}\, g(k^3)\,(E_{\underline{k}}\partial_{k^3})\,f(k^3),\end{aligned}$$

wherein we have used that f vanishes for $|k^3| \to \infty$ while g remains bounded. Therefore we can write (2.140) as

$$(K_{03} \triangleright \phi)(t,\underline{x}) = \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{(2\pi)^3\,2E_{\underline{k}}} \Big(\big(E_{\underline{k}}\partial_{k^3}\phi^{+}_{\underline{k}}\big)\,\mathrm{e}^{-\mathrm{i}(Et-\underline{k}\,\underline{x})} + \big(E_{\underline{k}}\partial_{k^3}\overline{\phi^{-}_{\underline{k}}}\,\big)\,\mathrm{e}^{+\mathrm{i}(Et-\underline{k}\,\underline{x})}\Big). \tag{2.139}$$

We have thus found the momentum representation of $(K_{03} \triangleright \phi)$:

$$(K_{03} \triangleright \phi)^{\pm}_{\underline{k}} = E_{\underline{k}}\partial_{k^3}\phi^{\pm}_{\underline{k}}. \tag{2.140}$$

Now we can verify the invariance of the symplectic structure ω_t. Relation (2.59) tells us that to this end we need to show that

$$0 = \big(K_{03} \triangleright \omega\big)(\eta,\zeta) = -\omega\big(K_{03} \triangleright \eta,\,\zeta\big) - \omega\big(\eta,\,K_{03} \triangleright \zeta\big)\;. \tag{2.141}$$

Starting from (2.135) and using (2.140) we obtain (in the last line we use that both $\eta^{\pm}_{\underline{k}}$ and $\zeta^{\pm}_{\underline{k}}$ vanish for $|\underline{k}| \to \infty$, and hence the derivative ∂_{k^3} is an antisymmetric operator in this integral as well, and changing its direction in the first two terms yields the final result):

$$\begin{aligned}\big(K_{03}\triangleright\omega\big)(\eta,\zeta) &= -\tfrac{\mathrm{i}}{2}\int\limits_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{(2\pi)^3\,2E_{\underline{k}}}\Big(\overline{(K_{03}\triangleright\eta)^-_{\underline{k}}}\,\zeta^+_{\underline{k}} - (K_{03}\triangleright\eta)^+_{\underline{k}}\,\overline{\zeta^-_{\underline{k}}} + \overline{\eta^-_{\underline{k}}}\,(K_{03}\triangleright\zeta)^+_{\underline{k}} - \eta^+_{\underline{k}}\,\overline{(K_{03}\triangleright\zeta)^-_{\underline{k}}}\Big)\\ &= -\tfrac{\mathrm{i}}{2}\int\limits_{\mathbb{R}^3}\frac{\mathrm{d}^3k}{(2\pi)^3\,2}\Big((\partial_{k^3}\overline{\eta^-_{\underline{k}}})\,\zeta^+_{\underline{k}} - (\partial_{k^3}\eta^+_{\underline{k}})\,\overline{\zeta^-_{\underline{k}}} + \overline{\eta^-_{\underline{k}}}\,(\partial_{k^3}\zeta^+_{\underline{k}}) - \eta^+_{\underline{k}}\,(\partial_{k^3}\overline{\zeta^-_{\underline{k}}})\Big)\\ &= 0.\end{aligned}$$

We have thus verified the invariance of the symplectic structure ω_t on equal-time hyperplanes in Minkowski spacetime, using cartesian coordinates. The technique we use consists in rewriting the boost generator K_{03} in its momentum space representation. This provides us with the momentum representation of the solution $K_{03}\triangleright\phi$. The antisymmetry of K_{03} in momentum space integrals then yields the invariance of ω_t. We will apply the same technique for ω_t and ω_r in spherical coordinates, however, the expressions become considerably less elegant there.

Minkowski time-interval region

We start with the symplectic structure (2.96) on the equal-time plane:

$$\omega_t(\eta,\,\zeta) \;=\; +\mathrm{i}\int\limits_0^\infty\!\mathrm{d}p\sum_{l,m_l}E_p\left\{\overline{\eta^-_{plm_l}}\,\zeta^+_{plm_l} - \eta^+_{plm_l}\,\overline{\zeta^-_{plm_l}}\right\}. \tag{2.142}$$

For an infinitesimal translation $1+\epsilon T_3$ invariance means that $\omega_t(\eta,\zeta) \overset{!}{=} ((1+\epsilon T_3)\triangleright\omega_t)\,(\eta,\zeta)$. Up to linear order in ϵ we have $((1+\epsilon T_3)\triangleright\omega_t)\,(\eta,\zeta) = \omega_t(\eta,\zeta)+\omega_t(-\epsilon T_3\triangleright\eta,\zeta)+\omega_t(\eta,-\epsilon T_3\triangleright\zeta)$. Hence invariance is to require that

$$0 \overset{!}{=} \omega_t(T_3\triangleright\eta,\zeta)+\omega_t(\eta,T_3\triangleright\zeta). \tag{2.143}$$

Showing that this actually holds is straightforward: starting with (2.142), plugging in the actions (2.131), shifting $l \to l+1$ in all the terms containing χ_--factors and applying relation (A.18): $\chi^{(2)}_-(l+1,m_l) = \chi^{(2)}_+(l,m_l)$ yields a zero sum. Analoguously, for boosts the invariance amounts to

$$\begin{aligned}0 &\overset{!}{=} \omega_t(K_{03}\triangleright\eta,\zeta)+\omega_t(\eta,K_{03}\triangleright\zeta)\\ &= +\mathrm{i}\int\limits_0^\infty\!\mathrm{d}p\sum_{l,m_l}\Big\{\overline{\big(K_{03}\triangleright\eta\big)^-_{plm_l}}\,E_p\zeta^+_{plm_l} - \big(K_{03}\triangleright\eta\big)^+_{plm_l}\,E_p\overline{\zeta^-_{plm_l}}\\ &\qquad\qquad + E_p\overline{\eta^-_{plm_l}}\,\big(K_{03}\triangleright\zeta\big)^+_{plm_l} - E_p\eta^+_{plm_l}\,\overline{\big(K_{03}\triangleright\zeta\big)^-_{plm_l}}\Big\}.\end{aligned}$$

That this actually holds can be checked by plugging in the actions (2.128), leading to an integral over 16 terms. Eight of these contain no derivative and cancel already after appropriately shifting $l \to l\pm1$. The other eight contain a derivative of the form $(E_pf)(\partial_pE_pg)$. Reverting the derivative to the left in four of these terms plus shifting $l \to l\pm1$ then cancels these terms as well, giving us zero. For reverting the derivatives, we note that we can use again the condition, that the involved functions vanish for $p=0$ and $p\to\infty$. Then, the rule for reverting the derivatives is simply

$$\overrightarrow{\partial_pE_p} \;\to\; -\overleftarrow{\partial_pE_p}, \tag{2.144}$$

that is: $(E_pf)(\partial_pE_pg) \to -(\partial_pE_pf)(E_pg)$. We have thus shown also in spherical coordinates, that the symplectic structure on the Minkowski equal-time plane is invariant under the actions of all Minkowski isometries.

Minkowski rod region

We continue with the symplectic structure (2.116) on the hypercylinder:

$$\omega_r(\eta, \zeta) = \int dE \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{16\pi} \left\{ \eta^a_{Elm_l} \zeta^b_{-E,l,-m_l} - \eta^b_{Elm_l} \zeta^a_{-E,l,-m_l} \right\} . \tag{2.145}$$

For the translation T_3 invariance means again that

$$0 \overset{!}{=} \omega_r(T_3 \triangleright \eta, \zeta) + \omega_r(\eta, T_3 \triangleright \zeta). \tag{2.146}$$

That this actually holds can be seen as follows: starting with (2.145), plugging in the actions (2.131) shifting $l \rightarrow l{+}1$ in the terms containing χ_--factors and applying (A.18): $\chi_-^{(2)}(l{+}1, m_l) = \chi_+^{(2)}(l, m_l)$ yields zero. Analoguously, for boosts the invariance amounts to

$$\begin{aligned} 0 &\overset{!}{=} \omega_r(K_{03} \triangleright \eta, \zeta) + \omega_r(\eta, K_{03} \triangleright \zeta) \\ &= \int dE \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{16\pi} \Big\{ \big(K_{03} \triangleright \eta\big)^a_{Elm_l} \zeta^b_{-E,l,-m_l} - \big(K_{03} \triangleright \eta\big)^b_{Elm_l} \zeta^a_{-E,l,-m_l} \\ &\qquad + \eta^a_{Elm_l} \big(K_{03} \triangleright \zeta^b\big)_{-E,l,-m_l} - \eta^b_{Elm_l} \big(K_{03} \triangleright \zeta\big)^a_{-E,l,-m_l} \Big\}. \end{aligned}$$

This holds as well: plugging into this the action (2.132), and letting in the terms of the second row above act the derivatives to the left following rule (B.53): $\overrightarrow{\mathrm{i}p^2 \, \partial_E} \rightarrow -\overleftarrow{\mathrm{i}p^2 \, \partial_E} \mp 2\mathrm{i}E$, we obtain sixteen terms, which consist of eight pairs whose terms nicely cancel each other.

We have thus verified that the symplectic structure ω_r on the Minkowski hypercylinders is indeed invariant under the actions of all isometries.

2.6 Anti de Sitter Spacetime (AdS): Classical Klein-Gordon Theory

2.6.1 Introduction

Three classic spacetimes of constant scalar curvature in Mathematical Physics are Minkowski spacetime $\mathbb{R}^{1,d}$, de Sitter $\mathrm{dS}_{1,d}$ and Anti de Sitter $\mathrm{AdS}_{1,d}$ (wherein d is the spatial dimension). They all have constant (zero, positive and negative) curvature. The particular interest for field theory on these spacetimes is due to their high degree of symmetry: each of them possesses the maximum number of $(d{+}1)(d{+}2)/2$ (linear independent) Killing vector fields, that is, spacetime isometries, and therefore they are called maximally symmetric spacetimes.

Apart from QFT on curved spacetime, current research related to AdS concerns mainly two topics: Black Holes and String Theory. In classical General Relativity, Bizoń and Rostworowski in [12] find evidence that pure AdS is unstable. That is, an initial AdS spacetime coupled to a massless scalar field (with no-flux boundary condition on the timelike boundary) develops an apparent horizon under arbitrarily small initial values of the scalar field. By contrast, Holzegel and Smulevici in [39] find that Schwarzschild-AdS spacetimes (the black hole solutions of the Einstein equation with negative cosmological constant in the vacuum) are asymptotically stable (that is, small perturbations of Schwarzschild-AdS initial data evolute to black holes, with the metric on the black hole exterior approaching a Schwarzschild-AdS spacetime for large times). Yagdjian and Galstian in [77] investigate the limit of vanishing black hole mass in Schwarzschild-AdS spacetime (that is: pure AdS), and find the solution of the Cauchy problem for the Klein-Gordon (KG) equation $(\Box{-}m^2)\phi(x) = f(x)$ with source term $f(x)$. AdS has been one of the most studied spacetimes in String Theory since the late 90's. This was caused by the famous conjecture of Maldacena [44], about a correspondence between type IIB string theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$ background and four-dimensional $\mathcal{N} = 4$ Super Yang-Mills theory on this spacetime's boundary $\partial(\mathrm{AdS}_5 \times \mathbb{S}^5) = \partial\mathrm{AdS}_5 = \mathbb{R} \times \mathbb{S}^3$. Further, in [75] Witten argues that a version of this correspondence is related to the thermodynamics of AdS black holes.

Despite AdS being such an object of interest, we found in the literature only the standard symplectic structure for standard Klein-Gordon solutions (well defined and bounded on all of AdS). Its time-independence is well known [7]. Some studies have been done of solutions that are not regular on all of space, e.g. [9]. These nonstandard solutions are well defined and bounded on (rod respectively tube) hypercylinder regions, see Section 2.6.3. However, we have found no mention of a symplectic structure for these nonstandard solutions in the literature. Neither have we found addressed the issues of isometry actions on the solutions, nor the isometry-invariance of the symplectic structure(s). We close this gap by introducing a natural symplectic structure for the nonstandard Klein-Gordon solutions, and showing the isometry invariance of both standard and new symplectic structure. To this end we calculate the actions of isometries on the solutions, and as a byproduct find some contiguous relations for hyperspherical harmonics and Jacobi polynomials. Moreover, we compare our results to the corresponding cases for Klein-Gordon theory on Minkowski spacetime. We find correspondences between the flat limit of AdS Killing vectors, field expansions and symplectic structures and the respective Minkowski counterparts. In particular, we give the symplectic structure for hypercylinder surfaces Σ_ρ, which turns out to be independent of the radius ρ. The hypercylinder regions and surfaces are of interest for the conjectured AdS/CFT correspondence, because the AdS boundary is a hypercylinder surface, and its neighborhood is a tube region. Another byproduct is the Wronskian for the involved hypergeometric functions.

In the following we give an overview of past results concerning Klein-Gordon theory on AdS. The earliest publication on Quantum Theory on spacetimes with constant curvature that we found in the literature dates back to 1935 and is by Dirac [28]. He studies scalar and electron wave equations and Maxwell equations for deSitter $\mathrm{dS}_{1,3}$ and Anti-deSitter $\mathrm{AdS}_{1,3}$ spacetime but without giving solutions. Next we mention an article by Fronsdal from 1965 [34]. Therein, he conjectures that *"a physical theory in flat space is obtainable as the limit of a physical theory in a curved space"*, with limit being understood as that of zero curvature. In the spirit of this we compare (the limit of large curvature radius, that is: zero curvature, of) our results for AdS Klein-Gordon theory with the corresponding Minkowski results.

The earliest solution of the Klein-Gordon equation for AdS that we could spot is in the article [69] by Limic, Niederle and Raczka from 1966. Although written with hypergeometric functions, their solutions are what we call AdS-Jacobi modes. These modes have a discrete set of frequencies, dubbed "magic frequencies" in [8]. In their next article [43] they also present one of the nonstandard hypergeometric solutions, which we call hypergeometric S^a-modes. In [35] Fronsdal considers Klein-Gordon theory on $\mathrm{AdS}_{1,3}$ and constructs wave functions using the Jacobi solutions found in [69]. Moreover, he includes a beautiful section about the geometry of AdS and provides historical references. Avis, Isham and Storey in [7] study Klein-Gordon theory on $\mathrm{AdS}_{1,3}$ as well, with another clarifying discussion of AdS geometry. They introduce an "inner product" that is actually the standard symplectic structure for Klein-Gordon solutions on AdS. Although this symplectic structure is defined using an equal-time surface Σ_t, it turns out to be time-independent. In order to set up a covariant canonical quantization, the authors use both Jacobi and hypergeometric S^a-modes. Instead of using Cauchy data on an equal-time surface, above (3.16) they also determine a solution on a (time-interval) region by its field values at times $t_1 = 0$ and $t_2 = \pi$. This method is applied often in the General Boundary Formulation, see e.g. [55] and (12) in [22]. More about AdS geometry and its Penrose diagram can be found in Sections V-VII of [68] by Podolsky and Griffiths, in the Chapter "AdS" of Bengtsson's [10], in Section 2.2 of [2] by Aharony et al., and in [29].

In [13, 14] Breitenlohner and Freedman for $\mathrm{AdS}_{1,3}$ study the energy-momentum tensor and the energy functional for a Klein-Gordon solution at fixed time t. They find that the energy is positive only for the Jacobi solutions (which we denote by $J^{(+)}_{nl}(\rho)$ and $J^{(-)}_{nl}(\rho)$, and for the $J^{(-)}_{nl}(\rho)$ an "improved" version of the usual energy momentum tensor must be employed). In [46] their result is generalized in a detailed presentation by Mezincescu and Townsend for $\mathrm{AdS}_{1,d}$ of arbitrary spatial dimension

d. We also mention the works [15] of Burgess and Lutken and [33] of Dullemond and van Beveren about the Feynman propagator for scalar fields on AdS.

In Section 3.2 of [9] Balasubramanian, Kraus and Lawrence show how to find more types of Klein-Gordon solutions on AdS. In particular, they distinguish solutions which depend on $\sin^2\rho$ from those which depend on $\cos^2\rho$ (therein $\rho \in [0, \pi/2)$ is a compact version of a radial coordinate on AdS). We make use of their idea, because the former characterizes solutions according to their behavior on the time axis $\rho \equiv 0$ and the latter according to their behaviour near the timelike boundary of AdS at $\rho \equiv \pi/2$. Further, they give a list of Klein-Gordon solutions that is nearly complete (beware: small typo in their equation (30)). This concludes our short overview of results about classical Klein-Gordon theory on AdS.

For completeness, let us mention here also some works on quantized Klein-Gordon theory on AdS that are closely related to the classical theory. In [37], Giddings proposes an S-matrix for Klein-Gordon fields on AdS spacetime using canonical quantization. It is well known that in AdS no temporally asymptotical free states exist, due to the periodic convergence of timelike geodesics. Giddings suggests to avoid this problem by placing states on the timelike boundary of AdS. This boundary is a hypercylinder, and its neighborhood is what we call a tube region. Giddings' proposal uses as classical ingredients not only the Jacobi modes, but also the hypergeometric S^a-modes, which highlights the necessity for studying *all* classical solutions and not only those which are bounded on all of space. We review Giddings' work in Section 3.3.8 of the following Chapter about Quantized Theories. Gary and Giddings in [36] investigate the relation between flat space S-matrix and the AdS/CFT correspondence. In Section 3.1 therein they discuss the flat limit of AdS, where its curvature radius R_{AdS} tends towards infinity. We shall study the flat limit both for classical and quantum objects.

Last but not least we mention the work of Dorn et al., who in Section 3 of [32] develop a quantization for particle dynamics for $\text{AdS}_{1,d}$. They construct a Schrödinger wave function representation, and obtain as energy eigenvalues what we call magic frequencies, a result which relates classical and quantized theory rather nicely. Moreover, they construct a covariant quantization which is equivalent to their Schrödinger representation.

2.6.2 Basic AdS geometry and flat limit

This section briefly summarizes the geometry of AdS, with some more details given in Appendix C.1.1. We denote by AdS what is more precisely denoted as $\text{CAdS}_{1,d}$, that is, $(1+d)$-dimensional Anti-deSitter spacetime with Lorentzian signature in the universal covering version. AdS then has the topology of $\mathbb{R}^{1+d}$ and no closed timelike curves. Where not explicitly stated otherwise, we shall only consider AdS with odd spatial dimension $d \geq 3$.

We use global coordinates with the time coordinate $t \in (-\infty, +\infty)$, a radial coordinate $\rho \in [0, \frac{\pi}{2})$, and denote the $(d-1)$ angular coordinates on $\mathbb{S}^{d-1}$ collectively by $\Omega := (\theta_1, \ldots, \theta_{d-1})$. In contrast to Minkowski spacetime, AdS at $\rho = \pi/2$ has a timelike boundary, which we denote by ∂AdS. Its topology is that of a hypercylinder: $\partial\text{AdS} = \mathbb{R} \times \mathbb{S}^{d-1}$. With R_{AdS} denoting the curvature radius of AdS, its metric writes

$$\mathrm{d}s^2_{\text{AdS}} = \frac{R^2_{\text{AdS}}}{\cos^2\rho}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho\, \mathrm{d}s^2_{\mathbb{S}^{d-1}}\right). \tag{2.147}$$

For the precise form of coordinates and metric on the unit sphere $\mathbb{S}^{d-1}$ see Appendix A.3. There is also another version $\tilde\rho$ of the radial coordinate ρ, with both related through $\cos\rho = 1/\cosh\tilde\rho$. This $\tilde\rho$ ranges over $[0, \infty)$ and is the metric distance of the point $(t, \tilde\rho, \Omega)$ respectively $(t, \rho(\tilde\rho), \Omega)$ from the time axis. For the limit of large curvature radius R_{AdS}, we introduce the rescaled global coordinates

$$r := R_{\text{AdS}}\,\rho \qquad r \in [0, \tfrac{\pi}{2}\, R_{\text{AdS}}) \tag{2.148}$$

$$\tau := R_{\text{AdS}}\, t \qquad \tau \in (-\infty, +\infty). \tag{2.149}$$

Then, for large R_{AdS} the AdS metric (2.147) approximates the Minkowski metric

$$\mathrm{d}s^2_{\text{AdS}} \approx -\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\, \mathrm{d}s^2_{\mathbb{S}^{d-1}} = \mathrm{d}s^2_{\text{Mink}}. \tag{2.150}$$

Therefore the large-R_{AdS} limit is also called flat limit. For more details about the flat limit of AdS spacetime see Appendix C.1.1. The Laplace-Beltrami operator on AdS is given by

$$\begin{aligned}\Box_{\text{AdS}} &:= \frac{1}{\sqrt{|g|}}\,\partial_\mu\sqrt{|g|}\,g^{\mu\nu}\partial_\nu \\ &= R_{\text{AdS}}^{-2}\left\{-\cos^2\rho\,\partial_t^2+\cos^2\rho\,\partial_\rho^2+\frac{(d-1)}{\tan\rho}\,\partial_\rho+\tan^{-2}\rho\,\Box_{\mathbb{S}^{d-1}}\right\}, \end{aligned} \tag{2.151}$$

and its flat limit is the Laplace-Beltrami on Minkowski spacetime. On AdS we have the following Killing vector fields with latin lowercases in $1,\ldots,d$ (see Appendix B.1.1 for the K_{AB} notation with $A,B\in 0,1,\ldots,d,d{+}1$, and see Appendix C.1.2 for the derivation of the Killing vectors):

$$K_{d+1,0} = \partial_t \tag{2.152}$$

$$K_{jk} = \xi_j\partial_{\xi_k}-\xi_k\partial_{\xi_j} \tag{2.153}$$

$$K_{0j} = -\xi_j\cos t\,\sin\rho\,\partial_t-\xi_j\sin t\,\cos\rho\,\partial_\rho-\frac{\sin t}{\sin\rho}\left(\partial_{\xi_j}-\xi_j\xi_i\,\partial_{\xi_i}\right) \tag{2.154}$$

$$K_{d+1,j} = -\xi_j\sin t\,\sin\rho\,\partial_t+\xi_j\cos t\,\cos\rho\,\partial_\rho+\frac{\cos t}{\sin\rho}\left(\partial_{\xi_j}-\xi_j\xi_i\,\partial_{\xi_i}\right). \tag{2.155}$$

Therein, the ξ_j are the constrained cartesian coordinates on the unit sphere $\mathbb{S}^{d-1}$ with $\underline{\xi}^2=1$. The Killing vectors are of three different types: only one translation $K_{d+1,0}$, plus $d(d-1)/2$ spatial rotations K_{jk}, and $(2d)$ boosts K_{0j} and $K_{d+1,j}$. In total we thus have $(d+1)(d+2)/2$ Killing vectors on $\text{AdS}_{1,d}$, making it a maximally symmetric space(time). For comparison: in $(1+d)$-dimensional Minkowski spacetime we have $(d+1)$ translations, $(d^2-d)/2$ rotations and d boosts, giving the same total of $(d+1)(d+2)/2$ Killing vectors, which makes it a maximally symmetric spacetime as well. The above Killing vectors are the generators of the isometry group $\text{SO}(2,d)$ of $\text{AdS}_{1,d}$, whose Lie algebra $\mathbf{so}(2,d)$ writes as in (B.4):

$$[K_{AB},K_{CD}] = -\eta_{AC}\,K_{BD}+\eta_{BC}\,K_{AD}-\eta_{BD}\,K_{AC}+\eta_{AD}\,K_{BC}\,. \tag{2.156}$$

Between the Killing vectors of AdS and Minkowski spacetime there exists a correspondence: we can first switch to the coordinates $\tau=R_{\text{AdS}}t$ and $r=R_{\text{AdS}}\rho$, and then take the flat limit $R_{\text{AdS}}\to\infty$. This gives us

$$K_{d+1,0} \xrightarrow[\text{lim.}]{\text{flat}} R_{\text{AdS}}\,\partial_\tau \tag{2.157}$$

$$K_{jk} \xrightarrow[\text{lim.}]{\text{flat}} \xi_j\partial_{\xi_k}-\xi_k\partial_{\xi_j} \tag{2.158}$$

$$K_{0j} \xrightarrow[\text{lim.}]{\text{flat}} -\xi_j\,r\,\partial_\tau-\xi_j\,\tau\,\partial_r-\frac{\tau}{r}\left(\partial_{\xi_j}-\xi_j\xi_i\,\partial_{\xi_i}\right) \tag{2.159}$$

$$K_{d+1,j} \xrightarrow[\text{lim.}]{\text{flat}} R_{\text{AdS}}\Big(\xi_j\,\partial_r+\frac{1}{r}\left(\partial_{\xi_j}-\xi_j\xi_i\,\partial_{\xi_i}\right)\Big)\,. \tag{2.160}$$

Comparing these to (B.7)-(B.10), the resulting AdS-Minkowski Killing vector correspondence is given in Table 2.161.

AdS (flat limit)	$\to$	**Minkowski**
$R_{\text{AdS}}^{-1}K_{d+1,0}$	$\to$	T_0
$R_{\text{AdS}}^{-1}K_{d+1,j}$	$\to$	T_j
K_{jk}	$\to$	K_{jk}
K_{0j}	$\to$	K_{0j}

Table 2.161: Correspondence between Killing vectors on AdS and Minkowski

The Killing vectors of Minkowski spacetime generate the Poincaré algebra (with greek lowercase indices in $0, 1, \ldots, d$): the translations commute among themselves:

$$[T_\mu, T_\nu] \,=\, 0, \tag{2.162}$$

and the boosts and rotations form the Lorentz algebra $\mathbf{so}\,(1, d)$ as in (B.4):

$$[K_{\alpha\beta}, K_{\mu\nu}] \,=\, -\eta_{\alpha\mu}\, K_{\beta\nu} + \eta_{\beta\mu}\, K_{\alpha\nu} - \eta_{\beta\nu}\, K_{\alpha\mu} + \eta_{\alpha\nu}\, K_{\beta\mu}\,. \tag{2.163}$$

The commutation relations between translations and the generators of the Lorentz algebra are:

$$[T_\alpha, K_{\mu\nu}] \,=\, \eta_{\alpha\mu} T_\nu - \eta_{\alpha\nu} T_\mu. \tag{2.164}$$

The relations (2.164) and (2.163) are already contained in $\mathbf{so}\,(2, d)$ in (2.156) above, and thus are not a result arising from the flat limit, but are rather conserved by it. By contrast, relation (2.162) is not contained in $\mathbf{so}\,(2, d)$, and results from the flat limit by eliminating the t-dependence in (2.155): for small t resp. large R_{AdS} the first term is $\sim R_{\text{AdS}}^{-1}$, and in the second and third term $\cos t \,=\, \cos \frac{\tau}{R_{\text{AdS}}} \;\rightarrow\; 1$. This makes the flat limit of the Anti de Sitter algebra $\mathbf{so}\,(2, d)$ into the Poincaré algebra[2].

Regions of AdS

We recall three types of regions on AdS, on which different types of Klein-Gordon solutions are allowed. The first type of region denoted by $\mathbb{M}^{\text{AdS}}_{[t_1,t_2]}$ is the time-interval region and consists of a time interval $[t_1, t_2]$ times all of space. The second, denoted by $\mathbb{M}^{\text{AdS}}_{\rho_0}$, is the solid hypercylinder or rod region, and consists of all of time times a solid ball $\mathbb{B}_{\rho_0}$ of radius ρ_0 in space. The third is denoted by $\mathbb{M}^{\text{AdS}}_{[\rho_1,\rho_2]}$ and called a pierced hypercylinder or tube region: it consists of all of time times a spherical shell $\mathbb{B}_{[\rho_1,\rho_2]}$ with inner radius ρ_1 and outer radius ρ_2 in space.

Two of these regions arise naturally as (infinitesimal or finite) neighborhoods of hypersurfaces (submanifolds) of AdS. The time-interval region is a neighborhood of an equal-time hyperplane Σ_{t_0}, and the tube region is a neighborhood of an equal-radius hypercylinder Σ_{ρ_0} (if the neighborhood is chosen big enough, such that it extends inwards until the time axis $\rho \equiv 0$, then the tube becomes a rod region). In particular, neighborhoods of the boundary hypercylinder $\partial\text{AdS} = \Sigma_{\rho=\pi/2}$ are tube regions. We remark that the "region" of all of AdS can be obtained in two ways: the first is the limit $t_0 \to \infty$ of the time-interval region $\mathbb{M}^{\text{AdS}}_{[-t_0,+t_0]}$, and the second is the limit $\rho_0 \to \pi/2$ of the rod region $\mathbb{M}^{\text{AdS}}_{\rho_0}$.

Our three regions are generically not type-invariant under isometries. For example, after applying an isometry, what previously was a rod region will not be a rod in the new coordinates, but some deformed version of it. In particular, our three regions are only type-invariant under time translations and spatial rotations, but not under boosts. The reason for this is that our choice of regions depends on the coordinate system and is thus not geometric. However, this is not unusual: even the time-interval regions of Minkowski spacetime depend on the choice of a coordinate system.

2.6.3 Klein-Gordon solutions on AdS

With m denoting the field's mass, the action for a free, real scalar field $\phi(x)$ living in AdS is

$$S[\phi] \,=\, \int \mathrm{d}^{d+1}x\, \sqrt{|g|}\;\; \tfrac{1}{2}\,[-g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2], \tag{2.165}$$

and its Euler-Lagrange equation is the free Klein-Gordon equation

$$0 \,=\, (-\Box_{\text{AdS}} + m^2)\,\phi. \tag{2.166}$$

[2] We thank Alejandro Corichi (CCM-UNAM) for pointing us to study how the flat limit relates these algebras.

Below we list the solutions which are well defined and bounded on the respective AdS regions. For a review on how to find these solutions see Appendix C.2.2. On tube regions $\mathbb{M}^{\text{AdS}}_{[\rho_1,\rho_2]}$ the form of some solutions depends on whether the spatial dimension d is odd or even, and the form of other solutions on whether the quantity ν is integer or not, with $\nu = \sqrt{d^2/4 + m^2 R^2_{\text{AdS}}}$. However, the form of the solutions on time-interval regions $\mathbb{M}^{\text{AdS}}_{[t_1,t_2]}$ is always the same, ditto for the rod regions $\mathbb{M}^{\text{AdS}}_{\rho_0}$. Unless stated otherwise we shall assume that d is odd and ν noninteger (see Appendix C.2.3 for the other cases). The most general solutions of the Klein-Gordon equation on AdS are four types of modes which we call hypergeometric a and b-modes of type S respectively C (quantities relating to the S^a and S^b-modes carry superscripts S, e.g.: $\mu^{(S,a)}$, while quantities relating to the C-modes carry a C, e.g.: $\mu^{(C,a)}$):

$$\begin{aligned} \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) &= \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, S^a_{\omega l}(\rho) & \mu^{(C,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) &= \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, C^a_{\omega l}(\rho) \\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) &= \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, S^b_{\omega l}(\rho) & \mu^{(C,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) &= \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, C^b_{\omega l}(\rho)\,. \end{aligned} \tag{2.167}$$

Therein, $Y^{m_l}_{\underline{l}}(\Omega)$ denote the hyperspherical harmonics, see Appendix A.4. With $F(a,\,b;\,c;\,x)$ denoting Gauss's hypergeometric function, we use the radial functions

$$S^a_{\omega l}(\rho) \;=\; \sin^l\rho \quad \cos^{\tilde{m}_+}\rho\;\; F(\alpha^{S,a},\,\beta^{S,a};\,\gamma^{S,a};\,\sin^2\rho) \tag{2.168}$$
$$S^b_{\omega l}(\rho) \;=\; -(\sin\rho)^{2-l-d}\cos^{\tilde{m}_+}\rho\; F(\alpha^{S,b},\,\beta^{S,b};\,\gamma^{S,b};\,\sin^2\rho) \tag{2.169}$$
$$C^a_{\omega l}(\rho) \;=\; \sin^l\rho\,\cos^{\tilde{m}_+}\rho\; F(\alpha^{C,a},\,\beta^{C,a};\,\gamma^{C,a};\,\cos^2\rho) \tag{2.170}$$
$$C^b_{\omega l}(\rho) \;=\; \sin^l\rho\,\cos^{\tilde{m}_-}\rho\; F(\alpha^{C,b},\,\beta^{C,b};\,\gamma^{C,b};\,\cos^2\rho). \tag{2.171}$$

For the properties of these radial functions, see the next section. The above hypergeometric parameters are given by

$$\begin{aligned} \alpha^{S,a} &= \alpha^{C,a} = \tfrac{1}{2}(l+\widetilde{m}_+-\omega) & \alpha^{S,b} &= \alpha^{S,a}-\gamma^{S,a}+1 & \alpha^{C,b} &= \alpha^{C,a}-\gamma^{C,a}+1 \\ \beta^{S,a} &= \beta^{C,a} = \tfrac{1}{2}(l+\widetilde{m}_++\omega) & \beta^{S,b} &= \beta^{S,a}-\gamma^{S,a}+1 & \beta^{C,b} &= \beta^{C,a}-\gamma^{C,a}+1 \\ \gamma^{S,a} &= l+\tfrac{d}{2} \qquad \gamma^{C,a} = 1+\nu & \gamma^{S,b} &= 2-\gamma^{S,a} & \gamma^{C,b} &= 2-\gamma^{C,a}\,. \end{aligned} \tag{2.172}$$

With m denoting the field's mass, we further use

$$\widetilde{m}_\pm \;=\; \tfrac{d}{2} \pm \nu \qquad\qquad \nu \;=\; \sqrt{d^2/4 + m^2 R^2_{\text{AdS}}} \qquad\qquad \begin{matrix} \tilde{m}_+ > 0 & \forall\,\nu \geq 0 \\ \tilde{m}_- > 0 & \forall\,\nu \in (0, d/2) \end{matrix}\,.$$

The value of m^2 for which ν vanishes is called Breitenlohner-Freedman mass $m^2_{\text{BF}} := -d^2/(4R^2_{\text{AdS}})$. Whenever the frequency ω is one of the discrete values that were dubbed magic frequencies in [8]: $\omega^\pm_{nl} = 2n + l + \widetilde{m}_\pm$ (nonnegative for $\widetilde{m}_\pm \geq 0$), then the S^a-modes take on a special form. The hypergeometric function then can be written as a Jacobi polynomial, and therefore we call the two discrete sets of modes below Jacobi modes:

$$\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega) \;=\; \mu^{(S,a)}_{\omega^\pm_{nl}\underline{l}m_l}(t,\rho,\Omega) \;=\; \mathrm{e}^{-\mathrm{i}\omega^\pm_{nl}t}\, Y^{m_l}_{\underline{l}}(\Omega)\, J^{(\pm)}_{nl}(\rho)\,. \tag{2.173}$$

We call $\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)$ ordinary and $\mu^{(-)}_{n\underline{l}m_l}(t,\rho,\Omega)$ exceptional Jacobi modes. Moreover, we call $\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)$ the positive frequency modes, and the negative frequency modes are $\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}$. The exceptional AdS-Jacobi modes are only well defined for $\nu \in (0,1)$, and for this case $\widetilde{m}_- > 0$. By $P^{(a,b)}_n(x)$ we denote the Jacobi polynomials, by $(a)_n$ the Pochhammer symbols, and with that we write

$$J^{(\pm)}_{nl}(\rho) \;:=\; \frac{n!}{(l+d/2)_n}\,\sin^l\rho\,\cos^{\tilde{m}_+}\rho\; P^{(l+d/2-1,\pm\nu)}_n(\cos 2\rho). \tag{2.174}$$

We sketch how to find all these modes in Appendix C.2.2. In Appendix C.2.3 we give a complete list of solutions including the complementary cases of even d and integer ν. The Jacobi modes usually are the only modes used in the literature, e.g. [14], [46], [7] and [15]. The earliest mention of the Jacobi modes that we could spot is equation (4.8) of the first article [69] by Limic, Niederle and Raczka. The correspondence to our notation is given by $H^q_p = \text{AdS}_{1,d}$ for $(q,p) = (2,d)$, with AdS coordinates

$\tilde{\varphi}^1 = t$ and $\theta = \tilde{\rho}$ and thus $\tanh\theta = \sin\rho$ while $\cosh\theta = 1/\cos\rho$. The parameters are $\tilde{m}_1 = -\omega$, with $L = -\tilde{m}_\pm$ and $\lambda = -m^2 R^2_{\text{AdS}}$. Then, their relation (4.5) corresponds to our magic frequencies. The first mention of the S^a-modes we found in equation (2.5) of [43]. The correspondence to our notation is the same as above plus $\Lambda^2 = \lambda - (d^2/4)$, that is, $-\Lambda^2 = \nu^2$. The authors of [14], [46] and [7] mention both S^a and S^b-modes but then discard the latter, and use only the special case of Jacobi modes of the S^a-modes. Giddings in [37] then actually makes use of *all S^a-modes*. The first appearance of the $C^{a,b}$-modes we found in [9], wherein the S^b-modes are discarded once more.

Dorn et al. in Section 3 of [32] find another physical meaning of the magic frequencies. Constructing a Schrödinger wave function representation for particle dynamics for $\text{AdS}_{1,d}$, they obtain as energy eigenvalues precisely the magic frequencies: $\omega^\pm_{nl} = E^\pm_0 + 2n + l$ with ground state energy $E^\pm_0 = \tilde{m}_\pm = d/2 \pm \sqrt{d^2/4 + m^2 R^2_{\text{AdS}}}$. To see this, note that their N is our d, their $\mathcal{M}^2$ our m^2, and combine their expressions (3.13), (4.3) and $\tilde{a} = (N-1)/4N$ below (4.13).

Properties of the AdS Klein-Gordon solutions

The S^a and S^b-modes form one pair of linear independent solutions, and the C^a and C^b-modes form another. They are related through ("on" stands for odd-noninteger):

$$\begin{pmatrix} S^a_{\omega l} \\ S^b_{\omega l} \end{pmatrix} = M^{\text{on}}_{\omega l} \begin{pmatrix} C^a_{\omega l} \\ C^b_{\omega l} \end{pmatrix} , \tag{2.175}$$

see Appendix C.2.4 for the elements of the matrix $M^{\text{on}}_{\omega l}$. The hypergeometric S^a-modes (except for the magic frequencies) and S^b-modes and the C^b-modes are evanescent modes: when approaching the boundary $\rho = \frac{\pi}{2}$ they grow exponentially with metric distance $\tilde{\rho}$ from the time axis. (This holds for $\tilde{m}_- < 0$, that is, positive mass square m^2. For $\tilde{m}_- = 0$ their value for large $\tilde{\rho}$ becomes some finite constant, and for $\tilde{m}_- > 0$ they decay exponentially with $\tilde{\rho}$). The C^a-modes are also evanescent: they decay exponentially with $\tilde{\rho}$ when approaching the boundary. On the time axis $\rho \equiv 0 \equiv \tilde{\rho}$ the S^a-modes are regular, like the Bessel modes on Minkowski spacetime. However, the C^a and C^b-modes and S^b-modes are singular there: they behave like $\tilde{\rho}^{-l-(d-2)}$, that is, inverse power of metric distance, like the Neumann modes on Minkowski spacetime. (For a discussion of the question why we need to consider all classical solutions, see Section 2.3.) The S^a and S^b-modes and the C^a and C^b-modes provide us two different parametrizations of Klein-Gordon solutions. The S^a and S^b-modes parametrize solutions according to their behaviour near the time axis $\rho \equiv 0$, on which the former are regular and the latter diverge. The C^a and C^b-modes parametrize solutions according to their behaviour near the timelike boundary of AdS at $\rho \equiv \pi/2$, on which the former are regular and the latter diverge.

The Jacobi modes are well defined and bounded both on time axis and boundary. Thus they are the only modes that are L^2-normalizable on an equal-time surface. For later use, in Appendix C.2.5 we calculate the following normalization constant for all $\nu \geq 0$ (that is: all masses $m^2 \geq m^2_{\text{BF}}$ above the Breitenlohner-Freedman mass):

$$\mathcal{N}^\pm_{nl} := \int_0^{\pi/2} \mathrm{d}\rho \, \tan^{d-1}\rho \, \left(J^{(\pm)}_{nl}(\rho)\right)^2 = \frac{n! \, \Gamma(\gamma^{\text{S}})^2 \, \Gamma(n \pm \nu + 1)}{2\omega^\pm_{nl} \, \Gamma(n + \gamma^{\text{S}}) \, \Gamma(n \pm \nu + \gamma^{\text{S}})} . \tag{2.176}$$

Radial behaviour of the AdS Klein-Gordon solutions

In this section we have a closer look at the behaviour of the radial solutions $S^{a,b}_{\omega l}(\rho)$ and $C^{a,b}_{\omega l}(\rho)$ in (2.168), always for the case of odd spatial dimension $d \geq 3$ and noninteger ν. On the time axis $\rho = 0$ both $\cos\rho$ and the hypergeometric functions with the $\sin^2\rho$-argument take the value 1, while $\sin\rho$

itself vanishes. Thus

$$S^a_{\omega l}(\rho \to 0) \approx \rho^l \approx \begin{cases} 1 & l = 0 \\ 0 & l \geq 1 \end{cases} \tag{2.177}$$

$$S^b_{\omega l}(\rho \to 0) \approx -(\rho)^{-(l+d-2)} \; . \tag{2.178}$$

That is, $S^a_{\omega l}$ is regular on the time axis while $S^b_{\omega l}$ diverges there for $d \geq 3$. Since according to (C.58) $C^a_{\omega l}$ and $C^b_{\omega l}$ are linear combinations of $S^a_{\omega l}$ and $S^b_{\omega l}$, the former generically diverge on the time axis. The only exceptions occur for the magic frequencies: for ω^+_{nl} the function $C^a_{\omega l}$ becomes regular on the time axis, while $C^b_{\omega l}$ becomes regular there for ω^-_{nl} (because for these frequencies they become Jacobi polynomials see also Appendix C.2.4).

Near spatial infinity $\rho = \pi/2$ both $\sin\rho$ and the hypergeometric functions with $\cos^2\rho$-argument take the value 1, while $\cos\rho$ vanishes. Since $\cos\rho = \sin(\pi/2-\rho)$ we have

$$C^a_{\omega l}(\rho \to \pi/2) \approx \cos^{\tilde{m}_+}\rho \approx (\pi/2-\rho)^{\tilde{m}_+} \tag{2.179}$$

$$C^b_{\omega l}(\rho \to \pi/2) \approx \cos^{\tilde{m}_-}\rho \approx (\pi/2-\rho)^{\tilde{m}_-} \; . \tag{2.180}$$

That is, $C^a_{\omega l}$ remains regular while approaching spatial infinity. $C^b_{\omega l}$ remains finite only for non-negative $\tilde{m}_-$, that is for $m^2 \leq 0$, else it diverges. As linear combinations of $C^a_{\omega l}$ and $C^b_{\omega l}$, generically $S^a_{\omega l}$ and $S^b_{\omega l}$ diverge near spatial infinity unless $m^2 \leq 0$. Exceptions are the frequencies $\pm\omega = -(l+d-2) + \tilde{m}_+ + 2n$ with $n \in \mathbb{N}_0$, for which $S^b_{\omega l}$ remains finite, and the magic frequencies ω^+_{nl}, for which $S^a_{\omega l}$ remains finite (because it becomes a Jacobi polynomial, see again C.2.4).

The behaviour of the radial functions for the generic cases is sketched in Figure 2.181.

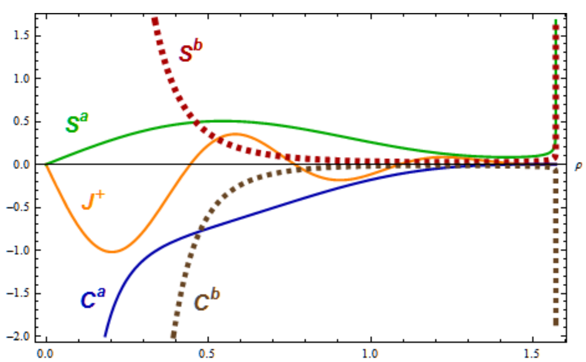


Figure 2.181: Typical behaviour of the radial functions for spatial dimension $d = 3$ and $m^2 = R^2_{\text{AdS}} = 1$: $\mathrm{J}^+ = -8\,J^{(+)}_{3,1}(\rho)$, $\mathrm{S}^a = 1.5\,S^a_{4,1}(\rho)$, $\mathrm{S}^b = -0.02\,S^b_{4,3}(\rho)$, $\mathrm{C}^a = -1.5\,C^a_{4,1}(\rho)$, $\mathrm{C}^b = -0.01\,C^b_{4,4}(\rho)$.

We conclude this section by taking a look at the Jacobi solutions (C.57):

$$J^{(+)}_{nl}(\rho) := \frac{n!}{(\gamma^{\mathrm{S}})_n} \sin^l\rho \cos^{\tilde{m}_+}\rho \, P_n^{(\gamma^{\mathrm{S}}-1,+\nu)}(\cos 2\rho) \qquad \text{for } \omega = \pm\omega^+_{nl}$$

$$J^{(-)}_{nl}(\rho) := \frac{n!}{(\gamma^{\mathrm{S}})_n} \sin^l\rho \cos^{\tilde{m}_-}\rho \, P_n^{(\gamma^{\mathrm{S}}-1,-\nu)}(\cos 2\rho) \qquad \text{for } \omega = \pm\omega^-_{nl} \; .$$

At the boundary $\rho \to \pi/2$ we have $\sin\rho \to 1$, and $\cos 2\rho \to -1$. According to AS [22.3.1]

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^{n-k} (x+1)^k, \qquad \binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\,\Gamma(a-b+1)} \; , \tag{2.182}$$

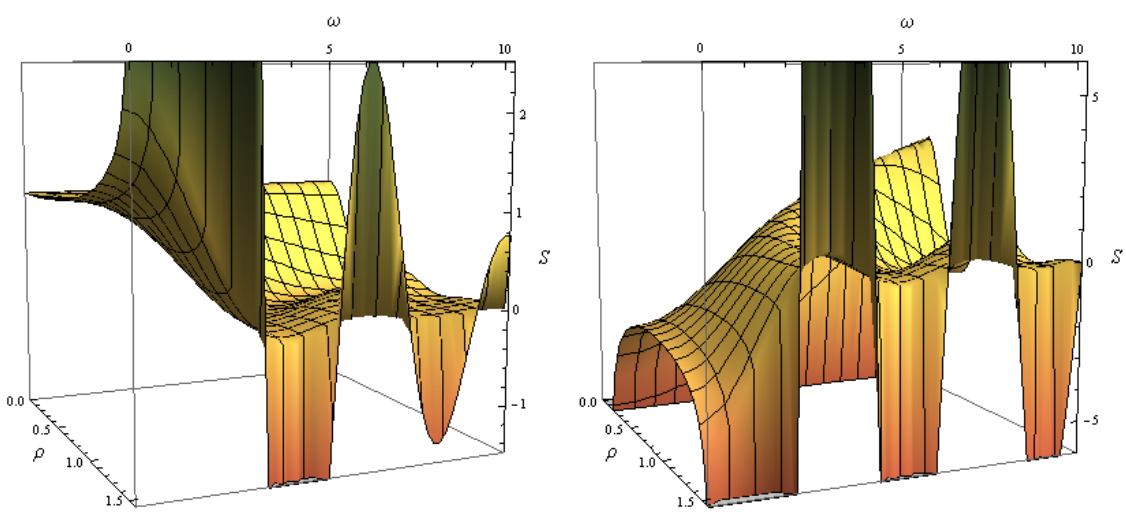
and thus for $x \to -1$ the Jacobi polynomials approach the finite value

$$P_n^{(\alpha,\beta)}(x \to -1) \to (-1)^n \, \frac{\Gamma(n+\beta+1)}{n!\,\Gamma(\beta+1)} \; ,$$

because only the $(k=0)$-term in the sum (2.182) survives. Hence near the boundary the Jacobi solutions $J^{(\pm)}_{nl}(\rho)$ behave like $\cos^{\tilde{m}_\pm}\rho$:

$$
\begin{aligned}
J^{(+)}_{nl}(\rho\to\pi/2) &\approx \cos^{\tilde{m}_+}\rho \; \frac{(-1)^n\,\Gamma(n+\nu+1)}{(\gamma^{\mathrm{S}})_n\;\Gamma(1+\nu)} && \text{for } \omega \;=\; \pm\omega^{+}_{nl} \\
J^{(-)}_{nl}(\rho\to\pi/2) &\approx \cos^{\tilde{m}_-}\rho \; \frac{(-1)^n\,\Gamma(n-\nu+1)}{(\gamma^{\mathrm{S}})_n\;\Gamma(1-\nu)} && \text{for } \omega \;=\; \pm\omega^{-}_{nl}\,.
\end{aligned}
\qquad (2.183)
$$

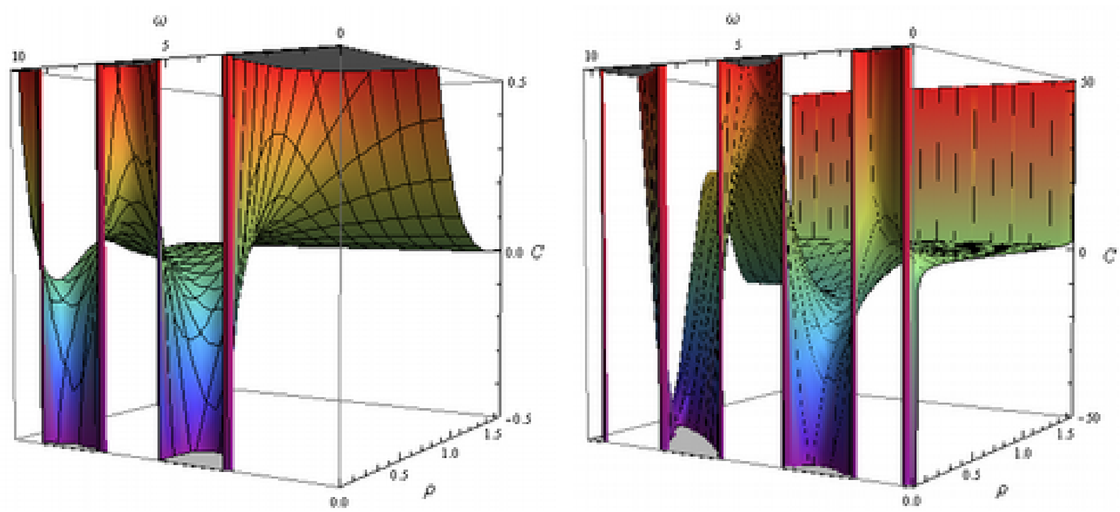
Since $\widetilde{m}_+ > 0$, the ordinary Jacobi solutions $J^{(+)}_{nl}(\rho\to\pi/2)$ remain regular at the boundary. And since we can only make use of the exceptional Jacobi solutions $J^{(-)}_{nl}(\rho)$ for $\nu<1$, which with $d\geq 3$ lets $\widetilde{m}_- > 0$, we see that $J^{(-)}_{nl}(\rho\to\pi/2)$ remains regular at the boundary, too. In Figure 2.184 we plot the radial functions $S^a_{\omega l}(\rho)$ and $S^b_{\omega l}(\rho)$. We see that at the time axis $\rho=0$ the former is regular and the latter diverges. At the boundary $\rho=\frac{\pi}{2}$ both diverge except for a discrete set of frequencies. For $S^a_{\omega l}(\rho)$, these are the magic frequencies $\pm\omega^-_{nl}$ from (C.44). For $S^b_{\omega l}(\rho)$ they are the frequencies that make $(M^{\mathrm{on}}_{\omega l})_{22}$ vanish in (C.58), that is, those that make $\alpha^{S,b}$ or $\beta^{S,b}$ take nonpositive integer values.



(a) $S^a_{\omega l}(\rho)$ (b) $S^b_{\omega l}(\rho)$

Figure 2.184: 3D plots of radial S-functions for $l=0$, $d=3$, $m=R_{\mathrm{AdS}}=1$, $\tilde{m}_+\approx 3.3$

The same thing (but on the time axis) happens for the radial C-functions, which we plot in 2.185. We see that at the boundary $\rho=\frac{\pi}{2}$ the function $C^a_{\omega l}(\rho)$ is regular and $C^b_{\omega l}(\rho)$ diverges. At the time axis $\rho=0$ both diverge except for a discrete set of frequencies. For $C^a_{\omega l}(\rho)$, these are again the magic frequencies $\pm\omega^-_{nl}$ from (C.44), while for $C^b_{\omega l}(\rho)$ they are the frequencies that make $(M^{\mathrm{no}}_{\omega l})_{22}$ vanish in (C.58), that is, those that make $(1-\alpha^{S,b})$ or $(1-\beta^{S,b})$ take nonpositive integer values.



(a) $C^a_{\omega l}(\rho)$ (b) $C^b_{\omega l}(\rho)$

Figure 2.185: 3D plots of radial C-functions for $l = 0$, $d = 3$, $m = R_{\text{AdS}} = 1$, $\tilde{m}_+ \approx 3.3$

2.6.4 Rod and tube: Solutions and structures

For the tube region $\mathbb{M}^{\text{AdS}}_{[\rho_1,\rho_2]} = \mathbb{R}\times[\rho_1,\rho_2]\times\mathbb{S}^{d-1}$, that is: the cartesian product of all of time and a spherical shell, we need Klein-Gordon solutions that are bounded for all of time while in space we only need them bounded on $[\rho_1, \rho_2]$. Thus we can use all four hypergeometric modes here, with the frequency ω being real. We expand an arbitrary complex(ified) Klein-Gordon solution (see e.g. Section 2.3 in [58]) on the tube region as an integral over these modes, where we call the upper line S-expansion and the lower line C-expansion:

$$\phi(t,r,\Omega) = \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{\omega\underline{l}m_l}\; \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^{b}_{\omega\underline{l}m_l}\; \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\} \tag{2.186}$$

$$= \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{C,a}_{\omega\underline{l}m_l}\; \mu^{(C,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^{C,b}_{\omega\underline{l}m_l}\; \mu^{(C,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\}\ . \tag{2.187}$$

If and only if $\overline{\phi^{S,a}_{\omega\underline{l}m_l}} = \phi^{S,a}_{-\omega,\underline{l},-m_l}$ and the same for $\phi^{S,b}_{\omega\underline{l}m_l}$ (respectively for $\phi^{C,a}_{\omega\underline{l}m_l}$ and $\phi^{C,b}_{\omega\underline{l}m_l}$), then the solution $\phi(t,\rho,\Omega)$ is real. Since (2.186) = (2.187), with (C.58) we have the following relation between the S and C-momentum representations of the Klein-Gordon solutions:

$$\begin{pmatrix} \phi^{S,a}_{\omega\underline{l}m_l} \\ \phi^{S,b}_{\omega\underline{l}m_l} \end{pmatrix} = (M^{\text{no}}_{\omega l})^\top \begin{pmatrix} \phi^{C,a}_{\omega\underline{l}m_l} \\ \phi^{C,b}_{\omega\underline{l}m_l} \end{pmatrix} \qquad \begin{pmatrix} \phi^{C,a}_{\omega\underline{l}m_l} \\ \phi^{C,b}_{\omega\underline{l}m_l} \end{pmatrix} = (M^{\text{on}}_{\omega l})^\top \begin{pmatrix} \phi^{S,a}_{\omega\underline{l}m_l} \\ \phi^{S,b}_{\omega\underline{l}m_l} \end{pmatrix}. \tag{2.188}$$

Therein, $^\top$ denotes the transposed matrix. We can also use the modified momentum representation $(\tilde{\phi}^{\text{F},a}_{\tilde{\omega}\underline{l}m_l}, \tilde{\phi}^{\text{F},b}_{\tilde{\omega}\underline{l}m_l})$, (where the label F stands for flat) in the S-expansion:

$$\begin{aligned} \phi^{S,a}_{\omega\underline{l}\,m_l} &= R^{-1}_{\text{AdS}}\tilde{\phi}^{\text{F},a}_{\tilde{\omega}\underline{l}m_l}\ \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\ \frac{(p^{\mathbb{R}}_{\omega})^l}{(2l+d-2)!!} \\ \phi^{S,b}_{\omega\underline{l}\,m_l} &= R^{-1}_{\text{AdS}}\tilde{\phi}^{\text{F},b}_{\tilde{\omega}\underline{l}m_l}\ \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\ \frac{(2l+d-4)!!}{(p^{\mathbb{R}}_{\omega})^{(l+1)}}\ . \end{aligned} \tag{2.189}$$

Then, the flat limit of the S-expansion for $d = 3$ yields the Minkowski tube expansion (2.107), see Appendix C.2.8:

$$\phi(t,r,\Omega) \xrightarrow[\text{lim.}]{\text{flat}} \int \mathrm{d}\tilde{\omega} \sum_{l,m_l} \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{4\pi} \left\{ \tilde{\phi}^{\mathrm{F},a}_{\tilde{\omega} l m_l} \mathrm{e}^{-\mathrm{i}\tilde{\omega}\tau} Y^{m_l}_l(\Omega)\, \check{\jmath}_{\tilde{\omega} l}(r) + \tilde{\phi}^{\mathrm{F},b}_{\tilde{\omega} l m_l} \mathrm{e}^{-\mathrm{i}\tilde{\omega}\tau} Y^{m_l}_l(\Omega)\, \check{n}_{\tilde{\omega} l}(r) \right\} . \tag{2.190}$$

For the rod region $\mathbb{M}^{\text{AdS}}_{\rho_0} = \mathbb{R} \times [0, \rho_0] \times \mathbb{S}^{d-1}$, that is: the cartesian product of all of time and a solid ball, we need Klein-Gordon solutions that are bounded for all of time while in space we only need them bounded on $[0, \rho_0]$. Therefore, the hypergeometric S^a-modes (2.167) alone span the space of Klein-Gordon solutions for the rod region. We expand any Klein-Gordon solution on the rod region as an integral over these modes, which we call rod expansion (again, $\phi(t, \rho, \Omega)$ is real iff $\phi^{S,a}_{\omega \underline{l} m_l} = \overline{\phi^{S,a}_{-\omega,\underline{l},-m_l}}$)

$$\phi(t,r,\Omega) = \int \mathrm{d}\omega \sum_{\underline{l},m_l} \phi^{S,a}_{\omega\underline{l}m_l}\ \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) . \tag{2.191}$$

Knowing the expansions of the solutions, we can evaluate structures on the spaces of solutions. As on Minkowski spacetime, our hypercylinders Σ_ρ are oriented inwards, hence $\operatorname{sign}\Sigma_\rho = -1$. We use the rescaled coordinates of Appendix C.1.1

$$\begin{aligned} r &:= R_{\text{AdS}}\, \rho & r &\in [0, \tfrac{\pi}{2}\ R_{\text{AdS}}) \\ \tau &:= R_{\text{AdS}}\, t & \tau &\in (-\infty, +\infty) \end{aligned} \tag{2.192}$$

and parameters

$$\begin{aligned} \widetilde{\omega} &:= \omega/R_{\text{AdS}} & \widetilde{\omega} &\in (-\infty, +\infty) \\ p^{\mathbb{R}}_\omega &:= \sqrt{|\omega^2 - m^2 R^2_{\text{AdS}}|} & p^{\mathbb{R}}_\omega,\ \tilde{p}^{\mathbb{R}}_{\tilde{\omega}} &\in [0, \infty) \\ \tilde{p}^{\mathbb{R}}_{\tilde{\omega}} &:= \sqrt{|\tilde{\omega}^2 - m^2|} & p^{\mathbb{R}}_\omega &= R_{\text{AdS}}\ \tilde{p}^{\mathbb{R}}_{\tilde{\omega}} . \end{aligned} \tag{2.193}$$

The symplectic potential from (2.54) turns out as (with $\mathrm{d}^{d-1}\Omega = \mathrm{d}\Omega \sqrt{|g_{\mathbb{S}^{d-1}}|}$)

$$\begin{aligned} \theta^{\Sigma_\rho}_\phi(\eta) = &-\int \mathrm{d}t\, \mathrm{d}^{d-1}\Omega\ R^{d-1}_{\text{AdS}} \tan^{d-1}\rho\ \ \eta(t,\rho,\Omega)\, (\partial_\rho\, \phi)(t,\rho,\Omega) \\ = &-2\pi R^{d-1}_{\text{AdS}} \tan^{d-1}\rho \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{\omega\underline{l}m_l}\, \eta^{S,a}_{-\omega,\underline{l},-m_l}\, (S^a_{\omega\underline{l}m_l} \partial_\rho S^a_{\omega\underline{l}m_l})(\rho) + \phi^{aS,}_{\omega\underline{l}m_l}\, \eta^{S,b}_{-\omega,\underline{l},-m_l}\, (S^b_{\omega\underline{l}m_l} \partial_\rho S^a_{\omega\underline{l}m_l})(\rho) \\ &+ \phi^{S,b}_{\omega\underline{l}m_l}\, \eta^{S,a}_{-\omega,\underline{l},-m_l}\, (S^a_{\omega\underline{l}m_l} \partial_\rho S^b_{\omega\underline{l}m_l})(\rho) + \phi^{S,b}_{\omega\underline{l}m_l}\, \eta^{S,b}_{-\omega,\underline{l},-m_l}\, (S^b_{\omega\underline{l}m_l} \partial_\rho S^b_{\omega\underline{l}m_l})(\rho) \Big\} \\ = &-2\pi R^{d-1}_{\text{AdS}} \tan^{d-1}\rho \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{C,a}_{\omega\underline{l}m_l}\, \eta^{C,a}_{-\omega,\underline{l},-m_l}\, (C^a_{\omega\underline{l}m_l} \partial_\rho C^a_{\omega\underline{l}m_l})(\rho) + \phi^{C,a}_{\omega\underline{l}m_l}\, \eta^{C,b}_{-\omega,\underline{l},-m_l}\, (C^b_{\omega\underline{l}m_l} \partial_\rho C^a_{\omega\underline{l}m_l})(\rho) \\ &+ \phi^{C,b}_{\omega\underline{l}m_l}\, \eta^{C,a}_{-\omega,\underline{l},-m_l}\, (C^a_{\omega\underline{l}m_l} \partial_\rho C^b_{\omega\underline{l}m_l})(\rho) + \phi^{C,b}_{\omega\underline{l}m_l}\, \eta^{C,b}_{-\omega,\underline{l},-m_l}\, (C^b_{\omega\underline{l}m_l} \partial_\rho C^b_{\omega\underline{l}m_l})(\rho) \Big\} . \end{aligned}$$

The symplectic structure (2.55) becomes

$$\omega_\rho(\eta,\zeta) = \frac{1}{2} \int \mathrm{d}t\, \mathrm{d}^{d-1}\Omega\ R^{d-1}_{\text{AdS}} \tan^{d-1}\rho\ \left(\eta\, \partial_\rho \zeta - \zeta\, \partial_\rho \eta\right) \tag{2.194}$$

$$= +\pi R^{d-1}_{\text{AdS}} \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \eta^{S,a}_{\omega\underline{l}m_l}\, \zeta^{S,b}_{-\omega,\underline{l},-m_l} - \eta^{S,b}_{\omega\underline{l}m_l}\, \zeta^{S,a}_{-\omega,\underline{l},-m_l} \Big\} (2l + d - 2) \tag{2.195}$$

$$= +\pi R^{d-1}_{\text{AdS}} \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \eta^{C,a}_{\omega\underline{l}m_l}\, \zeta^{C,b}_{-\omega,\underline{l},-m_l} - \eta^{C,b}_{\omega\underline{l}m_l}\, \zeta^{C,a}_{-\omega,\underline{l},-m_l} \Big\} (2\nu) . \tag{2.196}$$

We can read off that the S^a and S^b-modes form Lagrangian subspaces of the space of Klein-Gordon solutions on the AdS tube region. The C^a and C^b-modes form a different pair of Lagrangian subspaces. In both cases, the full space of Klein-Gordon solutions on the tube region is the direct sum of the Lagrangian (a)-subspace and the Lagrangian (b)-subspace. This holds for the real and complexified version of the solution space.

This will be important in Section 2.6.8, where we consider initial/boundary data: there, one subspace is related to the field values and the other one to the field derivatives/momenta. Moreover, Lagrangian subspaces corresponding to field values and momenta play an important role in the Schrödinger representation, see e.g. [61].

Recalling that the Klein-Gordon solutions on the rod region are purely S^a-modes, the symplectic structure associated to the boundary of any rod region vanishes. And further, since the AdS-Jacobi modes (the only allowed modes for time-interval regions) are merely special cases of the S^a-modes, the symplectic structure returns zero for any two such modes.

Next we calculate the flat limit of the symplectic structure in two different ways shown in the diagram below. Top left we have the symplectic structure ω_ρ^{AdS} on an AdS hypercylinder evaluated for two Klein-Gordon solutions η, ζ in its neighborhood. Bottom right we have the symplectic structure ω_r^{AdS} on a Minkowski hypercylinder evaluated for two Klein-Gordon solutions $\eta^{\text{M}}, \zeta^{\text{M}}$ on Minkowski spacetime, which are the flat limits of the AdS solutions η, ζ.

$$\begin{array}{ccc} \omega_\rho^{\text{AdS}}(\eta,\zeta) & \xrightarrow{\text{cont. flat lim.}} & \\ {\scriptstyle\int \mathrm{d}t}\Big\downarrow & & \Big\downarrow{\scriptstyle\int \mathrm{d}\tau} \\ & \xrightarrow{\text{cont. flat lim.}} & \omega_r^{\text{Mink}}(\eta^{\text{M}},\zeta^{\text{M}}) \end{array} \tag{2.197}$$

The first way is starting with (2.194), writing the solutions η and ζ as their S-expansions (2.186). Then, for $d = 3$, the flat limit is taken using the flat S-representation (C.94) and the flat limits (C.87):

$$\begin{aligned} &\int \mathrm{d}t \tan^{d-1}\rho \;\to\; \int \frac{\mathrm{d}\tau}{R_{\text{AdS}}}\,\frac{r^{d-1}}{R_{\text{AdS}}^{d-1}} &&\int \mathrm{d}\omega \;\to\; R_{\text{AdS}}\int \mathrm{d}\tilde\omega \\ &\phi^{S,a}_{\omega \underline{l}\, m_l} \;\to\; R_{\text{AdS}}^{-1}\tilde\phi^{\text{F},a}_{\tilde\omega \underline{l} m_l}\,\frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{(4\pi)}\,\frac{(p^{\mathbb{R}}_\omega)^l}{(2l+d-2)!!} &&\phi^{S,b}_{\omega \underline{l}\, m_l} \;\to\; R_{\text{AdS}}^{-1}\tilde\phi^{\text{F},b}_{\tilde\omega \underline{l} m_l}\,\frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{(4\pi)}\,\frac{(2l+d-4)!!}{(p^{\mathbb{R}}_\omega)^{l+d-2}} \\ &\tfrac{(p^{\mathbb{R}}_\omega)^l}{(2l+d-2)!!}\,S^a_{\omega l}(\rho) \;\to\; \check{j}_{\tilde\omega l}(r) &&\tfrac{(2l+d-4)!!}{(p^{\mathbb{R}}_\omega)^{l+d-2}}\,S^b_{\omega l}(\rho) \;\to\; \check{n}_{\tilde\omega l}(r) \end{aligned} \tag{2.198}$$

Only after that we integrate over the coordinates τ and Ω. This step is represented by $\int \mathrm{d}\tau$ in the diagram and results in the symplectic structure (2.116) of the Minkowski hypercylinder:

$$\omega_\rho(\eta,\zeta) \;\xrightarrow[\text{lim.}]{\text{flat}}\; \xrightarrow{d=3} \;\int \mathrm{d}\tilde\omega \sum_{l,m_l} \frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{16\pi}\left\{\tilde\eta^{\text{F},a}_{\tilde\omega l m_l}\tilde\zeta^{\text{F},b}_{-\tilde\omega,l,-m_l} - \tilde\eta^{\text{F},b}_{\tilde\omega l m_l}\tilde\zeta^{\text{F},a}_{-\tilde\omega,l,-m_l}\right\}. \tag{2.199}$$

As a consistency check, we now want to verify that going the other way in the diagram gives the same result. That is, after writing again the solutions η and ζ as their S-expansions we now first integrate over coordinates t and Ω, which is represented by the shorthand $\int \mathrm{d}t$ in the diagram and results in (2.195). Preparing the flat limit of this integral, from (2.198) we get

$$\eta^{S,a}_{\omega \underline{l}\, m_l}\zeta^{S,b}_{-\omega,\underline{l}\,,-m_l} \;\to\; \tilde\eta^{\text{F},a}_{\tilde\omega \underline{l} m_l}\tilde\zeta^{\text{F},b}_{-\tilde\omega,\underline{l},-m_l}\,\frac{(\tilde p^{\mathbb{R}}_{\tilde\omega})^{4-d}\;R_{\text{AdS}}^{-d}}{(4\pi)^2\,(2l+d-2)}\,, \tag{2.200}$$

and plugging it into (2.195) reproduces (2.199) already before applying the flat limit, which thus is trivial. Hence the above diagram indeed commutes, confirming that our flat limit of the symplectic structure is self-consistent.

Since for the hypercylinder surfaces there is no standard complex structure, at this point we can only consider symplectic potential and structure. In Sections 3.3.2-3.3.4 we continue from this point on, and construct complex structures for the hypercylinder. After this, in Section 3.3.5 consider the induced real g-products.

2.6.5 Time-interval: Solutions and structures

For the time-interval region $\mathbb{M}^{\text{AdS}}_{[t_1,t_2]} = [t_1,t_2]\times[0,\pi/2)\times\mathbb{S}^{d-1}$, that is: time-interval times all of space, we need Klein-Gordon solutions that are bounded on all of space. Thus we can only use Jacobi modes here, and expand any complex Klein-Gordon solution on the time-interval region as a sum of ordinary Jacobi modes, calling it (ordinary) Jacobi expansion:

$$\phi(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l}\Big\{\phi^+_{n\underline{l}m_l}\;\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\phi^-_{n\underline{l}m_l}}\;\overline{\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\}\;. \tag{2.201}$$

Only for $\nu\in(0,1)$ we can equivalently expand any solution using the exceptional AdS-Jacobi modes defined in (2.173), see e.g. (3.22) in [46]. Since these behave like the ordinary ones, we do not study them in this work. $\phi^+_{n\underline{l}m_l}$ determines the positive frequency part of the Klein-Gordon solution, and $\phi^-_{n\underline{l}m_l}$ the negative frequency part. If and only if $\phi^+_{n\underline{l}m_l} = \phi^-_{n\underline{l}m_l}$ then the solution $\phi(t,\rho,\Omega)$ is real. The ordinary Jacobi modes are propagating modes, well defined on the whole spacetime. Since the Jacobi modes are special cases of the S^a-modes, the space of Klein-Gordon solutions on time-interval regions is contained in the spaces of Klein-Gordon solutions on tube and rod regions as a subspace. Again, we can use a modified momentum representation $\tilde{\phi}^{\text{F},\pm}_{\tilde{p}\underline{l}m_l}$:

$$\phi^\pm_{n\underline{l}m_l} \;=\; \phi^\pm_{\omega^+_{nl}\underline{l}m_l} \;=\; \tilde{\phi}^{\text{F},\pm}_{\tilde{p}\underline{l}m_l}\,\frac{4\tilde{\omega}_{\tilde{p}}}{\sqrt{2\pi}}\;\frac{(p^{\mathbb{R}}_\omega)^l}{(2l+d-2)!!} \tag{2.202}$$

Then, in the flat limit for $d=3$ the ordinary Jacobi expansion becomes the Minkowski time-interval expansion (2.90), see Appendix C.2.8:

$$\phi(t,r,\Omega) \;\xrightarrow[\text{lim.}]{\text{flat}}\; \int_0^\infty\!\mathrm{d}\tilde{p}\sum_{l,m_l} 2\tilde{p}\,(2\pi)^{-1/2}\,j_l(\tilde{p}r)\Big\{\tilde{\phi}^{\text{F},+}_{\tilde{p}lm_l}\,\mathrm{e}^{-\mathrm{i}\tilde{\omega}_{\tilde{p}}\tau}\,Y^{m_l}_l(\Omega) + \overline{\tilde{\phi}^{\text{F},-}_{\tilde{p}lm_l}}\,\mathrm{e}^{\mathrm{i}\tilde{\omega}_{\tilde{p}}\tau}\,\overline{Y^{m_l}_l(\Omega)}\Big\}\;. \tag{2.203}$$

The expansion of the solution enables us to evaluate the structures on the space of solutions. As on Minkowski spacetime, our equal-time hypersurfaces are oriented pastwards, hence $\operatorname{sign}\Sigma_t = -1$. Using the expansion (2.201) in ordinary Jacobi modes, the symplectic potential for the equal-time plane is given by (with $\mathrm{d}^{d-1}\Omega \;=\; \mathrm{d}\Omega\,\sqrt{|g_{\mathbb{S}^{d-1}}|}$)

$$\theta^{\Sigma_t}_\phi(\eta) \;=\; \int_0^{\pi/2}\!\mathrm{d}\rho\int_{\mathbb{S}^{d-1}}\!\mathrm{d}^{d-1}\Omega\;R^{d-1}_{\text{AdS}}\tan^{d-1}\rho\;\;\eta(t,\rho,\Omega)\,(\partial_t\,\phi)(t,\rho,\Omega) \tag{2.204}$$

$$= +\mathrm{i}\sum_{n\underline{l}m_l}\omega^+_{nl}R^{d-1}_{\text{AdS}}\mathcal{N}^+_{nl}\Big\{-\phi^+_{n\underline{l}m_l}\eta^+_{n,\underline{l},-m_l}\mathrm{e}^{-2\mathrm{i}\omega^+_{nl}t} + \overline{\phi^-_{n\underline{l}m_l}}\;\;\eta^+_{n\underline{l}m_l} \tag{2.205}$$

$$+\;\overline{\phi^-_{n\underline{l}m_l}}\;\overline{\eta^-_{n,\underline{l},-m_l}}\,\mathrm{e}^{+2\mathrm{i}\omega^+_{nl}t} - \phi^+_{n\underline{l}m_l}\;\overline{\eta^-_{n\underline{l}m_l}}\;.\Big\}$$

For the expansion in exceptional Jacobi modes, we get the same expression up to replacing normalization constants and magic frequencies:

$$\mathcal{N}^+_{nl} \;\to\; \mathcal{N}^-_{nl} \qquad\qquad \omega^+_{nl} \;\to\; \omega^-_{nl}\;. \tag{2.206}$$

For the other structures we will simply write the expressions for both expansions in one equation using the label "±", with "+" referring to the ordinary and "−" to the exceptional modes expansion. Again we use the rescaled quantities of Appendix C.1.1:

$$\begin{aligned} r &:= R_{\text{AdS}}\,\rho & \qquad r &\in [0,\,\tfrac{\pi}{2}\,R_{\text{AdS}})\\ \tau &:= R_{\text{AdS}}\,t & \tau &\in (-\infty,+\infty)\\ \widetilde{\omega} &:= \omega/R_{\text{AdS}} & \tilde{\omega}_{\tilde{p}} &= \sqrt{\tilde{p}^2+m^2}\;. \end{aligned} \tag{2.207}$$

The symplectic structure (2.55) becomes

$$\omega_t(\eta,\zeta) = -\frac{1}{2}\int\limits_0^{\pi/2} \mathrm{d}\rho \int\limits_{\mathbb{S}^{d-1}} \mathrm{d}^{d-1}\Omega\; R^{d-1}_{\mathrm{AdS}} \tan^{d-1}\rho\, \big(\eta\,\partial_t\zeta - \zeta\,\partial_t\eta\big)(t,\rho,\Omega) \tag{2.208}$$

$$= +\mathrm{i}\sum_{n\underline{l}m_l} \omega^{\pm}_{nl} R^{d-1}_{\mathrm{AdS}} \mathcal{N}^{\pm}_{nl} \left\{\overline{\eta^{-}_{n\underline{l}m_l}}\,\zeta^{+}_{n\underline{l}m_l} - \eta^{+}_{n\underline{l}m_l}\,\overline{\zeta^{-}_{n\underline{l}m_l}}\right\} . \tag{2.209}$$

The first line is the coordinate representation of the symplectic structure, the second the momentum representation. From this result we can read off that the positive and negative frequency modes form Lagrangian subspaces of the complexified space of Klein-Gordon solutions on the AdS time-interval region, see definition (2.43). The full space of Klein-Gordon solutions on the time-interval region is the direct sum of both subspaces. For real solutions η,ζ (thus $\eta^{+}_{n\underline{l}m_l} = \eta^{-}_{n\underline{l}m_l}$, ditto for ζ) the symplectic structure gives real values.

Next we calculate the flat limit of the symplectic structure. This can be done in two ways which are shown in the diagram below. In its top left corner we have the symplectic structure ω^{AdS}_t on an AdS equal-time hypersurface evaluated for two Klein-Gordon solutions η,ζ in its neighborhood. In its bottom right corner we have the symplectic structure $\omega^{\mathrm{AdS}}_\tau$ on a Minkowski equal-time plane evaluated for two Klein-Gordon solutions $\eta^{\mathrm{M}},\zeta^{\mathrm{M}}$ on Minkowski spacetime, which are the flat limits of the AdS solutions η,ζ.

$$\begin{array}{ccc}
\omega^{\mathrm{AdS}}_t(\eta,\zeta) & \xrightarrow{\text{disc. flat lim.}} & \\
{\scriptstyle\int \mathrm{d}\rho}\downarrow & & \downarrow{\scriptstyle\int \mathrm{d}r} \\
 & \xrightarrow{\text{disc. flat lim.}} & \omega^{\mathrm{Mink}}_\tau(\eta^{\mathrm{M}},\zeta^{\mathrm{M}})
\end{array}$$

The first way is starting with (2.208), writing the solutions η and ζ as their Jacobi expansions (2.201). Then, for $d=3$, the flat limit is taken using results from Appendix C.2.8, and only after that we integrate over the coordinates r and Ω using (2.87). This step is represented by the shorthand $\int \mathrm{d}r$ in the diagram. This calculation is straightforward and for $d=3$ results in the symplectic structure (2.96) of the Minkowski equal-time plane:

$$\omega_t(\eta,\zeta) \;\xrightarrow[\text{lim.}]{\text{flat}}\; +\mathrm{i}\int\limits_0^\infty \mathrm{d}\tilde{p} \sum_{l,m_l} \tilde{\omega}_{\tilde{p}} \left\{\overline{\tilde{\eta}^{\mathrm{F},-}_{\tilde{p}lm_l}}\;\tilde{\zeta}^{\mathrm{F},+}_{\tilde{p}lm_l} - \tilde{\eta}^{\mathrm{F},+}_{\tilde{p}lm_l}\,\overline{\tilde{\zeta}^{\mathrm{F},-}_{\tilde{p}lm_l}}\right\} . \tag{2.210}$$

As a consistency check, we now want to verify that going the other way in the diagram gives the same result. That is, after writing again the solutions η and ζ as their Jacobi expansions we now first integrate over coordinates ρ and Ω using (C.67). This step is represented by the shorthand $\int \mathrm{d}\rho$ in the diagram and results in (2.209). Now we need to take the flat limit of this sum: using results from Appendix C.2.8, we obtain for $d=3$

$$\omega_t(\eta,\zeta) \;\xrightarrow[\text{lim.}]{\text{flat}}\; = +\mathrm{i}\int\limits_0^\infty \mathrm{d}\tilde{p} \sum_{l,m_l} \omega^{\pm}_{nl}\mathcal{N}^{\pm}_{nl}\,\frac{4}{\pi}\,\frac{(p^{\mathbb{R}}_{\omega})^{2l+1}}{((2l+d-2)!!)^2}\,\tilde{\omega}_{\tilde{p}} \left\{\overline{\tilde{\eta}^{\mathrm{F},-}_{\tilde{p}lm_l}}\,\tilde{\zeta}^{\mathrm{F},+}_{\tilde{p}lm_l} - \tilde{\eta}^{\mathrm{F},+}_{\tilde{p}lm_l}\,\overline{\tilde{\zeta}^{\mathrm{F},-}_{\tilde{p}lm_l}}\right\}.$$

In order to make this coincide with (2.210), it remains to show that the flat limit of $\omega^{\pm}_{nl}\mathcal{N}^{\pm}_{nl}\,\frac{4}{\pi}\,(p^{\mathbb{R}}_{\omega})^{2l+1}/((2l+d-2)!!)^2$ is 1. This can be done as in the calculation following (C.446), using that for odd k we have $k!! = \Gamma(\frac{k}{2}+1)\,2^{\frac{k+1}{2}}/\sqrt{\pi}$ and from $\omega^{+}_{nl} := \tilde{m}_+ + 2n + l$ replacing n by $\frac{1}{2}(\omega^{+}_{nl} - \tilde{m}_+ - l)$. Thus the above diagram indeed commutes, confirming that our flat limit of the symplectic structure is self-consistent.

The calculation of the flat limit of the symplectic structure indicates that as a shorthand we can take the flat limit of the momentum representation of the symplectic structure and related structures by substituting

$$\sum_{n\underline{l}m_l} \omega^{\pm}_{nl} R^{d-1}_{\text{AdS}} \mathcal{N}^{\pm}_{nl} \;\to\; \int_0^\infty \mathrm{d}\tilde{p} \sum_{\underline{l},m_l} \tilde{\omega}_{\tilde{p}} \qquad\qquad t \;\to\; \tau \tag{2.211}$$
$$\phi^{\pm}_{n\underline{l}m_l} \;\to\; \tilde{\phi}^{\mathrm{F},\pm}_{\tilde{p}\underline{l}m_l} \qquad\qquad \omega^{+}_{nl} \;\to\; \tilde{\omega}_{\tilde{p}}\ .$$

For the AdS hypercylinders we can use the same standard complex structure as for the Minkowski hypercylinder (for both ordinary and exceptional modes):

$$\big(J_t\,\phi\big)^{\pm}_{n\underline{l}m_l} \;=\; -\mathrm{i}\,\phi^{\pm}_{n\underline{l}m_l}\ . \tag{2.212}$$

Hence the real g-product is given by

$$\begin{aligned} \mathrm{g}_t(\eta,\zeta) \;&=\; 2\omega_t(\eta, J_{\Sigma_t}\zeta) \\ &=\; \sum_{n\underline{l}m_l} 2\omega^{\pm}_{nl} R^{d-1}_{\text{AdS}} \mathcal{N}^{\pm}_{nl} \Big\{\overline{\eta^{-}_{n\underline{l}m_l}}\,\zeta^{+}_{n\underline{l}m_l} + \eta^{+}_{n\underline{l}m_l}\,\overline{\zeta^{-}_{n\underline{l}m_l}}\Big\}\ , \end{aligned} \tag{2.213}$$

For $d=3$, its flat limit (for the ordinary Jacobi expansion) is the real g-product on the Minkowski equal-time plane:

$$\mathrm{g}_t(\eta,\zeta) \;\xrightarrow[\text{lim.}]{\text{flat}}\; + \int_0^\infty \mathrm{d}\tilde{p} \sum_{l,m_l} 2\tilde{\omega}_{\tilde{p}} \Big\{\overline{\tilde{\eta}^{\mathrm{F},-}_{\tilde{p}lm_l}}\ \tilde{\zeta}^{\mathrm{F},+}_{\tilde{p}lm_l} + \tilde{\eta}^{\mathrm{F},+}_{\tilde{p}lm_l}\overline{\tilde{\zeta}^{\mathrm{F},-}_{\tilde{p}lm_l}}\Big\}\ .$$

For real fields η,ζ the real g-product $\mathrm{g}_t(\eta,\zeta)$ gives real values. Moreover, for real fields $\phi\neq 0$ we have $\mathrm{g}_t(\phi,\phi) > 0$, that is, the real g-product is positive-definite. The complex inner product then becomes

$$\begin{aligned} \{\eta,\,\zeta\}_{\Sigma_t} \;&=\; \mathrm{g}_t(\eta,\zeta) + 2\mathrm{i}\,\omega_t(\eta,\zeta) \\ &=\; \sum_{n\underline{l}m_l} 4\omega^{\pm}_{nl} R^{d-1}_{\text{AdS}} \mathcal{N}^{\pm}_{nl} \Big\{\eta^{+}_{n\underline{l}m_l}\,\overline{\zeta^{-}_{n\underline{l}m_l}}\Big\}\ , \end{aligned} \tag{2.214}$$

Again, for $d=3$ its flat limit (for the ordinary Jacobi expansion) is the inner product of the Minkowski equal-time plane:

$$\{\eta,\,\zeta\}_{\Sigma_t} \;\xrightarrow[\text{lim.}]{\text{flat}}\; + \int_0^\infty \mathrm{d}\tilde{p} \sum_{l,m_l} 4\tilde{\omega}_{\tilde{p}} \Big\{\tilde{\eta}^{\mathrm{F},+}_{\tilde{p}lm_l}\overline{\tilde{\zeta}^{\mathrm{F},-}_{\tilde{p}lm_l}}\Big\}\ .$$

2.6.6 Isometry actions on Klein-Gordon solutions

In this section we consider the action of the isometry group $\mathrm{SO}\,(2,d)$ of $\mathrm{AdS}_{1,d}$. Uppercase Latin indices range as $A = 0,1,...,d,(d{+}1)$, and lowercase Latin indices as $k=1,...,d$. The generators of the Lie algebra $\mathbf{so}\,(2,d)$ are the Killing vectors K_{AB} of Section 2.6.2: $K_{AB} = \big(X_A\,\partial_B - X_B\,\partial_A\big)$. This choice is the same as in (4.18) in [26] up to an overall sign. The (representations of) finite group elements are denoted by g. The Lie algebra $\mathbf{so}\,(2,d)$ is determined by the same Lie bracket (B.4), which coincides with (4.21) in [26] up to an overall sign:

$$[K_{AB},\,K_{CD}] \;=\; -\eta_{AC}\,K_{BD} + \eta_{BC}\,K_{AD} - \eta_{BD}\,K_{AC} + \eta_{AD}\,K_{BC}\ . \tag{2.215}$$

For the various combinations of time translation, rotations and boosts this Lie bracket writes as

$$\begin{array}{lll} [K_{d+1,0},\,K_{jk}] = 0 & [K_{0j},\,K_{0k}] = \eta_{00}K_{kj} & [K_{0q},\,K_{jk}] = \eta_{jq}K_{0k} - \eta_{kq}K_{0j} \\ [K_{d+1,j},\,K_{d+1,k}] = \eta_{d+1,d+1}K_{kj} & [K_{d+1,0},\,K_{0k}] = \eta_{00}K_{d+1,k} & [K_{d+1,q},\,K_{jk}] = \eta_{jq}K_{d+1,k} - \eta_{kq}K_{d+1,j} \\ [K_{d+1,k},\,K_{d+1,0}] = \eta_{d+1,d+1}K_{0k} & [K_{0k},\,K_{d+1,j}] = \eta_{jk}K_{d+1,0} & [K_{jk},\,K_{pq}] = \eta_{kp}K_{jq} - \eta_{jp}K_{kq} \\ & & \qquad + \eta_{jq}K_{kp} - \eta_{kq}K_{jp}. \end{array} \tag{2.216}$$

On solution space, we denote the (infinitesimal) action of a generator K_{AB} respectively of a (finite) group element k on a field $\phi(x)$ by $K_{AB} \triangleright \phi$ respectively $k \triangleright \phi$. Requiring the transformed field at transformed coordinates to agree with the original field at original coordinates, we get for the transformed field at the original coordinates:

$$(k \triangleright \phi)(x) \;=\; \phi(k^{-1}x), \qquad\qquad \Big(K_{AB} \triangleright \phi\Big)(x) \;=\; (-K_{AB}\phi)(x). \tag{2.217}$$

Therein, $K_{AB}\phi$ means letting the generator (Killing vector) act as differential operator on the solution (function) $\phi(x)$. In the following sections we calculate these actions for the time translation, rotations and boosts for Klein-Gordon solutions on the time-interval and tube regions. Our goal will always be to transcribe the action from the coordinate representation to the momentum representation. We thus start from a field expansion over modes $\mu^{(S,a/S,b)}_{\omega\underline{l}m_l}(x)$ of momentum $(\omega, \underline{l}, m_l)$ wherein $\phi^{S,a/S,b}_{\omega\underline{l}m_l}$ are the momentum representation of the Klein-Gordon solution. (The corresponding constructions for the (C,a)-modes and (C,b)-modes are done after the S-modes in the same way.) What we want to find is an explicit expression for the momentum representations $(k \triangleright \phi)^{a,b}_{\omega\underline{l}m_l}$ of the transformed solution, such that we can directly write the transformed solution $(k \triangleright \phi)(x)$ in the original coordinates as in:

$$(k \triangleright \phi)(x) \;=\; \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ (k \triangleright \phi)^a_{\omega\underline{l}m_l}\, \mu^{(a)}_{\omega\underline{l}m_l}(x) + (k \triangleright \phi)^b_{\omega\underline{l}m_l}\, \mu^{(b)}_{\omega\underline{l}m_l}(x) \Big\} \;. \tag{2.218}$$

Action of time translations on Klein-Gordon solutions

Infinitesimal time translations arise nicely from the finite ones, thus we only deal with the latter here. Denoting finite time translations by $k_{\Delta t}: \quad t \;\to\; t + \Delta t$, its action on the hypergeometric modes is

$$\Big(k_{\Delta t} \triangleright \mu^{(a)}_{\omega\underline{l}m_l}\Big)(t,\rho,\Omega) \;=\; \mathrm{e}^{\mathrm{i}\omega\Delta t}\, \mu^{(a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \qquad \Big(k_{\Delta t} \triangleright \mu^{(b)}_{\omega\underline{l}m_l}\Big)(t,\rho,\Omega) \;=\; \mathrm{e}^{\mathrm{i}\omega\Delta t}\, \mu^{(b)}_{\omega\underline{l}m_l}(t,\rho,\Omega)\;, \tag{2.219}$$

and the action on the Jacobi modes as a special case of S^a-modes is the same:

$$\big(k_{\Delta t} \triangleright \mu^{(+)}_{n\underline{l}m_l}\big)(t,\rho,\Omega) \;=\; \mathrm{e}^{\mathrm{i}\omega^+_{nl}\Delta t}\, \mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega) \quad \big(k_{\Delta t} \triangleright \overline{\mu^{(+)}_{n\underline{l}m_l}}\big)(t,\rho,\Omega) \;=\; \mathrm{e}^{-\mathrm{i}\omega^+_{nl}\Delta t}\, \overline{\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)}\;. \tag{2.220}$$

Applying these to the tube expansion (2.186) respectively time-interval expansion (2.201), we can read off

$$\big(k_{\Delta t} \triangleright \phi\big)^a_{\omega\underline{l}m_l} \;=\; \mathrm{e}^{\mathrm{i}\omega\Delta t}\, \phi^a_{\omega\underline{l}m_l} \qquad\qquad \big(k_{\Delta t} \triangleright \phi\big)^b_{\omega\underline{l}m_l} \;=\; \mathrm{e}^{\mathrm{i}\omega\Delta t}\, \phi^b_{\omega\underline{l}m_l} \tag{2.221}$$

$$\big(k_{\Delta t} \triangleright \phi\big)^+_{n\underline{l}m_l} \;=\; \mathrm{e}^{\mathrm{i}\omega^+_{nl}\Delta t}\, \phi^+_{n\underline{l}m_l} \qquad\qquad \overline{\big(k_{\Delta t} \triangleright \phi\big)^-_{n\underline{l}m_l}} \;=\; \mathrm{e}^{-\mathrm{i}\omega^+_{nl}\Delta t}\, \overline{\phi^-_{n\underline{l}m_l}}\;. \tag{2.222}$$

Action of rotations on Klein-Gordon solutions

Let $\hat{R}(\underline{\alpha})$ denote a finite rotation, with $\underline{\alpha}$ denoting the rotation angles. We recall that rotated spherical harmonics are a linear combination of unrotated ones, with elements of Wigner's D-matrix as coefficients:

$$Y^{m_l}_{(l,\underline{l})}(\hat{R}(\underline{\alpha})\Omega) = \sum_{\underline{l}',\, m'_l} Y^{m'_l}_{(l,\underline{l}')}(\Omega)\, \overline{\Big(D^l_{\underline{l},\underline{l}'}(\underline{\alpha})\Big)_{m_l m'_l}}\;,$$

see Appendix A.4. For the hypergeometric modes this induces the action

$$\mu^{(a,b)}_{\omega l\,\underline{l}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) = \sum_{\underline{l}',\, m'_l} \mu^{(a,b)}_{\omega l\,\underline{l}'\,m'_l}(t,\rho,\Omega)\, \overline{\Big(D^l_{\underline{l},\underline{l}'}(\underline{\alpha})\Big)_{m_l m'_l}}\;. \tag{2.223}$$

Since the Jacobi modes are special cases of the S^a-modes, the action is the same for them. For tube solution we can apply (C.106) to expansion (2.186), giving:

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{a,b}_{\omega l\, \underline{l}\, m_l} = \sum_{\underline{l}',\, m_l'} \phi^{a,b}_{\omega l\, \underline{l}'\, m_l'} \overline{\left(D^l_{\underline{l}',\underline{l}}(\underline{\alpha})\right)_{m_l' m_l}} \,. \tag{2.224}$$

For time-interval solutions, we apply (C.106) to expansion (2.201), yielding:

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{\pm}_{nl\, \underline{l}\, m_l} = \sum_{\underline{l}',\, m_l'} \phi^{\pm}_{nl\, \underline{l}'\, m_l'} \overline{\left(D^l_{\underline{l}',\underline{l}}(\underline{\alpha})\right)_{m_l' m_l}} \,. \tag{2.225}$$

Action of boosts on Klein-Gordon solutions

Here the goal is to calculate the action of the AdS boost generators $K_{d+1,j}$ and K_{0j} on the bounded Klein-Gordon modes on time-interval and tube regions. Since this is somewhat involved, here we only sum up the results of Appendix C.3.4 and Appendix C.3.5.

We only consider infinitesimal boosts. The effect of boosts on the AdS time axis is qualitatively different from its Minkowski counterpart. For seeing this, consider e.g. a boost with finite rapidity λ in the $(0, j)$-plane in the embedding space of AdS. The boosted coordinates are $X_0' = X_0 \cosh\lambda + X_j \sinh\lambda$ and $X_j' = X_0 \sinh\lambda + X_j \cosh\lambda$, with the other coordinates unchanged. Thus all points with $X_0 = X_j = 0$ are preserved under these boosts, while other points are moved a finite distance on the AdS-hyperboloid. This means that, in contrast to Minkowski spacetime, on AdS the boosts do not rotate the time axis but deform it periodically (into some timelike geodesic). Therefore, small boosts on AdS move the time axis only a small distance away from the unboosted one. In contrast, on Minkowski spacetime even an arbitrarily small boost for sufficiently large times separates the boosted time axis an arbitrary distance from the unboosted one.

Recall now that the S^a-modes are regular on the time axis $\rho \equiv 0$ while the S^b-modes diverge there. A finite boost moves the boosted time axis off the unboosted one. Therefore the S^b-modes of the boosted coordinates now have singularities off the unboosted time axis, and thus cannot be well defined linear combinations of the original S^a and S^b-modes (because these are regular off the unboosted time axis). Hence only for infinitesimal boosts there is a chance that we might find a well defined action on Klein-Gordon solutions. For the two d-boosts $K_{d+1,d}$ and K_{0d} we recall the expressions (2.154)

$$K_{d+1,d} = -\sin t\, \sin\rho\, \cos\theta_{d-1}\, \partial_t + \cos t\, \cos\rho\, \cos\theta_{d-1}\, \partial_\rho + \sin^2\theta_{d-1} \cos t/(\sin\rho)\, \partial_{\cos\theta_{d-1}} \tag{2.226}$$

$$K_{0d} = -\cos t\, \sin\rho\, \cos\theta_{d-1}\, \partial_t - \sin t\, \cos\rho\, \cos\theta_{d-1}\, \partial_\rho - \sin^2\theta_{d-1} \sin t/(\sin\rho)\, \partial_{\cos\theta_{d-1}} \,. \tag{2.227}$$

Letting these act directly on Jacobi or hypergeometric modes results in rather complicated expressions. However, in [32] Dorn et al. note below equation (3.18) that a complex linear combination of these boosts increases the frequency ω exactly by one. Inspired by their equation (2.7) we thus define the Z-generators of the complexified Lie algebra:

$$\begin{aligned} Z_d &:= K_{0d} + \mathrm{i} K_{d+1,d} \\ &= -\mathrm{e}^{\mathrm{i}t} \sin\rho\, \cos\theta_{d-1}\, \partial_t + \mathrm{i}\, \mathrm{e}^{\mathrm{i}t} \cos\rho\, \cos\theta_{d-1}\, \partial_\rho + \mathrm{i}\, \mathrm{e}^{\mathrm{i}t} \sin^{-1}\!\rho\, \sin^2\theta_{d-1}\, \partial_{\cos\theta_{d-1}} \\ \overline{Z_d} &:= K_{0d} - \mathrm{i} K_{d+1,d} \\ &= -\mathrm{e}^{-\mathrm{i}t} \sin\rho\, \cos\theta_{d-1}\, \partial_t - \mathrm{i}\, \mathrm{e}^{-\mathrm{i}t} \cos\rho\, \cos\theta_{d-1}\, \partial_\rho - \mathrm{i}\, \mathrm{e}^{-\mathrm{i}t} \sin^{-1}\!\rho\, \sin^2\theta_{d-1}\, \partial_{\cos\theta_{d-1}} \,. \end{aligned} \tag{2.228}$$

We first calculate the action of $\overline{Z_d}$ and Z_d on the modes, and then the action of the d-boosts $K_{d+1,d}$ and K_{0d}. It is enough to know the actions of these two boosts, because the actions of the other boosts can be obtained from the Lie brackets of these two boosts with some rotators: from (2.216) we have $[K_{dk},\, K_{0d}] = K_{0k}$ and $[K_{dk},\, K_{d+1,d}] = K_{d+1,k}$. We do not treat the Jacobi modes as special case of the S^a-modes in this section, because it is more useful to give the results in terms of n for them, as opposed to ω for the hypergeometric modes.

In the formula below, the boosts' actions result in a sum involving coefficients denoted by z and $\tilde{z}$. The tilde merely indicates that the $\tilde{z}$ determine the action of $\overline{Z_d}$, while the z determine that of Z_d. The coefficients are given explicitly in (C.258), (C.259) (C.266) and (C.267). Since for tube regions we have the ranges $\omega \in \mathbb{R}$ and $l \in \mathbb{N}_0$, for notational convenience for negative values of l we set to zero all $z_{\omega l}^{(S,a,b)--}$, $z_{\omega l}^{(S,a,b)-+}$, $\tilde{z}_{\omega l}^{(S,a,b)+-}$, $\tilde{z}_{\omega l}^{(S,a,b)++}$ and $\phi^{S,a/S,b}_{\omega,l,\tilde{\underline{l}},m_l}$. With that, the actions of the d-boosts write

$$\begin{aligned}\left(K_{0d}\triangleright\phi\right)^a_{\omega\underline{l}m_l} &= \tfrac{\mathrm{i}}{2}\tilde{z}^{(S,a)+-}_{\omega-1,l+1}\,\phi^a_{\omega-1,l+1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}\tilde{z}^{(S,a)++}_{\omega-1,l-1}\,\phi^a_{\omega-1,l-1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}z^{(S,a)--}_{\omega+1,l+1}\,\phi^a_{\omega+1,l+1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}z^{(S,a)-+}_{\omega+1,l-1}\,\phi^a_{\omega+1,l-1,\tilde{\underline{l}},m_l}\\ \left(K_{0d}\triangleright\phi\right)^b_{\omega\underline{l}m_l} &= \tfrac{\mathrm{i}}{2}\tilde{z}^{(S,b)+-}_{\omega-1,l+1}\,\phi^b_{\omega-1,l+1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}\tilde{z}^{(S,b)++}_{\omega-1,l-1}\,\phi^b_{\omega-1,l-1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}z^{(S,b)--}_{\omega+1,l+1}\,\phi^b_{\omega+1,l+1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}z^{(S,b)-+}_{\omega+1,l-1}\,\phi^b_{\omega+1,l-1,\tilde{\underline{l}},m_l}\end{aligned} \tag{2.229}$$

$$\begin{aligned}\left(K_{d+1,d}\triangleright\phi\right)^a_{\omega\underline{l}m_l} &= -\tfrac{1}{2}\tilde{z}^{(S,a)+-}_{\omega-1,l+1}\,\phi^a_{\omega-1,l+1,\tilde{\underline{l}},m_l} - \tfrac{1}{2}\tilde{z}^{(S,a)++}_{\omega-1,l-1}\,\phi^a_{\omega-1,l-1,\tilde{\underline{l}},m_l} + \tfrac{1}{2}z^{(S,a)--}_{\omega+1,l+1}\,\phi^a_{\omega+1,l+1,\tilde{\underline{l}},m_l} + \tfrac{1}{2}z^{(S,a)-+}_{\omega+1,l-1}\,\phi^a_{\omega+1,l-1,\tilde{\underline{l}},m_l}\\ \left(K_{d+1,d}\triangleright\phi\right)^b_{\omega\underline{l}m_l} &= -\tfrac{1}{2}\tilde{z}^{(S,b)+-}_{\omega-1,l+1}\,\phi^b_{\omega-1,l+1,\tilde{\underline{l}},m_l} - \tfrac{1}{2}\tilde{z}^{(S,b)++}_{\omega-1,l-1}\,\phi^b_{\omega-1,l-1,\tilde{\underline{l}},m_l} + \tfrac{1}{2}z^{(S,b)--}_{\omega+1,l+1}\,\phi^b_{\omega+1,l+1,\tilde{\underline{l}},m_l} + \tfrac{1}{2}z^{(S,b)-+}_{\omega+1,l-1}\,\phi^b_{\omega+1,l-1,\tilde{\underline{l}},m_l}.\end{aligned} \tag{2.230}$$

The above infinitesimal actions can be derived from those of $K_{d+1,d}$ and K_{0d} on the hypergeometric modes. Applying actions (C.270)-(C.273) to expansion (2.186) and shifting ω and l by ±1 depending on the respective term yields the above actions.

For the time-interval regions, the coefficients z and $\tilde{z}$ are given explicitly in (C.212), and (C.227). Since here we have the ranges $n, l \in \mathbb{N}_0$, now for notational convenience we set to zero all quantities where n or l take values outside this range: all ω^+_{nl}, $z^{(+)-+}_{nl}$, $z^{(+)0-}_{nl}$, $\tilde{z}^{(+)+-}_{nl}$, $\tilde{z}^{(+)0+}_{nl}$ and $\phi^{\pm}_{n,l,\tilde{\underline{l}},m_l}$ are set to zero for negative n or l. Then, the actions of the infinitesimal d-boosts write

$$\begin{aligned}\left(K_{0d}\triangleright\phi\right)^+_{n\underline{l}m_l} &= \tfrac{\mathrm{i}}{2}z^{(+)0-}_{n,l+1}\,\phi^+_{n,l+1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}z^{(+)-+}_{n+1,l-1}\,\phi^+_{n+1,l-1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}\tilde{z}^{(+)+-}_{n-1,l+1}\,\phi^+_{n-1,l+1,\tilde{\underline{l}},m_l} + \tfrac{\mathrm{i}}{2}\tilde{z}^{(+)0+}_{n,l-1}\,\phi^+_{n,l-1,\tilde{\underline{l}},m_l}\\ \overline{\left(K_{0d}\triangleright\phi\right)^-_{n\underline{l}m_l}} &= -\tfrac{\mathrm{i}}{2}z^{(+)0-}_{n,l+1}\,\overline{\phi^-_{n,l+1,\tilde{\underline{l}},m_l}} - \tfrac{\mathrm{i}}{2}z^{(+)-+}_{n+1,l-1}\,\overline{\phi^-_{n+1,l-1,\tilde{\underline{l}},m_l}} - \tfrac{\mathrm{i}}{2}\tilde{z}^{(+)+-}_{n-1,l+1}\,\overline{\phi^-_{n-1,l+1,\tilde{\underline{l}},m_l}} - \tfrac{\mathrm{i}}{2}\tilde{z}^{(+)0+}_{n,l-1}\,\overline{\phi^-_{n,l-1,\tilde{\underline{l}},m_l}}\end{aligned} \tag{2.231}$$

$$\begin{aligned}\left(K_{d+1,d}\triangleright\phi\right)^+_{n\underline{l}m_l} &= \tfrac{1}{2}z^{(+)0-}_{n,l+1}\,\phi^+_{n,l+1,\tilde{\underline{l}},m_l} + \tfrac{1}{2}z^{(+)-+}_{n+1,l-1}\,\phi^+_{n+1,l-1,\tilde{\underline{l}},m_l} - \tfrac{1}{2}\tilde{z}^{(+)+-}_{n-1,l+1}\,\phi^+_{n-1,l+1,\tilde{\underline{l}},m_l} - \tfrac{1}{2}\tilde{z}^{(+)0+}_{n,l-1}\,\phi^+_{n,l-1,\tilde{\underline{l}},m_l}\\ \overline{\left(K_{d+1,d}\triangleright\phi\right)^-_{n\underline{l}m_l}} &= \tfrac{1}{2}z^{(+)0-}_{n,l+1}\,\overline{\phi^-_{n,l+1,\tilde{\underline{l}},m_l}} + \tfrac{1}{2}z^{(+)-+}_{n+1,l-1}\,\overline{\phi^-_{n+1,l-1,\tilde{\underline{l}},m_l}} - \tfrac{1}{2}\tilde{z}^{(+)+-}_{n-1,l+1}\,\overline{\phi^-_{n-1,l+1,\tilde{\underline{l}},m_l}} - \tfrac{1}{2}\tilde{z}^{(+)0+}_{n,l-1}\,\overline{\phi^-_{n,l-1,\tilde{\underline{l}},m_l}}\end{aligned} \tag{2.232}$$

These can be derived from the action of $K_{d+1,d}$ and K_{0d} on the Jacobi modes: applying actions (C.232)-(C.235) to expansion (2.201) and shifting n, l by ±1 depending on the respective term yields the above actions.

2.6.7 Invariance of symplectic structures under isometries

In this section we show the invariance of the symplectic structures on the equal-time surfaces and on the hypercylinders under time translation, rotations and boosts. We use the explicit expressions we have worked out for the action of these isometries in the momentum representation of the Klein-Gordon solutions in Section 2.6.6. For the boosts we have calculated explicitly only the actions of the (infinitesimal) d-boosts on the solutions. Before checking that the d-boosts also leave the symplectic structures invariant, we remark that this is already sufficient for assuring that actually *all boosts* do so, because the remaining boosts arise as Lie brackets of the d-boosts and spatial rotations, see Section 2.4.3.

Invariance under time translations

We show the invariance of the symplectic structure only for finite time translations, with the invariance for infinitesimal time translations then holding automatically. We denote the corresponding element of SO$(2,d)$ by $k_{\Delta t}$. For both time-interval and tube regions according to (2.56) we then have the action on the symplectic structure as

$$\big(k_{\Delta t}\triangleright\omega\big)(\eta,\zeta) \;=\; \omega(k_{\Delta t}^{-1}\triangleright\eta,\, k_{\Delta t}^{-1}\triangleright\zeta) \;=\; \omega(k_{-\Delta t}\triangleright\eta,\, k_{-\Delta t}\triangleright\zeta)\;.$$

For the time-interval region we can now plug the action (2.222) of $k_{-\Delta t}$ in the momentum representation into the symplectic structure (2.209):

$$\begin{aligned}\big(k_{\Delta t}\triangleright\omega_t\big)(\eta,\zeta) \;&=\; \mathrm{i}\sum_{n\underline{l}m_l}\omega^{\pm}_{nl}R^{d-1}_{\mathrm{AdS}}\mathcal{N}^{\pm}_{nl}\Big\{\overline{\big(k_{-\Delta t}\triangleright\eta\big)^{-}_{n\underline{l}m_l}}\,\big(k_{-\Delta t}\triangleright\zeta\big)^{+}_{n\underline{l}m_l} - \big(k_{-\Delta t}\triangleright\eta\big)^{+}_{n\underline{l}m_l}\,\overline{\big(k_{-\Delta t}\triangleright\zeta\big)^{-}_{n\underline{l}m_l}}\Big\}\\
&=\; \mathrm{i}\sum_{n\underline{l}m_l}\omega^{\pm}_{nl}R^{d-1}_{\mathrm{AdS}}\mathcal{N}^{\pm}_{nl}\Big\{\mathrm{e}^{\mathrm{i}\omega^{\pm}_{nl}\Delta t}\,\overline{\eta^{-}_{n\underline{l}m_l}}\,\mathrm{e}^{-\mathrm{i}\omega^{\pm}_{nl}\Delta t}\,\zeta^{+}_{n\underline{l}m_l} - \mathrm{e}^{-\mathrm{i}\omega^{\pm}_{nl}\Delta t}\,\eta^{+}_{n\underline{l}m_l}\,\mathrm{e}^{\mathrm{i}\omega^{\pm}_{nl}\Delta t}\,\overline{\zeta^{-}_{n\underline{l}m_l}}\Big\}\\
&=\; \mathrm{i}\sum_{n\underline{l}m_l}\omega^{\pm}_{nl}R^{d-1}_{\mathrm{AdS}}\mathcal{N}^{\pm}_{nl}\Big\{\overline{\eta^{-}_{n\underline{l}m_l}}\,\zeta^{+}_{n\underline{l}m_l} - \eta^{+}_{n\underline{l}m_l}\,\overline{\zeta^{-}_{n\underline{l}m_l}}\Big\}\;,\end{aligned}$$

and thus the time-interval region's symplectic structure is invariant under time translations:

$$\big(k_{\Delta t}\triangleright\omega_t\big)(\eta,\zeta) \;=\; \omega_t(\eta,\zeta) \qquad\quad \forall\,\eta,\zeta\;. \tag{2.233}$$

For the tube region the calculation is the same: we can plug the action (2.221) of $k_{-\Delta t}$ in the momentum representation into the symplectic structure (2.195):

$$\begin{aligned}\big(k_{\Delta t}\triangleright\omega_\rho\big)(\eta,\zeta) \;&=\; \pi R^{d-1}_{\mathrm{AdS}}\int\!\mathrm{d}\omega\sum_{\underline{l},m_l}(2l+d-2)\Big\{\big(k_{-\Delta t}\triangleright\eta\big)^{S,a}_{\omega\underline{l}m_l}\,\big(k_{-\Delta t}\triangleright\zeta\big)^{S,b}_{-\omega,\underline{l},-m_l}\\
&\qquad\qquad - \big(k_{-\Delta t}\triangleright\eta\big)^{S,b}_{\omega\underline{l}m_l}\,\big(k_{-\Delta t}\triangleright\zeta\big)^{S,a}_{-\omega,\underline{l},-m_l}\Big\}\\
&=\; \pi R^{d-1}_{\mathrm{AdS}}\int\!\mathrm{d}\omega\sum_{\underline{l},m_l}(2l+d-2)\Big\{\mathrm{e}^{-\mathrm{i}\omega\Delta t}\,\eta^{S,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{\mathrm{i}\omega\Delta t}\,\zeta^{S,b}_{-\omega,\underline{l},-m_l} - \mathrm{e}^{-\mathrm{i}\omega\Delta t}\,\eta^{S,b}_{\omega\underline{l}m_l}\,\mathrm{e}^{\mathrm{i}\omega\Delta t}\,\zeta^{S,a}_{-\omega,\underline{l},-m_l}\Big\}\\
&=\; \pi R^{d-1}_{\mathrm{AdS}}\int\!\mathrm{d}\omega\sum_{\underline{l},m_l}(2l+d-2)\Big\{\eta^{S,a}_{\omega\underline{l}m_l}\,\zeta^{S,b}_{-\omega,\underline{l},-m_l} - \eta^{S,b}_{\omega\underline{l}m_l}\,\zeta^{S,a}_{-\omega,\underline{l},-m_l}\Big\}\;.\end{aligned}$$

Thus the tube region's symplectic structure is invariant under time translations:

$$\big(k_{\Delta t}\triangleright\omega_\rho\big)(\eta,\zeta) \;=\; \omega_\rho(\eta,\zeta) \qquad\quad \forall\,\eta,\zeta\;. \tag{2.234}$$

Invariance under rotations

Again we show the invariance of the symplectic structure only for finite rotations, with the invariance for infinitesimal ones then holding automatically. We denote the corresponding element of SO$(2,d)$ by $\hat{R}(\underline{\alpha})$. For both time-interval and tube regions according to (2.56) we then have the action on the symplectic structure as

$$\big(\hat{R}(\underline{\alpha})\triangleright\omega\big)(\eta,\zeta) \;=\; \omega(\hat{R}(\underline{\alpha})^{-1}\triangleright\eta,\, \hat{R}(\underline{\alpha})^{-1}\triangleright\zeta)\;.$$

For the time-interval region we can plug the action (2.225) of $\hat{R}(\underline{\alpha})^{-1}$ in the momentum representation into the symplectic structure (2.209):

$$
\begin{aligned}
\big(\hat{R}(\underline{\alpha})\triangleright\omega_t\big)(\eta,\zeta) \;&=\; \mathrm{i}\sum_{n\underline{l}m_l} \omega^{\pm}_{nl} R^{d-1}_{\mathrm{AdS}} \mathcal{N}^{\pm}_{nl} \Big\{ \overline{\big(\hat{R}(\underline{\alpha})^{-1}\triangleright\eta\big)^{-}_{n\underline{l}m_l}}\, \big(\hat{R}(\underline{\alpha})^{-1}\triangleright\zeta\big)^{+}_{n\underline{l}m_l} \\
&\qquad - \big(\hat{R}(\underline{\alpha})^{-1}\triangleright\eta\big)^{+}_{n\underline{l}m_l}\, \overline{\big(\hat{R}(\underline{\alpha})^{-1}\triangleright\zeta\big)^{-}_{n\underline{l}m_l}} \Big\} \\
&= \mathrm{i}\sum_{n\underline{l}m_l} \omega^{\pm}_{nl} R^{d-1}_{\mathrm{AdS}} \mathcal{N}^{\pm}_{nl} \sum_{\underline{\tilde{l}}' m_l'} \sum_{\underline{\tilde{l}}'' m_l''} \Big\{ \overline{\eta^{-}_{nl\,\underline{\tilde{l}}' m_l'}}\, \big(D^{l}_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l' m_l}\, \zeta^{+}_{nl\,\underline{\tilde{l}}'' m_l''}\, \overline{\big(D^{l}_{\underline{\tilde{l}}'',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l'' m_l}} \\
&\qquad - \eta^{+}_{nl\,\underline{\tilde{l}}' m_l'}\, \overline{\big(D^{l}_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l' m_l}}\, \overline{\zeta^{-}_{nl\,\underline{\tilde{l}}'' m_l''}}\, \big(D^{l}_{\underline{\tilde{l}}'',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l'' m_l} \Big\} \\
&= \mathrm{i}\sum_{n\underline{l}m_l} \omega^{\pm}_{nl} R^{d-1}_{\mathrm{AdS}} \mathcal{N}^{\pm}_{nl} \Big\{ \overline{\eta^{-}_{n\underline{l}m_l}}\, \zeta^{+}_{n\underline{l}m_l} - \eta^{+}_{n\underline{l}m_l}\, \overline{\zeta^{-}_{n\underline{l}m_l}} \Big\} \,.
\end{aligned}
$$

For the last equality we have used the completeness relation (A.43) for Wigner's D-matrix:

$$
\sum_{\underline{\tilde{l}},\, m_l} \big(D^{l}_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l' m_l}\; \overline{\big(D^{l}_{\underline{\tilde{l}}'',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l'' m_l}} \;=\; \delta_{\underline{\tilde{l}}',\underline{\tilde{l}}''}\, \delta_{m_l' m_l''} \,.
$$

Thus the time-interval region's symplectic structure is invariant under rotations:

$$
\big(\hat{R}(\underline{\alpha})\triangleright\omega_t\big)(\eta,\zeta) \;=\; \omega_t(\eta,\zeta) \qquad \forall\, \eta,\zeta \,. \tag{2.235}
$$

For the tube region the calculation is similar again: we can plug the actions (2.224) of $\hat{R}(\underline{\alpha})^{-1}$ in the momentum representation into the symplectic structure (2.195) and again apply the completeness relation (A.43) for Wigner's D-matrix:

$$
\begin{aligned}
\big(\hat{R}(\underline{\alpha})\triangleright\omega_\rho\big)(\eta,\zeta) \;&=\; \pi R^{d-1}_{\mathrm{AdS}} \int \mathrm{d}\omega \sum_{\underline{l},m_l} (2l+d-2) \Big\{ \big(\hat{R}(\underline{\alpha})^{-1}\triangleright\eta\big)^{S,a}_{\omega\underline{l}m_l}\, \big(\hat{R}(\underline{\alpha})^{-1}\triangleright\zeta\big)^{S,b}_{-\omega,\underline{l},-m_l} \\
&\qquad - \big(\hat{R}(\underline{\alpha})^{-1}\triangleright\eta\big)^{S,b}_{\omega\underline{l}m_l}\, \big(\hat{R}(\underline{\alpha})^{-1}\triangleright\zeta\big)^{S,a}_{-\omega,\underline{l},-m_l} \Big\} \\
&= \pi R^{d-1}_{\mathrm{AdS}} \int \mathrm{d}\omega \sum_{\underline{l},m_l} (2l+d-2) \sum_{\underline{\tilde{l}}' m_l'} \sum_{\underline{\tilde{l}}'' m_l''} \Big\{ \eta^{S,a}_{\omega l\,\underline{\tilde{l}}' m_l'}\, \overline{\big(D^{l}_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l' m_l}}\; \zeta^{S,b}_{-\omega,l,\underline{\tilde{l}}'',-m_l''}\, \big(D^{l}_{\underline{\tilde{l}}'',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l'' m_l} \\
&\qquad - \eta^{S,b}_{\omega l\,\underline{\tilde{l}}' m_l'}\, \overline{\big(D^{l}_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l' m_l}}\; \zeta^{S,a}_{-\omega,l,\underline{\tilde{l}}'',-m_l''}\, \big(D^{l}_{\underline{\tilde{l}}'',\underline{\tilde{l}}}(\underline{\alpha})\big)_{m_l'' m_l} \Big\} \\
&= \pi R^{d-1}_{\mathrm{AdS}} \int \mathrm{d}\omega \sum_{\underline{l},m_l} (2l+d-2) \Big\{ \eta^{S,a}_{\omega\underline{l}m_l}\, \zeta^{S,b}_{-\omega,\underline{l},-m_l} - \eta^{S,b}_{\omega\underline{l}m_l}\, \zeta^{S,a}_{-\omega,\underline{l},-m_l} \Big\} \,.
\end{aligned}
$$

Thus the tube region's symplectic structure is invariant under rotations:

$$
\big(\hat{R}(\underline{\alpha})\triangleright\omega_\rho\big)(\eta,\zeta) \;=\; \omega_\rho(\eta,\zeta) \qquad \forall\, \eta,\zeta \,. \tag{2.236}
$$

Invariance under boosts

We show the invariance of the symplectic structure only for infinitesimal d-boosts, that is, for the action of the $\mathbf{so}\,(2,d)$ generators K_{0d} and $K_{d+1,d}$. Since for both types of d-boosts the calculations are essentially the same (up to some factors of $\pm\mathrm{i}$), we consider in detail only the action of K_{0d}. For both time-interval and tube regions according to (2.57) we then have the action on the symplectic structure as

$$\begin{aligned}\big((\mathbb{1}+\varepsilon K_{0d})\triangleright\omega\big)(\eta,\zeta) &= \omega(\eta,\zeta)+\omega\big(-\varepsilon K_{0d}\triangleright\eta,\,\zeta\big)+\omega\big(\eta,-\varepsilon K_{0d}\triangleright\zeta\big)\\ \big(K_{0d}\triangleright\omega\big)(\eta,\zeta) &= +\omega\big(-K_{0d}\triangleright\eta,\,\zeta\big)+\omega\big(\eta,-K_{0d}\triangleright\zeta\big)\ .\end{aligned}$$

For the time-interval region we can plug the actions (2.231) of K_{0d} in the momentum representation into the symplectic structure (2.209). For simplifying the notation a little, we suppress the indices $\underline{\tilde{l}}$ in $\underline{l}=(l,\underline{\tilde{l}})$, and only write the top angular momentum number l. This is possible because the $\underline{\tilde{l}}$ are the same in all expressions wherein l appears, and results in the following expression:

$$\begin{aligned}&\omega_t\big(-K_{0d}\triangleright\eta,\,\zeta\big)+\omega_t\big(\eta,-K_{0d}\triangleright\zeta\big)\\ &= \mathrm{i}\,\tfrac{\mathrm{i}}{2}\sum_{n\underline{l}\,m_l}\omega^{\pm}_{nl}R^{d-1}_{\mathrm{AdS}}\mathcal{N}^{\pm}_{nl}\Big\{\zeta^{+}_{nlm_l}\Big(\overline{\eta^{-}_{n,l+1,m_l}}\;z^{(\pm)0-}_{n,l+1}+\overline{\eta^{-}_{n+1,l-1,m_l}}\;z^{(\pm)-+}_{n+1,l-1}+\overline{\eta^{-}_{n-1,l+1,m_l}}\;\tilde{z}^{(\pm)+-}_{n-1,l+1}+\overline{\eta^{-}_{n,l-1,m_l}}\;\tilde{z}^{(\pm)0+}_{n,l-1}\Big)\\ &\qquad+\overline{\zeta^{-}_{nlm_l}}\Big(\eta^{+}_{n,l+1,m_l}\;z^{(\pm)0-}_{n,l+1}+\eta^{+}_{n+1,l-1,m_l}\;z^{(\pm)-+}_{n+1,l-1}+\eta^{+}_{n-1,l+1,m_l}\;\tilde{z}^{(\pm)+-}_{n-1,l+1}+\eta^{+}_{n,l-1,m_l}\;\tilde{z}^{(\pm)0+}_{n,l-1}\Big)\\ &\qquad-\overline{\eta^{-}_{nlm_l}}\Big(\zeta^{+}_{n,l+1,m_l}\;z^{(\pm)0-}_{n,l+1}+\zeta^{+}_{n+1,l-1,m_l}\;z^{(\pm)-+}_{n+1,l-1}+\zeta^{+}_{n-1,l+1,m_l}\;\tilde{z}^{(\pm)+-}_{n-1,l+1}+\zeta^{+}_{n,l-1,m_l}\;\tilde{z}^{(\pm)0+}_{n,l-1}\Big)\\ &\qquad-\eta^{+}_{nlm_l}\Big(\overline{\zeta^{-}_{n,l+1,m_l}}\;z^{(\pm)0-}_{n,l+1}+\overline{\zeta^{-}_{n+1,l-1,m_l}}\;z^{(\pm)-+}_{n+1,l-1}+\overline{\zeta^{-}_{n-1,l+1,m_l}}\;\tilde{z}^{(\pm)+-}_{n-1,l+1}+\overline{\zeta^{-}_{n,l-1,m_l}}\;\tilde{z}^{(\pm)0+}_{n,l-1}\Big)\Big\}\ .\end{aligned}$$

Therein, the first and third line cancel each other, and the second and fourth line cancel each other, too. Since the calculation is the same, we only consider the first and third line. Equally colored terms cancel each other. To see this, we shift the indices such that they agree for each pair of equally colored terms. Then, we obtain for the sum of first and third line:

$$\begin{aligned}\tfrac{-1}{2}R^{d-1}_{\mathrm{AdS}}\sum_{n\underline{l}\,m_l}\Big\{&\overline{\eta^{-}_{n,l+1,m_l}}\,\zeta^{+}_{nlm_l}\,\omega^{\pm}_{nl}z^{(\pm)0-}_{n,l+1}\mathcal{N}^{\pm}_{nl} &&+\overline{\eta^{-}_{n+1,l-1,m_l}}\,\zeta^{+}_{nlm_l}\,\omega^{\pm}_{nl}z^{(\pm)-+}_{n+1,l-1}\mathcal{N}^{\pm}_{nl}\\ &+\overline{\eta^{-}_{n-1,l+1,m_l}}\,\zeta^{+}_{nlm_l}\,\omega^{\pm}_{nl}\tilde{z}^{(\pm)+-}_{n-1,l+1}\mathcal{N}^{\pm}_{nl} &&+\overline{\eta^{-}_{n,l-1,m_l}}\,\zeta^{+}_{nlm_l}\,\omega^{\pm}_{nl}\tilde{z}^{(\pm)0+}_{n,l-1}\mathcal{N}^{\pm}_{nl}\\ &-\overline{\eta^{-}_{n,l-1,m_l}}\,\zeta^{+}_{nlm_l}\,\omega^{\pm}_{n,l-1}z^{(\pm)0-}_{nl}\mathcal{N}^{\pm}_{n,l-1} &&-\overline{\eta^{-}_{n-1,l+1,m_l}}\,\zeta^{+}_{nlm_l}\,\omega^{\pm}_{n-1,l+1}z^{(\pm)-+}_{nl}\mathcal{N}^{\pm}_{n-1,l+1}\\ &-\overline{\eta^{-}_{n+1,l-1,m_l}}\,\zeta^{+}_{nlm_l}\,\omega^{\pm}_{n+1,l-1}\tilde{z}^{(\pm)-+}_{nl}\mathcal{N}^{\pm}_{n+1,l-1} &&-\overline{\eta^{-}_{n,l+1,m_l}}\,\zeta^{+}_{nlm_l}\,\omega^{\pm}_{n,l+1}\tilde{z}^{(\pm)0+}_{nl}\mathcal{N}^{\pm}_{n,l+1}\Big\}\end{aligned}$$

Using the following equalities

$$\begin{aligned}\omega^{\pm}_{nl}\mathcal{N}^{\pm}_{nl}\,z^{(\pm)0-}_{n,l+1,\tilde{l}} = \omega^{\pm}_{n,l+1}\mathcal{N}^{\pm}_{n,l+1}\,\tilde{z}^{(\pm)0+}_{nl\tilde{l}} \quad&\Longrightarrow\quad \omega^{\pm}_{n,l-1}\mathcal{N}^{\pm}_{n,l-1}\,z^{(\pm)0-}_{nl\tilde{l}} = \omega^{\pm}_{nl}\mathcal{N}^{\pm}_{nl}\,\tilde{z}^{(\pm)0+}_{n,l-1,\tilde{l}}\\ \omega^{\pm}_{nl}\mathcal{N}^{\pm}_{nl}\,z^{(\pm)-+}_{n+1,l-1,\tilde{l}} = \omega^{\pm}_{n+1,l-1}\mathcal{N}^{\pm}_{n+1,l-1}\,\tilde{z}^{(\pm)+-}_{nl\tilde{l}} \quad&\Longrightarrow\quad \omega^{\pm}_{n-1,l+1}\mathcal{N}^{\pm}_{n-1,l+1}\,z^{(\pm)-+}_{nl\tilde{l}} = \omega^{\pm}_{nl}\mathcal{N}^{\pm}_{nl}\,\tilde{z}^{(\pm)+-}_{n-1,l+1,\tilde{l}}\end{aligned} \tag{2.237}$$

we get the zero that we desire for the sum of first and third line. This shows that

$$0 = \omega_t\big(-K_{0d}\triangleright\eta,\,\zeta\big)+\omega_t\big(\eta,-K_{0d}\triangleright\zeta\big)\ .$$

Thus the time-interval region's symplectic structure is also invariant under infinitesimal d-boosts:

$$\big((\mathbb{1}+\varepsilon K_{0d})\triangleright\omega_t\big)(\eta,\zeta) = \big((\mathbb{1}+\varepsilon K_{d+1,d})\triangleright\omega_t\big)(\eta,\zeta) = \omega_t(\eta,\zeta) \qquad \forall\,\eta,\zeta\ . \tag{2.238}$$

We remark that the equalities (2.237) are short but not as trivial as they look, recalling the origins of the various ingredients: an integration constant $\mathcal{N}^{\pm}_{nl}$ of Jacobi polynomials, magic frequencies $\omega^{\pm}_{nl}$ and contiguous coefficients $z^{(\pm)}_{nl}$ of the AdS Klein-Gordon modes for the time-interval region, containing the raising and lowering coefficients $\chi_{\pm}$ of the hyperspherical harmonics. Checking the equalities is a bit lengthy but straightforward, plugging in the definitions (C.44), (C.67), (C.212), (C.213), (C.227) and (C.230) and using (A.27) gets the job done.

For the tube region the calculation is similar but with different coefficients. We can plug the actions (2.229) of K_{0d} in the momentum representation into the symplectic structure (2.195), resulting in the following expression (again suppressing the indices $\underline{\tilde{l}}$):

$$\begin{aligned}
&\omega_\rho\big(-K_{0d}\triangleright\eta,\,\zeta\big)+\omega_\rho\big(\eta,-K_{0d}\triangleright\zeta\big)\\
&=\pi R_{\text{AdS}}^{d-1}\int\mathrm{d}\omega\sum_{l,m_l}(2l+d-2)\\
&\quad\Big\{\zeta^{S,b}_{-\omega,l,-m_l}\Big(-\eta^{S,a}_{\omega-1,l+1,m_l}\tilde z^{(S,a)+-}_{\omega-1,l+1}-\eta^{S,a}_{\omega-1,l-1,m_l}\tilde z^{(S,a)++}_{\omega-1,l-1}-\eta^{S,a}_{\omega+1,l+1,m_l}z^{(S,a)--}_{\omega+1,l+1}-\eta^{S,a}_{\omega+1,l-1,m_l}z^{(S,a)-+}_{\omega+1,l-1}\Big)\\
&\quad-\zeta^{S,a}_{-\omega,l,-m_l}\Big(-\eta^{S,b}_{\omega-1,l+1,m_l}\tilde z^{(S,b)+-}_{\omega-1,l+1}-\eta^{S,b}_{\omega-1,l-1,m_l}\tilde z^{(S,b)++}_{\omega-1,l-1}-\eta^{S,b}_{\omega+1,l+1,m_l}z^{(S,b)--}_{\omega+1,l+1}-\eta^{S,b}_{\omega+1,l-1,m_l}z^{(S,b)-+}_{\omega+1,l-1}\Big)\\
&\quad+\eta^{S,a}_{\omega l m_l}\Big(+\zeta^{S,b}_{-(\omega-1),l+1,-m_l}\tilde z^{(S,b)+-}_{\omega-1,l+1}+\zeta^{S,b}_{-(\omega-1),l-1,-m_l}\tilde z^{(S,b)++}_{\omega-1,l-1}+\zeta^{S,b}_{-(\omega+1),l+1,-m_l}z^{(S,b)--}_{\omega+1,l+1}+\zeta^{S,b}_{-(\omega+1),l-1,-m_l}z^{(S,b)-+}_{\omega+1,l-1}\Big)\\
&\quad-\eta^{S,b}_{\omega l m_l}\Big(+\zeta^{S,a}_{-(\omega-1),l+1,-m_l}\tilde z^{(S,a)+-}_{\omega-1,l+1}+\zeta^{S,a}_{-(\omega-1),l-1,-m_l}\tilde z^{(S,a)++}_{\omega-1,l-1}+\zeta^{S,a}_{-(\omega+1),l+1,-m_l}z^{(S,a)--}_{\omega+1,l+1}+\zeta^{S,a}_{-(\omega+1),l-1,-m_l}z^{(S,a)-+}_{\omega+1,l-1}\Big)\Big\}.
\end{aligned}$$

Again, the first and third quarter cancel each other, as well as the second and fourth quarter, and we only consider the calculation for the first and third quarter. To see that equally colored terms cancel, we again shift the indices such that they agree for each pair of equally colored terms. Then, we obtain for the sum of first and third line:

$$\begin{aligned}
\pi R_{\text{AdS}}^{d-1}\int\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{&-\eta^{S,a}_{\omega-1,l+1,m_l}\zeta^{S,b}_{-\omega,l,-m_l}(2l+d-2)\tilde z^{(S,a)+-}_{\omega-1,l+1}-\eta^{S,a}_{\omega-1,l-1,m_l}\zeta^{S,b}_{-\omega,l,-m_l}(2l+d-2)\tilde z^{(S,a)++}_{\omega-1,l-1}\\
&-\eta^{S,a}_{\omega+1,l+1,m_l}\zeta^{S,b}_{-\omega,l,-m_l}(2l+d-2)z^{(S,a)--}_{\omega+1,l+1}-\eta^{S,a}_{\omega+1,l-1,m_l}\zeta^{S,b}_{-\omega,l,-m_l}(2l+d-2)z^{(S,a)-+}_{\omega+1,l-1}\\
&+\eta^{S,a}_{\omega+1,l-1,m_l}\zeta^{S,b}_{-\omega,l,-m_l}(2l+d-4)\tilde z^{(S,b)+-}_{\omega l}+\eta^{S,a}_{\omega+1,l+1,m_l}\zeta^{S,b}_{-\omega,l,-m_l}(2l+d)\tilde z^{(S,b)++}_{\omega l}\\
&+\eta^{S,a}_{\omega-1,l-1,m_l}\zeta^{S,b}_{-\omega,l,-m_l}(2l+d-4)z^{(S,b)--}_{\omega l}+\eta^{S,a}_{\omega-1,l+1,m_l}\zeta^{S,b}_{-\omega,l,-m_l}(2l+d)z^{(S,b)-+}_{\omega l}\Big\}\,,
\end{aligned}$$

and using the equalities

$$\begin{aligned}
\tilde z^{(S,a)+-}_{\omega-1,l+1,\tilde l}&=\frac{2l+d}{2l+d-2}\,z^{(S,b)-+}_{\omega l\tilde l} & \tilde z^{(S,a)++}_{\omega-1,l-1,\tilde l}&=\frac{2l+d-4}{2l+d-2}\,z^{(S,b)--}_{\omega l\tilde l}\\
z^{(S,a)--}_{\omega+1,l+1,\tilde l}&=\frac{2l+d}{2l+d-2}\,\tilde z^{(S,b)++}_{\omega l\tilde l} & z^{(S,a)-+}_{\omega+1,l-1,\tilde l}&=\frac{2l+d-4}{2l+d-2}\,\tilde z^{(S,b)+-}_{\omega l\tilde l}
\end{aligned}\tag{2.239}$$

we again get zero:

$$0\;=\;\omega_\rho\big(-K_{0d}\triangleright\eta,\,\zeta\big)+\omega_\rho\big(\eta,-K_{0d}\triangleright\zeta\big)\;.$$

Thus also the tube region's symplectic structure is invariant under infinitesimal d-boosts:

$$\big(K_{0d}\triangleright\omega_\rho\big)(\eta,\zeta)\;=\;\big(K_{d+1,d}\triangleright\omega_\rho\big)(\eta,\zeta)\;=\;\omega_\rho(\eta,\zeta)\qquad\forall\,\eta,\zeta\,.\tag{2.240}$$

Again the equalities (2.239) look short but are not so trivial. Checking them is again somewhat lengthy but straightforward, plugging in the definitions (C.258), (C.259), (C.266) and (C.267) and using again (A.27) is sufficient.

2.6.8 AdS Klein-Gordon solutions from initial/boundary data

In this section we develop evidence for one-to-one correspondences between initial/boundary data and solutions on the interior of our three types of AdS regions. For a complete analysis of this situation it would be necessary to specify the (equivalence) classes of solutions forming the solution spaces. Since we do not do this at this point, our calculations remain of formal nature, although we can derive explicit formulas.

AdS time-interval region

On time-interval regions any bounded free Klein-Gordon solution $\phi(t,\rho,\Omega)$ is a linear combination of Jacobi modes, because the spatial parts of these modes form an orthogonal system on equal-time hypersurfaces. *Independently of the boundary conditions* one chooses at spatial infinity, $\phi(t,\rho,\Omega)$ it is completely determined by its momentum representation $(\phi^{+}_{n\underline{l}\,m_l}, \phi^{-}_{n\underline{l}\,m_l})$. Therefore solutions on a time-interval region are determined by boundary data on an equal-time hypersurface Σ_{t_0}. This boundary hypersurface can be located at the (early or late) boundary of the time-interval, or anywhere between them inside the region. The necessary data are then the field configuration and the field derivative on this hypersurface. We recall that the Jacobi modes are regular everywhere, including the time axis $\rho = 0$ and spatial infinity $\rho = \frac{\pi}{2}$. The momentum representation of a solution $\phi(t,\rho,\Omega)$ can be calculated from the initial data on Σ_{t_0} by formally inverting the Jacobi expansion (2.201):

$$\begin{aligned} \phi^{+}_{n\underline{l}\,m_l} &= \int \mathrm{d}t \int \mathrm{d}^{d-1}\Omega\ \tan^{d-1}\!\rho\ \overline{Y^{m_l}_{\underline{l}}(\Omega)}\ J^{(+)}_{nl}(\rho)\ \Big(\hat{f}(t_0)\,\phi + \hat{d}(t_0)\,\partial_t\phi\Big)(t_0,\rho,\Omega) \\ \overline{\phi^{-}_{n\underline{l}\,m_l}} &= \int \mathrm{d}t \int \mathrm{d}^{d-1}\Omega\ \tan^{d-1}\!\rho\ \overline{Y^{m_l}_{\underline{l}}(\Omega)}\ J^{(+)}_{nl}(\rho)\ \Big(\overline{\hat{f}(t_0)}\,\phi + \overline{\hat{d}(t_0)}\,\partial_t\phi\Big)(t_0,\rho,\Omega)\ , \end{aligned} \tag{2.241}$$

with the operators $\hat{f}(t_0)$ and $\hat{d}(t_0)$ having $Y^{m_l}_{\underline{l}}(\Omega)\ J^{(\pm)}_{nl}(\rho)$ as eigenfunctions with eigenvalues

$$\begin{aligned} f_{n\underline{l}\,m_l}(t_0) &= \mathrm{e}^{\mathrm{i}\omega^{+}_{nl}t_0}\ \tfrac{1}{2}\left(R^{d-1}_{\mathrm{AdS}}\mathcal{N}^{\pm}_{nl}\right)^{-1/2} \\ d_{n\underline{l}\,m_l}(t_0) &= \mathrm{e}^{\mathrm{i}\omega^{+}_{nl}t_0}\ \tfrac{\mathrm{i}}{2\omega^{\pm}_{nl}}\left(R^{d-1}_{\mathrm{AdS}}\mathcal{N}^{\pm}_{nl}\right)^{-1/2} . \end{aligned} \tag{2.242}$$

Because $Y^{m_l}_{\underline{l}}(\Omega)\ J^{(\pm)}_{nl}(\rho)$ form a complete system on Σ_{t_0}, this pair of formulas provides a one-to-one correspondence between initial data $(\varphi(\rho,\Omega), \dot{\varphi}(\rho,\Omega))$ on Σ_{t_0} and bounded solutions $\phi(t,\rho,\Omega)$ on the interior of the time-interval. (The dot in $\dot{\varphi}$ is just a label, meaning that φ represents the field values of ϕ and $\dot{\varphi}$ the values of the derivative $\partial_t\phi$.)

AdS tube region: well-behaved cases

Solutions on a tube region are also determined by boundary data on a hypersurface: the hypercylinder Σ_{ρ_0}. Again, this boundary hypersurface can be located at the (inner or outer) boundary of the tube, or anywhere between them inside the region. Only when we consider initial/boundary data on the boundary $\rho_0 = \frac{\pi}{2}$ of AdS, then we need to proceed more carefully. In any case, since the tube Klein-Gordon solutions (like the time-interval solutions) are determined by *two* functions $\phi^{S,a}_{\omega\underline{l}m_l}$ and $\phi^{S,b}_{\omega\underline{l}m_l}$, the necessary data again consists of two pieces: the field configuration and the field derivative on an boundary hypersurface.

We recall that the bounded solutions for the tube regions are the hypergeometric S^a and S^b-modes which are regular everywhere, except for the time axis $\rho = 0$ where the S^b-modes diverge, and for spatial infinity $\rho = \frac{\pi}{2}$ where both S^a and S^b-modes diverge, see Section 2.6.3. The latter divergence occurs only for $\widetilde{m}_- < 0$, that is, for masses $(m^2 > 0) \Leftrightarrow (\nu > d/2)$. For $\widetilde{m}_- \geq 0$, that is, $(m^2 \leq 0) \Leftrightarrow (\nu \leq d/2)$ the field (and its derivative) remain regular at $\rho = \frac{\pi}{2}$. Therefore, except for the case of both the boundary hypersurface being located at spatial infinity $\rho = \frac{\pi}{2}$ and the mass square being positive $m^2 > 0$, we have a similar formula as for the time-interval region that determines the momentum representation of a free Klein-Gordon solution $\phi(t,\rho,\Omega)$ from its initial values and derivatives on Σ_{ρ_0}. This formal inversion of the S-expansion is given by

$$\phi^{S,a}_{\omega\underline{l}\,m_l} = \int_{\Sigma_{\rho_0}} \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\;\mathrm{e}^{\mathrm{i}\omega t}\,\overline{Y^{m_l}_{\underline{l}}(\Omega)}\;\frac{\tan^{d-1}\rho}{2\pi\,(2\hat{l}+d-2)}\Big(+(\partial_\rho S^b_{\hat{\omega}\hat{l}})(\rho_0)\,\phi(t,\rho_0,\Omega) - S^b_{\hat{\omega}\hat{l}}(\rho_0)\,(\partial_\rho\phi)(t,\rho_0,\Omega)\Big)$$
$$\phi^{S,b}_{\omega\underline{l}\,m_l} = \int_{\Sigma_{\rho_0}} \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\;\mathrm{e}^{\mathrm{i}\omega t}\,\overline{Y^{m_l}_{\underline{l}}(\Omega)}\;\frac{\tan^{d-1}\rho}{2\pi\,(2\hat{l}+d-2)}\Big(-(\partial_\rho S^a_{\hat{\omega}\hat{l}})(\rho_0)\,\phi(t,\rho_0,\Omega) + S^a_{\hat{\omega}\hat{l}}(\rho_0)\,(\partial_\rho\phi)(t,\rho_0,\Omega)\Big)\,. \tag{2.243}$$

An equivalent formula for the expansion in C-modes is

$$\phi^{C,a}_{\omega\underline{l}\,m_l} = \int_{\Sigma_{\rho_0}} \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\;\mathrm{e}^{\mathrm{i}\omega t}\,\overline{Y^{m_l}_{\underline{l}}(\Omega)}\;\frac{\tan^{d-1}\rho}{2\pi\,(2\nu)}\Big(+(\partial_\rho C^b_{\hat{\omega}\hat{l}})(\rho_0)\,\phi(t,\rho_0,\Omega) - C^b_{\hat{\omega}\hat{l}}(\rho_0)\,(\partial_\rho\phi)(t,\rho_0,\Omega)\Big)$$
$$\phi^{C,b}_{\omega\underline{l}\,m_l} = \int_{\Sigma_{\rho_0}} \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\;\mathrm{e}^{\mathrm{i}\omega t}\,\overline{Y^{m_l}_{\underline{l}}(\Omega)}\;\frac{\tan^{d-1}\rho}{2\pi\,(2\nu)}\Big(-(\partial_\rho C^a_{\hat{\omega}\hat{l}})(\rho_0)\,\phi(t,\rho_0,\Omega) + C^a_{\hat{\omega}\hat{l}}(\rho_0)\,(\partial_\rho\phi)(t,\rho_0,\Omega)\Big)\,. \tag{2.244}$$

The operators $\hat{\omega}$ and $\hat{l}$ have the eigenfunctions $\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)$ with eigenvalues ω and l. Again, since $\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)$ form a complete system on Σ_{ρ_0}, each pair of formulas provides a one-to-one correspondence between initial data $(\varphi(t,\Omega),\dot{\varphi}(t,\Omega))$ on Σ_{ρ_0} and bounded solutions $\phi(t,\rho,\Omega)$ on the interior of the tube. (The dot in $\dot{\varphi}$ is again a label, now indicating that φ represents the field values of ϕ and $\dot{\varphi}$ the values of the derivative $\partial_\rho\phi$ on Σ_{ρ_0}.)

AdS tube region: degeneracy problem

For the boundary hypersurface at spatial infinity $\rho = \frac{\pi}{2}$ with the mass square being positive $m^2 > 0$, we can try a version of the formula above, but since both $S^a_{\omega l}(\rho)$ and $S^b_{\omega l}(\rho)$ diverge for $\rho = \frac{\pi}{2}$ the "raw" boundary data will now be divergent, too. Knowing from Section 2.6.3 that both radial functions diverge like $\cos^{\tilde{m}_-}\rho$ (and their derivatives like $\cos^{\tilde{m}_--1}\rho$), we could try to use the *rescaled* boundary data $\varphi^\partial(t,\Omega) = \cos^{-\tilde{m}_-}\rho\;\phi(t,\rho,\Omega)|_{\rho=\pi/2}$ and $\dot{\varphi}^\partial(t,\Omega) = \cos^{1-\tilde{m}_-}\rho\;(\partial_\rho\phi(t,\rho,\Omega))|_{\rho=\pi/2}$. However, there is a degeneracy problem: according to (2.175) the S and C-modes are not linear independent:

$$\begin{pmatrix} C^a_{\omega l} \\ C^b_{\omega l}\end{pmatrix} = \begin{pmatrix} (M^{\mathrm{no}}_{11})_{\omega l} & (M^{\mathrm{no}}_{12})_{\omega l} \\ (M^{\mathrm{no}}_{21})_{\omega l} & (M^{\mathrm{no}}_{22})_{\omega l}\end{pmatrix}\begin{pmatrix} S^a_{\omega l} \\ S^b_{\omega l}\end{pmatrix}\,. \tag{2.245}$$

Recalling that $C^a_{\omega l}$ is the solution behaving like $\cos^{\tilde{m}_+}\rho$ near the boundary $\rho = \pi/2$, and thus vanishing there, the following two Klein-Gordon solutions have the same (with or without rescaling as above) boundary field values and derivatives:

$$\phi(t,r,\Omega) \;=\; \int\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\phi^{S,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)\,S^a_{\omega l}(\rho) + \phi^{S,b}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)\,S^b_{\omega l}(\rho)\Big\}$$
$$\phi'(t,r,\Omega) \;=\; \int\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\phi'^{S,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)\,S^a_{\omega l}(\rho) + \phi'^{S,b}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)\,S^b_{\omega l}(\rho)\Big\}$$

wherein

$$\phi'^{S,a}_{\omega\underline{l}m_l} \;=\; \phi^{S,a}_{\omega\underline{l}m_l} + \phi^0_{\omega\underline{l}m_l}\,(M^{\mathrm{no}}_{11})_{\omega l}$$
$$\phi'^{S,b}_{\omega\underline{l}m_l} \;=\; \phi^{S,b}_{\omega\underline{l}m_l} + \phi^0_{\omega\underline{l}m_l}\,(M^{\mathrm{no}}_{12})_{\omega l}\,.$$

Therein, $\phi^0_{\omega\underline{l}m_l}$ is arbitrary. Thus ϕ' is ϕ plus some C^a-modes determined by $\phi^0_{\omega\underline{l}m_l}$. Then, both ϕ and ϕ' are different on the interior of AdS, but have the same boundary field values and derivatives (with or without rescaling as above). Thus we run into a problem here: this boundary data cannot distinguish between ϕ and ϕ'.

If we use the C-expansion instead of the S-expansion, the problem is even more obvious. Both rescaled boundary data $\varphi^\partial(t,\Omega) = \cos^{-\tilde{m}_-}\rho\ \phi(t,\rho,\Omega)|_{\rho=\pi/2}$ and $\dot{\varphi}^\partial(t,\Omega) = \cos^{1-\tilde{m}_-}\rho\ (\partial_\rho\phi(t,\rho,\Omega))|_{\rho=\pi/2}$ now only depend on one half of the momentum representation, namely on $\phi^{C,b}_{\omega\underline{l}m_l}$. This boundary data is completely blind to the C^a-mode content of any Klein-Gordon solution, since these modes vanish on the boundary whereas the C^b-modes diverge. While the rescaling cures the divergence, it makes the vanishing even faster.

Since this combination of boundary field value and derivative works neither in S nor C-expansion, we conclude that it is not suitable for our purposes. This problem is solved in the following.

AdS tube region: higher twisted derivatives

In [73], Claude Warnick investigates boundary conditions for (asymptotically) AdS spacetimes. However, as noted below equations (3.4) and (4.6) therein, his method only works for a rather narrow range of mass: $\nu \in (0, 1)$ that is, $m^2 R^2_{\text{AdS}} \in (-d^2/4,\ -d^2/4+1)$. He introduces what he calls a twisted derivative ∂^α_r, with α a mass-dependent parameter, that is defined in the following way:

$$\partial^\alpha_r\, f(r) \;:=\; r^{-\alpha}\,\partial_r\,(r^\alpha\, f(r))\ .$$

In our coordinates this writes as

$$\partial^\alpha_{\cos\rho}\ f(\cos\rho) \;:=\; (\cos\rho)^{-\alpha}\,\partial_{\cos\rho}\,(\cos^\alpha\!\rho\, f(\cos\rho))\ .$$

Motivated by his work, in this section we construct a method that works for all mass values. To this end we define a higher order twisted derivative $\partial^{(\nu)}_\rho$ by

$$\partial^{(\nu)}_\rho\, f(\cos\rho) \;:=\; (\cos\rho)^{1+2\lfloor\nu\rfloor-2\nu}\,\partial_{\cos\rho}\left(\frac{1}{\cos\rho}\,\partial_{\cos\rho}\right)^{\lfloor\nu\rfloor}\Big\{(\cos\rho)^{-\tilde{m}_-}\, f(\cos\rho)\Big\}\ . \tag{2.246}$$

Therein, we denote by $\lfloor\nu\rfloor$ (read: floor) the largest integer number that is smaller than ν. That is, for all $\nu \in \mathbb{R}$ their floors fulfill:

$$\lfloor\nu\rfloor \in \mathbb{Z} \qquad\qquad \lfloor\nu\rfloor \le \nu \qquad\qquad \lfloor\nu\rfloor + 1 > \nu\ .$$

Thus our twisted derivative is of order $\lfloor\nu\rfloor+1$, that is, we employ a derivative, whose order depends on the value of the mass parameter m^2. In order to write its action on the radial functions $C^a_{\omega l}(\rho)$ and $C^b_{\omega l}(\rho)$, we perform a Taylor expansion of them near the boundary where $\rho \to \pi/2$ and thus $\cos\rho \to 0$. To this end we need the definition (C.25) of the hypergeometric function, and the Taylor expansion of $\sin^l\!\rho$ around $\rho = \pi/2$, which for $\rho \in [0, \pi/2]$ is given by

$$\sin^l\!\rho \;=\; (1-\cos^2\!\rho)^{l/2} \;=\; \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\,(\cos\rho)^{2j}\,(l/2+1-j)_j\ .$$

(We recall the definitions (C.26) and (C.29) for the usual $(\cdot)_\cdot$ and double $(\!(\cdot)\!)_\cdot$ Pochhammer symbols.) For even l this sum only has $(l/2+1)$ terms, while for odd l it has infinitely many terms. With these ingredients we find for the Taylor expansion of $C^a_{\omega l}$ and $C^b_{\omega l}$ near the boundary:

$$\begin{aligned} C^a_{\omega l}(\rho) \;&=\; \sum_{a=0}^{\infty}(\cos\rho)^{\tilde{m}_++2a}\, d^+_a \\ C^b_{\omega l}(\rho) \;&=\; \sum_{a=0}^{\infty}(\cos\rho)^{\tilde{m}_-+2a}\, d^-_a\ , \end{aligned} \tag{2.247}$$

wherein the coefficients are given by the finite sums

$$\begin{aligned} d_a^+ &= \sum_{b=0}^{a} \frac{(-1)^b}{b!}\,(l/2+1-b)_b\, \frac{(\alpha_+)_{a-b}\,(\beta_+)_{a-b}}{(\gamma_+^{\rm C})_{a-b}\,(a-b)!} \\ d_a^- &= \sum_{b=0}^{a} \frac{(-1)^b}{b!}\,(l/2+1-b)_b\, \frac{(\alpha_-)_{a-b}\,(\beta_-)_{a-b}}{(\gamma_-^{\rm C})_{a-b}\,(a-b)!}\ . \end{aligned} \tag{2.248}$$

Letting the twisted derivative act on the Taylor expanions of $C^a_{\omega l}$ and $C^b_{\omega l}$ results in

$$\partial_\rho^{(\nu)}\, C^a_{\omega l}(\rho) \;=\; \sum_{a=0}^{\infty} (\cos\rho)^{2a}\, d_a^+\, (\!(2\nu+2a-2\lfloor\nu\rfloor)\!)_{\lfloor\nu\rfloor+1} \tag{2.249}$$

$$\partial_\rho^{(\nu)}\, C^b_{\omega l}(\rho) \;=\; \sum_{a=0}^{\infty} (\cos\rho)^{-2\nu+2a}\, d_a^-\, (\!(2a-2\lfloor\nu\rfloor)\!)_{\lfloor\nu\rfloor+1} \tag{2.250}$$

$$= \sum_{a=\lfloor\nu\rfloor+1}^{\infty} (\cos\rho)^{-2\nu+2a}\, d_a^-\, (\!(2a-2\lfloor\nu\rfloor)\!)_{\lfloor\nu\rfloor+1}\ . \tag{2.251}$$

The first terms in (2.250) vanish, because for low values of a the double Pochhammer symbol contains a zero factor. This is caused by the factor $(1/\cos\rho)$ in the centre of the twisted derivative. Studying the limit $\rho\to\pi/2$, in (2.249) only the $(a=0)$-term survives, and thus the limit is finite:

$$\left[\partial_\rho^{(\nu)}\, C^a_{\omega l}\right]_{\rho\to\pi/2} \;=\; (\!(2\nu-2\lfloor\nu\rfloor)\!)_{\lfloor\nu\rfloor+1} \qquad \forall\,\nu\notin\mathbb{Z}\ . \tag{2.252}$$

If $\nu\in\mathbb{Z}$, then this limit vanishes, because the double Pochhammer symbol then contains a zero factor. The coefficient d_0^+ does not appear explicitly in this limit because $d_0^+=1$. By contrast,in (2.251) we always have $a\geq\lfloor\nu\rfloor+1$ and thus $a>\nu$. Hence the factor of $(\cos\rho)$ always appears with positive power. Therefore each summand vanishes in the limit $\rho\to\pi/2$ and we get

$$\left[\partial_\rho^{(\nu)}\, C^b_{\omega l}\right]_{\rho\to\pi/2} \;=\; 0\ . \tag{2.253}$$

To give an example, let the mass m^2 such that $\nu\in(2,3)$. Then our twisted derivative writes

$$\partial_\rho^{(\nu)}\, f(\cos\rho) \;=\; (\cos\rho)^{5-2\nu}\,\partial_{\cos\rho}\left(\frac{1}{\cos\rho}\,\partial_{\cos\rho}\right)^2\left\{(\cos\rho)^{-\tilde m_-}\, f(\cos\rho)\right\}\,,$$

and the limits of the radial functions are

$$\begin{aligned} \left[\partial_\rho^{(\nu)}\, C^a_{\omega l}\right]_{\rho\to\pi/2} &= (\!(2\nu-4)\!)_3 \;=\; (2\nu-4)\,(2\nu-2)\,(2\nu) \\ \left[\partial_\rho^{(\nu)}\, C^b_{\omega l}\right]_{\rho\to\pi/2} &= 0\ . \end{aligned}$$

We can now use this in order to find a relation between a Klein-Gordon solution and its boundary behaviour. First, we define $\varphi^{\partial-}$ as usual as rescaled boundary field value, and $\varphi^{\partial+}_{(\nu)}$ as the boundary value of the twisted derivative of the field:

$$\varphi^{\partial-}(t,\rho,\Omega) \;:=\; \left[\cos^{-\tilde m_-}\!\rho\;\phi(t,\rho,\Omega)\right]_{\rho\to\pi/2} \tag{2.254}$$

$$\varphi^{\partial+}_{(\nu)}(t,\rho,\Omega) \;:=\; \left[\partial_\rho^{(\nu)}\,\phi(t,\rho,\Omega)\right]_{\rho\to\pi/2}\ . \tag{2.255}$$

Plugging into this the C-expansion (2.187) of the solution

$$\phi(t,r,\Omega) \;=\; \int\!{\rm d}\omega \sum_{\underline{l},m_l}\left\{\phi^{C,a}_{\omega\underline{l}m_l}\;\mu^{(C,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^{C,b}_{\omega\underline{l}m_l}\;\mu^{(C,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega)\right\}\,,$$

and using (2.252), we can formally reverse this relation:

$$\phi^{C,a}_{\omega\underline{l}m_l} = \int \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\ \frac{\mathrm{e}^{\mathrm{i}\omega t}\,\overline{Y^{m_l}_{\underline{l}}(\Omega)}}{((2\nu-2\lfloor\nu\rfloor))_{\lfloor\nu\rfloor+1}}\,\varphi^{\partial+}_{(\nu)}(t,\Omega)/(2\pi) \tag{2.256}$$

$$\phi^{C,b}_{\omega\underline{l}m_l} = \int \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\ \mathrm{e}^{\mathrm{i}\omega t}\,\overline{Y^{m_l}_{\underline{l}}(\Omega)}\,\varphi^{\partial-}(t,\Omega)/(2\pi)\ . \tag{2.257}$$

We thus recover the full momentum representation as a function of the (rescaled and therefore finite) boundary data.

AdS rod region

Free Klein-Gordon solutions on a rod region are also determined by boundary data on a hypercylinder. Again, this boundary hypersurface can be located at the (one and only outer) boundary of the rod region, or anywhere in its interior. And again, when we consider boundary data on this outer boundary, and it is located at spatial infinity $\rho = \frac{\pi}{2}$, then we need to be more subtle.

We recall that the bounded solutions for the rod regions are only the hypergeometric S^a-modes which are regular everywhere, except for spatial infinity $\rho = \frac{\pi}{2}$ where they diverge, see Section 2.6.3. Thus the rod region's free Klein-Gordon solutions are determined by *only one* function $\phi^{S,a}_{\omega\underline{l}m_l}$ on momentum space, and therefore the necessary boundary data is now only (either) the field configuration (or equivalently the field derivative) on a hypercylinder. The field can be expanded as in (2.191):

$$\phi(t,r,\Omega) = \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \phi^{S,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)\,S^a_{\omega l}(\rho) \tag{2.258}$$

$$= \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \mathrm{e}^{-\mathrm{i}\omega t} Y^{m_l}_{\underline{l}}(\Omega)\phi^{S,a}_{\omega\underline{l}m_l}\Big[(M^{\mathrm{on}}_{11})_{\omega l}C^a_{\omega l}(\rho)+(M^{\mathrm{on}}_{12})_{\omega l}C^b_{\omega l}(\rho)\Big]\ . \tag{2.259}$$

If the boundary hypersurface Σ_{ρ_0} is not at the boundary: $\rho_0 < \frac{\pi}{2}$, we can formally invert the definition

$$\varphi^{\rho_0}(t,\Omega) := \phi(t,\rho_0,\Omega) = \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \phi^{S,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)\,S^a_{\omega l}(\rho_0)\ , \tag{2.260}$$

$$\Longrightarrow \qquad \phi^{S,a}_{\omega\underline{l}m_l} = \int \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\ \frac{1}{(2\pi)\,S^a_{\omega l}(\rho_0)}\,\mathrm{e}^{\mathrm{i}\omega t}\,\overline{Y^{m_l}_{\underline{l}}(\Omega)}\,\varphi^{\rho_0}(t,\Omega)\ . \tag{2.261}$$

The zeros of $S^a_{\omega l}(\rho_0)$ do not pose a problem, because the set of (ω, l) causing them is of zero measure in momentum space. Singularities caused by the zeros of $S^a_{\omega l}(\rho_0)$ are canceled by their second appearance when we expand the initial data in terms of S^a-modes as in (2.260). If the initial data is placed on the boundary $\rho_0 = \pi/2$, then only the $C^b_{\omega l}$-part of $S^a_{\omega l}$ survives, as in (2.259), and we can rescale as follows:

$$\varphi^{\partial}_{\rho_0}(t,\Omega) := \cos^{-\tilde{m}_-}\rho\ \phi(t,\rho\to\pi/2,\Omega) = \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \mathrm{e}^{-\mathrm{i}\omega t}Y^{m_l}_{\underline{l}}(\Omega)\,\phi^{S,a}_{\omega\underline{l}m_l}(M^{\mathrm{on}}_{12})_{\omega l}\ . \tag{2.262}$$

Thus finite boundary data $\varphi^{\partial}(t,\Omega)$ is given by the rescaled boundary field value. This can formally be inverted to

$$\phi^{S,a}_{\omega\underline{l}m_l} = \int \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\ \frac{1}{(2\pi)\,(M^{\mathrm{on}}_{12})_{\omega l}}\,\mathrm{e}^{\mathrm{i}\omega t}\,\overline{Y^{m_l}_{\underline{l}}(\Omega)}\,\varphi^{\partial}_{\rho_0}(t,\Omega)\ . \tag{2.263}$$

This formula recovers the momentum representation of a free Klein-Gordon solution on the rod region as a function of the (rescaled and theirefore finite) boundary field value.

Chapter 3

Quantized Theory

In this chapter we quantize the classical theory. Section 3.1 reviews the method of Holomorphic Quantization (HQ) which we shall use. Therein, we also recall the construction of the usual S-matrix in QFT, and how the GBF generalizes it. Then, we consider three general properties of the amplitudes, and how these can be realized in Holomorphic Quantization. The first property is invariance of the amplitudes under the actions of spacetime isometries, which induces the same invariance for the S-matrix. In HQ, this is ensured if the real g-product is invariant under the isometries' actions. This in turn is induced by the invariance of the symplectic structure (treated already in the previous chapter), together with the complex structure commuting with these actions (which we study in this chapter). The second property is called amplitude equivalence, meaning that on a spacetime we want the amplitudes of different regions to coincide. In HQ, for this to happen we need the real g-products on the boundaries of the regions to agree. The third property is the flat limit: we wish our AdS amplitudes to reproduce the Minkowski amplitudes in the flat limit. Again, in HQ this is ensured if the real g-product of AdS reproduces the g of Minkowski spacetime in this limit. As a reference for this limit, in Section 3.2 we review the Holomorphic Quantization of the real Klein-Gordon field on Minkowski spacetime.

In Section 3.3 we then proceed to AdS. We start with the usual AdS time-interval regions in Section 3.3.1. Due to the standard complex structure on equal-time hypersurfaces, this provides us with a reference for the amplitude equivalence. The remaining sections are dedicated mainly to AdS rod regions. In Section 3.3.2 we start with the most general form of the complex structure, and then simplify it by imposing invariance under spacetime isometries. This results in several possible forms of the complex structure. We further restrict these forms in Section 3.3.3 by requiring amplitude equivalence. This leaves us with two candidates for the complex structure, which we explore in Section 3.3.4. The properties of the real g-products induced by these candidates are studied in Section 3.3.5. Finally, we calculate the flat limits of these real g-products in Section 3.3.6. We find that one candidate's flat limit reproduces the Minkowski rod's amplitude only for a discrete subset of frequencies. The second candidate can be modified to do so for all frequencies.

We then relate our complex structures to previous results. In Section 3.3.7 we show that Colosi's form of the complex structure is equivalent to ours. After this, in Section 3.3.8 we survey the similarities and differences of our amplitudes with those of Giddings. We summarize our results and comment on them in the closing Chapter 4.

3.1 GBF and Quantum Field Theory

3.1.1 States in Holomorphic Quantization

In the early days of the GBF, only the Schrödinger Representation was used, see for example [22]. In Appendix A therein, the alternative Polynomial Representation is described. Also the Holomorphic Representation is mentioned there already, which we shall review now. We sum up here the Holomorphic Quantization (HQ) presented in Section 4 of [59] and Section 3 of [61]. HQ arises as a particular kind of Geometric Quantization, see e.g. Section 2 of [61] and Section 9.2 of [76]. As usual we let Σ a hypersurface on spacetime. We recall the real and complex inner products $\mathrm{g}_\Sigma(\cdot,\cdot)$ and $\{\cdot,\cdot\}_\Sigma$ on the space L_Σ of solutions which are well defined in a neighborhood of Σ, see Section 2.2.

The state space $\mathcal{H}_\Sigma$ of Holomorphic Quantization[1] is the complex Hilbert space $\mathrm{H}^2(\mathrm{L}_\Sigma, \nu_\Sigma)$ of holomorphic square-integrable functions on L_Σ with respect to the measure ν_Σ given by (32) in [61]:

$$\mathrm{d}\nu_\Sigma(\xi) \;=\; \mathrm{d}\mu_\Sigma(\xi)\;\exp\left(-\tfrac{1}{2}\mathrm{g}_\Sigma(\xi,\xi)\right).$$

Therein, μ_Σ is a (fictitious) Lebesgue measure on L_Σ normalized to

$$1 \overset{!}{=} \int\limits_{\xi\in\mathrm{L}_\Sigma}\mathrm{d}\mu_\Sigma(\xi)\;\exp\left(-\tfrac{1}{2}\mathrm{g}_\Sigma(\xi,\xi)\right).$$

Hence ν_Σ is a probability measure:

$$1 \;=\; \int\limits_{\xi\in\mathrm{L}_\Sigma}\mathrm{d}\nu_\Sigma(\xi).$$

The inner product of the quantum state space $\mathcal{H}_\Sigma$ is then

$$\langle\psi',\,\psi\rangle_\Sigma \;=\; \int\limits_{\xi\in\mathrm{L}_\Sigma}\mathrm{d}\nu_\Sigma(\xi)\;\overline{\psi'(\xi)}\,\psi(\xi).$$

A function f on the infinite-dimensional space L_Σ is holomorphic, if for all $\lambda,\alpha\in\mathrm{L}_\Sigma$ and $c\in\mathbb{C}$ we can write $f(\lambda+c\alpha) \;=\; f(\lambda)+c\,\mathrm{d}f_\lambda(\alpha)+o(|\alpha|)$. It is antiholomorphic, if instead we have $\overline{c}\,\mathrm{d}f_\lambda(\alpha)$ on the right hand side.

Of particular importance are the coherent states, which generate a dense subspace $\mathcal{H}_\Sigma^{\mathrm{coh}}\subset\mathcal{H}_\Sigma$. Normalized coherent states in the Holomorphic Quantization map solutions near hypersurfaces Σ to complex numbers as given by:

$$K_\Sigma^\xi(\phi) \;:=\; \mathcal{N}_\Sigma^\xi\;\exp\left(\tfrac{1}{2}\,\{\xi,\,\phi\}_\Sigma\right) \qquad\qquad \mathcal{N}_\Sigma^\xi \;:=\; \exp\left(-\tfrac{1}{4}\,\{\xi,\,\xi\}_\Sigma\right) \;=\; \exp\left(-\tfrac{1}{4}\mathrm{g}_\Sigma(\xi,\xi)\right). \tag{3.1}$$

$\xi(x)$ is a solution near Σ which we call the characterizing solution of the coherent state. Here we shall always assume that $\mathcal{N}_\Sigma^\xi\in\mathbb{R}$, since $\xi\in\mathrm{L}_\Sigma$ is a real solution. In order to relate the Holomorphic Representation to the more established Schrödinger one, in [61] a one-to-one correspondence is proven between the spaces of coherent states $\mathcal{H}_\Sigma^{\mathrm{coh}}$ in the Holomorphic Representation and $\tilde{\mathcal{H}}_\Sigma^{\mathrm{coh}}$ in the Schrödinger Representation. This is achieved by explicitly constructing an isometric isomorphism (Proposition (3.1) therein) $\mathcal{B}:\;\tilde{\mathcal{H}}_\Sigma^{\mathrm{coh}}\;\to\;\mathcal{H}_\Sigma^{\mathrm{coh}}$. The normalized coherent states have a number of useful properties, of which we only mention the reproducing property

$$\begin{aligned}\langle K_\Sigma^\xi,\,\psi_\Sigma\rangle_\Sigma \;&=\; \psi_\Sigma(\xi)\cdot\mathcal{N}_\Sigma^\xi &&\forall\;\xi\in\mathrm{L}_\Sigma\\ \langle\psi_\Sigma,\,K_\Sigma^\xi\rangle_\Sigma \;&=\; \overline{\psi_\Sigma(\xi)}\cdot\mathcal{N}_\Sigma^\xi &&\forall\;\psi_\Sigma\in\mathcal{H}_\Sigma,\end{aligned} \tag{3.2}$$

and the completeness relation

$$\left\langle\psi_\Sigma,\;\psi'_\Sigma\right\rangle_\Sigma \;=\; \int\limits_{\mathrm{L}_\Sigma}\mathrm{d}\nu_\Sigma(\xi)\;\left\langle\psi_\Sigma,\;K_\Sigma^\xi\right\rangle_\Sigma\left\langle K_\Sigma^\xi,\;\psi'_\Sigma\right\rangle_\Sigma/(\mathcal{N}_\Sigma^\xi)^2 \qquad\qquad \forall\;\psi_\Sigma,\,\psi'_\Sigma\in\mathcal{H}_\Sigma. \tag{3.3}$$

[1] As discussed in Section 3.3 of [59], the states of HQ are actually functions not on L_Σ, but rather on the space $\hat{\mathrm{L}}_\Sigma$. This space can be understood as an extended space of solutions in a neighborhood of Σ, including distributions. $\hat{\mathrm{L}}_\Sigma$ is the algebraic dual of the topological dual of L_Σ. However, since the values of a state ψ on L_Σ completely fix ψ on $\hat{\mathrm{L}}_\Sigma$, we shall continue to use the simplified notation of [61] and write just L_Σ.

Next we consider one-particle states, which we shall denote by p (for particle) as in Section 4.2 of [60]. Since they are continuous, complex-linear maps $\mathrm{L}_\Sigma \to \mathbb{C}$, they form the topological dual of L_Σ. Hence, by the Riesz Representation Theorem each such map is represented by its associated element $\xi \in \mathrm{L}_\Sigma$ through the inner product: $p^\xi_\Sigma \sim \{\xi, \cdot\}_\Sigma$. Complex linearity here implies

$$p^\xi_\Sigma(J_\Sigma \phi) = \mathrm{i}\, p^\xi_\Sigma(\phi) \qquad \forall\, \xi, \phi \in \mathrm{L}_\Sigma, \tag{3.4}$$

because J_Σ is linear and $J_\Sigma^2 = -\mathbb{1}_\Sigma$. Since the complex inner product $\{\cdot, \cdot\}_\Sigma$ is conjugate linear in the first and complex linear in the second argument in the sense of (2.9), it is straightforward to verify that (3.4) is indeed fulfilled, see also Equation (2.10). For a one-particle state on a hypersurface Σ in the Holomorphic Representation we thus write its wave function(al) as

$$\begin{aligned} p^\xi_\Sigma :\ \mathrm{L}_\Sigma &\to \mathbb{C} && p^\xi_\Sigma \in \mathcal{H}_\Sigma \\ p^\xi_\Sigma(\phi) &= \tfrac{1}{\sqrt{2}}\, \{\xi, \phi\}_\Sigma && \xi, \phi \in \mathrm{L}_\Sigma. \end{aligned} \tag{3.5}$$

An n-particle state is represented by a linear combination of products of n wave functions of this type. The creation operator $a^{\xi\dagger}_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$ for a particle of type ξ then acts on a wave function by multiplication:

$$\big(a^{\xi\dagger}_\Sigma \psi_\Sigma\big)(\phi) = p^\xi_\Sigma(\phi) \cdot \psi_\Sigma(\phi) \qquad \xi, \phi \in \mathrm{L}_\Sigma,\ \psi_\Sigma, a^{\xi\dagger}_\Sigma \psi_\Sigma \in \mathcal{H}_\Sigma. \tag{3.6}$$

The corresponding annihilation operator is the adjoint of the creator. Using the reproducing property (3.2) of the normalized coherent states, the action $a^\xi_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$ of the annnihilator writes as

$$\big(a^\xi_\Sigma \psi_\Sigma\big)(\phi) = \big\langle K^\phi_\Sigma,\, a^\xi_\Sigma \psi_\Sigma \big\rangle_\Sigma / \mathcal{N}^\phi_\Sigma = \big\langle a^{\xi\dagger}_\Sigma K^\phi_\Sigma,\, \psi_\Sigma \big\rangle_\Sigma / \mathcal{N}^\phi_\Sigma \qquad \xi, \phi \in \mathrm{L}_\Sigma,\ \ \psi_\Sigma, a^\xi_\Sigma \psi_\Sigma \in \mathcal{H}_\Sigma. \tag{3.7}$$

Therefore, the annihilator's action on a coherent state turns out to be essentially by multiplication:

$$\begin{aligned} \big(a^\xi_\Sigma K^\lambda_\Sigma\big)(\phi) &= \big\langle a^{\xi\dagger}_\Sigma K^\phi_\Sigma,\, K^\lambda_\Sigma \big\rangle_\Sigma = \big\langle p^\xi_\Sigma \cdot K^\phi_\Sigma,\, K^\lambda_\Sigma \big\rangle_\Sigma \qquad \xi, \lambda, \phi \in \mathrm{L}_\Sigma \\ &= \tfrac{1}{\sqrt{2}}\, \{\lambda, \xi\}_\Sigma \cdot K^\lambda_\Sigma(\phi) \\ \Longrightarrow \quad a^\xi_\Sigma K^\lambda_\Sigma &= \overline{p^\xi_\Sigma(\lambda)} \cdot K^\lambda_\Sigma\,. \end{aligned} \tag{3.8}$$

The action of the creators allows us to write the coherent states as an exponential of a creator acting on the vacuum state $\psi^{\mathrm{vac}}_\Sigma \equiv 1$:

$$K^\xi_\Sigma(\phi) = \mathcal{N}^\xi_\Sigma \exp\big(\tfrac{1}{2}\, \{\xi, \phi\}_\Sigma\big) = \mathcal{N}^\xi_\Sigma \exp\big(\tfrac{1}{\sqrt{2}} a^{\xi\dagger}_\Sigma\big)\, \psi^{\mathrm{vac}}_\Sigma(\phi). \tag{3.9}$$

3.1.2 Standard amplitudes and S-Matrix

In this section we summarize the construction of the standard S-matrix. That is, for a moment we put aside the terminology of the GBF (that is, regions, boundaries and hypersurfaces) and the Holomorphic Quantization, and consider techniques frequently used in textbooks. Standard QFT uses only *one single* state space $\mathcal{H}$ which is the Fock space over the space of free 1-particle states. The states in $\mathcal{H}$ are thought of as describing the system on all of space at some time t, that is, on an equal-time hypersurface Σ_t. The details of this interpretation vary somewhat according to which of the following three pictures is used.

In the Schrödinger picture (labeled by S), the states $|\,\psi_{\mathrm{S}(t)}\rangle$ are labeled by the time t, whereas operators $\hat{O}_{\mathrm{S}}$ acting on the states are independent of time. In the Heisenberg picture, (labeled by H) it is just the other way round: the states $|\,\psi_{\mathrm{H}}\rangle$ are independent of time while the operators $\hat{O}_{\mathrm{H}(t)}$ depend on time. For some reference time, which for simplicity we set to to zero here, the states and operators coincide: $|\,\psi_{\mathrm{H}}\rangle = |\,\psi_{\mathrm{S}(0)}\rangle$ and $\hat{O}_{\mathrm{H}(0)} = \hat{O}_{\mathrm{S}}$. Further, $|\,\psi_{\mathrm{S}(t)}\rangle = \mathcal{U}_{t,0}|\,\psi_{\mathrm{H}}\rangle$ and $\hat{O}_{\mathrm{H}(t)} = \mathcal{U}^\dagger_{t,0} \hat{O}_{\mathrm{S}} \mathcal{U}_{t,0}$, wherein the unitary time-evolution operator $\mathcal{U}_{t,0}$ from time 0 to time t is constructed from the *full* interacting Hamiltonian.

In the Dirac picture (also called interaction picture, and thus labeled by I), both states and operators are time-dependent. Here, the time-evolution of the states is determined by the *interaction* part of the Hamiltonian, while the evolution of the operators is determined by the *free* part of the Hamiltonian. (Hence for a free theory the Dirac picture coincides with the Heisenberg picture.) However, all three pictures have in common that there is only one state space in which the states live and the operators act.

The S-matrix $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a unitary operator on the state space. More precisely, it is the limit of large times $t \to \infty$ of the time-evolution operator $\mathcal{U}^{\mathrm{I}}_{t,-t}$ constructed from the interaction part of the Hamiltonian. The S-matrix is the main tool for making predictions in perturbative QFT. Using the interaction picture (Dirac picture), its matrix elements are obtained as large-time limit

$$\mathcal{S}_{\eta,\zeta} \sim \lim_{t\to\infty} {}_t\langle \zeta | \mathcal{U}^{\mathrm{I}}_{t,-t} | \eta \rangle_{-t}, \tag{3.10}$$

see for example [41, 45, 67]. Usually it is assumed that the interaction is such that $|\,\eta\rangle_{-t}$ and ${}_t\langle\,\zeta\,|$ are free states for $t > t_0$ for some fixed time t_0. When taking the limit $t \to \infty$, the states remain asymptotically free while the interaction can now be "switched on" on all of spacetime.

There is an alternative way to describe this. Physical interactions are not switched on and off by *time*, but often decrease with the *distance* between interacting particles. Consider for example particles which scatter in some region V and then separate. When their separation becomes large, the state can be considered free and the particles are moving along timelike geodesics. For later times the state remains free iff spacetime geometry is such that the separation remains large. In Minkowski spacetime this is assured because the causal geodesics are just straight lines. However, in curved spacetime this cannot be taken for granted. In particular, on AdS the timelike geodesics emanating from a point reconverge periodically (see Figure 1.4), which brings the scattering particles close together again and again. Hence the state does not become asymptotically free for large times, and the interpretation of the asymptotic amplitudes as an S-matrix is not justified. This is one reason for generalizing the construction of S-matrices in the following sections.

Let us recall explicitly how the S-matrix is obtained using sources, the Feynman propagator and coherent states as summarized in Section 2 of [31]. We use notation suggesting a real scalar field. Here, we work in the standard setting described above: states live in the one and only state space $\mathcal{H}$ and live on equal-time hyperplanes Σ_t in the sense that they describe the system at time t on all of space. In this setting, the spaces L_t of classical solutions in a neighborhood of Σ_t each coincide with the space L of global classical solutions. We can identify here the phase space of the free field theory with the space L of global solutions. Each solution $\xi \in \mathrm{L}$ has an associated normalized coherent state K^ξ in the Hilbert space $\mathcal{H}$ of the free theory. It arises from the vacuum state ψ^{vac} by exponentiating the corresponding creator $a^\dagger_\xi$ as in (3.9):

$$K^\xi \sim \exp\left(\tfrac{1}{\sqrt{2}} a^\dagger_\xi\right) \psi^{\mathrm{vac}}. \tag{3.11}$$

To the free action we can add a source term D_μ given by

$$D_\mu(\phi) = \int \mathrm{d}x\ \mu(x)\, \phi(x). \tag{3.12}$$

Let K^{η_1} an initial coherent state at time t_1, and K^{ζ_2} a final one at t_2. Then, the transition amplitude for the theory with source between these states is given by (see Equation (52) in [22], which compares directly to Equation (9-86) in Itzykson's & Zuber's QFT book [41], wherein the special solution $\hat{\xi}(x)$ is called the classical asymptotic field):

$$\left\langle K^{\zeta_2}, \mathcal{U}^{\mu}_{t_2,t_1} K^{\eta_1} \right\rangle_{\mathcal{H}} = \left\langle K^{\zeta_2}, K^{\eta_1} \right\rangle_{\mathcal{H}} \exp\left(\mathrm{i}\int \mathrm{d}x\, \mu(x)\, \hat{\xi}(x)\right) \exp\left(\tfrac{\mathrm{i}}{2}\int \mathrm{d}x \int \mathrm{d}y\ \mu(x)\, G_F(x,y)\, \mu(y)\right). \tag{3.13}$$

$\mathcal{U}^{\mu}_{t_2,t_1}$ is again the unitary time evolution operator (for a theory with source μ), and G_F is the Feynman propagator. $\langle K^{\zeta_2}, K^{\eta_1}\rangle_{\mathcal{H}}$ is the transition amplitude of the free theory (without source). $\hat{\xi}$

is the element of the complexified phase space $\mathrm{L}^{\mathbb{C}}$ which using the definitions in (2.16) writes as (see Equation (39) in [22] and the end of Section 5.1 in [59]):

$$\begin{aligned}\hat{\xi} &= \tfrac{1}{2}(\eta_1+\zeta_2) - \tfrac{\mathrm{i}}{2}(-J\eta_1+J\zeta_2) = \mathrm{P}^-\eta_1 + \mathrm{P}^+\zeta_2 \\ &= \xi^{\mathrm{R}} - \mathrm{i}\xi^{\mathrm{I}}.\end{aligned} \tag{3.14}$$

$J : \mathrm{L} \to \mathrm{L}$ is here the standard complex structure (2.97) on equal-time planes, which multiplies positive energy solutions with $-\mathrm{i}$ and negative energy solutions with $+\mathrm{i}$, and $\mathrm{P}^\pm$ are the polarization projectors (2.74). In other words, $\hat{\xi}$ coincides with η_1 in its negative energy component and with ζ_2 in its positive energy component.

Since the coherent states are parametrized in terms of global solutions (interaction picture), the transition amplitude (3.13) remains independent of t_1 and t_2, as long as the source is contained completely in the interval $[t_1, t_2]$. This makes taking the limit $t_1 \to -\infty$ with $t_2 \to +\infty$ trivial. Next we consider an interaction that contributes to the action through a potential term:

$$D_V(\phi) = \int \mathrm{d}x\, V(\phi(x)). \tag{3.15}$$

The corresponding S-matrix[2] $\mathcal{U}^V$ is then formally written using functional derivatives as:

$$\left\langle K^{\zeta_2}, \mathcal{U}^V_{t_2,t_1} K^{\eta_1} \right\rangle = \exp\left(\mathrm{i}\int \mathrm{d}x\, V\big(-\mathrm{i}\tfrac{\delta}{\delta\mu(x)}\big)\right) \left\langle K^{\zeta_2}, \mathcal{U}^\mu_{t_2,t_1} K^{\eta_1} \right\rangle \Big|_{\mu=0}. \tag{3.16}$$

After this review in standard notation, we can rewrite the expressions in the notation of the GBF introduced throughout Section 1.2. Instead of representing transition amplitudes through evolution operators $\mathcal{U}_{t_2,t_1} : \mathcal{H} \to \mathcal{H}$ and inner products $\langle \cdot\,, \cdot \rangle_{\mathcal{H}}$, we write them directly as the amplitude maps $\rho : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathbb{C}$ of Core Axiom (T4), which fulfill:

$$\rho(\psi_1 \otimes \psi_2) = \langle \psi_2, \mathcal{U}_{t_2,t_1}\psi_1 \rangle_{\mathcal{H}}. \tag{3.17}$$

Here, $\mathcal{H}_1$ is the state space of initial states at time t_1, and $\mathcal{H}_2$ of final states at t_2. We shall write here ρ, ρ^μ and ρ^V for the amplitude maps of the free theory, the theory with source μ, and the theory with interaction potential V, respectively. The state space $\mathcal{H} = \mathcal{H}_1$ is the Fock space over L, where L is equipped with the Hilbert space structure of the 1-particle space. $\mathcal{H}_2$ is the Fock space over $\overline{\mathrm{L}}$, which denotes L with reversed complex structure and complex conjugated inner product, compare to Classical Axiom (C2). (If we think of L and $\overline{\mathrm{L}}$ as identified, then we can view all state spaces as identified: $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$.) Correspondingly, the tensor product state space

$$\mathcal{H}_\partial := \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{3.18}$$

is the Fock space over the complex Hilbert space

$$\mathrm{L}_\partial := \mathrm{L} \oplus \overline{\mathrm{L}}. \tag{3.19}$$

What is more, the product of a coherent state in $\mathcal{H}_1$ and a coherent state in $\mathcal{H}_2$ is a coherent state in $\mathcal{H}_\partial$. It is parametrized by an element $\xi_{12} = (\eta_1, \zeta_2)$ in L_∂ and generated by an operator of the form (3.11) on $\mathcal{H}_\partial$. We write

$$K^{\xi_{12}} = K^{(\eta_1,\zeta_2)} = K^{\eta_1} \otimes K^{\zeta_2}. \tag{3.20}$$

With this notation, formula (3.13) for the amplitude with source takes the form

$$\rho^\mu(K^{\xi_{12}}) = \rho(K^{\xi_{12}}) \exp\left(\mathrm{i}\int \mathrm{d}x\, \mu(x)\, \hat{\xi}(x)\right) \exp\left(\tfrac{\mathrm{i}}{2}\int \mathrm{d}x \int \mathrm{d}y\, \mu(x)\, G_F(x,y)\, \mu(y)\right). \tag{3.21}$$

[2] Strictly speaking, the S-matrix arises as the limit $t_1 \to -\infty$ with $t_2 \to +\infty$ of this expression. However, for the reason commented above, this is trivial.

Similarly, the S-matrix formula (3.16) rewrites as

$$\rho^V(\psi) = \exp\Big(\mathrm{i}\int\!\mathrm{d}x\, V\big(-\mathrm{i}\tfrac{\delta}{\delta\mu(x)}\big)\Big)\,\rho^\mu(\psi)\,\Big|_{\mu=0}\,. \tag{3.22}$$

This shows how the amplitudes for theories with source μ or interaction potential V can be obtained from the free amplitude. While considering only time-interval regions, we thus have rewritten the standard expressions for amplitudes in the GBF's notation. In the following section we shall extend this to more general regions, and later in this chapter we construct the free amplitudes for time-interval and rod regions on Minkowski and AdS spacetime.

3.1.3 GBF amplitudes in Holomorphic Quantization

As the notation suggests, the formulas (3.21) and (3.22) apply much beyond the context given in the previous section. Underlying this are the GBF's generalized notions of amplitude, observable and S-matrix [50, 55]. In particular, amplitudes can be associated to general spacetime regions whose dimension is $d+1$, see Section 1.2.2, and not only to the time-interval regions $[t_1,t_2]\times\mathbb{R}^d$ (resp. $[-\infty,+\infty]\times\mathbb{R}^d$) that we considered throughout the previous section. In the special case of a region $[t_1,t_2]\times\mathbb{R}^d$, the boundary decomposes into two connected components, one at t_1 and one at t_2. The boundary state space correspondingly decomposes into a tensor product of two Hilbert spaces $\mathcal{H}_1\otimes\mathcal{H}_2$. The GBF's amplitude map is then a conventional transition amplitude and corresponds to an operator. In general, however, a single Hilbert (or Krein[3]) space accommodates both incoming and outgoing particles [52].

It turns out that formula (3.21) provides the amplitude for a free bosonic field theory with a source μ in an *arbitrary* region $\mathbb{M}$ in an *arbitrary* spacetime, given that its ingredients are defined [58, 66]. We proceed to explain this, starting with a free classical field theory on an unspecified spacetime, recalling briefly the notation of Section 2.2. The second variation of the action yields the *symplectic structure* $\omega_\Sigma : \mathrm{L}_\Sigma\times\mathrm{L}_\Sigma \to \mathbb{R}$, which we assume to be non-degenerate. This makes L_Σ into a symplectic vector space: the *phase space* on Σ. We denote by $\mathrm{L}_\mathbb{M}$ the real vector space of solutions of the equations of motion in $\mathbb{M}$. Remarkably, the subspace $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}} := r_\mathbb{M}(\mathrm{L}_\mathbb{M}) \subseteq \mathrm{L}_{\partial\mathbb{M}}$ is generically a *Lagrangian subspace*. That is, the symplectic form $\omega_{\partial\mathbb{M}}$ vanishes on $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$, and $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ is a maximal subspace with this property, see (2.43).

To quantize the theory we need a compatible complex structure J_Σ on L_Σ for each hypersurface Σ. That is, J_Σ must satisfy $J_\Sigma^2 = -\mathbb{1}_\Sigma$ and be compatible with the symplectic structure: $\omega_\Sigma(J_\Sigma\phi_1, J_\Sigma\phi_2) = \omega_\Sigma(\phi_1,\phi_2)$. Then, as introduced in Section 2.2,

$$g_\Sigma(\phi_1,\phi_2) := 2\omega_\Sigma(\phi_1, J_\Sigma\phi_2) \qquad\text{and}\qquad \{\phi_1,\phi_2\}_\Sigma := g_\Sigma(\phi_1,\phi_2) + 2\mathrm{i}\omega_\Sigma(\phi_1,\phi_2) \tag{3.23}$$

define a real and a complex inner product on L_Σ, respectively. In standard quantization the complex structure is such that g_Σ, and hence also $\{\cdot,\cdot\}_\Sigma$, is positive-definite. However, a quantization where

[3] For completeness, we include the definition of a Krein space (with the usual notation adapted to fit our notation). Let L a complex vector space with an indefinite inner product $\{\cdot,\cdot\}$. (In our context we assume that it arises from ω and J as in (3.23), although in the general theory of Krein spaces this is not necessary.) Its positive definite subspace is defined by $\mathrm{L}_{++} := \big\{\lambda\in\mathrm{L}\;\big|\;\{\lambda,\lambda\}>0\big\}$ and its negative definite subspace by $\mathrm{L}_{--} := \big\{\lambda\in\mathrm{L}\;\big|\;\{\lambda,\lambda\}<0\big\}$. Then, L is called Krein space, if there exists a decomposition $\mathrm{L} = \mathrm{L}_+\oplus\mathrm{L}_-$ (called fundamental decomposition), which respects the complex structure J on L (that is: $J\mathrm{L}_\pm = \mathrm{L}_\pm$) and has $\mathrm{L}_+\subseteq\mathrm{L}_{++}\cup\{0\}$ and $\mathrm{L}_-\subseteq\mathrm{L}_{--}\cup\{0\}$. Further, let $Q_\pm$ the linear projectors onto $\mathrm{L}_\pm$ (that is: $Q_\pm=\mathbb{1}$ on $\mathrm{L}_\pm$ whereas $Q_\pm=0$ on $\mathrm{L}_\mp$), which commute with J. Then, $Q := Q_+ - Q_-$ is called the metric operator or fundamental symmetry of L. For Krein spaces, it fulfills $Q^2=\mathbb{1}$ and thus $Q^3=Q$. The metric operator induces the positive definite inner product $\{\cdot,\cdot\}^Q := \{\cdot,Q\cdot\} = \{\cdot,Q_+\cdot\}-\{\cdot,Q_-\cdot\}$ on L, which is called Hilbert inner product.

If $\omega(Q_+\cdot,Q_-\cdot) = \omega(\cdot,\cdot)$, then also $\omega(Q\,\cdot,Q\,\cdot) = \omega(\cdot,\cdot)$, that is: the metric operator Q is compatible with the symplectic structure. Then, $J^Q := QJ$ is a new complex structure on L with $(J^Q)^2 = -\mathbb{1}$ (because J and Q commute and $Q^2=\mathbb{1}$ and $J^2=-\mathbb{1}$) and also $\omega(J^Q\cdot,J^Q\cdot) = \omega(\cdot,\cdot)$. This new complex structure induces a positive definite real g-product $\mathrm{g}^Q(\cdot,\cdot) := \omega(\cdot,J^Q\cdot) = \mathrm{g}(\cdot,Q\,\cdot)$, which relates to $\{\cdot,\cdot\}^Q$ as in (3.23).

However, if the original complex structure J commutes with the action of some spacetime isometry K on $\mathrm{L}_{\partial\mathbb{M}}$, then the new complex structure J^Q *does not* necessarily inherit this commutation property! Since J commutes with both Q and K, we have $[J^Q,K]\equiv[QJ,K] = J[Q,K]$. Hence J^Q commutes with K, if and only if Q commutes with K.

these structures are indefinite is perfectly consistent [62]. With the complex inner product (and upon completion), L_Σ is a complex Hilbert space (or Krein space in the indefinite case). This is the "one-particle" space. The state space is the bosonic Fock space over L_Σ. (In the indefinite case, the Fock space is also indefinite and a Krein space.)

Given a spacetime region $\mathbb{M}$, the fact that $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}} \subseteq \mathrm{L}_{\partial\mathbb{M}}$ is Lagrangian has an important consequence: $\mathrm{L}_{\partial\mathbb{M}}$ decomposes as a direct sum $\mathrm{L}_{\partial\mathbb{M}} = \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}} \oplus J_{\partial\mathbb{M}}\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ over $\mathbb{R}$ (for any fixed $J_{\partial\mathbb{M}}$), see Lemma (2.11). For $\xi \in \mathrm{L}_{\partial\mathbb{M}}$ this writes as the decomposition (which is unique for any fixed $J_{\partial\mathbb{M}}$)

$$\xi = \xi^{\mathrm{R}} + J_{\partial\mathbb{M}}\xi^{\mathrm{I}} \tag{3.24}$$

with $\xi^{\mathrm{R}}, \xi^{\mathrm{I}} \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$. We recall that each such boundary solution $\xi \in \mathrm{L}_{\partial\mathbb{M}}$ has an associated element, generalizing (3.14),

$$\hat{\xi} \;:=\; \xi^{\mathrm{R}} - \mathrm{i}\xi^{\mathrm{I}} \tag{3.25}$$

in the complexified subspace of interior solutions $(\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}})^{\mathbb{C}} \subseteq \mathrm{L}^{\mathbb{C}}_{\partial\mathbb{M}}$. For example, for a time-interval region with the notation of (2.16), this element writes as (the end of Section 5.1 in [59]):

$$\hat{\xi}_{12} \;=\; \xi^{\mathrm{R}}_{12} - \mathrm{i}\xi^{\mathrm{I}}_{12}. \tag{3.26}$$

For a rod region, with the notation of (2.19), this element writes as (see Section 5.3 in [59]):

$$\hat{\xi}_{0} \;:=\; \xi^{\mathrm{R}}_{0} - \mathrm{i}\xi^{\mathrm{I}}_{0}. \tag{3.27}$$

Remarkably, using these ingredients, the amplitude map for the region $\mathbb{M}$ can be expressed in closed form. Given $\xi \in \mathrm{L}_{\partial\mathbb{M}}$, the amplitude of the associated (normalized) coherent state $K^{\xi} \in \mathcal{H}_{\partial\mathbb{M}}$ is calculated in Section 4.3 of [59], resulting in

$$\rho_{\mathbb{M}}(K^{\xi}) \;=\; \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\Big(\tfrac{1}{4} g_{\partial\mathbb{M}}(\hat{\xi},\hat{\xi})\Big) \;=\; \exp\Big(-\tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{I}},\, \xi^{\mathrm{I}}) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{R}},\, \xi^{\mathrm{I}})\Big). \tag{3.28}$$

However, the constructions that are necessary to make the path integration rigorous, are rather deep (see the entire Section 3 in [59]). Therefore, here we only motivate the result, and postpone the explicit calculation of (3.28) to Section 3.1.4. The amplitude for any boundary state $\psi_{\partial\mathbb{M}}$ is defined as the path integral

$$\rho_{\mathbb{M}}(\psi_{\partial\mathbb{M}}) \;=\; \int\limits_{\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}} \mathrm{d}\nu(\phi)\; \psi_{\partial\mathbb{M}}(\phi). \tag{3.29}$$

We recall that $\mathrm{L}_{\mathbb{M}}$ is the space of classical solutions on the interior of $\mathbb{M}$, and $\mathrm{L}_{\partial\mathbb{M}}$ is the space of solutions in a neighborhood of $\partial\mathbb{M}$. The map $r_{\mathbb{M}}$ of Classical Axiom (C5) allows us to see the interior solutions as a subset of boundary solutions: $r_{\mathbb{M}}\mathrm{L}_{\mathbb{M}} =: \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}} \subset \mathrm{L}_{\partial\mathbb{M}}$. The measure $\mathrm{d}\nu(\phi)$ is a probability measure like the one above, but on $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$. Further, $\psi_{\partial\mathbb{M}}$ is assumed to be integrable: $\psi_{\partial\mathbb{M}} \in \mathcal{L}^1(\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}, \nu)$.

The coherent states are integrable in this sense, and for them the integral (3.29) yields precisely (3.28). Hence Core Axiom (T4) is fulfilled and its amplitude subspace is here $\mathcal{H}^{\mathrm{coh}}_{\partial\mathbb{M}}$. Also Core Axiom (T5a) about the amplitude for disjoint unions is met, because the measure for a disjoint region is the product measure. Further, in Theorem (4.5) of [59] it is proven that also Core Axiom (T5b) about gluing holds, provided that the gluing data satisfy an integrability condition. Proposition (4.3) therein proves that Core Axiom (T3x) about slice regions holds as well.

For time-interval regions, we can plug relations (2.16) into the amplitude formula (3.28) and obtain the amplitude for a coherent boundary state $K^{(\eta_1,\zeta_2)}_{\partial\mathbb{M}_{[t_1,t_2]}} \;=\; K^{\eta_1}_{\Sigma_1} \otimes \overline{K^{\zeta_2}_{\Sigma_2}}$:

$$\begin{aligned} \rho_{[t_1,t_2]}\Big(K^{\eta_1}_{\Sigma_1} \otimes \overline{K^{\zeta_2}_{\Sigma_2}}\Big) \;&=\; \exp\Big(-\tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}_{[t_1,t_2]}}(\xi^{\mathrm{I}}_{12},\, \xi^{\mathrm{I}}_{12}) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{\partial\mathbb{M}_{[t_1,t_2]}}(\xi^{\mathrm{R}}_{12},\, \xi^{\mathrm{I}}_{12})\Big) \\ &=\; \exp\Big(-\mathrm{g}_{\Sigma_1}(\xi^{\mathrm{I}}_{12},\, \xi^{\mathrm{I}}_{12}) - \mathrm{i}\mathrm{g}_{\Sigma_1}(\xi^{\mathrm{R}}_{12},\, \xi^{\mathrm{I}}_{12})\Big) \qquad (3.30) \\ &=\; \exp\Big(-\tfrac{1}{4}\mathrm{g}_{\Sigma_1}(\eta_1,\, \eta_1) - \tfrac{1}{4}\mathrm{g}_{\Sigma_1}(\zeta_2,\, \zeta_2) + \tfrac{1}{2}\mathrm{g}_{\Sigma_1}(\eta_1,\, \zeta_2)\Big). \qquad (3.31) \end{aligned}$$

For rod regions, without using information about the modes (which ones are regular on the rod's interior and which are not), it is not possible to give an explicit construction of ξ_0^{R} and ξ_0^{I} (as in (2.16) for the time-interval region), and we can only write (3.28) adapted to the rod region's notation:

$$\rho_{r_0}\big(\overline{K_{\Sigma_0}^{\xi_0}}\big) \;=\; \exp\Big(-\tfrac{1}{2}\mathrm{g}_{r_0}(\xi_0^{\mathrm{I}},\,\xi_0^{\mathrm{I}}) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{r_0}(\xi_0^{\mathrm{R}},\,\xi_0^{\mathrm{I}})\Big). \tag{3.32}$$

Even though we consider a free theory where everything is supposed to be simple, the result (3.28) is still striking: It applies irrespective of the shape of the spacetime region $\mathbb{M}$ and without even specifying what kind of spacetime we are actually in. Of course, restrictions on both are hidden in the assumptions we have made on the various ingredients, in particular the complex structure. We are still very far from a general understanding of these restrictions in terms of spacetime structure and field theory.

If we add to the action a linear functional D on field configurations $K_{\mathbb{M}}$ in $\mathbb{M}$, i.e., $D : K_{\mathbb{M}} \to \mathbb{R}$ linear, then the equations of motion in $\mathbb{M}$ are modified. Introducing a source term μ via (3.12) is a special case of this. There is a special solution β_D of these modified equations with the property that its restriction to the boundary lies in $J_{\partial\mathbb{M}}\mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}} \subseteq \mathrm{L}_{\partial\mathbb{M}}$. The amplitude for the thus modified theory can be shown to take the following form for coherent states [58, 66],

$$\rho_{\mathbb{M}}^{D}(K^{\xi}) = \rho_{\mathbb{M}}(K^{\xi})\,\exp\Big(\mathrm{i}D(\hat{\xi})\Big)\,\exp\Big(\tfrac{\mathrm{i}}{2}D(\beta_D) - \tfrac{1}{2}g_{\partial\mathbb{M}}(\beta_D,\beta_D)\Big). \tag{3.33}$$

For the special case (3.12), the expression (3.33) turns into the expression (3.21) with the three factors on the right hand side in exact correspondence. For the third factor, this can be seen [66] through the relation between complex structure and Feynman propagator [42].

We return to the setting of Section 3.1.2 to see how it fits into the framework just presented. The vector space L_∂ may be viewed as the space of solutions of the classical equations of motion in a neighborhood of the boundary of the spacetime region $[t_1,t_2]\times\mathbb{R}^d$ (or $[-\infty,\infty]\times\mathbb{R}^d$). Because of the Cauchy property, such a solution is equivalent to a pair (η_1,ζ_2) of two (generically distinct) global solutions. Hence, $\mathrm{L}_\partial = \mathrm{L}_1 \oplus \mathrm{L}_2$ with each summand equivalent to L. The space L_1 inherits the complex structure J of L, while L_2 has the opposite complex structure, due to its opposite orientation as a boundary component of the region. Combining the two yields the complex structure

$$\begin{aligned} J_\partial :\; \mathrm{L}_\partial \;&\to\; \mathrm{L}_\partial \\ J_\partial(\eta_1,\zeta_2) \;&=\; (J\eta_1, -J\zeta_2), \end{aligned} \tag{3.34}$$

see also Section 2.3. The space of solutions inside the region is again equivalent to the space L of global solutions. Thus, given $\xi \in \mathrm{L}_\partial$, the global solution $\hat{\xi} = \xi^{\mathrm{R}} - \mathrm{i}\xi^{\mathrm{I}}$ is an element of $\mathrm{L}^{\mathbb{C}}$. As is straightforward to verify now, it is precisely given by formula (3.14).

Given a quantum field theory and a spacetime region $\mathbb{M}$, the availability of a compatible notion of source amplitude in $\mathbb{M}$ via formula (3.33) or (3.21) depends crucially on the availability of a suitable complex structure on $\partial\mathbb{M}$. This is a given for standard QFTs and spacelike hypersurfaces in Minkowski spacetime. There, invariance under Poincaré transformations determines a complex structure essentially uniquely. For some relevant results for time-like hypersurfaces (not necessarily explicitly using the language of a complex structure), see [52, 53, 59, 24].

3.1.4 Calculation of the amplitude for coherent states

In this section we show the explicit calculation of the amplitude (3.28) for coherent states, following the proof of Proposition 4.2 in [59]. Our goal is thus to obtain Equation (44) of [59], that is:

$$\rho_{\mathbb{M}}(K^{\xi}) = \mathcal{N}_{\partial\mathbb{M}}^{\xi}\exp\Big(\tfrac{1}{4}g_{\partial\mathbb{M}}(\hat{\xi},\hat{\xi})\Big) = \mathcal{N}_{\partial\mathbb{M}}^{\xi}\exp\Big(\tfrac{1}{4}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{R}},\xi^{\mathrm{R}}) - \tfrac{1}{4}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{I}},\xi^{\mathrm{I}}) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{R}},\xi^{\mathrm{I}})\Big), \tag{3.35}$$

with the general amplitude (3.29) as point of departure:

$$\rho_{\mathbb{M}}(\psi_{\partial\mathbb{M}}) \;=\; \int_{\mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}} \mathrm{d}\nu(\phi)\;\psi_{\partial\mathbb{M}}(\phi).$$

First, for $\phi \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$, coherent states act on it as follows:

$$K^{\xi}_{\partial\mathbb{M}}(\phi) = \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\Big(\tfrac{1}{2}\left\{\xi^{\mathrm{R}} + J_{\partial\mathbb{M}}\xi^{\mathrm{I}},\, \phi\right\}_{\partial\mathbb{M}}\Big) = \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\Big(\tfrac{1}{2}\left\{\xi^{\mathrm{R}},\, \phi\right\}_{\partial\mathbb{M}} - \tfrac{\mathrm{i}}{2}\left\{\xi^{\mathrm{I}},\, \phi\right\}_{\partial\mathbb{M}}\Big)$$
$$= \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\Big(\tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{R}},\phi) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{I}},\phi)\Big).$$

Therein, we use that $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ is a Lagrangian subspace, and thus the symplectic structure $\omega_{\partial\mathbb{M}}$ vanishes on it. The coherent state is thus an almost translation invariant function on $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$. According to Definition (3.9) in [59], a function $f:\ \mathrm{L} \to \mathbb{C}$ on a separable Hilbert space L with complex inner product $\{\cdot,\cdot\}$ is called almost translation invariant, if there exists a closed subspace $\mathrm{C} \subseteq \mathrm{L}$ of finite codimension n, such that $f(\phi+\chi) = f(\phi)$ for all $\phi \in \mathrm{L}$ and all $\chi \in \mathrm{C}$. Let us denote the subspace of $\mathrm{L}_{\partial\mathbb{M}}$ which is spanned by ξ^{R} and ξ^{I} as $\mathrm{C}^{\xi}_{\partial\mathbb{M}}$. This subspace has dimension $n = 2$ if $\xi^{\mathrm{R,I}}$ are linear independent, which we shall assume here. (If they are linear dependent, then $n = 1$ and the calculation simplifies, while the final result remains the same.) Then, the coherent state is translation invariant with respect to the orthogonal complement of $\mathrm{C}^{\xi}_{\partial\mathbb{M}}$ in $\mathrm{L}_{\partial\mathbb{M}}$, which has codimension $n = 2$. This can be quickly seen as follows: let χ any element of this orthogonal complement, that is, $\mathrm{g}_{\partial\mathbb{M}}(\chi,\xi^{\mathrm{R,I}}) = 0$. Then,

$$K^{\xi}_{\partial\mathbb{M}}(\phi+\chi) = \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\Big(\tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{R}},\phi+\chi) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{I}},\phi+\chi)\Big) = \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\Big(\tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{R}},\phi) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{\partial\mathbb{M}}(\xi^{\mathrm{I}},\phi)\Big)$$
$$= K^{\xi}_{\partial\mathbb{M}}(\phi).$$

This allow us to apply Proposition (3.10) of [59], which essentially states that for almost translation invariant f as above, we have

$$\int_{\mathrm{L}} \mathrm{d}\nu(\phi)\ f(\phi) = \int_{\mathbb{R}^n} \mathrm{d}\nu_Q(x)\ \tilde{f}(z),$$

if the auxiliary function $\tilde{f} \in \mathcal{L}^1(\mathbb{R}^n,\nu_Q)$ is integrable. Therein, we use a basis $\{\xi_i\}_{i=1,\ldots,n}$ of the above orthogonal complement, and with that we define according to Equation (12) and Proposition (3.10) in [59]:

$$\tilde{f}:\ \mathbb{R}^n \to \mathbb{C} \qquad\qquad Q_{ij} := \tfrac{1}{2}\mathrm{g}_{\mathrm{L}}(\xi_i,\xi_j)$$
$$\tilde{f}(x) := f\Big(\sum_{i=1}^{n} x_i\xi_i\Big) \qquad\qquad \mathrm{d}\nu_Q(x) := \mathrm{d}\mu(x)\sqrt{\frac{\det Q}{\pi^n}}\, \exp\left(-x^{\mathrm{T}}Q\,x\right)$$

(The factor of $\frac{1}{2}$ in the definition of Q stems from comparing with Equation (22) in [59].) Q is thus a real, positive definite, symmetric matrix. Proposition (3.10) thus enables us to evaluate the infinite-dimensional integral of almost translation invariant functions $f:\ \mathrm{L} \to \mathbb{C}$ via a finite-dimensional integral of the auxiliary function $\tilde{f}:\ \mathbb{R}^n \to \mathbb{C}$.

In our case, $n = 2$ with $\xi_1 = \xi^{\mathrm{R}}$ and $\xi_2 = \xi^{\mathrm{I}}$. Further, we have $Q_{ij} = \mathrm{g}_{\partial\mathbb{M}}(\xi_i,\xi_j) \in \mathbb{R}$. That is, here Q is indeed a real, symmetric, positive definite (2,2)-matrix as stated above. The auxiliary version of the coherent states then writes as

$$\tilde{K}^{\xi}_{\partial\mathbb{M}}(x_1,x_2) := K^{\xi}_{\partial\mathbb{M}}(x_1\xi^{\mathrm{R}}+x_2\xi^{\mathrm{I}}) = \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\Big(\tfrac{1}{2}\left\{\xi^{\mathrm{R}}+J_{\partial\mathbb{M}}\xi^{\mathrm{I}},\, x_1\xi^{\mathrm{R}}+x_2\xi^{\mathrm{I}}\right\}_{\partial\mathbb{M}}\Big)$$
$$= \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\left(x_1Q_{11} - \mathrm{i}x_1Q_{21} + x_2Q_{12} - \mathrm{i}x_2Q_{22}\right).$$

With these preparations, we can now calculate the holomorphic amplitude of the coherent states:

$$\rho_{\mathbb{M}}(K^{\xi}_{\partial\mathbb{M}}) = \int_{\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}} \mathrm{d}\nu(\phi)\ K^{\xi}_{\partial\mathbb{M}}(\phi) = \int_{\mathbb{R}^2} \mathrm{d}\nu_Q(x_1,x_2)\ \tilde{K}^{\xi}_{\partial\mathbb{M}}(x_1,x_2) \tag{3.36}$$

$$= \mathcal{N}^{\xi}_{\partial\mathbb{M}}\sqrt{\frac{\det Q}{\pi^2}} \int_{\mathbb{R}^2} \mathrm{d}^2x\ \exp\left(-\tfrac{1}{2}x^{\mathrm{T}}Ax + B^{\mathrm{T}}x\right), \tag{3.37}$$

wherein $A = 2Q$ and $B = (\frac{1}{2}Q_{11} - \frac{\mathrm{i}}{2}Q_{21},\ \frac{1}{2}Q_{12} - \frac{\mathrm{i}}{2}Q_{22})$. For the above Gaussian integral with linear term we can use that

$$\int_{\mathbb{R}^n} \mathrm{d}^2 x\ \exp\left(-\tfrac{1}{2}x^{\mathrm{T}}Ax + B^{\mathrm{T}}x\right) \ = \ \sqrt{\tfrac{(2\pi)^n}{\det A}}\ \exp\left(\tfrac{1}{2}B^{\mathrm{T}}A^{-1}B\right). \tag{3.38}$$

This formula holds for real, symmetric, positive definite A and complex B, as can be verified by completing the squares $-\frac{1}{2}x^{\mathrm{T}}Ax + B^{\mathrm{T}}x \ = \ -\frac{1}{2}(A^{1/2}x - A^{-1/2}B)^{\mathrm{T}}(A^{1/2}x - A^{-1/2}B) + \frac{1}{2}B^{\mathrm{T}}A^{-1}B$ and making a change of variables $y \ := \ A^{1/2}x$ such that $\mathrm{d}^n x \ = \ \mathrm{d}^n y\,(\det\ A)^{-1/2}$. The resulting integral has y added with an imaginary term in the exponential. The imaginary part of this integral vanishes because it is an odd function, and (3.38) is the real part, which arises from integral 3.896.4 in [38]:

$$\int_0^\infty \mathrm{d}x\ \cos(bx)\ \mathrm{e}^{-\beta x^2} \ = \ \tfrac{1}{2}\sqrt{\tfrac{\pi}{\beta}}\ \mathrm{e}^{-b^2/(4\beta)}.$$

Evaluating (3.37) with (3.38) yields the desired result (3.35):

$$\rho_{\mathbb{M}}(K^{\xi}_{\partial\mathbb{M}}) \ = \ \mathcal{N}^{\xi}_{\partial\mathbb{M}} \exp\left(\tfrac{1}{2}\,Q_{11} - \tfrac{1}{2}\,Q_{22} - \mathrm{i}\,Q_{12}\right). \tag{3.39}$$

3.1.5 Unitarity and evolution

A nice feature of Holomorphic Quantization is that it includes evolution and unitarity in a rather natural way, as shown in Section 4.5 of [59]. For evolution to make sense, we need an "initial" hypersurface Σ_1 (not necessarily Cauchy) and a "final" Σ_2. Initial here only means that we consider evolution starting on Σ_1 and ending up on Σ_2. Evolution is thus taken in a generalized sense here, and not necessarily in time, see the examples below. We thus consider regions $\mathbb{M}$ for which Σ_1 and $\overline{\Sigma}_2$ form part of the boundary $\partial\mathbb{M}$, and with a one-to-one correspondence between the spaces of classical solutions L_{Σ_1} and L_{Σ_2}. (Hence we only consider regions with at least two boundary components, which excludes regions with a connected boundary, as for example the rod region.) The easiest example is Σ_1 and Σ_2 being equal-time hypersurfaces or Cauchy surfaces. This is the case in the standard setting of a time-interval region in Minkowski spacetime. However, the GBF also considers more general evolution. For example, in Section III of [53] evolution is considered between two hypersurfaces in Minkowski spacetime, on which the spatial coordinate x^1 is constant, respectively. And Section V therein treats evolution between two Minkowski hypercylinders, each of constant radius.

Let thus $\mathbb{M}$ be a region such that its boundary decomposes as a disjoint union $\partial\mathbb{M} \ = \ \Sigma_1 \cup \overline{\Sigma}_2$. Then, we have the linear maps

$$r_1 : \ \mathrm{L}_{\mathbb{M}} \ \to \ \mathrm{L}_{\Sigma_1} \qquad\qquad r_1 \ = \ \hat{P}_{\Sigma_1} \circ r_{\mathbb{M}},$$
$$r_2 : \ \mathrm{L}_{\mathbb{M}} \ \to \ \mathrm{L}_{\Sigma_2} \qquad\qquad r_2 \ = \ \hat{P}_{\Sigma_2} \circ r_{\mathbb{M}},$$

wherein $\hat{P}_{\Sigma_1} : \ \mathrm{L}_{\partial\mathbb{M}} \ \to \ \mathrm{L}_{\Sigma_1}$ is the projector onto the subspace L_{Σ_1} (whose action is just cutting away the component in L_{Σ_2}). We assume here that $r_{1,2}$ are homeomorphisms, and define the classical evolution map

$$\mathcal{T}^{\mathbb{M}}_{21} : \ \mathrm{L}_{\Sigma_1} \ \to \ \mathrm{L}_{\Sigma_2} \qquad\qquad \mathcal{T}^{\mathbb{M}}_{21} \ := \ r_2 \circ r_1^{-1}. \tag{3.40}$$

Let now $\eta, \zeta \in \mathrm{L}_{\mathbb{M}}$, and

$$\eta_1 \ := \ r_1\eta \qquad\qquad \eta_2 \ := \ r_2\eta \ = \ \mathcal{T}^{\mathbb{M}}_{21}\eta_1,$$
$$\zeta_1 \ := \ r_1\zeta \qquad\qquad \zeta_2 \ := \ r_2\zeta \ = \ \mathcal{T}^{\mathbb{M}}_{21}\zeta_1.$$

Then, $\eta_{\partial\mathbb{M}} := (\eta_1, \eta_2) \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ and $\zeta_{\partial\mathbb{M}} := (\zeta_1, \zeta_2) \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$. According to Classical Axiom (C5), $\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ is Lagrangian, thus we have

$$0 = \omega_{\partial\mathbb{M}}(\eta_{\partial\mathbb{M}}, \zeta_{\partial\mathbb{M}}) = \omega_{\Sigma_1}(\eta_1, \zeta_1) - \omega_{\Sigma_2}(\eta_2, \zeta_2),$$

and thus the classical evolution map preserves the symplectic structure:

$$\omega_{\Sigma_2}\left(\mathcal{T}^{\mathbb{M}}_{21}\eta_1, \mathcal{T}^{\mathbb{M}}_{21}\zeta_1\right) = \omega_{\Sigma_1}(\eta_1, \zeta_1). \tag{3.41}$$

If and only if moreover the classical evolution commutes with the complex structure as in

$$J_{\Sigma_2} \circ \mathcal{T}^{\mathbb{M}}_{21} = \mathcal{T}^{\mathbb{M}}_{21} \circ J_{\Sigma_1}, \tag{3.42}$$

then we get

$$\begin{aligned} \mathrm{g}_{\Sigma_2}\left(\mathcal{T}^{\mathbb{M}}_{21}\eta_1, \mathcal{T}^{\mathbb{M}}_{21}\zeta_1\right) &= \omega_{\Sigma_2}\left(\mathcal{T}^{\mathbb{M}}_{21}\eta_1, J_{\Sigma_2}\mathcal{T}^{\mathbb{M}}_{21}\zeta_1\right) = \omega_{\Sigma_2}\left(\mathcal{T}^{\mathbb{M}}_{21}\eta_1, \mathcal{T}^{\mathbb{M}}_{21}J_{\Sigma_1}\zeta_1\right) = \omega_{\Sigma_1}(\eta_1, J_{\Sigma_1}\zeta_1) \\ &= \mathrm{g}_{\Sigma_1}(\eta_1, \zeta_1). \end{aligned} \tag{3.43}$$

This implies that also

$$\left\{\mathcal{T}^{\mathbb{M}}_{21}\eta_1, \mathcal{T}^{\mathbb{M}}_{21}\zeta_1\right\}_{\Sigma_2} = \left\{\eta_1, \zeta_1\right\}_{\Sigma_1}, \tag{3.44}$$

that is, the classical evolution map $\mathcal{T}^{\mathbb{M}}_{21}$ is unitary. Proposition (4.6) in [59] then constructs a linear quantum evolution map $\mathcal{U}^{\mathbb{M}}_{21} : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$

$$\left(\mathcal{U}^{\mathbb{M}}_{21}\psi_1\right)(\phi_2) = \rho_{\mathbb{M}}\left(\psi_1 \otimes \overline{K^{\phi_2}_{\Sigma_2}}\right)/\mathcal{N}^{\phi_2}_{\Sigma_2} \qquad \forall\, \psi_1 \in \mathcal{H}_{\Sigma_1}, \quad \forall\, \phi_2 \in \mathrm{L}_{\Sigma_2}. \tag{3.45}$$

As proven below, this quantum evolution map relates the amplitude and the inner product through

$$\rho_{\mathbb{M}}\left(\psi_1 \otimes \iota_{\Sigma_2}\psi_2\right) = \left\langle \psi_2, \mathcal{U}^{\mathbb{M}}_{21}\psi_1\right\rangle_{\Sigma_2} \qquad \forall\, \psi_1 \in \mathcal{H}_{\Sigma_1}, \quad \forall\, \psi_2 \in \mathcal{H}_{\Sigma_2}. \tag{3.46}$$

Moreover, if the classical evolution is unitary, then the quantum evolution is unitary, and they relate through

$$\left(\mathcal{U}^{\mathbb{M}}_{21}\psi_1\right)(\phi_2) = \psi_1(\mathcal{T}^{\mathbb{M}}_{12}\phi_2) \qquad \forall\, \psi_1 \in \mathcal{H}_{\Sigma_1}, \quad \forall\, \phi_2 \in \mathrm{L}_{\Sigma_2}. \tag{3.47}$$

At first sight, this might be taken to suggest that quantum evolution were determined by classical evolution. However, there are differences already for the free case. Classically, we can have any solution ϕ_2 near Σ_2, and then the solution near Σ_1 must be its preimage $\mathcal{T}^{\mathbb{M}}_{12}\phi_2$. On the quantum level, the amplitude for coherent states $K^{\mathcal{T}^{\mathbb{M}}_{12}\phi_2}_{\Sigma_1} \otimes \overline{K^{\phi_2}_{\Sigma_2}}$ determined by such solutions is unity, see (3.30) with $\xi^{\mathrm{I}} \equiv 0$. By contrast, coherent states whose solutions are not related through classical evolution $K^{\eta_1}_{\Sigma_1} \otimes \overline{K^{\zeta_2}_{\Sigma_2}}$ have a nonvanishing amplitude, albeit exponentially damped by the classically forbidden term ξ^{I}. We can view this as tunneling.

In particular, for coherent states the quantum evolution becomes

$$\mathcal{U}^{\mathbb{M}}_{21}K^{\xi_1}_{\Sigma_1} = K^{\mathcal{T}^{\mathbb{M}}_{21}\xi_1}_{\Sigma_2} \qquad \forall\, \xi_1 \in \mathrm{L}_{\Sigma_1}. \tag{3.48}$$

Relation (3.46) is proven as follows:

$$\begin{aligned} \left\langle \psi_2, \mathcal{U}^{\mathbb{M}}_{21}\psi_1\right\rangle_{\Sigma_2} &= \int_{\mathrm{L}_{\Sigma_2}} \mathrm{d}\nu_{\Sigma_2}(\phi_2)\, \overline{\psi_2(\phi_2)}\, \rho_{\mathbb{M}}\left(\psi_1 \otimes \overline{K^{\phi_2}_{\Sigma_2}}\right)/\mathcal{N}^{\phi_2}_{\Sigma_2} = \int_{\mathrm{L}_{\Sigma_2}} \mathrm{d}\nu_{\Sigma_2}(\phi_2)\, \overline{\psi_2(\phi_2)} \int_{\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}} \mathrm{d}\nu_{\partial\mathbb{M}}^{\mathrm{int}}(\xi)\, \psi_1(\xi)\, \overline{K^{\phi_2}_{\Sigma_2}}(\xi)/\mathcal{N}^{\phi_2}_{\Sigma_2} \\ &= \int_{\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}} \mathrm{d}\nu_{\partial\mathbb{M}}^{\mathrm{int}}(\xi)\, \psi_1(\xi) \int_{\mathrm{L}_{\Sigma_2}} \mathrm{d}\nu_{\Sigma_2}(\phi_2)\, \overline{\psi_2(\phi_2)}\, K^{\xi}_{\Sigma_2}(\phi_2)/\mathcal{N}^{\xi}_{\Sigma_2} = \int_{\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}} \mathrm{d}\nu_{\partial\mathbb{M}}^{\mathrm{int}}(\xi)\, \psi_1(\xi) \left\langle \psi_2, K^{\xi}_{\Sigma_2}\right\rangle_{\Sigma_2}/\mathcal{N}^{\xi}_{\Sigma_2} \\ &= \int_{\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}} \mathrm{d}\nu_{\partial\mathbb{M}}^{\mathrm{int}}(\xi)\, \psi_1(\xi)\, \overline{\psi_2(\xi)} = \rho_{\mathbb{M}}\left(\psi_1 \otimes \iota_{\Sigma_2}\psi_2\right). \end{aligned}$$

In order to prove (3.47) for unitary classical evolution, note that here we can identify the real vector spaces of classical solutions $\mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}$ and L_{Σ_1}, such that the measures coincide $\nu_{\substack{\mathrm{int}\\ \partial\mathbb{M}}} = \nu_{\Sigma_1}$. Using $\xi_{1,2}$ to indicate the solutions near $\Sigma_{1,2}$ induced by $\xi \in \mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}$, we then get

$$
\begin{aligned}
\big(\mathcal{U}_{21}^{\mathbb{M}}\psi_1\big)(\phi_2) = \rho_{\mathbb{M}}\big(\psi_1 \otimes K_{\overline{\Sigma}_2}^{\phi_2}\big)/\mathcal{N}_{\overline{\Sigma}_2}^{\phi_2} &= \int_{\mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}} \mathrm{d}\nu_{\substack{\mathrm{int}\\ \partial\mathbb{M}}}(\xi)\ \psi_1(\xi_1)\, K_{\overline{\Sigma}_2}^{\phi_2}(\xi_2)/\mathcal{N}_{\overline{\Sigma}_2}^{\phi_2} \\
&= \int_{\mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}} \mathrm{d}\nu_{\substack{\mathrm{int}\\ \partial\mathbb{M}}}(\xi)\ \psi_1(\xi_1)\ \exp\Big(\tfrac{1}{2}\,\{\phi_2,\,\xi_2\}_{\overline{\Sigma}_2}\Big)\,\exp\Big(\tfrac{1}{4}\,\{\phi_2,\,\phi_2\}_{\overline{\Sigma}_2}\Big) \\
&= \int_{\mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}} \mathrm{d}\nu_{\substack{\mathrm{int}\\ \partial\mathbb{M}}}(\xi)\ \psi_1(\xi_1)\ \overline{\exp\Big(\tfrac{1}{2}\,\big\{\mathcal{T}_{21}^{\mathbb{M}}\phi_2,\,\mathcal{T}_{21}^{\mathbb{M}}\xi_2\big\}_{\Sigma_1}\Big)}\ \overline{\exp\Big(\tfrac{1}{4}\,\big\{\mathcal{T}_{21}^{\mathbb{M}}\phi_2,\,\mathcal{T}_{21}^{\mathbb{M}}\phi_2\big\}_{\Sigma_1}\Big)} \\
&= \int_{\mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}} \mathrm{d}\nu_{\substack{\mathrm{int}\\ \partial\mathbb{M}}}(\xi)\ \psi_1(\xi_1)\ \overline{K_{\Sigma_1}^{\mathcal{T}_{21}^{\mathbb{M}}\phi_2}(\xi_1)}/\mathcal{N}_{\Sigma_1}^{\mathcal{T}_{21}^{\mathbb{M}}\phi_2} \;=\; \int_{\mathrm{L}_{\partial\mathbb{M}}^{\mathrm{int}}} \mathrm{d}\nu_{\Sigma_1}(\xi)\ \psi_1(\xi_1)\ \overline{K_{\Sigma_1}^{\mathcal{T}_{21}^{\mathbb{M}}\phi_2}(\xi_1)}/\mathcal{N}_{\Sigma_1}^{\mathcal{T}_{21}^{\mathbb{M}}\phi_2} \\
&= \Big\langle K_{\Sigma_1}^{\mathcal{T}_{21}^{\mathbb{M}}\phi_2},\ \psi_1\Big\rangle_{\Sigma_1}/\mathcal{N}_{\Sigma_1}^{\mathcal{T}_{21}^{\mathbb{M}}\phi_2} \\
&= \psi_1(\mathcal{T}_{21}^{\mathbb{M}}\phi_2).
\end{aligned}
$$

Finally, (3.48) proves by

$$
\begin{aligned}
\big(\mathcal{U}_{21}^{\mathbb{M}} K_{\Sigma_1}^{\xi_1}\big)(\phi_2) \;&=\; K_{\Sigma_1}^{\xi_1}(\mathcal{T}_{21}^{\mathbb{M}}\phi_2) \;=\; \exp\Big(\tfrac{1}{2}\ \big\{\xi_1,\,\mathcal{T}_{21}^{\mathbb{M}}\phi_2\big\}_{\Sigma_1}\Big)\exp\Big(-\tfrac{1}{4}\ \{\xi_1,\,\xi_1\}_{\Sigma_1}\Big) \\
&=\; \exp\Big(\tfrac{1}{2}\ \big\{\mathcal{T}_{21}^{\mathbb{M}}\xi_1,\,\phi_2\big\}_{\Sigma_1}\Big)\exp\Big(-\tfrac{1}{4}\ \big\{\mathcal{T}_{21}^{\mathbb{M}}\xi_1,\,\mathcal{T}_{21}^{\mathbb{M}}\xi_1\big\}_{\Sigma_2}\Big) \\
&=\; K_{\Sigma_2}^{\mathcal{T}_{21}^{\mathbb{M}}\xi_1}(\phi_2).
\end{aligned}
$$

Now suppose that we consider a spacetime with a system Σ_λ of connected hypersurfaces, such that for any pair $(\Sigma_{\lambda_1}, \Sigma_{\lambda_2})$ the solution spaces L_1 and L_2 are related through a linear homeomorphism $\mathcal{T}_{21} : \ \mathrm{L}_1 \ \to \ \mathrm{L}_2$ as above. The Classical Axioms (C1)-(C7) ensure the uniqueness of this map. The most important example for such a situation are Cauchy hypersurfaces in a globally hyperbolic spacetime. For defining a complete quantization it is now sufficient to fix a complex structure J_{Σ_0} on some hypersurface Σ_0 (compatible with the symplectic structure ω_{Σ_0}). Then, transporting J_{Σ_0} with the classical evolution maps as in

$$J_{\Sigma_\lambda} \;=\; \mathcal{T}_{\lambda 0}^{\mathbb{M}} \circ J_{\Sigma_0} \circ \mathcal{T}_{0\lambda}^{\mathbb{M}} \tag{3.49}$$

defines a complex structure J_{Σ_λ} for each Σ_λ, which is compatible with condition (3.42). Further, the symplectic structures ω_{Σ_λ} are compatible with classical evolution as well, as discussed above. Hence the real and complex inner products are compatible, too, making the classical evolution maps $\mathcal{T}_{21}$ unitary. In turn, this makes the quantum evolution maps $\mathcal{U}_{21}^{\mathbb{M}}$ unitary for any two hypersurfaces $\Sigma_{1,2}$. This agrees with previous results obtained in [23], wherein the Schrödinger-Feynman quantization is used.

3.1.6 Vacuum

The simplest choice of a vacuum state $\psi_\Sigma^{\mathrm{Vac}}$ for any hypersurface Σ is setting $\psi_\Sigma^{\mathrm{Vac}}(\phi) \equiv 1$ for all $\phi \in \mathrm{L}_\Sigma$. That is, $\psi_\Sigma^{\mathrm{Vac}}$ is the coherent state K_Σ^0 with characterizing solution $0 \in \mathrm{L}_\Sigma$. This choice complies with all the Vacuum Axioms (V1)-(V5) of Section 1.2.3.

However, in the case of a global background (see Section 1.2.4), a different construction is possible. By global background we mean that all regions and hypersurfaces are submanifolds (of codimension 0 respectively 1) of some fixed manifold B. Then, any global classical solution (well defined on all of

B) restricts to a solution on any region and near any hypersurface. The normalized coherent states with these characterising solutions then form a vacuum, which meets all Vacuum Axioms. In Section 4.6 of [59] this is made precise through the following three properties. Although they are inspired by having a global background, this is actually not necessary. A set $\{\phi_\Sigma\}$ which assigns a $\phi_\Sigma \in \mathrm{L}_\Sigma$ to any hypersurface is called a global solution iff it satisfies the following conditions.

(1) For any hypersurface Σ in B, we have $\phi_{\overline{\Sigma}} = \phi_\Sigma$.

(2) If a hypersurface $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n$ decomposes into a disjoint union of hypersurfaces, then its assigned solution has n components: $\phi_\Sigma = (\phi_{\Sigma_1}, \ldots, \phi_{\Sigma_n})$.

(3) For any region $\mathbb{M}$ in B, its boundary's assigned solution is an interior solution: $\phi_{\partial\mathbb{M}} \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$.

Proposition (4.8) in [59] then associates a family of vacuum states $\{\psi^{\mathrm{Vac}}_\Sigma\}$ to any global solution $\{\phi_\Sigma\}$, by choosing as the vacuum on any hypersurface Σ the normalized coherent state whose characterizing solution is the global solution ϕ_Σ assigned to Σ. That is: $\psi^{\mathrm{Vac}}_\Sigma = K^{\phi_\Sigma}_\Sigma$. (Hence the simple choice $\psi^{\mathrm{Vac}}_\Sigma \equiv 1$ is the special case where the vacuum is induced by the trivial global solution $\phi_\Sigma \equiv 0$ for all Σ.) Then, Vacuum Axiom (V2) $\psi^{\mathrm{Vac}}_{\overline{\Sigma}} = \overline{\psi^{\mathrm{Vac}}_\Sigma}$ is fulfilled due to (1) when using Classical Axiom (C2): $\{\eta, \zeta\}_{\overline{\Sigma}} = \overline{\{\eta, \zeta\}_\Sigma}$ with the coherent states' definition (3.1). Vacuum Axiom (V3) about decomposition can be seen to hold using (2) with Classical Axiom (C3) and Core Axiom (T2). Finally, Vacuum Axiom (V5) on the unit amplitude of the vacuum holds because of (3), when evaluating the amplitude formula (3.28). Since $\phi_{\partial\mathbb{M}} \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$, we have $\phi^{\mathrm{I}}_{\partial\mathbb{M}} \equiv 0$, and hence the vacuum amplitude becomes unity:

$$\rho_{\mathbb{M}}\big(K^{\phi_{\partial\mathbb{M}}}_{\partial\mathbb{M}}\big) \;=\; \exp\Big(-\tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}}\big(\phi^{\mathrm{I}}_{\partial\mathbb{M}},\, \phi^{\mathrm{I}}_{\partial\mathbb{M}}\big) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{\partial\mathbb{M}}\big(\phi^{\mathrm{R}}_{\partial\mathbb{M}},\, \phi^{\mathrm{I}}_{\partial\mathbb{M}}\big)\Big) \;=\; 1.$$

3.1.7 Observables

Although in this thesis we will not work with observables in QFT on AdS, at least we want to provide a simple example for observables fulfilling the Observable Axioms of Section 1.2.6. Unfortunately, for the Holomorphic Quantization there is no such simple example, because it views observables as functions on phase space (represented as the space of solutions). By contrast, the method of Schrödinger-Feynman Quantization (SFQ) views observables as functions on configuration space. This simplifies the situation for gluing regions. Let us denote by $\mathrm{K}_{\mathbb{M}}$ the space of field configurations on the region $\mathbb{M}$ (not just field configurations of classical solutions, but all possible field configurations). Let further $\mathbb{M}_1$ and $\mathbb{M}_2$ two adjacent regions, which we can glue to yield a new region $\mathbb{M}$. Then for the configuration spaces we simply have $\mathrm{K}_{\mathbb{M}} = \mathrm{K}_{\mathbb{M}_1} \times \mathrm{K}_{\mathbb{M}_2}$. That is, the observables on the two regions "decouple", meaning that each two classical observables on $\mathbb{M}_1$ and $\mathbb{M}_2$ induce a classical observable on $\mathbb{M}$ and vice versa. In contrast, for the solution spaces we have $\mathrm{L}_{\mathbb{M}} \subseteq \mathrm{L}_{\mathbb{M}_1} \times \mathrm{L}_{\mathbb{M}_2}$. Here, each two classical observables on $\mathbb{M}_1$ and $\mathbb{M}_2$ induce a classical observable on $\mathbb{M}$ *but not* vice versa.

Therefore, we sketch here the Weyl observables of Section 4.5 in [66] in Schrödinger-Feynman Quantization. As for Holomorphic Quantization, we consider linear field theory here. We represent a linear, classical, real observable $D_{\mathbb{M}}$ on a spacetime region $\mathbb{M}$ as a functional

$$D_{\mathbb{M}}:\;\; \mathrm{K}_{\mathbb{M}} \;\to\; \mathbb{R}.$$

A simple example for this is a linear interaction with some source field μ:

$$D^{\mu}_{\mathbb{M}}(\phi) \;=\; \int_{\mathbb{M}} \mathrm{d}x\; \mu(x)\, \phi(x).$$

For any $D_{\mathbb{M}}$, its associated classical Weyl observable is

$$W_{\mathbb{M}}:\;\; \mathrm{K}_{\mathbb{M}} \;\to\; \mathbb{C} \qquad\qquad W_{\mathbb{M}} \;:=\; \exp\,(\mathrm{i}D_{\mathbb{M}}). \tag{3.50}$$

(We shall always use $F_{\mathbb{M}} : \mathrm{K}_{\mathbb{M}} \to \mathbb{C}$ for general classical observables, $D_{\mathbb{M}}$ for such that are both linear and real, and $W_{\mathbb{M}}$ for the Weyl observables $\exp(\mathrm{i}D_{\mathbb{M}})$. However, we shall write the label $\mathbb{M}$ only when there is enough space to do so.) We consider the class of observables which consists in complex linear combinations of Weyl observables (these form an algebra). For later use, for an observable $F_{\mathbb{M}}$ and any interior solution $\xi \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$, we define the associated translated observable $F^{\xi}_{\mathbb{M}}$ through

$$F^{\xi}_{\mathbb{M}} : \ \mathrm{K}_{\mathbb{M}} \to \mathbb{C} \qquad\qquad F^{\xi}_{\mathbb{M}}(\phi) := F_{\mathbb{M}}(\phi+\xi). \tag{3.51}$$

We note that for a linear observable $D_{\mathbb{M}}$ the translated observable $D^{\xi}_{\mathbb{M}}$ generically is not linear any more due to $D^{\xi}_{\mathbb{M}}(\phi) = D_{\mathbb{M}}(\xi) + D_{\mathbb{M}}(\phi)$. By contrast, the Weyl observable associated to $D^{\xi}_{\mathbb{M}}$ (which is just the translated Weyl observable $W^{\xi}_{\mathbb{M}}$ of $D_{\mathbb{M}}$) is again in the algebra of Weyl observables due to

$$W^{\xi}_{\mathbb{M}}(\phi) = \underbrace{\mathrm{e}^{\mathrm{i}D(\xi)}}_{\in\,\mathbb{C}}\, W_{\mathbb{M}}(\phi).$$

We now quantize Weyl observables using SFQ, and start with the path integral for the SF amplitude

$$\rho_{\mathbb{M}}(\psi_{\partial\mathbb{M}}) = \int_{\mathrm{K}_{\mathbb{M}}} \mathrm{d}\mu(\phi)\, \mathrm{e}^{\mathrm{i}S_{\mathbb{M}}(\phi)} \psi_{\partial\mathbb{M}}(\phi|_{\partial\mathbb{M}}).$$

Therein, $S_{\mathbb{M}}(\phi)$ is the classical action, and $\mu(\phi)$ is a (fictitious) translation-invariant measure on $\mathrm{K}_{\mathbb{M}}$. For a classical observable $F_{\mathbb{M}} : \mathrm{K}_{\mathbb{M}} \to \mathbb{C}$, a natural modification of this amplitude is

$$\rho^{F}_{\mathbb{M}}(\psi_{\partial\mathbb{M}}) := \int_{\mathrm{K}_{\mathbb{M}}} \mathrm{d}\mu(\phi)\, \mathrm{e}^{\mathrm{i}S_{\mathbb{M}}(\phi)} \psi_{\partial\mathbb{M}}(\phi|_{\partial\mathbb{M}})\, F_{\mathbb{M}}(\phi). \tag{3.52}$$

This is the Schrödinger-Feynman quantization of $F_{\mathbb{M}}$ in Section 4.1 of [60] and Section 4.5 of [66]. As in Holomorphic Quantization, this integral can be evaluated for coherent states[4]: For observables $F_{\mathbb{M}}$ which are complex linear combinations of Weyl observables, Proposition (4.4) in [66] proves the Coherent Factorization Property (CFP):

$$\rho^{F}_{\mathbb{M}}(K^{\xi}_{\partial\mathbb{M}}) = \rho_{\mathbb{M}}(K^{\xi}_{\partial\mathbb{M}}) \cdot \rho^{F^{\hat{\xi}}}_{\mathbb{M}}(K^{0}_{\partial\mathbb{M}}) \qquad\qquad \forall\, \xi \in \mathrm{L}_{\partial\mathbb{M}}. \tag{3.53}$$

Therein, $\hat{\xi} := \xi^{\mathrm{R}} - \mathrm{i}\xi^{\mathrm{I}}$ as in (3.25), with $\xi = \xi^{\mathrm{R}} + J_{\partial\mathbb{M}}\xi^{\mathrm{I}}$ wherein $\xi^{\mathrm{R}}, \xi^{\mathrm{I}} \in \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$. The coherent state $K^{0}_{\partial\mathbb{M}}$ is the simplest choice of a vacuum on $\partial\mathbb{M}$, as discussed in Section 3.1.6. Next, we cite two results, which are necessary in order to show that the Schrödinger-Feynman Quantization of Weyl observables fulfills the Observable Axioms of Section 1.2.6. First, for Weyl observables the Coherent Factorization Property (3.53) becomes more explicit, as proven in Proposition (4.3) of [66]:

$$\rho^{W}_{\mathbb{M}}(K^{\xi}_{\partial\mathbb{M}}) = \rho_{\mathbb{M}}(K^{\xi}_{\partial\mathbb{M}}) \cdot \rho^{W}_{\mathbb{M}}(K^{0}_{\partial\mathbb{M}}) \cdot W(\hat{\xi}) \qquad\qquad \forall\, \xi \in \mathrm{L}_{\partial\mathbb{M}}, \tag{3.54}$$

with further

$$\rho^{W}_{\mathbb{M}}(K^{0}_{\partial\mathbb{M}}) = \exp\left(\tfrac{\mathrm{i}}{2} D_{\mathbb{M}}(\eta_D) - \tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}}(\eta_D, \eta_D)\right) \tag{3.55}$$

wherein the special boundary solution $\eta_D \in J_{\partial\mathbb{M}}\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ is uniquely determined. Note that $\eta_D \in J_{\partial\mathbb{M}}\mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$ implies that $\eta_D \notin \mathrm{L}^{\mathrm{int}}_{\partial\mathbb{M}}$, saying that it is a boundary solution, never an interior solution.

Now let us see how these results relate to the Observable Axioms. (O1) becomes satisfied by taking as observable maps the complex linear combinations of Weyl observables. Then, (O2a) can be seen to hold by setting

$$\rho^{W_1}_{\mathbb{M}_1} \diamond \rho^{W_2}_{\mathbb{M}_2} := \rho^{W_1 \otimes W_2}_{\mathbb{M}}. \tag{3.56}$$

[4] In the usual way of writing the Schrödinger representation, coherent states are characterized not by solutions near a hypersurface Σ, but by field configurations on Σ. However, this representation can be formulated in a way, which allows characterization by solutions, because the formulation is sensitive only to the solutions' configuration on Σ.

(O2a) is then met: using (3.54) from Proposition (4.3) and Core Axiom (T5a), we get

$$\begin{aligned}\Big(\rho_{\mathbb{M}_1}^{W_1}\diamond\rho_{\mathbb{M}_2}^{W_2}\Big)(K_{\partial\mathbb{M}_1}^{\xi_1}\otimes K_{\partial\mathbb{M}_2}^{\xi_2}) &= \rho_{\mathbb{M}}^{W_1\otimes W_2}(K_{\partial\mathbb{M}_1}^{\xi_1}\otimes K_{\partial\mathbb{M}_2}^{\xi_2})\\ &= \rho_{\mathbb{M}}(K_{\partial\mathbb{M}_1}^{\xi_1}\otimes K_{\partial\mathbb{M}_2}^{\xi_2})\cdot\rho_{\mathbb{M}}^{W_1\otimes W_2}(K_{\partial\mathbb{M}}^{0})\cdot\big(W_1\otimes W_2\big)(\hat{\xi}_1,\hat{\xi}_2)\\ &= \rho_{\mathbb{M}_1}(K_{\partial\mathbb{M}_1}^{\xi_1})\cdot\rho_{\mathbb{M}_2}(K_{\partial\mathbb{M}_2}^{\xi_2})\cdot W_1(\hat{\xi}_1)\cdot W_2(\hat{\xi}_2)\cdot\\ &\quad\cdot\exp\Big(\tfrac{\mathrm{i}}{2}D_1(\eta_{D_1})-\tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}_1}(\eta_{D_1},\eta_{D_1})+\tfrac{\mathrm{i}}{2}D_2(\eta_{D_2})-\tfrac{1}{2}\mathrm{g}_{\partial\mathbb{M}_2}(\eta_{D_2},\eta_{D_2})\Big)\\ &= \rho_{\mathbb{M}_1}^{W_1}(K_{\partial\mathbb{M}_1}^{\xi_1})\cdot\rho_{\mathbb{M}_2}^{W_2}(K_{\partial\mathbb{M}_2}^{\xi_2}).\end{aligned}$$

For Axiom (O2b), we summarize Proposition (4.2) in [66]. It treats the gluing of observables in the setting of Core Axiom (T5b): $\mathbb{M}_G$ is a region with its boundary decomposing as a disjoint union $\partial\mathbb{M}_G = \Sigma_N\cup\Sigma_G\cup\overline{\Sigma'_G}$, where Σ'_G is a copy of Σ_G. Then, $\mathbb{M}_N$ denotes the region resulting from gluing $\mathbb{M}_G$ with itself along Σ_G and $\overline{\Sigma'_G}$, with $\partial\mathbb{M}_N = \Sigma_N$.

We note, that $\mathrm{K}_{\mathbb{M}_N}\subseteq\mathrm{K}_{\mathbb{M}_G}$, since the possible configurations on the glued region $\mathbb{M}_N$ are just those configurations of the original region $\mathbb{M}_G$, which coincide on the gluing hypersurfaces Σ_G and Σ'_G. Therefore, any observable F_G on $\mathbb{M}_G$ induces an observable F_N on $\mathbb{M}_N$ through

$$F_N(\phi_N) := F_G(\phi_N) \qquad \forall\,\phi_N\in\mathrm{K}_{\mathbb{M}_N}.$$

Therein, on the right hand side ϕ_N is viewed as a configuration on $\mathbb{M}_G$ which coincides on Σ_G and Σ'_G. We now consider the two linear observables

$$D_G:\ \mathrm{K}_{\mathbb{M}_G}\to\mathbb{R} \qquad\qquad D_N:\ \mathrm{K}_{\mathbb{M}_N}\to\mathbb{R},$$

wherein D_N is induced by D_G. The associated Weyl observables are

$$W_G = \exp(\mathrm{i}D_G) \qquad\qquad W_N = \exp(\mathrm{i}D_N).$$

We now rewrite (1.36) as in Section 4.4 of [59] and Proposition (4.2) of [66]. That is, instead of summing over an orthonormal basis $\{\xi_G^i\}$ of $\mathcal{H}_{\Sigma_G}$, we write an integral over coherent states as in the completeness relation (3.3). Further, since the coherent states generate a dense subset, we only consider the axiom for these. In Proposition (4.2) the following rewritten version of (O2b) is proven (generalizing Theorem (4.5) of [59]):

$$\begin{aligned}\big(\diamond_{\Sigma_G}\rho_{\mathbb{M}_G}^{W_G}\big)(K_{\Sigma_N}^{\phi})\cdot c(\mathbb{M}_G,\Sigma_G,\overline{\Sigma'_G}) &\equiv \rho_{\mathbb{M}_N}^{W_N}(K_{\Sigma_N}^{\phi})\cdot c(\mathbb{M}_G,\Sigma_G,\overline{\Sigma'_G})\\ &= \int_{\mathrm{L}_{\Sigma_G}}\mathrm{d}\nu_{\Sigma_G}(\xi)\ \rho_{\mathbb{M}_G}^{W_G}(K_{\Sigma_N}^{\phi}\otimes K_{\Sigma_G}^{\xi}\otimes\iota_{\Sigma_G}K_{\Sigma_G}^{\xi})/(\mathcal{N}_{\Sigma_G}^{\xi})^2.\end{aligned} \tag{3.57}$$

We have thus sketched how (the algebra of complex linear combinations of) Weyl observables realizes the Observable Axioms of the GBF. Weyl observables are simple enough for allowing us to obtain an explicit formula for their quantization. However, they also generate a much larger class of observables via functional differentiation as described in detail in Section 4.9 of [66]. This is justified by the linearity of holomorphic quantization, which lets it commute with the functional derivative. The relation

$$D = -\mathrm{i}\,\tfrac{\partial}{\partial\lambda}\,\exp(\mathrm{i}\lambda D)|_{\lambda=0},$$

for $F := \exp(\mathrm{i}\lambda D)$ gives us

$$\rho_{\mathbb{M}}^{D} = -\mathrm{i}\,\tfrac{\partial}{\partial\lambda}\,\rho_{\mathbb{M}}^{F}|_{\lambda=0}.$$

This can be evaluated using (3.54) from Proposition (4.7). In Section 4.9 of [66], corresponding formulas are derived for products of polynomials of linear observables with Weyl observables. As a beautiful detail, let us mention here only that if on a time-interval region on Minkowski spacetime we take as observable the two-point-function D_1D_2 which results as the product of $D_1(\phi) := \phi(x_1)$ with $D_2(\phi) := \phi(x_2)$, then the corresponding observable map recovers precisely the Feynman propagator $-\mathrm{i}G_F(x_1,x_2)$.

3.1.8 Isometries and complex structure

As recalled in Section 3.1.2, a key ingredient in the quantization of a classical field theory is the complex structure J, see e.g. [4]. It has to be compatible with the symplectic structure encoding classical dynamics: $\omega(\eta,\zeta)\overset{!}{=}\omega(J\eta,\,J\zeta)$, yielding the inner product (2.4) on the space L of classical solutions, which in turn determines the Fock space of states.

To make spacetime symmetries of the classical theory also symmetries of the quantum theory, the complex structure should be invariant under these. In Section 1.2.4 we discuss the action of spacetime transformations k on regions, hypersurfaces and the associated objects in Quantum Theory. For Minkowski and AdS the relevant transformations are their respective spacetime isometries. Axiom (SG5) requires the amplitude of a region to remain invariant under their actions. For the amplitudes (3.28) of Holomorphic Quantization this can be achieved as follows. The holomorphic amplitudes are determined by the real g-product and by the decomposition (3.24). The former is determined by the symplectic structure $\omega_{\partial\mathbb{M}}$ and the complex structure $J_{\partial\mathbb{M}}$, and the latter only by the complex structure. Hence invariance of the holomorphic amplitudes is assured by invariance of $\mathrm{g}_{\partial\mathbb{M}}$, which in turn depends on two conditions:

(1) that the symplectic structure is k-invariant: $\quad \omega_{k\triangleright\partial\mathbb{M}}(k\triangleright\eta, k\triangleright\zeta) \;=\; \omega_{\partial\mathbb{M}}(\eta,\zeta), \qquad$ and

(2) that the complex structure commutes with k: $\quad J_{k\triangleright\partial\mathbb{M}}\,(k\triangleright\phi) \;=\; k\triangleright(J_{\partial\mathbb{M}}\,\phi).$

With these conditions holding, we get a k-invariant $\mathrm{g}_{\partial\mathbb{M}}$:

$$\begin{aligned}\mathrm{g}_{k\triangleright\partial\mathbb{M}}(k\triangleright\eta, k\triangleright\zeta) \;&=\; \omega_{k\triangleright\partial\mathbb{M}}\big(k\triangleright\eta, J_{k\triangleright\partial\mathbb{M}}(k\triangleright\zeta)\big) \;=\; \omega_{k\triangleright\partial\mathbb{M}}\big(k\triangleright\eta, k\triangleright(J_{\partial\mathbb{M}}\zeta)\big) \;=\; \omega_{\partial\mathbb{M}}(\eta, J_{\partial\mathbb{M}}\zeta)\\ &=\; \mathrm{g}_{\partial\mathbb{M}}(\eta,\zeta)\,.\end{aligned}$$

Hence we will need to verify whether these two conditions are fulfilled. In the previous chapter, Condition (1) is studied for Minkowski spacetime in Section 2.5.5, and for AdS in Section 2.6.7. The present chapter deals with condition (2). In Minkowski spacetime with its isometry group of Poincaré transformations, the complex structure J_t in (3.74) on an equal-time hypersurface Σ_t fulfills this requirement, and is essentially uniquely determined by it. For spacelike hypersurfaces in more general curved spacetimes the situation is more complicated [4].

In the standard approach, the space L_t (of classical solutions well defined near Σ_t) where the complex structure lives, is thought of as a space of global solutions. Since isometries map global solutions to global solutions, it is then clear how isometries act and what isometry invariance of the complex structure means. In contrast, in the GBF the complex structure is seen as intrinsically associated to the hypersurface and to solutions in its neighborhood. However, not all isometries "preserve neighborhoods", that is, map solutions in a neighborhood of some hypersurface Σ to solutions that again are well defined on that neighborhood. Thus, in the GBF there is a straightforward action only for isometries that map the hypersurface to itself. If we restrict to infinitesimal isometries, however, the fact that solutions are defined not only on the hypersurface itself, but on a neighborhood, is enough to make their actions well defined. (In this way the isometry invariance of symplectic structures on hypersurfaces on AdS was understood in [30].)

The complex structure J_r in (3.77) for Klein-Gordon theory on the hypercylinder in Minkowski spacetime has this same essential uniqueness property, as exhibited implicitly in [53] and explicitly in [59] for propagating solutions. This gives additional motivation for pursuing the same isometry invariance criterion for selecting reasonable complex structures on AdS spacetime. For the AdS time-interval region, this is done in Section 3.3.1, and for the AdS rod region in Section 3.3.2.

3.1.9 Generalized S-Matrices for Minkowski and AdS spacetimes

We are interested here in the particular case of spacetime regions extended to infinity to cover all of spacetime. In this case the interacting theory can be described perturbatively through formula (3.22). As recalled above, the usual S-matrix in Minkowski space is obtained by taking a time-interval region $[t_1, t_2]$ and sending the boundaries to infinity, $t_1 \to -\infty$ and $t_2 \to +\infty$. However, this is not the only possibility for covering all of spacetime. A particularly compelling setup is the rod region $\mathbb{M}_r = \mathbb{R}_t \times \mathbb{B}^d_r$. Physically, we are considering an experiment that is spatially confined, but may run continuously. We are injecting and detecting particles from a distance r from the center, but at any time. The asymptotic idealization is achieved by letting r go to infinity, moving out from the interaction region, where it is well justified to consider particles as free. One might even argue, that actually this setup is more physically compelling than the usual asymptotics in time. It was shown in [22] for Klein-Gordon theory on Minkowski spacetime, that the resulting asymptotic amplitude is in fact precisely equivalent to the usual S-matrix.

Physically, the equivalence is based on a correspondence between asymptotic classical solutions. In the standard S-matrix setting, these solutions are pairs of global solutions, one at early times and one at late times. We have already identified this space of solutions as $\mathrm{L}_{\partial T} = \mathrm{L}_1 \oplus \mathrm{L}_2$, (but note the slight change of notation). In the hypercylinder setting, let L_r the space of solutions in a neighborhood of the hypercylinder Σ_r. Then, the asymptotic solutions live in the space $\mathrm{L}_{\partial R}$, arising as the limit of L_r, when r goes to infinity. In this case there is a subtlety. In addition to the usual propagating solutions, L_r contains evanescent solutions (that is, solutions showing exponential behavior in space, and thus well-defined in a neighborhood of the hypercylinder with finite radius r, while diverging for $r \to \infty$). The evanescent solutions are absent, however, in $\mathrm{L}_{\partial R}$. Taking this into account, there is a one-to-one correspondence between the elements of $\mathrm{L}_{\partial R}$ and those of $\mathrm{L}_{\partial T}$. More precisely, there is an equivalence between $\mathrm{L}_{\partial R}$ and $\mathrm{L}_{\partial T}$ *as symplectic vector spaces*. For this equivalence to survive quantization, we need to choose corresponding complex structures on $\mathrm{L}_{\partial R}$ and $\mathrm{L}_{\partial T}$. Then $\mathrm{L}_{\partial R}$ and $\mathrm{L}_{\partial T}$ are equivalent *as complex Hilbert (or Krein) spaces*. Consequently, the state spaces of the quantum theory, i.e., the Fock spaces over $\mathrm{L}_{\partial R}$ and $\mathrm{L}_{\partial T}$ will also be equivalent as complex Hilbert (or Krein) spaces. What is more, as a consequence of the classical correspondence, the amplitudes will be the same, without and with sources. For later use, we refer to this equivalence as *amplitude equivalence*. For examples of amplitude equivalence in curved space times, see [17, 18, 19, 24].

The availability of different asymptotic regimes in a given spacetime becomes particularly interesting when they are inequivalent. This is the case for AdS spacetime. As is well known, the conventional S-matrix approach fails due to the lack of temporally asymptotically free states. From the present perspective this manifests itself as follows. The space L_t of admissible solutions in a neighborhood of the equal-time hypersurface at time t is rather small and admits only discrete energy levels. There is a larger continuum of solutions, but these do not decay sufficiently fast at spatial infinity to be normalizable. The negative curvature of AdS makes the solutions behave akin to being in a box potential: only those solutions that vanish at radial infinity (the "wall of the box") are normalizable.

On the other hand, it was shown in [30] that a hypercylinder geometry leads to a very different picture. The space of admissible solutions L_ρ in a neighborhood of the hypercylinder of radius ρ contains a full continuum of solutions. This suggests to build the physical S-matrix in AdS on the asymptotic hypercylinder geometry rather than the conventional asymptotic time-interval geometry [20]. The requisite classical theory was developed in Chapter 2. The main ingredient for quantization is the complex structure, which is the main focus of the present chapter whose results are published in [31].

3.1.10 Minkowski limit and amplitude equivalence

Overview

Since QFT in Minkowski spacetime is much better understood than in curved spacetime, we shall make use of the fact that Minkowski spacetime arises from AdS in a flat limit, in the sense of Section 2.6.2. Concretely, we shall require that the flat limit of the AdS amplitudes reproduces the respective Minkowski amplitudes. From (3.28) and (3.33) it is easy to see that this holds, if for Klein-Gordon solutions η, ζ on AdS the flat limit of the real product $\mathrm{g}^{\mathrm{AdS}}(\eta, \zeta)$ is the same as the Minkowski real product $\mathrm{g}^{\mathrm{Mink}}(\tilde{\eta}, \tilde{\zeta})$ of the solutions' flat limits $\tilde{\eta}, \tilde{\zeta}$. In turn, this holds if the limit of the AdS complex structure is the Minkowski complex structure for the relevant class of hypersurfaces, as sketched in diagrams (3.85) and (3.103). This turns out to make sense for both equal-time hypersurfaces and hypercylinders, yielding a limit both on rod regions and on time-interval regions, which we work out in Sections 3.3.1 and 3.3.6 respectively.

As mentioned in Section 3.1.9, the standard asymptotic time-interval geometry and the asymptotic rod geometry lead to equivalent amplitudes in Minkowski spacetime [21, 22]. On the other hand, the corresponding geometries are inequivalent in AdS. In particular, the relevant asymptotic solution spaces are rather different. The space $\mathrm{L}_\rho^{\mathrm{AdS}}$ of solutions on the boundary hypercylinder of a rod region is continuous, while the space $\mathrm{L}_{t_1}^{\mathrm{AdS}} \oplus \mathrm{L}_{t_2}^{\mathrm{AdS}}$ of solutions on the boundary of a time-interval is much smaller and discrete. In particular, there can be no amplitude equivalence. However, it turns out that there is a suitable embedding of the discrete space $\mathrm{L}_{t_1}^{\mathrm{AdS}} \oplus \mathrm{L}_{t_2}^{\mathrm{AdS}}$ into the continuous one $\mathrm{L}_\rho^{\mathrm{AdS}}$. This allows for a weak form of amplitude equivalence, which we work out in Section 3.3.3. What is more, this *weak amplitude equivalence* can in the flat limit be brought into congruence with the (strong) amplitude equivalence in Minkowski spacetime. The ensuing relations between amplitudes (and thus complex structures) are illustrated in Figure 3.1.

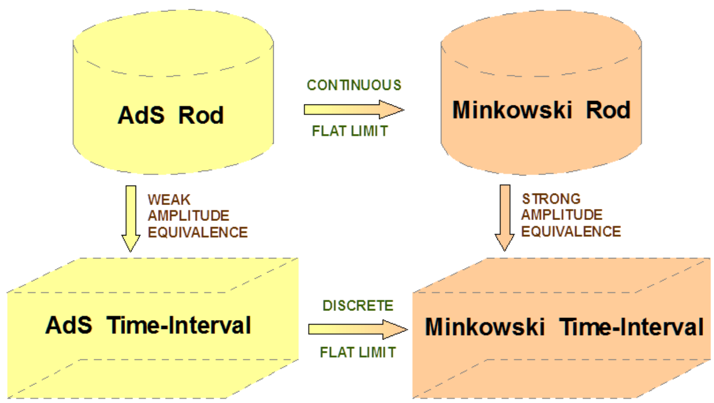


Figure 3.1: Amplitude relations for time-interval and rod regions on AdS and Minkowski spacetime.

Amplitude equivalence

We proceed to explain in more detail how amplitude equivalence relates complex structures. Suppose we have two spacetime regions $\mathbb{M}$, $\mathbb{N}$ to describe the same physics in the interior of both. (This is illustrated for the example of $\mathbb{M}$ being a time-interval and $\mathbb{N}$ a rod region in Figure 3.2.)

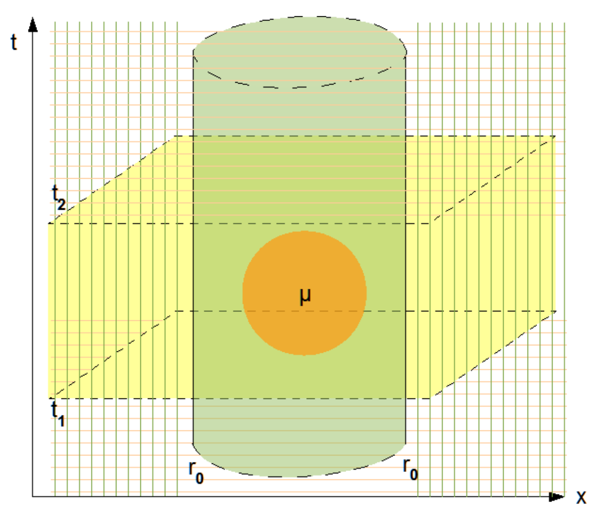


Figure 3.2: Time-interval and rod regions with source μ.

In general terms, an amplitude equivalence is then a map $A: \mathcal{H}_{\partial\mathbb{M}} \to \mathcal{H}_{\partial\mathbb{N}}$ which is defined on some subspace of $\mathcal{H}_{\partial\mathbb{M}}$ and fulfills $\rho_{\mathbb{N}}(A\psi_{\partial\mathbb{M}}) = \rho_{\mathbb{M}}(\psi_{\partial\mathbb{M}})$ for all states $\psi_{\partial\mathbb{M}}$ in this subspace. The properties of amplitude equivalence maps (domain, bijectivity, isomorphism, linearity) depend on the theory under consideration, the shape of the regions and the geometry of spacetime. For example, we expect linearity properties to depend on the number of boundary components of the regions.

Comparing solutions outside the regions (or asymptotically) should yield an equivalence E between the boundary solution spaces $\mathrm{L}_{\partial\mathbb{M}}$ and $\mathrm{L}_{\partial\mathbb{N}}$, for which we write $E: \mathrm{L}_{\partial\mathbb{M}} \to \mathrm{L}_{\partial\mathbb{N}}$. For "outside" solutions η, ζ, the symplectic structures on the boundaries should be the same under this equivalence:

$$\omega_{\partial\mathbb{M}}(\eta,\zeta) = \omega_{\partial\mathbb{N}}\big(E(\eta), E(\zeta)\big). \tag{3.58}$$

While this equality follows straightforwardly from Lagrangian field theory for compact regions, the case of non-compact regions is less trivial. The free holomorphic amplitude (from which the interaction amplitudes can be obtained) is determined by the real inner product g. To obtain equality of the real g-products on $\partial\mathbb{M}$ and $\partial\mathbb{N}$ under the equivalence, that is:

$$\mathrm{g}_{\partial\mathbb{M}}(\eta,\zeta) \;=\; \mathrm{g}_{\partial\mathbb{N}}\big(E(\eta), E(\zeta)\big), \tag{3.59}$$

we require in addition for all outside solutions $\eta, \zeta \in \mathrm{L}_{\partial\mathbb{M}}$

$$\begin{aligned} \mathrm{g}_{\partial\mathbb{M}}(\eta,\zeta) \;=\; \omega_{\partial\mathbb{M}}(\eta, J_{\partial\mathbb{M}}\zeta) &\overset{(3.58)}{=} \omega_{\partial\mathbb{N}}\big(E(\eta), E(J_{\partial\mathbb{M}}\zeta)\big) \\ &\overset{!}{=} \omega_{\partial\mathbb{N}}\big(E(\eta), J_{\partial\mathbb{N}}E(\zeta)\big) \;=\; \mathrm{g}_{\partial\mathbb{N}}\big(E(\eta), E(\zeta)\big). \end{aligned} \tag{3.60}$$

This means that the complex structures on $\mathrm{L}_{\partial\mathbb{M}}$ and $\mathrm{L}_{\partial\mathbb{N}}$ need to be related by the same map between $\mathrm{L}_{\partial\mathbb{M}}$ and $\mathrm{L}_{\partial\mathbb{N}}$ that establishes the equivalence. That is, for all outside solutions $\zeta \in \mathrm{L}_{\partial\mathbb{M}}$ we need

$$E(J_{\partial\mathbb{M}}\zeta) \;\overset{!}{=}\; J_{\partial\mathbb{N}}E(\zeta). \tag{3.61}$$

A trouble with this setting is that the decompositions (3.62) of the spaces $\mathrm{L}_{\partial\mathbb{M}}$ and $\mathrm{L}_{\partial\mathbb{N}}$ depend themselves on the choices of complex structures. This is so because $\mathrm{L}_{\partial\mathbb{M}}$ and $\mathrm{L}_{\partial\mathbb{N}}$ are supposed to be Hilbert (or Krein) spaces with their inner products. Thus, the complex structure itself determines to some extent what is the nature of the elements in $\mathrm{L}_{\partial\mathbb{M}}$ and $\mathrm{L}_{\partial\mathbb{N}}$. Usually, these are some kind of L^2 spaces, i.e., the elements are equivalence classes of square integrable functions.

On the other hand, the spaces $\mathrm{L}_{\mathbb{M}}$ and $\mathrm{L}_{\mathbb{N}}$ of free solutions in the interior of $\mathbb{M}$ and of $\mathbb{N}$, should coincide by assumption[5]: $\mathrm{L}_{\mathbb{M}} = \mathrm{L}_{\mathbb{N}}$. Hence, here we do not need to write explicitly an equivalence map E, since here E is the identity map. Moreover, these must give rise to Lagrangian subspaces of $\mathrm{L}_{\partial\mathbb{M}}$ and $\mathrm{L}_{\partial\mathbb{N}}$, see Classical Axiom (C5). We also recall the following decomposition from Section 2.2

$$\mathrm{L}_{\partial\mathbb{M}} = (r_{\mathbb{M}}\mathrm{L}_{\mathbb{M}}) \oplus J_{\partial\mathbb{M}}\,(r_{\mathbb{M}}\mathrm{L}_{\mathbb{M}}), \tag{3.62}$$

and the corresponding one for $\mathbb{N}$. We thus see, that it is sufficient to require the equality (3.60) for interior solutions, that is, elements of $\mathrm{L}_{\mathbb{M}} = \mathrm{L}_{\mathbb{N}}$:

$$\omega_{\partial\mathbb{M}}(\eta, J_{\partial\mathbb{M}}\zeta) = \omega_{\partial\mathbb{N}}(\eta, J_{\partial\mathbb{N}}\zeta) \qquad\qquad \forall\, \eta,\zeta \in \mathrm{L}_{\mathbb{M}} = \mathrm{L}_{\mathbb{N}}. \tag{3.63}$$

[5]The simplest example for this is real Klein-Gordon theory with mass parameter m on Minkowski spacetime, with $\mathbb{M}$ a time-interval region and $\mathbb{N}$ a rod region. The solutions on the interior of the time-interval are plane waves with energy $E^2 > m^2$. Their spatial part can be rewritten as spherical Bessel function times spherical harmonics, conserving the energy $E^2 > m^2$. The inner solutions of the rod region are precisely these "Bessel modes" (while the "Neumann" modes are only bounded near the rod's boundary but nut on the whole interior). However, there are additional inner solutions on the interior of the rod region called "modified Bessel modes" with $E^2 < m^2$ (again, the "modified Neumann modes" are not bounded on the whole interior). In order to describe the same physics in both regions and have $\mathrm{L}_{\mathbb{M}} = \mathrm{L}_{\mathbb{N}}$, we need to exclude the modified Bessel modes.

For the example of Figure 3.2 this is precisely condition (3.73). Then, with $\eta^{\mathrm{R,I}}, \zeta^{\mathrm{R,I}} \in \mathrm{L}_{\mathbb{M}} = \mathrm{L}_{\mathbb{N}}$ interior solutions, we let

$$\begin{aligned}
\eta_{\partial\mathbb{M}} &= \eta^{\mathrm{R}} + J_{\partial\mathbb{M}}\eta^{\mathrm{I}} \\
\eta_{\partial\mathbb{N}} &= E(\eta_{\partial\mathbb{M}}) = E(\eta^{\mathrm{R}} + J_{\partial\mathbb{M}}\eta^{\mathrm{I}}) = E(\eta^{\mathrm{R}}) + E(J_{\partial\mathbb{M}}\eta^{\mathrm{I}}) = \eta^{\mathrm{R}} + J_{\partial\mathbb{N}}E(\eta^{\mathrm{I}}) \\
&= \eta^{\mathrm{R}} + J_{\partial\mathbb{N}}\eta^{\mathrm{I}} \\
\zeta_{\partial\mathbb{M}} &= \zeta^{\mathrm{R}} + J_{\partial\mathbb{M}}\zeta^{\mathrm{I}} \\
\zeta_{\partial\mathbb{N}} &= E(\zeta_{\partial\mathbb{M}}) \\
&= \zeta^{\mathrm{R}} + J_{\partial\mathbb{N}}\zeta^{\mathrm{I}},
\end{aligned} \tag{3.64}$$

with thus $\eta_{\partial\mathbb{M}}, \zeta_{\partial\mathbb{M}} \in \mathrm{L}_{\partial\mathbb{M}}$ and $\eta_{\partial\mathbb{N}}, \zeta_{\partial\mathbb{N}} \in \mathrm{L}_{\partial\mathbb{N}}$ solutions near the respective boundaries. We then get the following real g-products for the boundary solutions:

$$\begin{aligned}
\mathrm{g}_{\partial\mathbb{M}}(\eta_{\partial\mathbb{M}},\, \zeta_{\partial\mathbb{M}}) &= \omega_{\partial\mathbb{M}}(\eta_{\partial\mathbb{M}},\, J_{\partial\mathbb{M}}\zeta_{\partial\mathbb{M}}) \\
&= \omega_{\partial\mathbb{M}}(\eta^{\mathrm{R}}+J_{\partial\mathbb{M}}\eta^{\mathrm{I}},\, J_{\partial\mathbb{M}}\zeta^{\mathrm{R}}-\zeta^{\mathrm{I}}) \\
&= \omega_{\partial\mathbb{M}}(\eta^{\mathrm{R}}, J_{\partial\mathbb{M}}\zeta^{\mathrm{R}}) - \underbrace{\omega_{\partial\mathbb{M}}(\eta^{\mathrm{R}}, \zeta^{\mathrm{I}})}_{0} + \underbrace{\omega_{\partial\mathbb{M}}(\eta^{\mathrm{I}}, \zeta^{\mathrm{R}})}_{0} - \omega_{\partial\mathbb{M}}(J_{\partial\mathbb{M}}\eta^{\mathrm{I}}, \zeta^{\mathrm{I}}) \\
&= \omega_{\partial\mathbb{M}}(\eta^{\mathrm{R}}, J_{\partial\mathbb{M}}\zeta^{\mathrm{R}}) + \omega_{\partial\mathbb{M}}(\eta^{\mathrm{I}}, J_{\partial\mathbb{M}}\zeta^{\mathrm{I}}) \\
&\overset{(3.63)}{=} \omega_{\partial\mathbb{N}}(\eta^{\mathrm{R}}, J_{\partial\mathbb{N}}\zeta^{\mathrm{R}}) + \omega_{\partial\mathbb{N}}(\eta^{\mathrm{I}}, J_{\partial\mathbb{N}}\zeta^{\mathrm{I}}) \\
&= \omega_{\partial\mathbb{N}}(\eta_{\partial\mathbb{N}},\, J_{\partial\mathbb{N}}\zeta_{\partial\mathbb{N}}) \\
&= \mathrm{g}_{\partial\mathbb{N}}(\eta_{\partial\mathbb{N}},\, \zeta_{\partial\mathbb{N}}).
\end{aligned}$$

That is, requirement (3.59) is satisfied for arbitrary boundary solutions $\eta_{\partial\mathbb{M}}, \zeta_{\partial\mathbb{M}}$:

$$\mathrm{g}_{\partial\mathbb{M}}(\eta_{\partial\mathbb{M}}, \zeta_{\partial\mathbb{M}}) = \mathrm{g}_{\partial\mathbb{N}}\big(E(\eta_{\partial\mathbb{M}}), E(\zeta_{\partial\mathbb{M}})\big). \tag{3.65}$$

Suppose in particular that the complex structure on $\mathrm{L}_{\partial\mathbb{M}}$ is given, and we wish to construct an equivalent one on $\mathrm{L}_{\partial\mathbb{N}}$. Once we have chosen a complement of $r_{\mathbb{N}}(\mathrm{L}_{\mathbb{N}})$ in $\mathrm{L}_{\partial\mathbb{N}}$, this equivalent complex structure on $\mathrm{L}_{\partial\mathbb{N}}$ is completely determined by equation (3.60) respectively (3.61) on $\mathrm{L}_{\mathbb{M}} = \mathrm{L}_{\mathbb{N}}$.

Amplitude equivalence for tube and rod regions

As a first example for amplitude equivalence, let us consider tube and rod regions. Since these regions appear naturally in the same situations, we now ask: For which states do the amplitudes of these regions coincide? We recall that the calculations for tube regions are identical to those of time-interval regions if we replace the foliation parameter t by the radius r. In the following we study situations where the complex structure J_r and the symplectic structure $\omega_r(\cdot,\cdot)$ associated to a hypercylinder Σ_r are actually independent of the radius. Nevertheless we keep the label r in order to remind us that they refer to hypercylinders. Let us consider a tube region $\mathbb{M}_{[r_1,r_2]}$ with boundary hypercylinders $\Sigma_1 \sqcup \overline{\Sigma}_2$. For two solutions η_1 near Σ_1 and ζ_2 near $\overline{\Sigma}_2$, the amplitude of the associated coherent boundary state $K^{\eta_1}_{\Sigma_1} \otimes \overline{K^{\zeta_2}_{\Sigma_2}}$ is then given by (3.31):

$$\rho_{[r_1,r_2]}\Big(K^{\eta_1}_{\Sigma_1} \otimes \overline{K^{\zeta_2}_{\Sigma_2}}\Big) \;=\; \exp\Big(-\mathrm{g}_r(\xi^{\mathrm{I}}_{12},\, \xi^{\mathrm{I}}_{12}) - \mathrm{i}\mathrm{g}_r(\xi^{\mathrm{R}}_{12},\, \xi^{\mathrm{I}}_{12})\Big). \tag{3.66}$$

The question is now: Which boundary state $\overline{K^{\xi_0}_{\Sigma_0}}$ of a rod region $\mathbb{M}_{r_0}$ gives rise to the same amplitude? to this end we compare the above tube amplitude (3.66) to the rod amplitude (3.32):

$$\begin{aligned}
\rho_{r_0}\big(\overline{K^{\xi_0}_{\Sigma_0}}\big) \;&=\; \exp\Big(-\tfrac{1}{2}\mathrm{g}_r(\xi^{\mathrm{I}}_0,\, \xi^{\mathrm{I}}_0) - \tfrac{\mathrm{i}}{2}\mathrm{g}_r(\xi^{\mathrm{R}}_0,\, \xi^{\mathrm{I}}_0)\Big) \qquad (3.67)\\
\overset{!}{=}\; \rho_{[r_1,r_2]}\big(K^{\eta_1}_{\Sigma_1} \otimes \overline{K^{\zeta_2}_{\Sigma_2}}\big) \;&=\; \exp\Big(-\mathrm{g}_r(\xi^{\mathrm{I}}_{12},\, \xi^{\mathrm{I}}_{12}) - \mathrm{i}\mathrm{g}_r(\xi^{\mathrm{R}}_{12},\, \xi^{\mathrm{I}}_{12})\Big).
\end{aligned}$$

The amplitudes coincide precisely if both real and imaginary parts of the exponent agree for tube and rod amplitude. Since all ξ involved are real, this means that we need

$$\tfrac{1}{2}\mathrm{g}_r(\xi_0^{\mathrm{I}},\,\xi_0^{\mathrm{I}}) \stackrel{!}{=} \mathrm{g}_r(\xi_{12}^{\mathrm{I}},\,\xi_{12}^{\mathrm{I}}) \qquad\qquad \tfrac{1}{2}\mathrm{g}_r(\xi_0^{\mathrm{R}},\,\xi_0^{\mathrm{I}}) \stackrel{!}{=} \mathrm{g}_r(\xi_{12}^{\mathrm{R}},\,\xi_{12}^{\mathrm{I}}).$$

Therefore, for any given $(\eta_1,\,\zeta_2)$ respectively $(\xi_{12}^{\mathrm{R}},\,\xi_{12}^{\mathrm{I}})$ such that $\mathrm{g}_r(\xi_{12}^{\mathrm{I}},\xi_{12}^{\mathrm{I}}) \neq 0$ there is an infinite number of $(\xi_0^{\mathrm{R}},\,\xi_0^{\mathrm{I}})$ that makes the amplitudes coincide. We can construct them by starting from two arbitrary solutions $(\tilde{\xi}_0^{\mathrm{R}},\,\tilde{\xi}_0^{\mathrm{I}})$ fulfilling two requirements. First, they must be well defined on the whole rod region. Second, we need $\mathrm{g}_r(\tilde{\xi}_0^{\mathrm{I}},\tilde{\xi}_0^{\mathrm{I}}) \neq 0$ and $\mathrm{g}_r(\tilde{\xi}_0^{\mathrm{R}},\tilde{\xi}_0^{\mathrm{I}}) \neq 0$. Then, rescaling the two solutions

$$\xi_0^{\mathrm{I}} \;=\; \tilde{\xi}_0^{\mathrm{I}}\sqrt{\frac{\mathrm{g}_r(\xi_{12}^{\mathrm{I}},\xi_{12}^{\mathrm{I}})}{\frac{1}{2}\,\mathrm{g}_r(\tilde{\xi}_0^{\mathrm{I}},\tilde{\xi}_0^{\mathrm{I}})}} \qquad\qquad \xi_0^{\mathrm{R}} \;=\; \tilde{\xi}_0^{\mathrm{R}}\,\frac{\mathrm{g}_r(\xi_{12}^{\mathrm{R}},\xi_{12}^{\mathrm{I}})}{\frac{1}{2}\,\mathrm{g}_r(\tilde{\xi}_0^{\mathrm{R}},\xi_0^{\mathrm{I}})}\,. \tag{3.68}$$

(mind the tildes) makes the amplitudes coincide, as can be checked by plugging the above $(\xi_0^{\mathrm{R}},\xi_0^{\mathrm{I}})$ into the first line of the condition (3.67). (Here we assume a positive-definite g_r. If this is not the case, then care must be taken with signs.) This infinite number of solutions $(\xi_0^{\mathrm{R}},\,\xi_0^{\mathrm{I}})$ does not tell us anything physical in particular.

However, there is a more natural correspondence. In the case that the solutions $(\eta_1,\,\zeta_2)$ are such that the induced $(\xi_{12}^{\mathrm{R}},\,\xi_{12}^{\mathrm{I}})$ are well defined on the whole rod region, we can make rod and tube amplitudes coincide by simply setting

$$\begin{aligned} \xi_0^{\mathrm{I}} &= \tfrac{1}{\sqrt{2}}(-J_r\eta_1 + J_r\zeta_2) & \qquad \xi_0^{\mathrm{R}} &= \tfrac{1}{\sqrt{2}}(\eta_1+\zeta_2) \\ &= \sqrt{2}\,\xi_{12}^{\mathrm{I}} & &= \sqrt{2}\,\xi_{12}^{\mathrm{R}}. \end{aligned} \tag{3.69}$$

Due to bilinearity of g_r and ω_r, the relations (3.69) make the amplitudes of tube and rod regions coincide also for complex η_1 and ζ_2. The factors of $\sqrt{2}$ appear because g_r is bilinear, and the number of boundary components of the tube region is 2 times that of the rod region. The solution ξ_0 characterizing the state $\overline{K_{\Sigma_0}^{\xi_0}}$ near $\overline{\Sigma}_0$ induced by these $\xi_0^{\mathrm{I,R}}$ is thus simply

$$\xi_0 \;=\; \xi_0^{\mathrm{R}} - J_r\xi_0^{\mathrm{I}} \;=\; \sqrt{2}\,\zeta_2. \tag{3.70}$$

It appears that the solution η_1 is lost somehow. However, the condition that the induced $(\xi_{12}^{\mathrm{R}},\,\xi_{12}^{\mathrm{I}})$ are well defined on the whole rod region induces a relation between η_1 and ζ_2. That is, η_1 and ζ_2 are not independent here, and thus η_1 appears in (3.70) implicitly. As an example for this, let us consider Klein-Gordon theory on Minkowski spacetime. Expansion (2.108) for a complex solution on a tube region gives us

$$\begin{aligned} \eta_1(t,r,\Omega) &= \int\!\mathrm{d}E\sum_{l,m_l}\Big\{\eta^{1,a}_{Elm_l}\mu^{(a)}_{Elm_l}(t,r,\Omega) + \eta^{1,b}_{Elm_l}\mu^{(b)}_{Elm_l}(t,r,\Omega)\Big\} \\ \zeta_2(t,r,\Omega) &= \int\!\mathrm{d}E\sum_{l,m_l}\Big\{\zeta^{2,a}_{Elm_l}\mu^{(a)}_{Elm_l}(t,r,\Omega) + \zeta^{2,b}_{Elm_l}\mu^{(b)}_{Elm_l}(t,r,\Omega)\Big\}\,. \end{aligned}$$

We recall that the a-modes are regular on the time-axis $r=0$ while the b-modes diverge there. The condition that $(\xi_{12}^{\mathrm{R}},\,\xi_{12}^{\mathrm{I}})$ be regular on rod regions thus means that they may not contain b-modes. From (2.16) we have

$$\xi_{12}^{\mathrm{R}} \;=\; \tfrac{1}{2}(\eta_1+\zeta_2) \qquad\qquad \xi_{12}^{\mathrm{I}} \;=\; \tfrac{1}{2}(-J_r\eta_1 + J_r\zeta_2).$$

The condition that $\xi_{12}^{\mathrm{R}} = \frac{1}{2}(\eta_1+\zeta_2)$ may not contain b-modes implies that $\zeta^{2,b}_{Elm_l} = -\eta^{1,b}_{Elm_l}$. Using as an example the complex structure (2.119), that is: $(J_r\phi)^a_{Elm_l} = -\phi^b_{Elm_l}$ while $(J_r\phi)^b_{Elm_l} = +\phi^a_{Elm_l}$, we get

$$-(J_r\eta_1)(t,r,\Omega) \;=\; \int\!\mathrm{d}E\sum_{l,m_l}\Big\{+\eta^{1,b}_{Elm_l}\mu^{(a)}_{Elm_l}(t,r,\Omega)-\eta^{1,a}_{Elm_l}\mu^{(b)}_{Elm_l}(t,r,\Omega)\Big\}$$

$$(J_r\zeta_2)(t,r,\Omega) \;=\; \int\!\mathrm{d}E\sum_{l,m_l}\Big\{-\zeta^{2,b}_{Elm_l}\mu^{(a)}_{Elm_l}(t,r,\Omega)+\zeta^{2,a}_{Elm_l}\mu^{(b)}_{Elm_l}(t,r,\Omega)\Big\}\,.$$

Hence the condition that $\xi^{\mathrm{I}}_{12} = \frac{1}{2}(-J_r\eta_1 + J_r\zeta_2)$ may not contain b-modes implies that $\zeta^{2,a}_{Elm_l} = +\eta^{1,a}_{Elm_l}$. This example illustrates that the condition of having $(\xi^{\mathrm{R}}_{12},\, \xi^{\mathrm{I}}_{12})$ regular on rod regions makes ζ_2 determined completely by η_1. The thereby induced relation between η_1 and ζ_2 depends on the chosen complex structure J_r.

We have thus seen a first example of amplitude equvalence. Since the hypersurfaces that bound tube and rod regions are of the same type (hypercylinders), the same complex structure J_r is involved in tube and rod amplitudes. Hence the equivalence essentially consists in an unsurprising factor of $\sqrt{2}$, which relates the characterising solutions of the coherent states on the tube's and rod's boundaries.

Amplitude equivalence for time-interval and rod regions

Next let us consider rod and time-interval regions. Since time-interval regions include the origin $r = 0$, solutions that are well defined on time-interval regions are naturally well defined on rod regions as well. Here we consider global solutions only, that is, solutions that are well defined for all times and on all of space. Therefore, if η_1 respectively ζ_2 are global solutions near Σ_1 at $t = t_1$ respectively Σ_2 at $t = t_2$, then the induced solutions (2.16)

$$\xi^{\mathrm{R}}_{12} \;=\; \tfrac{1}{2}(\eta_1+\zeta_2) \qquad\qquad \xi^{\mathrm{I}}_{12} \;=\; \tfrac{1}{2}(-J_t\eta_1+J_t\zeta_2).$$

are global as well, and hence they are well defined on rod regions. In the previous subsection we saw that for tube regions this is an additional requirement which implies a relation between η_1 and ζ_2, but for time-interval regions it is fulfilled naturally. For simplicity we assume again that symplectic structures and complex structures are independent of time respectively radius, and use the labels t and r merely in order to indicate to which type of hypersurface they refer. We thus call two complex structures J_t and J_r amplitude-equivalent, if they make the following amplitudes coincide for all global solutions η_1 and ζ_2:

$$\begin{aligned}\rho_{[t_1,t_2]}\big(K^{\eta_1}_{\Sigma_1}\otimes\overline{K^{\zeta_2}_{\Sigma_2}}\big) \;&=\; \exp\Big(-\mathrm{g}_t(\xi^{\mathrm{I}}_{12},\,\xi^{\mathrm{I}}_{12})-\mathrm{i}\mathrm{g}_t(\xi^{\mathrm{R}}_{12},\,\xi^{\mathrm{I}}_{12})\Big)\\ \stackrel{!}{=}\;\rho_{r_0}\big(\overline{K^{\xi_0}_{\Sigma_0}}\big) \;&=\; \exp\Big(-\tfrac{1}{2}\mathrm{g}_r(\xi^{\mathrm{I}}_{12},\,\xi^{\mathrm{I}}_{12})-\tfrac{\mathrm{i}}{2}\mathrm{g}_r(\xi^{\mathrm{R}}_{12},\,\xi^{\mathrm{I}}_{12})\Big),\end{aligned} \tag{3.71}$$

wherein now we have (mind where we use J_t and where J_r!)

$$\xi_0 \;=\; \xi^{\mathrm{R}}_{12} - J_r\xi^{\mathrm{I}}_{12}. \tag{3.72}$$

Comparing (3.71) to (3.65), we here have $\mathrm{g}_{\partial\mathbb{M}} = 2\mathrm{g}_t$ due to $\partial\mathbb{M} = \Sigma_1\cup\overline{\Sigma}_2$ and $\mathrm{g}_{\Sigma 1} = \mathrm{g}_{\overline{\Sigma}2}$, while $\mathrm{g}_{\partial\mathbb{N}} = \mathrm{g}_r$ since the rod's boundary is one single hypersurface Σ_0. And comparing (3.72) to (3.64), we have here $(\eta_1,\zeta_2) = \xi_{\partial\mathbb{M}}$ while $\xi_0 = \xi_{\partial\mathbb{N}}$, with $\xi_{\partial\mathbb{N}} = E(\xi_{\partial\mathbb{M}})$ as discussed above.

Equation (3.72) is a natural way to relate complex structures on equal-time hypersurfaces to those on hypercylinders. Relation (3.72) means that the solution ξ_0 (well-defined near the rod's boundary) and the solution (η_1,ζ_2) (well-defined near the time-interval's boundary) share the *same* global solution ξ^{R}_{12} as part which is well-defined on the whole region respectively. Further, they also share the *same* global solution ξ^{I}_{12} which induces the respective parts which are not well-defined on the whole region, but only near the boundaries. For the time-interval this part is induced by J_t, for the rod by J_r. The negative sign before J_r is due again to the opposite orientations of Σ_r and the rod's boundary.

Plugging (3.72) into (3.71), we can read off the following: amplitude-equivalence of complex structures for global solutions is actually the same as inducing the same real g-product for all global solutions up to a factor of $\frac{1}{2}$. This factor again results from the circumstance that the time-interval has 2 times the number of boundary components of the rod. Ergo, J_t and J_r are amplitude-equivalent for global solutions, precisely if for all global solutions η, ζ they induce

$$\mathrm{g}_t(\eta, \zeta) = \tfrac{1}{2}\, \mathrm{g}_r(\eta, \zeta). \tag{3.73}$$

This is just requirement (3.63) written for time-interval and rod regions. Counting boundary components, imposing this equality from the outset would be another natural way to relate complex structures on equal-time surfaces and hypercylinders. Here we have motivated it further by showing that it makes *time-interval* amplitudes coincide with rod amplitudes.

3.2 Minkowski Spacetime: HQ of Klein-Gordon field

3.2.1 Time-interval regions

For time-interval regions we already dispose of all necessary ingredients to apply Holomorphic Quantization to the classical theory. This has been carried out earlier in Section 5.1 of [59] using cartesian coordinates and plane wave modes. In order to establish an equivalence of amplitudes between time-intervals and rod regions, it is more useful to work in spherical coordinates for both types of regions. Classical solutions near an equal-time plane Σ_t can be written as the mode expansion (2.90)

$$\phi(t,r,\Omega) \;=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 2p\,(2\pi)^{-1/2}\, j_l(pr) \Big\{ \phi^+_{plm_l}\, \mathrm{e}^{-\mathrm{i}E_p t}\, Y_l^{m_l}(\Omega) + \overline{\phi^-_{plm_l}}\, \mathrm{e}^{\mathrm{i}E_p t}\, \overline{Y_l^{m_l}(\Omega)} \Big\}\,.$$

The symplectic structure induced by the Lagrange density for these solutions is given by (2.96)

$$\omega_t \;=\; +\mathrm{i} \int_0^\infty \mathrm{d}p \sum_{l,m_l} E_p \left\{ \overline{\eta^-_{plm_l}}\, \zeta^+_{plm_l} \;-\; \eta^+_{plm_l}\, \overline{\zeta^-_{plm_l}} \right\}\,.$$

Showing that this symplectic structure is invariant under the actions of time-translations and spatial rotations can be done in the way as for the AdS hypercylinders in Section 2.6.7. In Section 2.5.5 we show that ω_t is also invariant under spatial translations and boosts. For the complex structure on the space $\mathrm{L}_t^{\mathrm{Mink}}$ of solutions near Σ_t there is the standard choice (2.97)

$$\big(J_{\Sigma_t}\phi\big)^{\pm}_{plm_l} \;=\; -\mathrm{i}\, \phi^{\pm}_{plm_l}\,. \tag{3.74}$$

In Appendix B.3.2 we show that it commutes with spatial translations and boosts. Again, commutation with time-translations and spatial rotations can be shown easily as for the AdS case in Appendix C.4. From symplectic and complex structure we can build real and complex inner products as in (2.98) and (2.99):

$$\mathrm{g}_t\big(\eta,\, \zeta\big) \;=\; 2\omega_t\big(\eta,\, J_t\zeta\big) \;=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 2E_p \left\{ \overline{\eta^-_{plm_l}}\, \zeta^+_{plm_l} \;+\; \eta^+_{plm_l}\, \overline{\zeta^-_{plm_l}} \right\} \tag{3.75}$$

$$\{\eta,\, \zeta\}_t \;=\; \mathrm{g}_t\big(\eta,\, \zeta\big) + 2\mathrm{i}\, \omega_t\big(\eta,\, \zeta\big) \;=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 4E_p\, \eta^+_{plm_l}\, \overline{\zeta^-_{plm_l}}\,. \tag{3.76}$$

These are the structures that determine the free amplitude (3.28) for coherent holomorphic states (3.1). For time-intervals this amplitude formula becomes (3.30):

$$\begin{aligned}\rho_{[t_1,t_2]}(K^{\eta_1}_{\Sigma_1} \otimes \overline{K^{\zeta_2}_{\Sigma_2}}) &= \exp\Big(-\mathrm{g}_{t_1}(\xi^{\mathrm{I}}_{12},\, \xi^{\mathrm{I}}_{12}) - \mathrm{i}\mathrm{g}_{t_1}(\xi^{\mathrm{R}}_{12},\, \xi^{\mathrm{I}}_{12})\Big)\\ &= \exp\Big(-\tfrac{1}{4}\mathrm{g}_{t_1}(\eta_1,\, \eta_1) - \tfrac{1}{4}\mathrm{g}_{t_1}(\zeta_2,\, \zeta_2) + \tfrac{1}{2}\big\{\eta_1,\, \zeta_2\big\}_{t_1}\Big).\end{aligned}$$

3.2.2 Rod regions

For rod regions we also dispose of the necessary ingredients for Holomorphic Quantization. Classical solutions near a hypercylinder Σ_r can be written as the mode expansion (2.107)

$$\phi(t, r, \Omega) \;=\; \int \mathrm{d}E \sum_{l,m_l} \frac{p^{\mathbb{R}}_E}{4\pi} \Big\{\phi^a_{Elm_l} \mathrm{e}^{-\mathrm{i}Et} Y^{m_l}_l(\Omega)\, \check{j}_{El}(r) + \phi^b_{Elm_l} \mathrm{e}^{-\mathrm{i}Et} Y^{m_l}_l(\Omega)\, \check{n}_{El}(r)\Big\}\,.$$

The symplectic structure induced by the Lagrange density for these solutions is given by (2.116):

$$\omega_r\big(\eta,\, \zeta\big) \;=\; \int \mathrm{d}E \sum_{l,m_l} \frac{p^{\mathbb{R}}_E}{16\pi} \Big\{\eta^a_{Elm_l}\, \zeta^b_{-E,l,-m_l} - \eta^b_{Elm_l}\, \zeta^a_{-E,l,-m_l}\Big\}\;.$$

Showing that this symplectic structure is invariant under the actions of time-translations and spatial rotations can be done in the way as for the AdS hypercylinders in Section 2.6.7. In Section 2.5.5 we show that ω_r is also invariant under spatial translations and boosts.

For the complex structure on the space $\mathrm{L}^{\mathrm{Mink}}_r$ of solutions near Σ_r there is no standard choice. In Section 5.3 of [59] a complex structure is introduced, which we denote by J^{pos}_r since it induces a positive-definite real inner product $\mathrm{g}^{\mathrm{pos}}_r(\cdot,\cdot)$. It acts as calculated in (B.71) for all energies $E \in \mathbb{R}$:

$$(J^{\mathrm{pos}}_r\phi)^a_{Elm_l} \;=\; +\phi^b_{Elm_l} \qquad\qquad (J^{\mathrm{pos}}_r\phi)^b_{Elm_l} \;=\; -\phi^a_{Elm_l}. \tag{3.77}$$

In Appendix B.3.1 we discuss that J^{pos}_r commutes with time-translations and spatial rotations, but not with spatial translations and with boosts. We also construct a slightly different complex structure J^{iso}_r in (B.90) which commutes with all isometries of Minkowski spacetime:

$$\begin{aligned}\big(J^{\mathrm{iso}}_r\phi\big)^a_{Elm_l} &= -\,\phi^b_{Elm_l} & \big(J^{\mathrm{iso}}_r\phi\big)^b_{Elm_l} &= +\,\phi^a_{Elm_l} & E^2 &> m^2\\ \big(J^{\mathrm{iso}}_r\phi\big)^a_{Elm_l} &= -(-1)^l\,\phi^b_{Elm_l} & \big(J^{\mathrm{iso}}_r\phi\big)^b_{Elm_l} &= +(-1)^l\,\phi^a_{Elm_l} & E^2 &< m^2\,.\end{aligned} \tag{3.78}$$

The properties of these two complex structures are summarized in Table 3.79. The induced real product $\mathrm{g}^{\mathrm{iso}}_r$ is positive-definite for modes with $E^2 > m^2$. When $E^2 < m^2$, then $\mathrm{g}^{\mathrm{iso}}_r$ is positive-definite for modes with even l and negative-definite for odd l.

Minkowski	J^{iso}_r	J^{pos}_r
commute with time-translations	✓	✓
commute with spatial translations	✓	-
commute with spatial rotations	✓	✓
commute with boosts	✓	-
strong amplitude equivalence	✓	✓
induced real inner product g_r	indefinite	positive-definite

Table 3.79: Properties of complex structures for Minkowski hypercylinder

In Appendix B.4 we find that both J_r^{pos} and J_r^{iso} induce the amplitude equivalence between time-intervals and rod regions which we introduced in Section 3.1.10. That is, the real products induced by the two complex structures both fulfill (3.73) for global classical solutions η, ζ:

$$g_t(\eta,\zeta) = \tfrac{1}{2}\, g_r^{\text{pos}}(\eta,\zeta) \qquad\qquad g_t(\eta,\zeta) = \tfrac{1}{2}\, g_r^{\text{iso}}(\eta,\zeta)\,. \tag{3.80}$$

This is natural, since global solutions consist of modes with $E^2 > m^2$, and for these energies the complex structures coincide. (By strong equivalence we emphasize that this equivalence holds for all these frequencies. By contrast, on AdS only a weak equivalence holds, that is, it is valid only for a discrete subset of energies.) From symplectic and complex structure we build again the inner products (2.120) and (B.91):

$$g_r^{\text{pos}}(\eta,\,\zeta) \;=\; 2\omega_r(\eta,\,J_r^{\text{pos}}\zeta) \;=\; \int\!\mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{8\pi} \left\{\eta^a_{Elm_l}\,\zeta^a_{-E,l,-m_l} + \eta^b_{Elm_l}\,\zeta^b_{-E,l,-m_l}\right\}$$

$$g_r^{\text{iso}}(\eta,\zeta) \;=\; 2\omega_r(\eta,\,J_r^{\text{iso}}\zeta) \;=\; \int\!\mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{8\pi}\,(\pm 1)^l \left\{\eta^a_{Elm_l}\zeta^a_{-E,l,-m_l} + \eta^b_{Elm_l}\zeta^b_{-E,l,-m_l}\right\}\,.$$

(Therein, in ± 1 the upper sign holds for $E^2 > m^2$ and the lower sign for $E^2 < m^2$.) These are the structures that determine the free amplitude (3.32) for coherent holomorphic states (3.1):

$$\rho_{r_0}\big(\overline{K^{\xi_0}_{\Sigma_0}}\big) \;=\; \exp\Big(-\tfrac{1}{2} g_{r_0}(\xi_0^{\text{I}},\,\xi_0^{\text{I}}) - \tfrac{\mathrm{i}}{2} g_{r_0}(\xi_0^{\text{R}},\,\xi_0^{\text{I}})\Big).$$

3.3 Anti de Sitter Spacetime: HQ of Klein-Gordon field

For time-interval regions on AdS, the amplitudes ρ_t are determined by the real inner product g_t via (3.28). The boundary of this region are equal-time hypersurfaces, and g_t is determined by the complex structure J_t associated to these hypersurfaces. Here, there is a long-known standard choice for J_t and therefore in Section 3.3.1 we only need to check that it commutes with the isometries and has the correct flat limit.

For the rod region, the amplitudes ρ_ρ are determined by the real inner product g_ρ by (3.28) as well. The boundary of this region is a hypercylinder, and g_ρ is determined by the complex structure J_ρ associated to this hypersurface. Since here there is no standard choice, we need to construct it. With this goal, in Section 3.3.2 we implement the requirements of Sections 3.1.8 and 3.1.10 in the following sequence. First, we impose commutation of J_ρ with the isometries' actions, because this already fixes the form of J_ρ to a great degree. Using this preliminary form we implement a weak version of amplitude equivalence, because this completely fixes the action of J_ρ on the Jacobi modes (the modes with magic frequencies ω^+_{nl}). Then, we present two choices how to extend the action of J_ρ to all real frequencies ω (that is, two ways of completely fixing J_ρ) and study their properties.

3.3.1 Time-interval regions

For time-interval regions we already dispose of all necessary ingredients to apply Holomorphic Quantization to the classical theory, because there is a standard choice for the complex structure J_t. Classical solutions near an equal-time plane Σ_t can be written as the Jacobi expansion (2.201)

$$\phi(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l} \left\{\phi^+_{n\underline{l}m_l}\,\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\phi^-_{n\underline{l}m_l}}\;\overline{\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)}\right\}\,.$$

The symplectic structure induced by the Lagrange density for these solutions is given by (2.209)

$$\omega_t(\eta,\zeta) \;=\; +\mathrm{i}\sum_{n\underline{l}m_l} \omega^\pm_{nl} R^{d-1}_{\text{AdS}} \mathcal{N}^\pm_{nl} \left\{\overline{\eta^-_{n\underline{l}m_l}}\,\zeta^+_{n\underline{l}m_l} - \eta^+_{n\underline{l}m_l}\,\overline{\zeta^-_{n\underline{l}m_l}}\right\}\,.$$

In Section 2.6.7 we show that this symplectic structure is invariant under the actions of all Ads isometries (time-translation, spatial rotations, and boosts). For the complex structure on the space $\mathrm{L}_t^{\mathrm{AdS}}$ of solutions near Σ_t there is the standard choice (2.212)

$$\big(J_t\,\phi\big)^{\pm}_{n\underline{l}m_l} \;=\; -\mathrm{i}\,\phi^{\pm}_{n\underline{l}m_l}\,. \tag{3.81}$$

In Appendix C.4.1 we verify that it commutes with the action of all AdS isometries. From symplectic and complex structure we can build real and complex inner products as in (2.213) and (2.214):

$$\mathrm{g}_t(\eta,\zeta) \;=\; \sum_{n\underline{l}m_l} 2\omega^{\pm}_{nl} R^{d-1}_{\mathrm{AdS}} \mathcal{N}^{\pm}_{nl} \left\{ \overline{\eta^{-}_{n\underline{l}m_l}}\,\zeta^{+}_{n\underline{l}m_l} + \eta^{+}_{n\underline{l}m_l}\,\overline{\zeta^{-}_{n\underline{l}m_l}} \right\} \tag{3.82}$$

$$\{\eta,\,\zeta\}_t \;=\; \sum_{n\underline{l}m_l} 4\omega^{\pm}_{nl} R^{d-1}_{\mathrm{AdS}} \mathcal{N}^{\pm}_{nl} \left\{ \eta^{+}_{n\underline{l}m_l}\,\overline{\zeta^{-}_{n\underline{l}m_l}} \right\}. \tag{3.83}$$

These are the structures that determine the free amplitude (3.28) for coherent holomorphic states (3.1). For time-intervals this amplitude formula becomes (3.30):

$$\begin{aligned} \rho_{[t_1,t_2]}(K^{\eta_1}_{\Sigma_1}\otimes\overline{K^{\zeta_2}_{\Sigma_2}}) \;&=\; \exp\Big(-\mathrm{g}_{t_1}(\xi^{\mathrm{I}}_{12},\,\xi^{\mathrm{I}}_{12}) - \mathrm{i}\mathrm{g}_{t_1}(\xi^{\mathrm{R}}_{12},\,\xi^{\mathrm{I}}_{12})\Big) \\ &=\; \exp\Big(-\tfrac{1}{4}\mathrm{g}_{t_1}(\eta_1,\,\eta_1) - \tfrac{1}{4}\mathrm{g}_{t_1}(\zeta_2,\,\zeta_2) + \tfrac{1}{2}\big\{\eta_1,\,\zeta_2\big\}_{t_1}\Big). \end{aligned}$$

The real g-product of the Ads equal-time surface is positive-definite for all real solutions $\phi\in\mathrm{L}_t^{\mathrm{AdS}}$ (that is, for all Jacobi modes): Let ϕ such a solution and hence $\phi^{-}_{n\underline{l}m_l} = \phi^{+}_{n\underline{l}m_l}$. Then (3.82) returns

$$\mathrm{g}_t(\phi,\phi) \;=\; \sum_{n\underline{l}m_l} 4\omega^{+}_{nl} R^{d-1}_{\mathrm{AdS}} \mathcal{N}^{+}_{nl} \left|\phi^{+}_{n\underline{l}m_l}\right|^2, \tag{3.84}$$

which is positive (since $\mathcal{N}^{+}_{nl}$ is always positive). The same happens for the real g-product on a Minkowski equal-time plane: it is positive for all modes well-defined on all of space.

The complex structure J_t and its induced real g-product relate to their Minkowski counterparts via the flat limit. In (2.197) we show that the flat limit of the AdS symplectic structure ω_t yields the Minkowski ω_t. In Appendix C.7.1 we discuss that the complex structure J_t and the process of taking the flat limit commute as in the diagram below.

$$\begin{array}{ccc} \phi^{\mathrm{AdS}} & \xrightarrow{\;J_t^{\mathrm{AdS}}\;} & J_t^{\mathrm{AdS}}\phi^{\mathrm{AdS}} \\ {\scriptstyle \text{flat lim.}}\big\downarrow & & \big\downarrow{\scriptstyle \text{flat lim.}} \\ \phi^{\mathrm{Mink}} & \xrightarrow{\;J_t^{\mathrm{Mink}}\;} & J_t^{\mathrm{Mink}}\phi^{\mathrm{Mink}} \end{array} \tag{3.85}$$

We then proceed with calculating the flat limit of the induced g_t and find that it yields the Minkowski g_t.

3.3.2 Rod regions: isometry invariance

For rod regions we also dispose of the necessary ingredients for Holomorphic Quantization. Classical solutions near a hypercylinder Σ_ρ can be written as the mode expansion (2.186)

$$\phi(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{\omega\underline{l}m_l}\;\mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^{b}_{\omega\underline{l}m_l}\;\mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\}$$

The symplectic structure induced by the Lagrange density for these solutions is given by (2.195):

$$\omega_\rho(\eta,\zeta) \;=\; \pi R_{\text{AdS}}^{d-1} \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \eta^{S,a}_{\omega\underline{l}m_l}\, \zeta^{S,b}_{-\omega,\underline{l},-m_l} - \eta^{S,b}_{\omega\underline{l}m_l}\, \zeta^{S,a}_{-\omega,\underline{l},-m_l} \Big\}\, (2l{+}d{-}2)\,.$$

In Section 2.6.7 we show that ω_ρ is invariant under the actions of all AdS isometries (time-translations, spatial rotations, and boosts.) For the complex structure on the space $\mathrm{L}_\rho^{\text{AdS}}$ of solutions near Σ_ρ there is no standard choice. Our goal is to construct this complex structure such that it realizes various properties. As a minimal requirement, any complex structure J_ρ must fulfill three properties, which we call essential properties: it must be linear and map real solutions to real solutions, its square must be minus unity $J_\rho^2 = -\mathbb{1}$, and it must be compatible with the symplectic structure: $\omega_\rho(J_\rho\eta,\, J_\rho\zeta) \;=\; \omega_\rho(\eta,\zeta)$. In addition to these essentials, we aim to construct J_ρ such that it also commutes with all isometries of AdS, induces a positive-definite real g-product and behaves well in the flat limit.

We start in Appendix C.4.2 from the most general expression (C.321) of a linear operator that acts in the momentum representation of solutions. We start by requiring this general form to commute with time-translations and spatial rotations, because this quickly simplifies the general form, resulting in (C.330):

$$\begin{aligned} \big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} &= j^{S,aa}_{\omega l}\, \phi^{S,a}_{\omega\underline{l}m_l} + j^{S,ab}_{\omega l}\, \phi^{S,b}_{\omega\underline{l}m_l} \\ \big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} &= j^{S,ba}_{\omega l}\, \phi^{S,a}_{\omega\underline{l}m_l} + j^{S,bb}_{\omega l}\, \phi^{S,b}_{\omega\underline{l}m_l}. \end{aligned}$$

Therein, the four j-factors are complex functions of frequency ω and angular momentum l and determine the complex structure completely. Next we implement the essential properties mentioned above, which results in (C.358):

$$\begin{aligned} \big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} &= j^{S,aa}_{\omega l}\, \phi^{S,a}_{\omega\underline{l}m_l} + j^{S,ab}_{\omega l}\, \phi^{S,b}_{\omega\underline{l}m_l} \\ \big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} &= j^{S,ba}_{\omega l}\, \phi^{S,a}_{\omega\underline{l}m_l} - j^{S,aa}_{\omega l}\, \phi^{S,b}_{\omega\underline{l}m_l} \end{aligned} \tag{3.86}$$

$$\begin{aligned} \big(j^{S,aa}_{\omega l}\big)^2 &= -j^{S,ab}_{\omega l}\, j^{S,ba}_{\omega l} - 1 \geq 0 \\ j^{S,ab}_{-\omega,l} &= j^{S,ab}_{\omega l} \qquad\qquad j^{S,ba}_{-\omega,l} = j^{S,ba}_{\omega l}. \end{aligned} \tag{3.87}$$

We call this the nondiagonal form, since it holds for nonvanishing $j^{S,ab}$ and $j^{S,ba}$, see Appendix C.4.4. The essential properties imply that the three j-factors are now real and symmetric w.r.t. the frequency: $j^{\cdot\cdot}_{-\omega,l} = j^{\cdot\cdot}_{+\omega,l}$. Further, the real g-product induced by J_ρ becomes positive-definite for all modes with frequencies and angular momenta for which

$$j^{S,ab}_{\omega l} < 0 \qquad\qquad j^{S,ba}_{\omega l} > 0\,. \tag{3.88}$$

We then consider the boosts and find that J_ρ commutes with them if the two conditions (C.370) are met:

$$\begin{aligned} j^{S,ab}_{\omega-1,l+1} &\overset{!}{=} -j^{S,ab}_{\omega l}\; \frac{(\widetilde{m}_+{+}\omega{-}l{-}d)\,(\widetilde{m}_+{-}\omega{+}l)}{(2l{+}d)\,(2l{+}d{-}2)} \\ j^{S,ab}_{\omega+1,l+1} &\overset{!}{=} -j^{S,ab}_{\omega l}\; \frac{(\widetilde{m}_+{-}\omega{-}l{-}d)\,(\widetilde{m}_+{+}\omega{+}l)}{(2l{+}d)\,(2l{+}d{-}2)}\,. \end{aligned} \tag{3.89}$$

Similar conditions hold for $j^{S,ba}$, which can be obtained from the above by setting $j^{S,ba} = 1/j^{S,ab}$. There exists various solutions to these conditions, for example those listed in (C.372). J_ρ is determined by choosing one solution of the respective conditions for $j^{S,ab}$ and one for $j^{S,ba}$, and then $(j^{S,aa}_{\omega l})^2 = -j^{S,ab}_{\omega l}\, j^{S,ba}_{\omega l} - 1$.

For completeness we include that for vanishing $j^{S,ab}$ or $j^{S,ba}$, we find the following form of the complex structure in (C.351):

$$\begin{aligned} & \mathbb{R}\mathrm{e}\, j^{S,ab}_{\omega l} = 0 \qquad \Longleftrightarrow \qquad \mathbb{R}\mathrm{e}\, j^{S,ba}_{\omega l} = 0 \\ \Longrightarrow \quad & j^{S,aa}_{\omega l} = j^{S,bb}_{\omega l} = \pm \mathrm{i} \quad \text{with} \quad j^{S,aa}_{-\omega,l} = \overline{j^{S,aa}_{\omega,l}} = -j^{S,aa}_{\omega,l}, \\ \Longrightarrow \quad & j^{S,ab}_{\omega l} = j^{S,ba}_{\omega l} = 0. \end{aligned} \tag{3.90}$$

We call it the diagonal form. It does not work for $\omega = 0$. Let us mention already that it fulfills the essential properties plus commutation with time-translation and spatial roations, but not with boosts (see Appendix C.4.6). It induces a real g-product that vanishes for the modes with all the frequencies for which this complex structure is chosen (see Appendix C.6). Further, choosing it for the magic frequencies would not induce amplitude equivalence. For these reasons, we shall not consider the diagonal form as a candidate for a complex structure in its own right. However, it is useful as an "emergency" choice in the case that for some discrete frequencies one of the elements $j^{S,ab}$ or $j^{S,ba}$ of the nondiagonal form becomes zero.

3.3.3 Amplitude equivalence for AdS

In order to determine which of these various solutions we should choose, in Appendix C.5 we require J_ρ to induce amplitude equivalence between time-intervals and rod regions as discussed in Section 3.1.10. Amplitude equivalence only involves global solutions, which for AdS are the Jacobi modes (that is, hypergeometric S^a-modes where the frequency is magic $\omega = \omega^+_{nl}$). Therefore this requirement can only fix our choice of J_ρ for the magic frequencies. Thus for AdS the amplitude equivalence only holds for a discrete subset of the frequency range $\omega \in \mathbb{R}$ allowed for modes on rod regions (respectively on a neighborhood of its boundary). For this reason we call it a *weak* equivalence. In (C.397) we find that this weak equivalence holds precisely if

$$j^{S,ba}_{\omega^+_{nl} l} = j^{S,ba}_{-\omega^+_{nl} l} = \frac{1}{\pi}\,\Gamma(\gamma^{\mathrm{S}})\Gamma(\gamma^{\mathrm{S}}-1)\,\frac{\Gamma(1-\alpha^{S,b})\;\Gamma(\beta^{S,b})}{\Gamma(1-\alpha^{S,a})\;\Gamma(\beta^{S,a})}\,. \tag{3.91}$$

The parameters α, β and γ^{S} are calculated from ω and l, see (C.376). In order to build a $j^{S,ba}$ which fulfills this condition, we now single out two candidate solutions of the boost conditions, which we call the α-version and β-version (C.402):

$$\begin{aligned} j^{S,ba,\alpha}_{\omega l} &= \frac{1}{\pi}\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)\,\frac{\Gamma(\alpha^{S,b})}{\Gamma(\alpha^{S,a})}\;\frac{\Gamma(1-\beta^{S,a})}{\Gamma(1-\beta^{S,b})} \\ j^{S,ba,\beta}_{\omega l} &= \frac{1}{\pi}\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)\,\frac{\Gamma(\beta^{S,b})}{\Gamma(\beta^{S,a})}\;\frac{\Gamma(1-\alpha^{S,a})}{\Gamma(1-\alpha^{S,b})}\,. \end{aligned} \tag{3.92}$$

Since switching the sign of ω corresponds to interchanging α and β-parameters, we have have the following relation, which will be important for realizing frequency symmetry $j^{S,ba}_{-\omega,l} = j^{S,ba}_{+\omega,l}$

$$j^{S,ba,\alpha}_{\omega,l} = j^{S,ba,\beta}_{-\omega,l}.$$

Now the only possibility for $j^{S,ba}_{\omega l}$ to fulfill (3.91) is to choose the β-version for positive magic frequencies $\omega = +\omega^+_{nl}$ and the α-version for the negative ones $\omega = -\omega^+_{nl}$ as in (C.404)

$$j^{S,ba}_{\omega^+_{nl},l} = j^{S,ba,\beta}_{\omega^+_{nl},l} \qquad\qquad j^{S,ba}_{-\omega^+_{nl},l} = j^{S,ba,\alpha}_{-\omega^+_{nl},l}. \tag{3.93}$$

We have thus fixed $j^{S,ba}_{\omega l}$ for the magic frequencies. Before extending this to the remaining frequencies and fixing also $j^{S,aa}_{\omega l}$ and $j^{S,ab}_{\omega l}$, we remark that already the existence of our choice (3.93) is quite nontrivial, because the factors related therein have rather different origins. The factor appearing in amplitude equivalence condition (3.91) stems from integrating a global solution over an equal-time hyperplane Σ_t, while the factors in the boost conditions (3.89) stem from boost compatibility of the complex structure J_ρ for more general solutions near a hypercylinder Σ_ρ.

3.3.4 Rod regions: candidates for J_ρ

Two-branched version $J_\rho^{\mathbf{two}}$

In Appendix C.5.1 we construct a first version of a complex structure J_ρ that we call two-branches choice, because for the negative-frequency branch we choose the α-version and for the positive-frequency branch the β-version. More over we choose an anti-diagonal matrix, that is, $j_{\omega l}^{S,aa,\text{two}} \equiv 0$. In (C.407) we define this complex structure as

$$J_{\omega l}^{\rho,\text{two}} = \begin{pmatrix} 0 & -1/j_{\omega l}^{S,ba,\text{two}} \\ j_{\omega l}^{S,ba,\text{two}} & 0 \end{pmatrix} \qquad j_{\omega l}^{S,ba,\text{two}} = \begin{cases} j_{\omega l}^{S,ba,\beta} & \omega > 0 \\ j_{\omega l}^{S,ba,\alpha} & \omega < 0 \end{cases}. \tag{3.94}$$

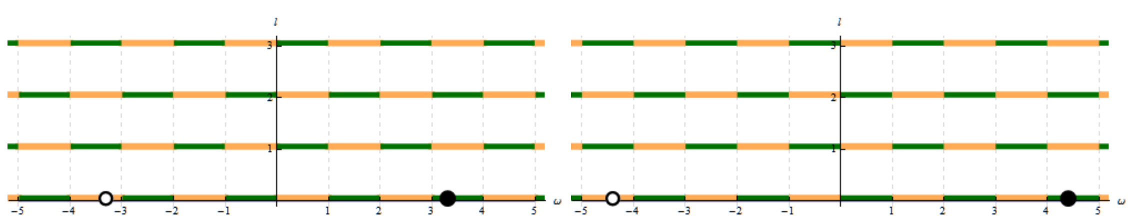
However, the gluing together of the two different versions at $\omega = 0$ causes the loss of commutation with the boosts' actions as we show in (C.412) (commutation with time-translations and spatial rotations remains preserved).

For the positive frequencies at which the β-version becomes zero or singular, and for the negative frequencies at which the α-version becomes zero or singular, we have to choose the diagonal form (3.90).

Interlaced version $J_\rho^{\mathbf{iso}}$

Therefore in Appendix C.5.2 we construct a second version J_ρ^{iso} which does commute with the boosts' actions and thus with all AdS isometries. The commutation with boosts holds only for values of $\tilde{m}_+$ and ν that are neither integer nor half-integer. We extend the choice (3.93) from its discrete set of frequencies $\omega = \pm\omega_{nl}^+$ to all $\omega \in \mathbb{R}$ for fixed l, for simplicity $l = 0$. The values for the other l are then determined completely by the boost conditions (3.89). This version is constructed by choosing frequency intervals for fixed l, on which we choose either the α or the β-version. This exploits the fact that the boost conditions (3.89) only relate the values of $j_{\omega l}^{S,ba}$ for frequency differences that are even integers (since we have to apply the conditions twice to get back the original l). Therefore we also call this version the interlaced version.

Before fixing these intervals, we recall that the extension must fulfill three properties: first, include the magic frequencies as in (3.93) in order to ensure amplitude equivalence. Second, be frequency-symmetric: $j_{\omega l}^{S,ba,\text{iso}} = j_{-\omega,l}^{S,ba,\text{iso}}$ in order to fulfill the essential properties and maintain commutation with time-translations and spatial rotations. Third, the pattern of interlaced intervals where we choose the α and β-versions must be translation-invariant for steps of 2 in ω-direction and l-direction. This is necessary in order to comply with the boost conditions (3.89), which relate $j_{\omega l}^{S,ba}$ to $j_{\omega\pm 2,l}^{S,ba}$ and $j_{\omega,l\pm 2}^{S,ba}$. Therefore, choosing the α-version for some frequency ω induces choosing the α-version for *all frequencies* $\omega \pm 2z$ with $z \in \mathbb{Z}$ (ditto for the β-version). The last two conditions imply that the interlaced intervals of the α and β-version alternate, and have the same length. Due to the step length of 2, this length can be set to values of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, ... For simplicity, we choose this length to be 1, see Figure 3.95 (a). Therein, we have ω on the horizontal axis and l on the vertical. Intervals on which we choose the α-version appear in orange (light gray in monochrome), and intervals with the β-version are dark green (darker gray).



(a) Case α (b) Case β

Figure 3.95: Interlaced complex structure J_ρ^{iso}: intervals in (ω, l)-space with α and β-version.

The first condition of amplitude equivalence determines which of these intervals are associated to the α respectively β-version. For $l = 0$, we associate the interval $(\lfloor \tilde{m}_+ \rfloor, \lceil \tilde{m}_+ \rceil)$ to the β-version. We use the standard notation of $\lfloor x \rfloor$ for the floor function (largest integer $\leq x$), and $\lceil x \rceil$ for ceiling (smallest integer $\geq x$). This choice already determines all other intervals: for $l = 0$ they alternate between α and β-version, and for higher l they are induced by the boost conditions. Thus the α and β-version alternate both horizontally (ω-direction) and vertically (l-direction), see Figure 3.95 (a). In the (ω, l)-plane let us denote by I_β the set of intervals associated to the β-version as described above, and by I_α the set of intervals associated to the α-version. Then, our interlaced choice J_ρ^{iso} writes as

$$j_{\omega l}^{S,ba,\text{iso}} = \begin{cases} j_{\omega l}^{S,ba,\alpha} & (\omega, l) \in I_\alpha \\ j_{\omega l}^{S,ba,\beta} & (\omega, l) \in I_\beta \end{cases}, \tag{3.96}$$

wherein $j_{\omega l}^{S,ba,\alpha}$ and $j_{\omega l}^{S,ba,\beta}$ are those of (3.92). This implies that two different patterns are possible for our choice: for "Case α": $\tilde{m}_+ \in (d{+}2n,\ d{+}2n{+}1)$ with $n \in \mathbb{N}_0$ we have the unit interval $\omega \in (0,1)$ for $l = 0$ associated to the α-version, see Figure 3.95 (a), while for "Case β": $\tilde{m}_+ \in (d{+}2n{+}1,\ d+2n{+}2)$ we have it associated to the β-version. see Figure 3.95 (b). The label of the case thus refers to which version occupies the unit interval $\omega \in (0,1)$ for $l = 0$. (We recall that d is odd, and that we only consider values of $\tilde{m}_+$ that are neither integer nor half-integer.) In both figures, the position of $\omega = +\tilde{m}_+$ is marked by a black disk, and that of $\omega = -\tilde{m}_+$ by a black circle. For $d = 3$ with $R_{\text{AdS}} = 1$, the example in Figure 3.95 (a) arises from Klein-Gordon mass $m = 1$ giving $\tilde{m}_+ \approx 3.3$, the example (b) from $m = 2.5$ giving $\tilde{m}_+ \approx 4.4$. In any case, our complex structure is constructed such that for $l = 0$ the black disk of $\omega = +\tilde{m}_+$ allways sits on a green (dark gray) interval of the β-version, and hence the circle of $\omega = -\tilde{m}_+$ on an orange (light gray) interval of the α-version.

Again, for the frequencies at which the α resp. β-version becomes zero or singular on the respective intervals, we have to choose the diagonal form (3.90).

We thus have fixed completely the element $j_{\omega l}^{S,ba}$ of our complex structure J_ρ^{iso} through interlacing intervals on which we choose the α respectively β-version. While not very elegant, this is physically motivated: it makes our complex structure fulfill the essential properties, commute *with all* isometry actions, and induce amplitude equivalence between time-interval and rod amplitudes. As for the two-branched version, we choose an anti-diagonal matrix for the interlaced version: $j_{\omega l}^{S,aa,\text{iso}} \equiv 0$.

3.3.5 Rod regions: induced real g_ρ

In Appendix C.6 we study the real g-products induced by our complex structures J_ρ^{two} and J_ρ^{iso}. Via (2.195), any anti-diagonal choice (that is: setting $j_{\omega l}^{S,aa} \equiv 0$) induces the real g-product

$$\begin{aligned} \mathrm{g}_\rho(\eta, \zeta) &= 2\omega_\rho(\eta, J_\rho\zeta) \\ &= 2\pi R_{\text{AdS}}^{d-1} \int \mathrm{d}\omega \sum_{\underline{l}, m_l} \left\{ \eta^a_{\omega \underline{l} m_l} \, (J_\rho\zeta)^b_{-\omega,\underline{l},-m_l} - \eta^b_{\omega \underline{l} m_l} \, (J_\rho\zeta)^a_{-\omega,\underline{l},-m_l} \right\} (2l{+}d{-}2) \\ &= 2\pi R_{\text{AdS}}^{d-1} \int \mathrm{d}\omega \sum_{\underline{l}, m_l} \left\{ \eta^a_{\omega \underline{l} m_l} \, \zeta^a_{-\omega,\underline{l},-m_l} \, j_{\omega l}^{S,ba} + \eta^b_{\omega \underline{l} m_l} \, \zeta^b_{-\omega,\underline{l},-m_l} / j_{\omega l}^{S,ba} \right\} (2l{+}d{-}2). \end{aligned} \tag{3.97}$$

For real solutions ϕ we have $\phi^a_{-\omega,\underline{l},-m_l} = \overline{\phi^a_{\omega \underline{l} m_l}}$ and $\phi^b_{-\omega,\underline{l},-m_l} = \overline{\phi^b_{\omega \underline{l} m_l}}$ and thus obtain

$$\mathrm{g}_\rho(\phi, \phi) = 2\pi R_{\text{AdS}}^{d-1} \int \mathrm{d}\omega \sum_{\underline{l}, m_l} \left\{ \left| \phi^a_{\omega \underline{l} m_l} \right|^2 j_{\omega l}^{S,ba} + \left| \phi^b_{\omega \underline{l} m_l} \right|^2 / j_{\omega l}^{S,ba} \right\} (2l{+}d{-}2). \tag{3.98}$$

We can read off that the real g-product is positive for modes with ω and such that $j_{\omega l}^{S,ba}$ is positive. The β-version $j_{\omega l}^{S,ba,\beta}$ is positive for all $\omega \geq (\tilde{m}_+{+}l)$ while the α-version is positive for all $\omega \leq -(\tilde{m}_+{+}l)$.

For all other frequencies, both versions alternate between intervals with positive and negative sign. (See for example Figure 3.99.) Therefore, the interlaced version $j^{S,ba,\mathrm{iso}}_{\omega l}$ alternates its sign quite frequently. However, it is positive for all magic frequencies $\pm\omega^+_{nl}$. For the two-branched version $j^{S,ba,\mathrm{two}}_{\omega l}$ the situation is simpler: due to its definition, $j^{S,ba,\mathrm{two}}_{\omega l}$ is positive for all frequencies with $|\omega| \geq (\tilde{m}_+ + l)$, while it alternates its sign for the remaining frequencies. In (C.436) we give explicit formulas that tell us where $j^{S,ba,\mathrm{two}}_{\omega l}$ and $j^{S,ba,\mathrm{iso}}_{\omega l}$ are positive and where negative.

Let us compare this to the real g-product of a Minkowski hypercylinder. There we also have two complex structures: J^{pos}_r and J^{iso}_r. Both induce positive real g-products for the propagating modes (which there are those with $|\omega| > m$). Moreover, $\mathrm{g}^{\mathrm{pos}}_r$ is positive for the evanescent modes as well, while $\mathrm{g}^{\mathrm{iso}}_r$ alternates sign for evanescent modes ($|\omega| < m$). On AdS, both the interlaced and two-branched version have similar properties: for the propagating modes (which here are those with magic frequencies $\pm\omega^+_{nl}$) the real g-products become positive, while for the evanescent modes (all other frequencies) their sign alternates.

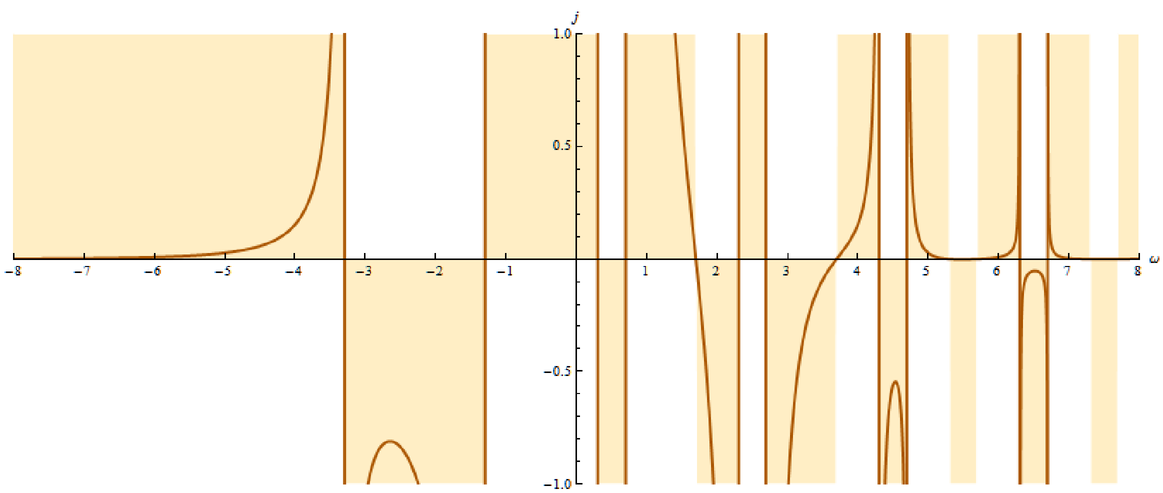


Figure 3.99: Typical plot of the α-version $j^{S,ba,\alpha}_{\omega l}$ (orange continuous curves): here $d = 3$, $m = 1$, $R_{\mathrm{AdS}} = 1$, $l = 2$, giving us $\tilde{m}_+ \approx 3.3$. The vertical orange lines are the poles of $j^{S,ba,\alpha}_{\omega,2}$. The background color indicates the sign: the α-version is positive where the orange background is above the ω-axis, and negative where the orange background lies below it.

3.3.6 Flat limits

In Appendix C.7.3 we study the flat limits of J^{iso}_ρ and the induced $\mathrm{g}^{\mathrm{iso}}_\rho$. We find that the interlaced complex structure commutes with the process of taking the flat limit, and that the flat limit of $\mathrm{g}^{\mathrm{iso}}_\rho$ is the Minkowski $\mathrm{g}^{\mathrm{iso}}_r$. However, this holds only for a discrete subset of frequencies on the AdS side (the magic frequencies and some heuristically chosen frequencies). As for the time-interval regions, these frequencies become dense in the flat limit.

Since this is not very satisfactory, in Appendix C.7.4 we study the flat limit of the two-branched version J^{two}_ρ (3.94). This leads us to modifying the two-branched version for low frequencies, resulting in a positive-definite version (C.473):

$$J^{\rho,\mathrm{pos}}_{\omega l} = \begin{pmatrix} 0 & -1/j^{S,ba,\mathrm{pos}}_{\omega l} \\ j^{S,ba,\mathrm{two}}_{\omega l} & 0 \end{pmatrix} \qquad j^{S,ba,\mathrm{pos}}_{\omega l} = \begin{cases} j^{S,ba,\beta}_{\omega l} & \omega \leq -\omega^{\mathrm{split}}_l \\ j^{S,ba,\alpha}_{\omega l} & \omega \geq +\omega^{\mathrm{split}}_l \\ j^{S,ba,\mathrm{obv}}_{\omega l} & \text{else} \end{cases}, \tag{3.100}$$

wherein the splitting frequency $\omega_l^{\text{split}} = \tilde{m}_+ + l$ defines which frequencies are "high" and which are "low". The "obvious" version (C.471)

$$j_{\omega l}^{S,ba,\text{obv}} = \tfrac{1}{\pi}\,\Gamma(\gamma^{\mathrm{S}})\,\Gamma(\gamma^{\mathrm{S}}-1)\left(\tfrac{1}{2}p_\omega^{\mathbb{R}}\right)^{-2l-1} \tag{3.101}$$

$$= (2l+d-2)!!\,(2l+d-4)!!\;2^{3-d}\left(p_\omega^{\mathbb{R}}\right)^{-2l-1} \tag{3.102}$$

is the simplest version which is positive and whose flat limit recovers the Minkowski J_r^{pos} (hence also the induced $\mathrm{g}_\rho^{\text{pos}}$ of the AdS hypercylinder recovers the Minkowski $\mathrm{g}_r^{\text{pos}}$) as in Diagram (3.85).

$$\begin{array}{ccc} \phi^{\text{AdS}} & \xrightarrow{\;J_\rho\;} & J_\rho\phi^{\text{AdS}} \\ \text{cont. flat lim.}\big\downarrow & & \big\downarrow\text{cont. flat lim.} \\ \phi^{\text{Mink}} & \xrightarrow{\;J_r^{\text{pos}}\;} & J_r^{\text{pos}}\phi^{\text{Mink}} \end{array} \tag{3.103}$$

In Figures 3.104 and 3.105 we plot our different versions of $j_{\omega l}^{S,ba}$. The positive $j_{\omega l}^{S,ba,\text{pos}}$ (thick, continuous orange curve) is discontinuous at $\omega = \pm\omega_l^{\text{split}}$. At $\omega = 0$ it is continuous (albeit not differentiable). At $\omega = \pm\tilde{m}_+$ it diverges as commented in Appendix C.7.4. (Thus for $l = 0$ we have $\omega_l^{\text{split}} = \tilde{m}_+$ and hence the divergence and discontinuity coincide.) For large $|\omega|$, we see that $j_{\omega l}^{S,ba,\text{two}}$ and $j_{\omega l}^{S,ba,\text{pos}}$ coincide and are positive, while for small $|\omega|$ only $j_{\omega l}^{S,ba,\text{pos}}$ is positive and $j_{\omega l}^{S,ba,\text{two}}$ alternates sign. Further, $j_{\omega l}^{S,ba,\text{iso}}$ coincides with $j_{\omega l}^{S,ba,\text{two}}$ only on half of the integer ω-intervals. This is so, because their respective associations of $j_{\omega l}^{S,ba,\alpha}$ and $j_{\omega l}^{S,ba,\beta}$ coincide for half of these intervals while they are opposite for the other half. We also see that all versions are frequency symmetric: $j_{\omega l}^{S,ba} = j_{-\omega,l}^{S,ba}$.

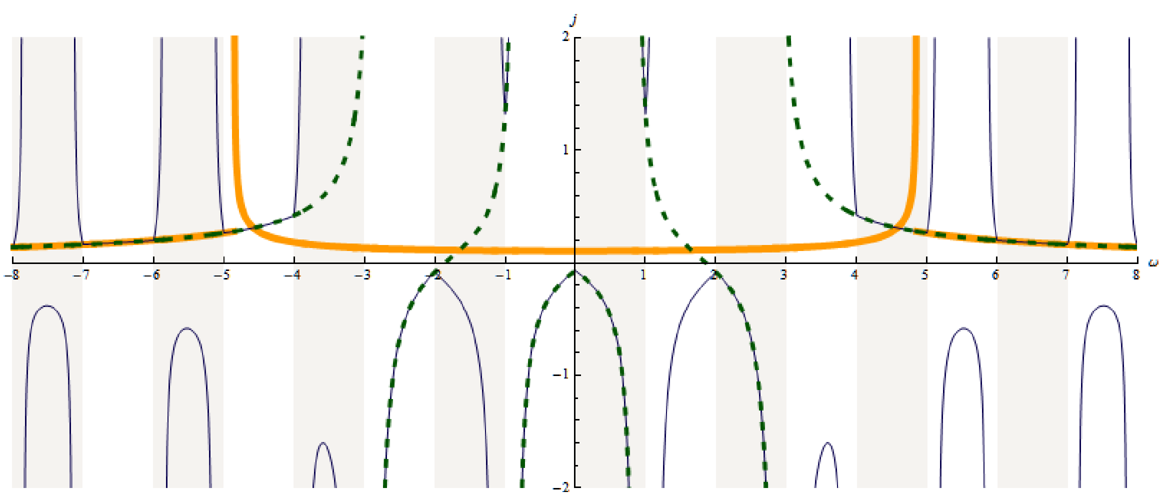


Figure 3.104: Typical plots of $j_{\omega l}^{S,ba,\text{iso}}$ (thin continuous curve, dark blue), $j_{\omega l}^{S,ba,\text{two}}$ (dashed curve, dark green) and $j_{\omega l}^{S,ba,\text{pos}}$ (thick continuous curve, orange). Here $d = 3$, $m = 3$, $R_{\text{AdS}} = 1$, $l = 0$, giving us $\tilde{m}_+ \approx 4.9$ and thus $\omega_l^{\text{split}} = \tilde{m}_+ + l \approx 4.9$. The light gray background shading merely serves to distinguish the integer intervals on the ω-axis.

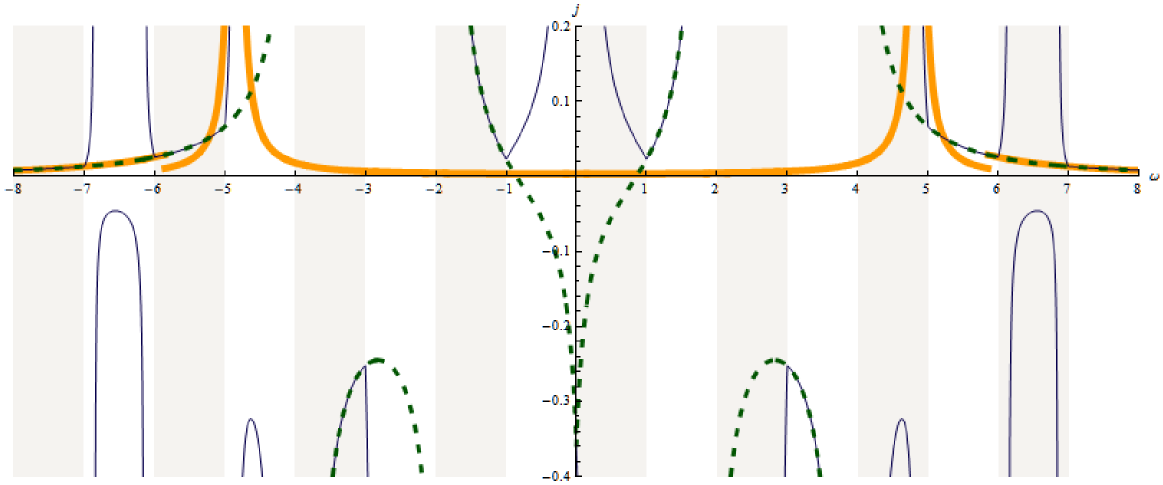


Figure 3.105: The same plot as Figure 3.104, but now for $l = 1$ and thus $\omega_l^{\text{split}} = \tilde{m}_+ + l \approx 5.9$.

The properties of our two complex structures are summarized in Table 3.106. The induced real product $\mathrm{g}_\rho^{\text{pos}}$ is positive-definite for all modes. By contrast, $\mathrm{g}_\rho^{\text{iso}}$ is indefinite (see Appendix C.6 for details) but at least it is positive-definite for modes with magic frequencies.

AdS	J_ρ^{iso}	J_ρ^{pos}
commute with time-translations	✓	✓
commute with spatial rotations	✓	✓
commute with boosts	✓	-
weak amplitude equivalence	✓	✓
induced real inner product g_ρ	indefinite	positive-definite
flat limit	J_ρ^{iso} only for a discrete subset of frequencies	J_ρ^{pos} for all real frequencies

Table 3.106: Properties of complex structures for AdS hypercylinder

Both J_ρ^{pos} and J_ρ^{iso} induce (a weak version of) the amplitude equivalence between time-intervals and rod regions which we introduced in Section 3.1.10. That is, the real products induced by the two complex structures both fulfill (3.73) for global classical solutions η, ζ:

$$\mathrm{g}_t(\eta,\zeta) = \tfrac{1}{2}\,\mathrm{g}_\rho^{\text{pos}}(\eta,\zeta) \qquad\qquad \mathrm{g}_t(\eta,\zeta) = \tfrac{1}{2}\,\mathrm{g}_\rho^{\text{iso}}(\eta,\zeta)\,. \tag{3.107}$$

This is natural, since global solutions consist of modes with magic frequencies, and for these the complex structures coincide. (By weak equivalence we emphasize that this equivalence holds only for the discrete magic frequencies. By contrast, on Minkowski spacetime a strong equivalence holds, that is, it is valid for the continuous range of frequencies $E^2 > m^2$.) Combining (3.97) with (3.100) yields the positive real g-product which is induced by J_ρ^{pos}:

$$\mathrm{g}_\rho^{\text{pos}}(\eta,\zeta) \;=\; 2\pi R_{\text{AdS}}^{d-1}\int \mathrm{d}\omega \sum_{\underline{l},m_l}\Big\{\eta^a_{\omega\underline{l}m_l}\,\zeta^a_{-\omega,\underline{l},-m_l}\,j_{\omega l}^{S,ba,\text{pos}} + \eta^b_{\omega\underline{l}m_l}\,\zeta^b_{-\omega,\underline{l},-m_l}/j_{\omega l}^{S,ba,\text{pos}}\Big\}\,(2l+d-2). \tag{3.108}$$

This product determines the free amplitude (3.32) for coherent holomorphic states (3.1):

$$\rho_{\rho_0}\big(\overline{K^{\xi_0}_{\Sigma_0}}\big) \;=\; \exp\Big(-\tfrac{1}{2}\mathrm{g}_{\rho_0}(\xi^{\mathrm{I}}_0,\,\xi^{\mathrm{I}}_0) - \tfrac{\mathrm{i}}{2}\mathrm{g}_{\rho_0}(\xi^{\mathrm{R}}_0,\,\xi^{\mathrm{I}}_0)\Big).$$

3.3.7 Relation with Colosi's complex structures

In order to connect our nondiagonal form (3.86) of the complex structure to previous work, we now show that it coincides with Colosi's form of the complex structure. In [16] Colosi derived the form of the vacuum operator A_Σ, which determines the vacuum wave function(al) of a hypersurface Σ in the Schrödinger representation of the quantum states (see also Section IV.B in [19]). This result is obtained in two different ways: first via Feynman path integral quantization, and then through canonical quantization. It is assumed, that spacetime can be foliated with the foliation parameter denoted by τ, and that the spacetime metric is block-diagonal w.r.t. τ, that is: $g_{\tau,\mu} = 0$ for all $\mu \neq \tau$. Further, it is asssumed that the Klein-Gordon equation can be solved via separation of variables, resulting in two different classes of modes (like spherical Bessel and Neumann modes for Minkowski spacetime). For short, we call these a-modes and b-modes and denote them by $\mu^{(a,b)}_{\omega\underline{l}m_l}$.

The vacuum operator is determined by two complex functions $c^{a,b}_{\omega\underline{l}m_l}$ on momentum space, obeying $c^{a,b}_{-\omega,\underline{l},-m_l} = \overline{c^{a,b}_{\omega\underline{l}m_l}}$ and $\mathbb{I}\mathrm{m}\,\big(\overline{c^a_{\omega\underline{l}m_l}}c^b_{\omega\underline{l}m_l}\big) \neq 0$. Since vacuum states and complex structures on a hypersurface are in one-to-one correspondences [61], fixing the vacuum through any choice of $c^{a,b}_{\omega\underline{l}m_l}$ correpsonds to fixing the complex structure. It turns out, that Colosi's vacuum operator A_Σ transcribes into the following complex structure, which we thus call Colosi's form:

$$\begin{pmatrix} (J_r\phi)^a_{\omega\underline{l}m_l} \\ (J_r\phi)^b_{\omega\underline{l}m_l} \end{pmatrix} \;=\; \frac{1}{\mathbb{I}\mathrm{m}\,\big(\overline{c^a_{\omega\underline{l}m_l}}c^b_{\omega\underline{l}m_l}\big)} \begin{pmatrix} +\mathbb{R}\mathrm{e}\,\big(\overline{c^a_{\omega\underline{l}m_l}}c^b_{\omega\underline{l}m_l}\big) & -\big|c^a_{\omega\underline{l}m_l}\big|^2 \\ +\big|c^b_{\omega\underline{l}m_l}\big|^2 & -\mathbb{R}\mathrm{e}\,\big(\overline{c^a_{\omega\underline{l}m_l}}c^b_{\omega\underline{l}m_l}\big) \end{pmatrix} \begin{pmatrix} \phi^a_{\omega\underline{l}m_l} \\ \phi^b_{\omega\underline{l}m_l} \end{pmatrix}. \tag{3.109}$$

The relation between j-factors and c-factors can be obtained by first writing:

$$c^a \;=\; r_a \mathrm{e}^{\mathrm{i}\varphi_a} \qquad c^b \;=\; r_b \mathrm{e}^{\mathrm{i}\varphi_b} \qquad\qquad \mathbb{I}\mathrm{m}\,\big(\overline{c^a}c^b\big) \neq 0 \iff (\varphi_b - \varphi_a) \neq \pm n\pi \quad \forall\, n \in \mathbb{N}_0. \tag{3.110}$$

Comparing to (3.109), we can read off that

$$j^{aa} \;=\; \frac{\cos(\varphi_b - \varphi_a)}{\sin(\varphi_b - \varphi_a)} \qquad j^{ab} \;=\; -\frac{r_a/r_b}{\sin(\varphi_b - \varphi_a)} \qquad j^{ba} \;=\; +\frac{r_b/r_a}{\sin(\varphi_b - \varphi_a)}. \tag{3.111}$$

That is, the j-form is determined by the two real functions j^{ab} and j^{ba}, while Colosi's form is determined by the two real functions $Q_r := (r_a/r_b)$ and $\Delta\varphi := (\varphi_b - \varphi_a)$. The equivalence between both forms is provided by

$$Q_r \;:=\; \frac{r_a}{r_b} \;=\; \sqrt{\frac{-j^{ab}}{j^{ba}}} \qquad\qquad \Delta\varphi \;:=\; (\varphi_b - \varphi_a) \;=\; \arcsin\sqrt{\frac{1}{-j^{ab}\,j^{ba}}}. \tag{3.112}$$

Further, note that we can combine the modes $\mu^{(a,b)}_{\omega\underline{l}m_l}$ of the real radial functions (2.167) to complex modes

$$\mu^{(+)}_{\omega\underline{l}m_l} \;:=\; c^a_{\omega,\underline{l}m_l}\mu^{(a)}_{\omega\underline{l}m_l} + c^b_{\omega\underline{l}m_l}\mu^{(b)}_{\omega\underline{l}m_l} \qquad\qquad \mu^{(-)}_{\omega\underline{l}m_l} \;:=\; \overline{\mu^{(+)}_{\omega\underline{l}m_l}}, \tag{3.113}$$

and expand classical solutions in these modes with $\phi^{\pm}_{\omega\underline{l}m_l}$ the corresponding momentum representation. We call this the frequency representation. Then, Colosi's form (3.109) induces the usual simple form (2.212), (2.118) of the complex structure:

$$\big(J_r\,\phi\big)^{\pm}_{\omega\underline{l}m_l} \;=\; -\mathrm{i}\,\phi^{\pm}_{\omega\underline{l}m_l}. \tag{3.114}$$

Now let us check whether Colosi's form fulfills the conditions that lead to the j-form (3.86) for AdS hypercylinders. (3.109) is clearly a nondiagonal form fulfilling $J^2 = -1$. In order for the j-form to be equivalent to Colosi's form, j^{ab} and j^{ba} need to have opposite sign. (3.109) also fulfills $(j^{aa})^2 = -j^{ab}j^{ba} - 1 \geq 0$. Since the $j^{S,\cdot\cdot}$-factors depend only on ω and l (due to essential properties and commutation with time translations and spatial rotations), the same must be required for the $c^{a,b}$-factors. Then, $c^{a,b}_{-\omega,l} = \overline{c^{a,b}_{\omega l}}$, and only frequency symmetry remains to be checked. This can be done using (3.111) or (3.109). In both ways, we quickly see that the complex conjugation caused by $\omega \rightarrow -\omega$ induces an overall minus sign for the complex structure, thus breaking frequency symmetry. Resolving this issue will be part of future work.

3.3.8 Giddings' radial S-matrix for AdS

For comparison, let us summarize Giddings' construction of a radial S-matrix for AdS in [37]. Since Giddings' and our approach rely on Klein-Gordon modes (although he uses only S^a-modes, while we include S^b-modes as well), one should expect that the results of both approaches relate rather well. In (38) therein he defined his boundary S-matrix S_∂ as (we write f^+ for Giddings' f and f^- for his f')

$$S_\partial[f_1^+, ..., f_m^+, f_1^-, ..., f_n^-] := \left\langle 0 \,\middle|\, \mathbf{T} \prod_{j=1}^{n} \hat{\alpha}^{\text{out}}_{f_j^-} \prod_{i=1}^{m} (\hat{\alpha}^{\text{in}}_{f_i^+})^\dagger \,\middle|\, 0 \right\rangle = {}_{\text{out}}\langle f_1^-, ..., f_n^- \,|\, f_1^+, ..., f_m^+ \rangle_{\text{in}}. \tag{3.115}$$

Therein, $|\, 0 \,\rangle$ is the vacuum state on the boundary, and $\mathbf{T}$ is the time-ordering operator. The $f^\pm$ functions define wave packets as given below. The labels $\pm$ indicate that f^+ is purely positive frequency, while f^- is purely negative frequency. In order to have the time-ordering well-defined, above it is required that all wave packets defined by the $f^\pm$ are non-overlapping in time, and further that the support of all f_j^- lies in the future of all f_i^+. If the $f^\pm$ do not fulfill these conditions, then the above definition can be taken without the time-ordering, but more care must be exercised with the interpretation.

An annihilator $\hat{\alpha}^{\text{in}}_{f^+}$ has a positive-frequency function f^+, and with some renormaliation factor Z^+ it is defined as below. A creator $(\hat{\alpha}^{\text{out}}_{f^-})^\dagger$ has a negative-frequency function f^-, and with renormalizer Z^- it is defined in Giddings' (34) as

$$\hat{\alpha}^{\text{in}}_{f^+} := \hat{\alpha}_{f^+} / Z^+ \qquad\qquad (\hat{\alpha}^{\text{out}}_{f^-})^\dagger := \hat{\alpha}_{f^-} / Z^-. \tag{3.116}$$

In Giddings' (24), asymptotic creation and annihilation operators for states on the boundary are defined as the following limit, wherein $\hat{\Phi}$ is the full interacting field:

$$\hat{\alpha}_f := \lim_{\rho \to \pi/2} \int_{\Sigma_\rho} \mathrm{d}\Sigma^\mu \left(\overline{\phi_f} \overleftrightarrow{\partial}_\mu \hat{\Phi} \right). \tag{3.117}$$

Since no mention is made of modes that diverge on the time-axis, the notation in (8) of [37] suggests that therein $\phi_{\omega \underline{l} m_l}(t, \rho, \omega)$ is our $\mu^{(a)}_{\omega \underline{l} m_l}(t, \rho, \omega)$, that is: the extension of Giddings' global modes (our Jacobi modes) $\phi_{n,\underline{l},m_l}$ from magic frequencies ω^+_{nl} to arbitrary frequency ω. Thus the solution ϕ_f consists only of hypergeometric S^a-modes:

$$\phi_f(t, \rho, \Omega) = \int \mathrm{d}\omega \sum_{\underline{l}, m_l} f_{\omega \underline{l} m_l} \, \mu^{(a)}_{\omega \underline{l} m_l}(t, \rho, \Omega), \tag{3.118}$$

which makes Giddings' f correspond to our $\phi^{S,a}$. However, Giddings' construction can be extended to comprise also S^b-modes, and then the mixing of S^a and S^b-modes must be taken care of. This is precisely what our complex structure J_ρ achieves. The operator $\hat{\alpha}_f$ acts on states as given in Giddings' (28):

$$\langle 0 \,|\, \hat{\alpha}_f \,|\, \Delta, n, \underline{l}, m_l \rangle \sim \overline{f_{\omega^+_{nl}, \underline{l}, m_l}} \tag{3.119}$$

$$\langle \Delta, n, \underline{l}, m_l \,|\, \hat{\alpha}_f \,|\, 0 \rangle \sim \overline{f_{-\omega^+_{nl}, \underline{l}, -m_l}}, \tag{3.120}$$

wherein $|\,\Delta,\, n, \underline{l}, m_l\rangle$ is a one-particle state with frequency ω^+_{nl} and angular quantum numbers $(\underline{l}, m_l)$. These kets form a representation of the AdS isometry group $\mathrm{SO}(2,d)$, with Δ the representation's weight. For a free field we have $\Delta = \tilde{m}_+$. Giddings observed for (3.119), that $\hat{\alpha}_f$ annihilates a one-particle boundary-state $|\,\Delta,\, n, \underline{l}, m_l\rangle$ if $f_{\omega\underline{l}m_l}$ is nonzero at $\omega = +\omega^+_{nl}$. In (3.120), $\hat{\alpha}_f$ creates the one-particle boundary-state $|\,\Delta,\, n, \underline{l}, m_l\rangle$ if $f_{\omega\underline{l}m_l}$ is nonzero at $\omega = -\omega^+_{nl}$. In addition to particle states, in (36) Giddings also defines in and out states of wave packets whose profile is determined by f-functions:

$$\Big|\, f_1^+, ..., f_m^+ \Big\rangle_{\rm in} := \prod_{i=1}^{m} (\hat{\alpha}^{\rm in}_{f_i^+})^\dagger \Big|\, 0 \Big\rangle \qquad\qquad \Big|\, f_1^-, ..., f_n^- \Big\rangle_{\rm out} := \prod_{j=1}^{n} (\hat{\alpha}^{\rm out}_{f_j^-})^\dagger \Big|\, 0 \Big\rangle. \tag{3.121}$$

For evaluating definition (3.115) of the boundary S-matrix, Giddings derived an LSZ-formula in (42) of [37]:

$$S_\partial[f_1^+, ..., f_m^+, f_1^-, ..., f_n^-] = \Big(\prod_{j=1}^{n} \int \mathrm{d}V_j^- \, \frac{\phi_{f_j^-}(x_j^-)}{Z_j^-} \Big) \Big(\prod_{i=1}^{m} \int \mathrm{d}V_i^+ \, \frac{\phi_{f_i^+}(x_i^+)}{Z_i^+} \Big) \Big\langle 0 \,\Big|\, \mathbf{T} \prod_{q=1}^{n} \hat{\Phi}(x_q^-) \prod_{p=1}^{m} \hat{\Phi}(x_p^+) \,\Big|\, 0 \Big\rangle_{\rm T}. \tag{3.122}$$

Therein, the multiple volume integrals $\int \mathrm{d}V^\pm$ are over all of AdS, $\mathbf{T}$ is again the time-ordering operator, and the subscript T indicates that the vacuum expectation value is calculated with the Truncated Green's function. The superscripts $\pm$ for the volume elements $\mathrm{d}V$ and related coordinates x are merely meant to mark which coordinate refers to which integration. In order to relate our S-matrix to Giddings', let us evaluate (3.122) for a simple case. Thus we consider only one incoming packet shaped by f^+, and one outgoing packet shaped by f^-. We consider only the free theory, and set the renormalizers $Z^\pm = 1$. Then,

$$S_\partial[f^+, f^-] = \int \mathrm{d}V^- \, \phi_{f^-}(x^-) \int \mathrm{d}V^+ \, \phi_{f^+}(x^+) \, \langle 0 \,|\, \mathbf{T} \hat{\Phi}(x^-) \hat{\Phi}(x^+) \,|\, 0 \rangle_{\rm T}. \tag{3.123}$$

We can replace $\langle 0 \,|\, \mathbf{T} \hat{\Phi}(x^-) \hat{\Phi}(x^+) \,|\, 0 \rangle_{\rm T}$ by the Feynman propagator $G_{\rm F}(x^+, x^-)$. For simplicity we consider only solutions ϕ_f composed of (ordinary) Jacobi modes (2.173),

$$\mu^{(+)}_{n\underline{l}m_l}(t, \rho, \Omega) \;=\; \mu^{(S,a)}_{\omega^+_{nl}\underline{l}m_l}(t, \rho, \Omega) \;=\; \mathrm{e}^{-\mathrm{i}\omega^+_{nl} t} \, Y^{m_l}_{\underline{l}}(\Omega) \, J^{(+)}_{nl}(\rho) \;. \tag{3.124}$$

We recall that despite using only Jacobi modes we are considering solutions near a hypercylinder, and that Jacobi modes are just S^a-modes with magic frequencies ω^+_{nl}. Therefore we can expand the fields using the hypergeometric expansion (2.186), keeping in mind that the solutions here are only S^a-modes with purely positive (respectively negative) frequencies:

$$\phi_{f^+}(t, \rho, \Omega) \;=\; \sum_{n\underline{l}m_l} f^+_{n\underline{l}m_l} \, \mu^{(S,a)}_{\omega^+_{nl}, \underline{l}, m_l}(t, \rho, \Omega) \qquad \phi_{f^-}(t, \rho, \Omega) \;=\; \sum_{n\underline{l}m_l} f^-_{n\underline{l}m_l} \, \mu^{(S,a)}_{-\omega^+_{nl}, \underline{l}, m_l}(t, \rho, \Omega) \tag{3.125}$$

Thus, Giddings $f^+_{n\underline{l}\,m_l}$ is our $\phi^{S,a}_{\omega^+_{nl}\underline{l}m_l}$, and Giddings $f^-_{n\underline{l}\,m_l}$ is our $\phi^{S,a}_{-\omega^+_{nl}, \underline{l}, m_l}$. For the Feynman propagator we can try Equation (80), from [19], which for AdS becomes

$$G_{\rm F}(x, x') \;=\; \tfrac{\mathrm{i}}{2} \sum_{n, \underline{l}, m_l} \frac{Y^{m_l}_{\underline{l}}(\Omega)\, \overline{Y^{m_l}_{\underline{l}}(\Omega')} \; J^{(+)}_{nl}(\rho) \, J^{(+)}_{nl}(\rho')}{\omega^+_{nl} \, (R^{d-1}_{\rm AdS} \mathcal{N}^+_{nl})^2} \Big(\theta(t - t') \, \mathrm{e}^{-\mathrm{i}\omega^+_{nl}(t - t')} + \theta(t' - t) \, \mathrm{e}^{-\mathrm{i}\omega^+_{nl}(t' - t)} \Big). \tag{3.126}$$

Then, $(\Box_x - m^2)\, G_{\rm F}(x, x') \;=\; -\delta^{(d+1)}(x - x')/\sqrt{|g(x)|}$. Since the support of the f^--packet is to the future of the f^+-packet, in $G_{\rm F}(x^+, x^-)$ we only keep the term $\theta(t^- - t^+)\, \mathrm{e}^{-\mathrm{i}\omega^+_{nl}(t^- - t^+)}$ and drop the term $\theta(t^+ - t^-)\, \mathrm{e}^{-\mathrm{i}\omega^+_{nl}(t^+ - t^-)}$. (However, the result actually does not depend on this.) Inserting now into (3.123) the expansions (3.125) and the Feynman propagator (3.126), we can first integrate over the angles Ω^+ and Ω^-, then over the times t^+ and t^-. This gives the following Kronecker deltas: δ_{l,l^+}, δ_{l,l^-}, $\delta_{m_l,-m_l^+}$, δ_{m_l,m_l^-}, δ_{n,n^+} and δ_{n,n^-}. Integrating then over ρ^+ with metric factor $R^{d-1}\tan^{d-1}\rho^+$ and ditto for ρ^- cancels the denominator $(R_{\mathrm{AdS}}^{d-1}\mathcal{N}_{nl}^+)^2$. Thus, we obtain

$$S_\partial[f^+,f^-] \sim \mathrm{i}\sum_{n,\underline{l},m_l} f^+_{n\underline{l}m_l}\, f^-_{n,\underline{l},-m_l}\,/\omega^+_{nl} \;=\; \mathrm{i}\sum_{n,\underline{l},m_l} \phi^{S,a}_{\omega^+_{nl}\underline{l}m_l}\,\phi^{S,a}_{-\omega^+_{nl},\underline{l},-m_l}\,/\omega^+_{nl}. \tag{3.127}$$

We did not take much care with the renormalizers $Z^\pm$, hence we do not consider the quotient $1/\omega^+_{nl}$, and focus only on the factor $\phi^{S,a}_{\omega^+_{nl}\underline{l}m_l}\,\phi^{S,a}_{-\omega^+_{nl},\underline{l},-m_l}$. This factor appears in our S-matrix as well (multiplied by the factor $j^{S,ba}_{\omega l}$ of the complex structure). The result (3.127) coming from Giddings' approach relates to our results as follows. Giddings considers states consisting of several wave packets, each shaped by its individual momentum function $f^\pm_{\omega\underline{l}\,m_l}$. We do not consider these packets individually, rather we view them combined as a "multiple-packet" shaped by a *single* momentum function $\phi^a_{\omega\underline{l}m_l}$ (which is just the sum over all $f^\pm_{\omega\underline{l}\,m_l}$). Further, Giddings considers multiple-particle states, while we consider coherent states. In Giddings' canonical approach coherent states can be written as an exponential of creators acting on the vacuum, instead of pure creators as in (3.121). Hence for coherent states we would obtain an S-matrix element like

$$S^{\mathrm{coh}}_\partial[f^+,f^-] \sim \exp\Big(\mathrm{i}\sum_{n,\underline{l},m_l} f^+_{n\underline{l}m_l}\, f^-_{n,\underline{l},-m_l}\,/\omega^+_{nl}\Big) \;=\; \exp\Big(\mathrm{i}\sum_{n,\underline{l},m_l} \phi^{S,a}_{\omega^+_{nl}\underline{l}m_l}\,\phi^{S,a}_{-\omega^+_{nl},\underline{l},-m_l}\,/\omega^+_{nl}\Big). \tag{3.128}$$

This resembles our S-matrix rather well. Therefore we conclude that Giddings' canonical approach and the GBF approach with Holomorphic Quantization are indeed compatible. However, for making this more precise one would need to know more accurately the details of Giddings' setup and perform a more careful analysis than the one sketched here. Like our approach, Giddings' is also applicable for interacting field theories via perturbation techniques. The GBF approach provides a rigorous framework, and facilitates to implement invariance of the S-matrix under isometry-actions. In comparison, Giddings did not discuss the $\mathrm{SO}(2,d)$-invariance of his boundary S-matrix S_∂, rather the abstract states $|\,\Delta,\, n, \underline{l}, m_l\rangle$ which he uses were taken to be a representation of $\mathrm{SO}(2,d)$ from the outset. However, the effects of time-ordering and the truncated Green's function on the $\mathrm{SO}(2,d)$-invariance of S_∂ were not elaborated on. Further, the notation of [37] suggests that (for both free and interacting theory) only the S^a-modes were taken into account, while our construction also includes the S^b-modes.

3.3.9 Restricting to one single scattering

The problem of how to construct wave functions resp. wave packets that scatter only once inside of AdS is rather nontrivial[6], see for example the detailed discussion in [36], and also [8]. (The problem is not due to using rod regions, which works perfectly well in Minkowski spacetime. Rather it is caused by the particular geometry of AdS with its reconverging timelike geodesics.) Therein, two ways of resolving this problem are brought forth. The first way is considering not a global AdS spacetime, but instead a finite AdS bubble which is somehow immersed in a spacetime which admits (temporally) asymptotically free states (for example Minkowski spacetime). However, this is rather a change of the setting than a solution to the original problem.

The second way is constructing a quantum-mechanical wave function $\psi(x)$ for point particles scattering on AdS, and requiring it to have compact support on the boundary hypercylinder $\overline{\Sigma}_0$ of some rod region of radius ρ_0. This support encodes where particles enter and leave the region. This boundary data is written as a function $f(t,\Omega)$. The wave function is then constructed using the Feynman propagator G_{F}:

$$\psi_f(t,\rho,\Omega) \;=\; \int_{\overline{\Sigma}_0} \mathrm{d}t'\,\mathrm{d}^{d-1}\Omega'\; f(t',\Omega')\; G_{\mathrm{F}}((t,\rho,\Omega),(t',\rho_0,\Omega')).$$

[6]We thank Axel Weber (IFM-UMSNH) for indicating us to discuss this issue.

(In case the boundary is located at infinity $\rho_0 = \frac{\pi}{2}$, a rescaled version of the Feynman propagator is used, called bulk-to-boundary propagator.) However, the wave packets induced by the compactly supported boundary data are found to be not sharply peaked (power law tails in both momentum and position space).

Using Holomorphic Quantization (HQ), we propose the following approach to this problem. Considering a rod region of finite radius ρ_0, we control where and when particles enter and leave the region through the quantum state on the boundary $\overline{\Sigma}_{\rho_0}$. That is, we need states that encode particles which enter and leave the rod region each at most once. (In other words: we want that particles having left the rod region not to return into it and scatter again.) This should be achieved by using holomorphic states, whose characterizing solution λ has compact support on the boundary. This support specifies where and when the particles and enter and leave. (We recall that λ needs to be well defined and bounded only near the boundary $\overline{\Sigma}_{r_0}$, but not on the whole rod region.)

Given such compact boundary data for the values of λ and its momentum $\partial_\rho \lambda$, the solution λ is determined uniquely through (2.243) as discussed in Section 2.6.8. This is based on the fact that the temporal and angular parts of our modes $e^{-i\omega t} Y_{\underline{l}}^{m_l}(\Omega)$ form an orthonormal basis for boundary data on hypercylinders $\Sigma_{\rho_0} = \mathbb{R}_t \times \mathbb{S}^d_{\rho_0}$. In turn, this is due to including all frequencies $\omega \in \mathbb{R}$ and not only frequencies above some mass threshold.

We can use coherent states $K^{\lambda}_{\overline{\Sigma}_{\rho 0}}$ whose amplitudes are given for general spacetimes and regions in Eqation (47) of [59]. For AdS, we have calculated their amplitudes in (3.108). The expectation value of a coherent state should be its characterizing solution λ and thus have compact support on the rod's boundary, corresponding to only one scattering.

We can also use multi-particle states, as given by Equation (17) in [62], which are symmetrized products of the 1-particle states (3.5). Their amplitudes are given in Equation (84) of [62], and are consistent with those of the coherent states as discussed in Section 12 therein.

In how far this approach actually realizes a single scattering, needs careful study and interpretation. In particular, in order to calculate GBF expectation values, subspaces of the state space need to be specified, which correspond to preparation and to observation. At present, it is not clear to the author, how these subspaces are to be chosen for some given boundary solution λ.

Chapter 4

Summary and outlook

The focus of this thesis is applying the General Boundary Formulation (GBF) of Quantum Theory to real Klein-Gordon QFT in Anti de Sitter spacetime (AdS). The goal is constructing an S-matrix for this theory, which is problematic using standard techniques due to the lack of the usual asymptotically free states on AdS.

The GBF is a rather recent formulation of Quantum Theory, which generalizes the standard formulation, while including it and reproducing its results. (That is, the GBF is not some particular quantum theory, but an axiomatic framework about how any quantum theory should be formulated.) By standard formulation we mean that quantum states "live on" (decribe the system on) equal-time or Cauchy hypersurfaces (in the Schrödinger picture), and amplitudes are calculated with the inner product of the theory's state space, which is a Hilbert space. The GBF generalizes this by associating a state space $\mathcal{H}_\Sigma$ to each hypersurface Σ of codimension one in spacetime. A priori, these hypersurfaces can be spacelike, timelike, lightlike, or mixed. Amplitudes are calculated for states living on the boundary $\partial\mathbb{M}$ of spacetime regions $\mathbb{M}$, which have codimension zero. That is, each region $\mathbb{M}$ has an associated amplitude map $\rho_\mathbb{M}: \ \mathcal{H}_{\partial\mathbb{M}} \to \mathbb{C}$ (which depends on the theory under consideration). The regions and their boundaries can be compact or noncompact. These are the basic ideas of the GBF. In addition, there are several sets of axioms concerning unions and gluings of regions, the vacuum state, spacetime transformations, a probability interpretation consistent with Born's rule, and observables with their expectation values. We describe these details of the GBF in Chapter 1.

After that introductory chapter, we study classical and quantized Klein-Gordon theory on three types of regions in AdS: time-interval regions, and two types of hypercylinder regions called rod and tube. Time-interval regions $[t_1, t_2] \times [0, \frac{\pi}{2})_\rho \times \mathbb{S}^{d-1}$ are neighborhoods of equal-time surfaces Σ_t. This type of regions is always used in the standard formulation of Quantum Theory. Tube regions $\mathbb{R}_t \times [\rho_1, \rho_2] \times \mathbb{S}^{d-1}$ are neighborhoods of equal-radius hypercylinders $\Sigma_\rho = \mathbb{R}_t \times \mathbb{S}^d_\rho$. Rod regions $\mathbb{R}_t \times [0, \rho] \times \mathbb{S}^{d-1}$ are the whole region (including the time axis) enclosed by Σ_ρ. In Chapter 2 we review the classical theory, and in Chapter 3 we consider the quantized theory for these regions. The quantization which we use is called Holomorphic Quantization (HQ), and is based on parametrizing states by classical solutions (instead of e.g. field configurations on some hypersurface as in the Schrödinger representation). Therefore, spaces of classical solutions and structures on them are crucial ingredients for this quantization, and we study them in Chapter 2.

While on AdS we can formulate the standard S-matrix with the usual states (which describe the system at some time t), these states are restricted to a set of modes with discrete frequencies only (called magic frequencies). A generalized notion of S-matrix in the spirit of the GBF is more promising. There, asymptotic states live on the timelike hypercylinder boundary $\Sigma_{\rho=\pi/2}$ at spatial infinity [20]. This hypercylinder is the boundary of the rod region of "all of AdS". In contrast to the usual states, the states on hypercylinders allow for a continuous set of modes including all frequencies $\omega \in \mathbb{R}$. As usual in field theory, in the quantization of the Klein-Gordon field there is an ambiguity, which can be conveniently parametrized in terms of a *complex structure* J_ρ. In standard quantization in Minkowski spacetime, this ambiguity is fixed by requiring the inner product to be positive-definite and to be invariant under isometries. The main focus of the present work thus becomes finding this complex structure for AdS, and in particular for (asymptotic) fields on a hypercylinder geometry, induced by the boundary of AdS. Given this complex structure, the quantization is completely determined and the generalized S-matrix can be computed, see Section 3.1.9.

Classical theory

In the classical part, whose results are published in [30], we first give a complete list of classical Klein-Gordon solutions on AdS spacetimes. These solutions are well known already up to two exceptional cases that we present to make the list complete. The AdS Jacobi modes are solutions which are well defined and bounded on time-interval regions. These modes only exist for discrete real energies/frequencies, and are available for use in the standard S-matrix. For tube regions, there are the hypergeometric modes of types S^a, S^b and C^a, C^b (not all independent), while on rod regions the hypergeometric S^a-modes are the only well defined and bounded solutions. These modes exist for all real frequencies, and can be used for the radial S-matrix. The so-called magic frequencies are a discrete subset, for which S^a and C^a-modes coincide and turn into the Jacobi modes. (This occurs because for certain parameters the hypergeometric function turns into a Jacobi polynomial.)

For large curvature radius $R_{\text{AdS}} \to \infty$ the curvature of AdS tends to zero and (a part of) AdS asymptotically becomes Minkowski spacetime. This is called flat limit or Minkowski limit. Between AdS and Minkowski spacetime, this limit relates the spacetime metric, its Laplace-Beltrami operator and Killing vectors. Between (the flat limit of the) Killing vectors of AdS and Minkowski spacetime we find a correspondence which among others relates Minkowski translations to a class of AdS boosts. For the AdS solutions, we find that the flat limit of the S^a-modes are the (spherical) Bessel modes on Minkowski spacetime, while the S^b-modes turn into the (spherical) Neumann modes.

We also find the actions of the Killing vector fields of AdS (that is, the generators of its isometry group's Lie algebra $\mathbf{so}(2,d)$) on the modes. For time translations this action is merely a phase factor. For rotations it is a sum over modes of same top angular momentum number l with elements of Wigner's D-matrix as coefficients. For infinitesimal d-boosts the action is a sum over contiguous modes with certain coefficients that we calculate explicitly. As a byproduct we find some contiguous relations for hyperspherical harmonics and Jacobi polynomials.

Next, for equal-time surfaces Σ_t and equal-radius hypercylinder Σ_ρ we give the symplectic structures determined naturally by the Lagrange density of the theory. The well known symplectic structure for time-interval regions is defined using an equal-time surface Σ_t, but turns out to be actually t-independent. Our new symplectic structure for tube regions is defined using an equal-radius hypercylinder Σ_ρ, and turns out to be ρ-independent. These independencies are not trivial, since usually they only hold for boundaries of compact regions (which our time-interval, rod and tube regions are not), respectively compactly supported solutions, (which our time-interval, rod and tube solutions are not). The symplectic structure for solutions on rod regions vanishes identically, since the S^a-modes form a Lagrangian subspace. Again, for regions with a connected boundary this usually holds only if the region is compact respectively for compactly supported solutions. As a byproduct we find the Wronskian for the hypergeometric functions involved. Then we proceed checking the invariance of the symplectic structures under the actions of the $\mathbf{so}(2,d)$-generators. It turns out, that both symplectic structures are invariant under all AdS isometries, that is, under time translation, spatial rotations and boosts.

We conclude the classical part finding one-to-one correspondences between initial data and classical solutions on the regions. For time-interval regions, this initial data consists of the values of the field and its derivative on an equal-time surface Σ_t, despite AdS not being globally hyperbolic. For tube regions, the initial data can be given on an equal-radius hypercylinder Σ_ρ that is inside of AdS. Then it consists again of field and derivative. If the hypercylinder is the boundary $\Sigma_{\rho=\pi/2}$ of AdS, then we need to rescale: the initial data consists now of rescaled field values together with a rescaled "twisted" derivative of the field. For rod regions, a one-to-one correspondence can be established with the field values as sufficient initial data on a hypercylinder. For the boundary hypercylinder we need to rescale again.

Quantized theory

In Chapter 3 we proceed to quantizing the classical theory, using the method of Holomorphic Quantization (HQ). The results of this chapter are published in [31]. A crucial structure in HQ is the real inner product $\mathrm{g}(\cdot,\cdot) \sim \omega(\cdot, J\cdot)$. Since the symplectic structure ω is determined completely by the classical Lagrangian of the theory, it is the complex structure J alone which determines the quantization. We consider these structures on different spaces of classical solutions: solutions in a neighborhood of an equal-time hypersurface Σ_t, and solutions near hypercylinders Σ_ρ.

For an equal-time hypersurface in AdS there is a standard complex structure J_t, given in (3.81) of Section 3.3.1. It is fixed by positive-definiteness and isometry invariance. This is analogous to standard quantization in Minkowski spacetime, but with the crucial difference that in AdS this works only for a subset of field modes with discrete frequencies, the magic frequencies. Consequently the standard S-matrix on AdS is restricted to these discrete modes.

For the AdS hypercylinder geometry (used for the generalized continuum-mode S-matrix) we find that there is no complex structure which is both isometry invariant and leads to a positive-definite inner product for the whole continuum of modes. However, there are complex structures J_ρ that partially satisfy these properties. Moreover, there are additional desirable properties for the complex structure that we take into account, see Section 3.1.10: One is an equivalence of J_ρ for the subset of modes with magic frequencies to the standard complex structure J_t on AdS equal-time hypersurfaces. We call this *weak amplitude equivalence*. The other is the recovery of known complex structures on Minkowski spacetime in the limit that AdS becomes flat and the solutions of the Klein-Gordon equation become solutions on Minkowski spacetime. We call this the *flat limit* for brevity.

We show that there is a *class* of complex structures on the AdS hypercylinder that is invariant under all isometries of AdS, see Section 3.3.2. Further imposing weak amplitude equivalence motivates an interlaced construction of a complex structure J_ρ^{iso} given in (C.423) of Section 3.3.4. It retains full isometry invariance and satisfies the weak amplitude equivalence as well. A disadvantage is that it leads to an indefinite inner product on the space of modes and thus also on the space of quantum states. The space of quantum states is thus a Krein space rather than a Hilbert space. However, this does not spoil the probability interpretation of quantum theory, see [62].

If instead we do not insist on full isometry invariance, but only on invariance with respect to isometries of the AdS hypercylinder (time-translations and spatial rotations), we obtain a complex structure J_ρ^{pos} given in (3.100) of Section 3.3.6, that yields a positive-definite inner product. What is more, in the flat limit J_ρ^{pos} reproduces a complex structure J_r^{pos} on the Minkowski hypercylinder [59] which is equivalent there to the standard quantization for propagating modes. Table 3.106 in Section 3.3.6 shows a summary of the properties of these complex structures. This is to be compared to complex structures in Minkowski spacetime, see Table 3.79 in Section 3.2.2.

We have identified the key requirements and desirable properties for complex structures on the AdS hypercylinder, surveyed the space of complex structures that satisfy much of these requirements, and identified candidates that are particularly interesting. This clearly shows that the GBF philosophy makes sense in the context of the S-matrix on AdS spacetimes, because it provides a consistent construction method. What remains to understand, from a physical perspective, is the differences between these candidates and their respective induced scattering theories. In this direction we only point out here the intimate relationship between complex structures and Feynman propagators [42, 66]. Much remains to be done in order to reach a satisfactory understanding of quantum field theory in AdS spacetime.

Motivated by the construction of a radial S-matrix working fine on AdS (and Minkowski spacetime), our next field of interest will be black hole spacetimes, starting with Schwarzschild as the simplest case. The radial setup is rather natural here: the spacetime is spherically symmetric, and even static like AdS and Minkowski. Since Schwarzschild becomes asymptotically flat for large radius, the states become asymptotically free there. Moreover, we should be able to use some flat limit here, too, since for vanishing black hole mass $M \to 0$ the Schwarzschild spacetime becomes Minkowski. However, a priori we cannot be sure here to find a form of amplitude equivalence between radial and temporal amplitudes, due to the singularity at the time-axis $r = 0$. This will depend on the radial behaviour of the modes, which so far is not known sufficiently.

Appendix A

Special Functions

A.1 General notation

We frequently refer to information provided by the DLMF (Digital Library of Mathematical Functions) [49], and denote equations in this source as in DLMF [15.10.1]. Equations etc. from the Handbook by Abramowitz and Stegun [1] are denoted as in AS [13.8.7].

Our metric's signature for Lorentzian spacetimes is $(+, -, \ldots, -)$.

A.2 Orthogonal polynomials and Dirac delta

In this short section we give a general relation between the Dirac delta "function" and orthogonal polynomials. Let $\{p_n\}_{n\,\in\,\mathbb{N}}$ a system of polynomials $p_n: \quad [a,b] \subset \mathbb{R} \;\to\; \mathbb{C}$ with n the degree of the polynomial. Let h_n denote some constants and $w(x)$ some weight function positive on (a,b). Then a scalar product for the functions on $[a,b]$ is defined by

$$\langle f,\, g\rangle_{w,a,b} \;:=\; \int_a^b \mathrm{d}x\; w(x)\; \overline{f(x)}\, g(x)\;.$$

If the following condition is fulfilled, then the system is called orthogonal on the intervall $[a,b]$, see e.g. DLMF [18.2.1+5].

$$\langle p_n,\, p_m\rangle_{w,a,b} \;:=\; \int_a^b \mathrm{d}x\; w(x)\; \overline{p_n(x)}\; p_m(x) \;=\; h_n\, \delta_{nm} \qquad \forall\; n, m \in \mathbb{N} \tag{A.1}$$

δ_{nm} is the Kronecker delta symbol. Integration over the continuous variable of orthogonal polynomials thus gives a delta for the discrete label. We now assume that the p_n form a complete basis for the functions on $[a,b]$. Further, we can write

$$w(x)\, p_n(x) \;=\; \sum_{m\in\mathbb{N}} w(x)\, p_m(x)\, \delta_{nm} \tag{A.2}$$

$$\overset{(A.1)}{=} \;\sum_{m\in\mathbb{N}} \int_a^b \mathrm{d}y\; w(x)\, p_m(x)\; w(y)\, p_n(y)\, \overline{p_m(y)}\, h_m^{-1} \tag{A.3}$$

$$= \int_a^b \mathrm{d}y\; w(y)\, p_n(y) \underbrace{\sum_{m\in\mathbb{N}} w(x)\, p_m(x)\, \overline{p_m(y)}\, h_m^{-1}}_{=:\, D(x,y)} \tag{A.4}$$

This equality holds for all p_n with $n \in \mathbb{N}$ and all $x \in (a,b)$, and hence $D(x,y)$ must be the Dirac delta $\delta(x-y)$. This is just a slight generalization of equation 6.3.11 of [47]. Summing over the discret label of orthogonal polynomials thus gives a delta for the continuous variable:

$$\sum_{m\in\mathbb{N}} w(x)\, p_m(x)\, \overline{p_m(y)}\, h_m^{-1} \;=\; \delta(x-y) \qquad \forall\; x,y \in (a,b)\;. \tag{A.5}$$

As an example, we apply this to the Jacobi polynomials $P_n^{(\alpha,\beta)}$. There the limits are $a = -1$ and $b = +1$, the weight function is $w(x) \;=\; (1-x)^\alpha (1+x)^\beta$ for $\alpha, \beta > -1$, and

$$h_n \;=\; \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\;\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\;\Gamma(n+\alpha+\beta+1)}\;, \tag{A.6}$$

see DLMF [18.3.1] and [18.2.1+5]. The only exceptional case where this is not well defined is $n = 0$ with $\alpha+\beta \;=\; -1$, where the Jacobi polynomial is identical to 1, and integration over the pure weight function results in

$$h_0^{\text{ex}} \;=\; \Gamma(\alpha+1)\,\Gamma(\beta+1) \;=\; \Gamma(\alpha+1)\,\Gamma(-\alpha) \;=\; -\pi/\sin(\alpha\pi)\;,$$

which is well defined due to $-1 < \alpha < 0$. For $x \in (-1,+1)$ we thus get

$$\sum_{n=0}^{\infty} \frac{(1-x)^\alpha(1+x)^\beta}{h_n}\, P_n^{(\alpha,\beta)}(x)\; P_n^{(\alpha,\beta)}(y) \;=\; \delta(x-y)\;. \tag{A.7}$$

Substituting $x \;=\; \cos 2\rho$ with $\rho \in (0,\,\frac{\pi}{2}\,)$ this transforms into

$$\sum_{n=0}^{\infty} \frac{(\sin^2\rho)^\alpha(\cos^2\rho)^\beta}{2^{-\alpha-\beta}\,h_n}\, P_n^{(\alpha,\beta)}(\cos 2\rho)\; P_n^{(\alpha,\beta)}(\cos 2\rho') \;=\; \delta(\cos 2\rho - \cos 2\rho') \;=\; \frac{\delta(\rho-\rho')}{2\sin 2\rho}\;. \tag{A.8}$$

A.3 Sphere

On the unit sphere $\mathbb{S}^{d-1}$ the metric tensor is denoted by $g_{\mathbb{S}^{d-1}}$. It is diagonal in standard (hyper)spherical coordinates given below, with $\underline{\xi} \;=\; (\xi_1,...,\xi_d)$ denoting the cartesian coordinates of $\mathbb{R}^d$ into which $\mathbb{S}^{d-1}$ is embedded. We use $(d-1)$ angles θ_i with $i \in \{1,...,(d-1)\}$. These we often denote collectively by $\Omega = (\theta_{d-1},...,\theta_1)$. On the two-sphere, traditionally θ_2 is just denoted by θ, and θ_1 by φ. We denote angles by upright thetas θ and varphis φ in order to distinguish them from the symplectic potential θ and field configurations φ. We shall choose our coordinates such that the d-dimensional case for $d = 3$ reproduces the standard spherical coordinates on $\mathbb{R}^3$, and thus for $d = 2$ the standard polar coordinates on the real plane $\mathbb{R}^2$.

$$\begin{aligned}
\xi_1 &\;=\; \cos\theta_1 \prod_{j=2}^{d-1}\sin\theta_j && \theta_1 \equiv \varphi \in [0,2\pi) \\
\xi_2 &\;=\; \sin\theta_1 \prod_{j=2}^{d-1}\sin\theta_j && \theta_i \in [0,\pi] \qquad \forall\; i \in \{2,...,(d-1)\} \\
\xi_k &\;=\; \cos\theta_{k-1} \prod_{j=k}^{d-1}\sin\theta_j && \forall\; k \in \{3,...,d\}
\end{aligned} \tag{A.9}$$

Note that from $\underline{\xi}^2 \;=\; 1$ follows (Einstein sum convention)

$$\xi_k\, \mathrm{d}\xi_k \;=\; 0\;.$$

In these coordinates the diagonal elements of $g_{\mathbb{S}^{d-1}}$ give the squared length element

$$\begin{aligned} \mathrm{d}s^2_{\mathbb{S}^{d-1}} &= \mathrm{d}\theta^2_{d-1} + \sin^2\theta_{d-1}\Big(\mathrm{d}\theta^2_{d-2} + \sin^2\theta_{d-2}\Big(\dots\ \big(\mathrm{d}\theta_2^2 + \sin^2\theta_2\,\mathrm{d}\theta_1^2\big)\ \dots\Big)\Big) \\ &= \sum_{k=1}^{d-1} \mathrm{d}\theta_k^2 \prod_{j=k+1}^{d-1} \sin^2\theta_j \\ \Rightarrow \qquad \sqrt{|g_{\mathbb{S}}|} &= (\sin\theta_{d-1})^{d-2}\,(\sin\theta_{d-2})^{d-3} \dots (\sin\theta_3)^2\,(\sin\theta_2)^1 \\ &= \prod_{j=2}^{d-1} (\sin\theta_j)^{j-1}\ . \end{aligned}$$

Defining $|g_{\mathbb{S}}| := |\det\,(g_{\mathbb{S}^{d-1}})_{\mu\nu}|$, we denote the volume element on $\mathbb{S}^{d-1}$ by $\mathrm{d}^{d-1}\Omega\ = \mathrm{d}\Omega\,\sqrt{|g_{\mathbb{S}}|}$, where $\mathrm{d}\Omega$ abbreviates $\mathrm{d}\theta_1 \dots \mathrm{d}\theta_{d-1}$.

A.4 Hyperspherical harmonics

A.4.1 Basics

The hyperspherical harmonics $Y_{\underline{l}}^{m_l}(\Omega)$ are the eigenfunctions of the Laplacian on a (hyper)sphere $\mathbb{S}^{d-1}$ embedded in $\mathbb{R}^d$ with $d \geq 3$. The eigenvalues depend only on the parameter l:

$$\Box_{\mathbb{S}^{d-1}} Y_{\underline{l}}^{m_l}(\Omega) \;=\; -l(l+d-2)\, Y_{\underline{l}}^{m_l}(\Omega)\ .$$

The $\underline{l}$ represents the multiindex $(l_{d-1}, \dots, l_2)$. We often write simply l for l_{d-1}. The m_l carries its subindex in order to distinguish it from the mass m. The indices take the following integer values:

$$\begin{aligned} l \equiv l_{d-1} &\in \{0,\,1,\,2\,,\dots\} \\ l_i &\in \{0, 1, 2, \dots, l_{i+1}\} \qquad\qquad \forall\, i \in \{2, \dots, (d-2)\} \\ m_l &\in \{-l_2, -l_2+1, \dots, l_2-1, l_2\}\ . \end{aligned} \tag{A.10}$$

The number l_j measures the angular momentum on the sphere $\mathbb{S}^j$. Thus we could as well denote m_l by l_1 and absorb it into the multiindex. But since the complex conjugated of a hyperspherical harmonic is easily expressed using m_l separately:

$$\overline{Y_{\underline{l}}^{m_l}(\Omega)} \;=\; Y_{\underline{l}}^{-m_l}(\Omega)\ , \tag{A.11}$$

we shall keep it as it is. The angle coordinatising $\mathbb{S}^1$ is special compared to the other angles: it is the only angle ranging over $[0, 2\pi)$ and being periodic, while the other range over $[0, \pi]$ and are not periodic. Also the angular momentum on $\mathbb{S}^1$ is special: it is oriented (signed), while the other angular momenta l_j are unsigned.

A.4.2 On the two-sphere

The spherical harmonics on the two-sphere belong to the most popular of the special functions and thus are very well known. With $\Omega = (\theta, \varphi)$ they write as

$$Y_l^{m_l}(\Omega) \;=\; \mathcal{N}_l^{m_l}\, \mathrm{e}^{\mathrm{i}m_l\varphi}\, P_l^{m_l}(\cos\theta)\ . \tag{A.12}$$

The $P_l^{m_l}$ are the associated Legendre polynomials (associated Legendre functions of the first kind with integer parameters). Since we wish to obtain an orthonormal system:

$$\int_{\mathbb{S}^2} \mathrm{d}^2\Omega\ Y_{l'}^{m_l'}(\Omega)\, \overline{Y_l^{m_l}(\Omega)} \;=\; \delta_{ll'}\, \delta_{m_l m_l'}\ , \tag{A.13}$$

(wherein the δ denote Kronecker deltas) we have to choose the standard normalisation

$$\mathcal{N}_l^{m_l} = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m_l)!}{(l+m_l)!}}\ . \tag{A.14}$$

This can be seen from AS [8.14.11+13]

$$\int_{-1}^{1} \mathrm{d}x\, P_l^{m_l}(x)\, P_{l'}^{m_l}(x) = \delta_{ll'}\, \frac{2}{2l+1}\, \frac{(l+m_l)!}{(l-m_l)!}$$

and

$$\int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{e}^{\mathrm{i}\varphi(m_l-m_l')} = (2\pi)\delta_{m_l m_l'}\ .$$

Later we will need to decompose spherical harmonics $Y_l^{m_l}$ and their derivative into a sum of spherical harmonics with the same m_l and $l\pm1$. As for the hypergeometric functions we shall call $Y_{l\pm1}^{m_l}$ contiguous spherical harmonics of $Y_l^{m_l}$. Now using AS [8.5.3+4]

$$\begin{aligned}
x\,P_l^{m_l}(x) &= \frac{l+m_l}{2l+1}\,P_{l-1}^{m_l}(x) + \frac{l+1-m_l}{2l+1}\,P_{l+1}^{m_l}(x)\\
(1-x)^2\,\frac{\mathrm{d}}{\mathrm{d}x}\,P_l^{m_l}(x) &= (l+m_l)\,P_{l-1}^{m_l}(x) - l\,x\,P_l^{m_l}(x)\\
&= \frac{(l+m_l)(l+1)}{2l+1}\,P_{l-1}^{m_l}(x) - \frac{l(l+1-m_l)}{2l+1}\,P_{l+1}^{m_l}(x)
\end{aligned}$$

we find the following relations for contiguous spherical harmonics (which are probably well known, although we did not find them in the literature):

$$\cos\theta\, Y_l^{m_l}(\Omega) = \chi_-^{(2)}(l,m_l)\, Y_{l-1}^{m_l}(\Omega) + \chi_+^{(2)}(l,m_l)\, Y_{l+1}^{m_l}(\Omega) \tag{A.15}$$

$$(1-\cos^2\theta)\,\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\, Y_l^{m_l}(\Omega) = \delta_-^{(2)}(l,m_l)\, Y_{l-1}^{m_l}(\Omega) + \delta_+^{(2)}(l,m_l)\, Y_{l+1}^{m_l}(\Omega) \tag{A.16}$$

with the raising and lowering coefficients

$$\begin{aligned}
\chi_-^{(2)}(l,m_l) &= \sqrt{\frac{l^2-m_l^2}{(2l-1)(2l+1)}} & \delta_-^{(2)}(l,m_l) &= (l+1)\,\chi_-^{(2)}(l,m_l)\\
\chi_+^{(2)}(l,m_l) &= \sqrt{\frac{(l+1)^2-m_l^2}{(2l+1)(2l+3)}} & \delta_+^{(2)}(l,m_l) &= -l\,\chi_+^{(2)}(l,m_l)\ .
\end{aligned} \tag{A.17}$$

Note that these coefficients behave very well in the following sense. First, for $l=0$ automatically $m_l=0$ and thus the lowering coefficients vanish in this case: $\chi_-^{(2)}(0,0) = \delta_-^{(2)}(0,0) = 0$, and therefore spherical harmonics with negative l do not appear. Second, the lowering coefficients also vanish whenever m_l has its extremal values $m_l=\pm l$, that is, $\chi_-^{(2)}(l,\pm l) = \delta_-^{(2)}(l,\pm l) = 0$, such that there appear no spherical harmonics $Y_{l-1}^{\pm l}$ where (the absolute value of) m_l is larger than l. Third, they are invariant under the sign change $m_l \to -m_l$. Raising and lowering coefficients are related through

$$\chi_-^{(2)}(l+1,m_l) = \chi_+^{(2)}(l,m_l)\ . \tag{A.18}$$

A.4.3 On higher-dimensional spheres

Although met a little less frequently in the literature, the spherical harmonics on the $(d-1)$-sphere are also well known (e.g. in chemical physics). From formula (3.101) in Avery's book [6] and formula (21) in [3] by Aquilanti et al. we can read off the product structure following below.

Therein, the spherical harmonics ${}^{(d-1)}Y_{\underline{l}}^{m_l}$ on the $(d-1)$-sphere are related to those ${}^{(d-2)}Y_{\underline{\tilde{l}}}^{m_l}$ on the $(d-2)$-sphere. In order to keep track of dimensions, in this section we shall therefore use a notation with an additional label: $\Omega_{d-1} = (\theta_{d-1}, ..., \theta_2, \theta_1 \equiv \varphi)$ and thus $\Omega_{d-2} = (\theta_{d-2}, ..., \theta_2, \theta_1 \equiv \varphi)$. Moreover $\underline{l} = (l_{d-1}, ..., l_2)$ and $\underline{\tilde{l}} = (l_{d-2}, ..., l_2)$, that is, $\underline{l}$ with the first entry cut off. Since we need them more often, we shall abbreviate as follows: $l := l_{d-1}$ and $\tilde{l} := l_{d-2}$. For $d = 3$ the latter is to be understood as $\tilde{l} := m_l$. Then,

$$ {}^{(d-1)}Y_{\underline{l}}^{m_l}(\Omega_{d-1}) = \mathcal{N}_{l,l_{d-2}}^{(d-1)} \; (\sin\theta_{d-1})^{l_{d-2}} \; C_{l-l_{d-2}}^{(l_{d-2}+d/2-1)}(\cos\theta_{d-1}) \; {}^{(d-2)}Y_{\underline{\tilde{l}}}^{m_l}(\Omega_{d-2}) \,, \tag{A.19} $$

wherein the $C_{l-l_{d-2}}^{(l_{d-2}+d/2-1)}$ denote the Gegenbauer (ultraspherical) polynomials, and the hyperspherical harmonics on the $(d-2)$-sphere are assumed to be orthonormal already. Since we also wish to obtain an orthonormal system on the $(d-1)$-sphere:

$$ \int_{\mathbb{S}^{d-1}} \mathrm{d}^{d-1}\Omega \; Y_{\underline{l}'}^{m_l'}(\Omega) \, \overline{Y_{\underline{l}}^{m_l}(\Omega)} = \delta_{\underline{l}\underline{l}'} \, \delta_{m_l m_l'} \,, \tag{A.20} $$

(wherein the δ denote Kronecker deltas) we have to fix the relative normalisation constants to

$$ \mathcal{N}_{l,l_{d-2}}^{(d-1)} = 2^{l_{d-2}+d/2-2} \, \Gamma(l_{d-2}+d/2-1) \sqrt{\frac{(l-l_{d-2})! \, (2l+d-2)}{\pi \, (l+l_{d-2}+d-3)}} \qquad (d-1) \geq 3 \,. \tag{A.21} $$

Up to notation (factorial versus Γ-function), our normalisation agrees with the one given by Aquilanti et al. (but apparently not with Avery's). Our normalisation can be derived using AS [22.2.3]

$$ \int_{-1}^{1} \mathrm{d}x \, (1-x^2)^{\alpha-1/2} \, C_n^{(\alpha)}(x) \, C_{n'}^{(\alpha)}(x) = \delta_{nn'} \, 2^{1-2\alpha} \frac{\pi \, \Gamma(2\alpha+n)}{n! \, (n+\alpha) \, (\Gamma(\alpha))^2} \qquad 0 \neq \alpha > -1/2 \,. $$

Since in our case $\alpha = l_{d-2} + d/2 - 1$ and $d \geq 4$, we can apply this formula. Starting from the well known spherical harmonics on $\mathbb{S}^2$ and using the recursion (A.19) we can now derive an explicit formula for the spherical harmonics on $\mathbb{S}^{d-1}$:

$$ Y_{\underline{l}}^{m_l}(\Omega) = \mathcal{N}_{\underline{l}}^{(d-1)} \left(\prod_{k=3}^{d-1} (\sin\theta_k)^{l_{k-1}} C_{l_k-l_{k-1}}^{(l_{k-1}+(k-1)/2)}(\cos\theta_k) \right) P_{l_2}^{m_l}(\cos\theta_2) \, \mathrm{e}^{\mathrm{i} m_l \theta_1} \,. \tag{A.22} $$

The full normalisation constant is given by a product of relative normalization constants starting at $\mathcal{N}_{l_2}^{m_l}$ (with $m_l \equiv l_1$):

$$ \mathcal{N}_{\underline{l}}^{(d-1)} = \prod_{k=2}^{d-1} \mathcal{N}_{l_k,l_{k-1}}^{(k)} \qquad\qquad \mathcal{N}_{l_2,l_1}^{(2)} = \mathcal{N}_{l_2}^{m_l} \,. \tag{A.23} $$

Now we want to work out the analogs of the contiguous relations (A.15) and (A.16) for the higher-dimensional cases. To this end, in AS[22.7.3] or the first line of table DLMF [18.9.1] we can find

$$ x \, C_n^{(\alpha)}(x) = \frac{2\alpha+n-1}{2(\alpha+n)} \, C_{n-1}^{(\alpha)}(x) + \frac{n+1}{2(\alpha+n)} \, C_{n+1}^{(\alpha)}(x) \,. $$

Further, we have AS [22.8.2]:

$$(1-x)^2\ \frac{\mathrm{d}}{\mathrm{d}x}\, C_n^{(\alpha)}(x) \;=\; (2\alpha+n-1)\, C_{n-1}^{(\alpha)}(x) - n\, x\, C_n^{(\alpha)}(x)$$
$$= \frac{(2\alpha+n)\,(2\alpha+n-1)}{2(\alpha+n)}\, C_{n-1}^{(\alpha)}(x) - \frac{n\,(n+1)}{2(\alpha+n)}\, C_{n+1}^{(\alpha)}(x)\ .$$

Using these, we find the following relations for contiguous hyperspherical harmonics

$$\cos\theta_{d-1}\, Y_{\underline{l}}^{m_l}(\Omega) \;=\; \chi_-^{(d-1)}(l,l_{d-2})\, Y_{(l-1,\underline{\tilde{l}})}^{m_l}(\Omega) + \chi_+^{(d-1)}(l,l_{d-2})\, Y_{(l+1,\underline{\tilde{l}})}^{m_l}(\Omega) \tag{A.24}$$
$$(1-\cos^2\theta_{d-1})\ \frac{\mathrm{d}}{\mathrm{d}\cos\theta_{d-1}}\, Y_{\underline{l}}^{m_l}(\Omega) \;=\; \delta_-^{(d-1)}(l,l_{d-2})\, Y_{(l-1,\underline{\tilde{l}})}^{m_l}(\Omega) + \delta_+^{(d-1)}(l,l_{d-2})\, Y_{(l+1,\underline{\tilde{l}})}^{m_l}(\Omega) \tag{A.25}$$

with the raising and lowering coefficients

$$\chi_-^{(d-1)}(l,l_{d-2}) \;=\; \sqrt{\frac{(l-l_{d-2})\,(l+l_{d-2}+d-3)}{(2l+d-4)\,(2l+d-2)}} \qquad \delta_-^{(d-1)}(l,l_{d-2}) \;=\; (l+d-2)\ \chi_-^{(d-1)}(l,l_{d-2}) \tag{A.26}$$
$$\chi_+^{(d-1)}(l,l_{d-2}) \;=\; \sqrt{\frac{(l-l_{d-2}+1)\,(l+l_{d-2}+d-2)}{(2l+d-2)\,(2l+d)}} \qquad \delta_+^{(d-1)}(l,l_{d-2}) \;=\; -l\ \chi_+^{(d-1)}(l,l_{d-2})\ .$$

Note first that for $d=3$ these coefficients reproduce exactly those for the two-sphere (with $l_1 \equiv m_l$). They also exhibit the same well-behavedness. Again for $l=0$ automatically $l_{d-2}=0$ and thus the lowering coefficients vanish in this case: $\chi_-^{(d-1)}(0,0) = \delta_-^{(d-1)}(0,0) = 0$, that is, hyperspherical harmonics with negative l do not appear. Further, again the lowering coefficients vanish if and only if $|l_{d-2}|$ has its top value $l_{d-2}=l$, that is, $\chi_-^{(d-1)}(l,l) = \delta_-^{(d-1)}(l,l) = 0$, such that there appear no hyperspherical harmonics $Y_{(l-1,l,l_{d-3}\ldots,l_2)}^{m_l}$ where l_{d-2} is bigger than l_{d-1}. (Recall that l_{d-2} is always nonnegative except for $d=3$ which we considered above.) The raising coefficients never vanish. A relation connecting raising and lowering coefficients which sometimes comes in handy is

$$\chi_-^{(d-1)}(l+1,l_{d-2}) \;=\; \chi_+^{(d-1)}(l,l_{d-2})\ . \tag{A.27}$$

From (A.2) with $w(x) \;\to\; \sqrt{|g_{\mathbb{S}^{d-1}}|}$ we then get a generalization of DLMF [1.17.25]:

$$\sum_{\underline{l},m_l} \overline{Y_{\underline{l}}^{m_l}(\Omega)}\ Y_{\underline{l}}^{m_l}(\Omega') \;=\; \frac{\delta^{(d-1)}(\Omega-\Omega')}{\sqrt{|g_{\mathbb{S}^{d-1}}|}}\ , \tag{A.28}$$

wherein $\delta^{(d-1)}(\Omega-\Omega') := \delta(\theta_1-\theta'_1)\cdot\ \ldots\ \cdot\delta(\theta_{d-1}-\theta'_{d-1})$.

A.4.4 Transformation under rotations: basics

An arbitrary spatial rotation on $\mathbb{R}^d$ is determined by $n = d(d-1)/2$ Euler angles $\underline{\alpha} = (\alpha_1,\ldots,\alpha_n)$ plus a choice of n axes[1] $\{\underline{e}_i\}_{i=1,\ldots,n}$ around which to rotate. That is, any rotation can be obtained by first rotating by an angle α_1 around axis $\underline{e}_1$, then rotating by an angle α_2 around axis $\underline{e}_2$, and so on until rotating by α_n around $\underline{e}_n$. As an example, in $\mathbb{R}^3$ we have $n=3$, and a frequent choice of axes is ZYZ (that is: $\underline{e}_1$ is the x^3-axis, $\underline{e}_2$ is the x^2-axis, and $\underline{e}_3$ is again the x^3-axis). The group of these rotations is the special orthogonal group SO(d).

In cartesian coordinates the rotation acts as a (d,d)-matrix $(\hat{R}(\underline{\alpha}))$ on the coordinate vector: $x^a \;\to\; x'^a = (\hat{R}(\underline{\alpha}))^b{}_a\, x^a$. The rotation matrix $(\hat{R}(\underline{\alpha}))$ is the product of the matrices $(\hat{R}_i(\alpha_i))$ with

[1] The "axes" for rotations in higher dimensions are actually hyperplanes of codimension 2, since all coordinates but two are held constant. For rotations in three dimensions this reduces to the usual axes which are lines of dimension 1. For simplicity, we shall use term "axes" also for higher dimensions.

$i = n, (n-1), ..., 1$, which each perform the rotation by an angle α_i around axis $\underline{e}_i$. In our example, the matrix $(\hat{R}_1(\alpha_1))$ is the well-known matrix of a rotation by α_1 around the x^3-axis:

$$(\hat{R}_1(\alpha_1)) = \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 \\ -\sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} .$$

The action of a rotation on spherical coordinates $\Omega \rightarrow \Omega' = \hat{R}\Omega$ cannot be expressed so nicely in matrix form, nevertheless it can be obtained in the following way: transform the spherical coordinates into cartesian ones, rotate the cartesian coordinates as described above, and transform them back into spherical coordinates.

A.4.5 Transformation under rotations: two-sphere

The action on hyperspherical harmonics is given by Wigner's D-matrix. In Chapter 15 of [74] it is originally defined for $\mathbb{R}^3$ using one-particle states $|\, l\, m_l \rangle$ with definite angular momentum quantum numbers l and m_l as

$$(D^l(\underline{\alpha}))_{m_l' m_l} := \langle l m_l' \,|\, \hat{R}(\underline{\alpha}) \,|\, l m_l \rangle \ .$$

$(D^l(\underline{\alpha}))$ is a $(2l{+}1, 2l{+}1)$-matrix. The one-particle states are orthonormal and fulfill a completeness relation:

$$\langle l\, m_l \,|\, l'\, m_l' \rangle = \delta_{ll'}\, \delta_{m_l m_l'} \tag{A.29}$$

$$\mathbb{1} = \sum_{l,\, m_l} |\, l\, m_l \rangle \, \langle l\, m_l \,| \ . \tag{A.30}$$

For our purposes the concrete form of Wigner's D-matrix is not important. Let us write it down anyway for the above example for the sake of completeness. There, Wigner's D-matrix is given by [74]

$$(D^l(\underline{\alpha}))_{m_l' m_l} = \mathrm{e}^{-\mathrm{i} m_l' \alpha_1} \, (d^{\,l}(\alpha_2))_{m_l' m_l} \, \mathrm{e}^{-\mathrm{i} m_l \alpha_3} \ , \tag{A.31}$$

with Wigner's small d-matrix given by

$$(d^{\,l}(\alpha_2))_{m_l' m_l} = \sqrt{(l{+}m_l')!\,(l{-}m_l')!\,(l{+}m_l)!\,(l{-}m_l)!} \sum_{s=\max(0,m_l-m_l')}^{\min(l+m_l,l-m_l')} (-1)^{s+m_l'-m_l} \frac{(\sin(\alpha_2/2))^{2s+m_l'-m_l}\ (\cos(\alpha_2/2))^{2l-2s-m_l'+m_l}}{s!\ (s{+}m_l'{-}m_l)!\ (l{-}m_l'{-}s)!\ (l{+}m_l{-}s)!} \ .$$

Note that the limits of the sum are precisely such that all factorials in it have a nonnegative argument. Let us denote a one-particle state with definite angular position Ω by $|\,\Omega\rangle$. The completeness relation is given by

$$\mathbb{1} = \int_{\mathbb{S}^2} \mathrm{d}^2\Omega \, |\,\Omega\rangle \, \langle \Omega \,| \ .$$

The (angular part of the) wavefunction in coordinate representation of our one-particle state $|\, l m_l \rangle$ is given by the spherical harmonics:

$$\langle \Omega \,|\, l m_l \rangle = Y_{\underline{l}}^{m_l}(\Omega) \ .$$

This allows us to write Wigner's D-matrix as

$$
\begin{aligned}
(D^l(\underline{\alpha}))_{m_l' m_l} &= \langle l m_l' \,|\, \hat{R}(\underline{\alpha}) \,|\, l m_l \rangle \\
&= \int \mathrm{d}^2\Omega \int \mathrm{d}^2\Omega' \, \langle l m_l' \,|\, \Omega \rangle \langle \Omega \,|\, \hat{R}(\underline{\alpha}) \,|\, \Omega' \rangle \langle \Omega' \,|\, l m_l \rangle \\
&= \int \mathrm{d}^2\Omega \int \mathrm{d}^2\Omega' \, \overline{Y_l^{m_l'}(\Omega)} \, \langle \Omega \,|\, \hat{R}(\underline{\alpha})\, \Omega' \rangle Y_l^{m_l}(\Omega') \\
&= \int \mathrm{d}^2\Omega' \, \overline{Y_l^{m_l'}(\hat{R}(\underline{\alpha})\Omega')} \, Y_l^{m_l}(\Omega') \\
&= \left\langle Y_l^{m_l'} \circ \hat{R}(\underline{\alpha}), \; Y_l^{m_l} \right\rangle_{\mathbb{S}^2} .
\end{aligned}
$$

The last line denotes the inner product of two functions on $\mathbb{S}^2$, and in between we have used

$$
\langle \Omega \,|\, \Omega' \rangle = \frac{\delta^{(2)}(\Omega, \Omega')}{\sqrt{g_{\mathbb{S}^2}}} .
$$

When considering the transformation of spherical harmonics under rotations, the crucial point is that the total spatial angular momentum l on $\mathbb{S}^2$ is conserved under rotations around the center of this sphere. Therefore $Y_l^{m_l'}(\hat{R}(\underline{\alpha})\Omega)$ can be expanded as a sum over $Y_l^{m_l}(\Omega)$, that is, over spherical harmonics with the same spatial angular momentum l see also Chapter 15 in [74]. In other words: the inner product of a rotated $Y_l^{m_l}(\hat{R}(\underline{\alpha})\Omega)$ and an unrotated spherical harmonic $Y_{l'}^{m_l'}(\Omega)$ vanishes except for $l' = l$. Thus we can write the rotated spherical harmonics as a linear combination of unrotated ones, with the coefficients provided by elements of Wigner's D-matrix:

$$
\begin{aligned}
Y_l^{m_l'}(\hat{R}(\underline{\alpha})\Omega) &= \int \mathrm{d}^2\Omega' \, Y_l^{m_l'}(\hat{R}(\underline{\alpha})\Omega') \, \frac{\delta^{(2)}(\Omega', \Omega)}{\sqrt{g_{\mathbb{S}^2}}} \\
&= \int \mathrm{d}^2\Omega' \sum_{m'', l''} Y_{l''}^{m_l''}(\Omega) \, \overline{Y_{l''}^{m_l''}(\Omega')} \, Y_l^{m_l'}(\hat{R}(\underline{\alpha})\Omega') \\
&= \sum_{m_l'', l''} Y_{l''}^{m_l''}(\Omega) \left\langle Y_{l''}^{m_l''}, \; Y_l^{m_l'} \circ \hat{R}(\underline{\alpha}) \right\rangle \\
&= \sum_{m_l} Y_l^{m_l}(\Omega) \left\langle Y_l^{m_l}, \; Y_l^{m_l'} \circ \hat{R}(\underline{\alpha}) \right\rangle \\
&= \sum_{m_l} Y_l^{m_l}(\Omega) \, \overline{(D^l(\underline{\alpha}))_{m_l' m_l}} .
\end{aligned}
$$

By complex conjugation we obtain from this relation

$$
\begin{aligned}
\overline{Y_l^{m_l'}(\hat{R}(\underline{\alpha})\Omega)} &= \sum_{m_l} \overline{Y_l^{m_l}(\Omega)} \, (D^l(\underline{\alpha}))_{m_l' m_l} \\
= Y_l^{-m_l'}(\hat{R}(\underline{\alpha})\Omega) &= \sum_{m_l} Y_l^{m_l}(\Omega) \, (D^l(\underline{\alpha}))_{m_l', -m_l} .
\end{aligned}
$$

Since for rotations $\hat{R}(\underline{\alpha}) \in \mathrm{SO}(3)$ we have $(\hat{R}(\underline{\alpha}))^{-1} = (\hat{R}(\underline{\alpha}))^{\top} = (\hat{R}(\underline{\alpha}))^{\dagger}$, we can also write

$$\begin{aligned}(D^l(\underline{\alpha}))_{m_l' m_l} &= \langle l m_l' \,|\, \hat{R}(\underline{\alpha}) \,|\, l m_l \rangle \\ &= \int \mathrm{d}^2\Omega \int \mathrm{d}^2\Omega' \,\langle l m_l' \,|\, \Omega \rangle \langle \Omega \,|\, \hat{R}(\underline{\alpha}) \,|\, \Omega' \rangle \langle \Omega' \,|\, l m_l \rangle \\ &= \int \mathrm{d}^2\Omega \int \mathrm{d}^2\Omega' \,\overline{Y_l^{m_l'}(\Omega)} \,\left\langle (\hat{R}(\underline{\alpha}))^{-1}\Omega \,\middle|\, \Omega' \right\rangle Y_l^{m_l}(\Omega') \\ &= \int \mathrm{d}^2\Omega \,\overline{Y_l^{m_l'}(\Omega)} \,Y_l^{m_l}((\hat{R}(\underline{\alpha}))^{-1}\Omega) \\ &= \left\langle Y_l^{m_l'},\, Y_l^{m_l} \circ (\hat{R}(\underline{\alpha}))^{-1} \right\rangle_{\mathbb{S}^2} ,\end{aligned}$$

and (mind the position of the prime!)

$$\begin{aligned}Y_l^{m_l'}((\hat{R}(\underline{\alpha}))^{-1}\Omega) &= \sum_{m_l} Y_l^{m_l}(\Omega) \left\langle Y_l^{m_l},\, Y_l^{m_l'} \circ (\hat{R}(\underline{\alpha}))^{-1} \right\rangle \\ &= \sum_{m_l} Y_l^{m_l}(\Omega)\, (D^l(\underline{\alpha}))_{m_l m_l'} .\end{aligned}$$

By complex conjugation we obtain from this relation

$$\begin{aligned}\overline{Y_l^{m_l'}(\hat{R}(\underline{\alpha})\Omega)} &= \sum_{m_l} \overline{Y_l^{m_l}(\Omega)}\; \overline{(D^l(\underline{\alpha}))_{m_l m_l'}} \\ = Y_l^{-m_l'}(\hat{R}(\underline{\alpha})\Omega) &= \sum_{m_l} Y_l^{m_l}(\Omega)\; \overline{(D^l(\underline{\alpha}))_{-m_l, m_l'}} .\end{aligned}$$

What will be most important for us is the following completeness relation fulfilled by Wigner's D-matrix:

$$\begin{aligned}\sum_{m_l} (D^l(\underline{\alpha}))_{m_l' m_l}\, \overline{(D^l(\underline{\alpha}))_{m_l'' m_l}} &= \sum_{m_l} \left\langle l\, m_l' \,\middle|\, \hat{R}(\underline{\alpha}) \,\middle|\, l\, m_l \right\rangle \left\langle l\, m_l \,\middle|\, (\hat{R}(\underline{\alpha}))^{\dagger} \,\middle|\, l\, m_l'' \right\rangle \\ &= \left\langle l\, m_l' \,\middle|\, \cancel{\hat{R}(\underline{\alpha})(\hat{R}(\underline{\alpha}))^{-1}} \,\middle|\, l\, m_l'' \right\rangle = \delta_{m_l' m_l''} \\ &= \int \mathrm{d}^2\Omega \,\left\langle l\, m_l' \,\middle|\, \Omega \right\rangle \left\langle \Omega \,\middle|\, l\, m_l'' \right\rangle \\ &= \int \mathrm{d}^2\Omega \,\overline{Y_l^{m_l'}(\Omega)}\; Y_l^{m_l''}(\Omega) \\ &= \delta_{m_l' m_l''} .\end{aligned} \tag{A.32}$$

A.4.6 Transformation under rotations: $(d-1)$-sphere

Except for the concrete values (A.31) which are not relevant for our calculations, the generalization of Wigner's D-matrix for rotations in $\mathbb{R}^d$ is straightforward:

$$\left(D^l_{\underline{\tilde{l}}', \underline{\tilde{l}}}(\underline{\alpha})\right)_{m_l' m_l} := \left\langle l\, \underline{\tilde{l}}' m_l' \,\middle|\, \hat{R}(\underline{\alpha}) \,\middle|\, l\, \underline{\tilde{l}}\, m_l \right\rangle , \tag{A.33}$$

wherein $|\, l\, \underline{\tilde{l}}\, m_l \rangle$ is a one-particle state with angular momentum quantum numbers $l, \underline{\tilde{l}}$ and m_l. $(D^l_{\underline{\tilde{l}}', \underline{\tilde{l}}}(\underline{\alpha}))$ is a $(2l'_{d-2}+1, 2l_{d-2}+1)$-matrix, i.e., generically not square. The one-particle states are orthonormal and fulfill a completeness relation:

$$\left\langle l\, \underline{\tilde{l}}\, m_l \,\middle|\, l'\, \underline{\tilde{l}}'\, m_l' \right\rangle = \delta_{l l'}\; \delta_{\underline{\tilde{l}}\, \underline{\tilde{l}}'}\; \delta_{m_l m_l'} \tag{A.34}$$

$$\mathbb{1} = \sum_{l, \underline{\tilde{l}}, m_l} \left|\, l\, \underline{\tilde{l}}\, m_l \right\rangle \left\langle l\, \underline{\tilde{l}}\, m_l \,\right| . \tag{A.35}$$

The coordinate completeness relation becomes

$$\mathbb{1} \;=\; \int\limits_{\mathbb{S}^{d-1}} \mathrm{d}^{d-1}\Omega \;|\,\Omega\rangle\;\langle\,\Omega\,|\;.$$

The (angular part of the) wavefunction in coordinate representation of our one-particle state $|\,l\,\tilde{\underline{l}}\,m_l\rangle$ is again given by the (hyper)spherical harmonics:

$$\langle\Omega\,|\,l\,\tilde{\underline{l}}\,m_l\rangle \;=\; Y^{m_l}_{(l,\tilde{\underline{l}})}(\Omega)\;.$$

This allows us to write Wigner's D-matrix as

$$\begin{aligned}\Big(D^{l}_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\Big)_{m_l' m_l} \;&=\; \int \mathrm{d}^{d-1}\,\Omega'\;\overline{Y^{m_l'}_{(l,\tilde{\underline{l}}')}(\hat{R}(\underline{\alpha})\Omega')}\;Y^{m_l}_{(l,\tilde{\underline{l}})}(\Omega')\\ &=\; \Big\langle Y^{m_l'}_{(l,\tilde{\underline{l}}')}\circ\hat{R}(\underline{\alpha}),\;Y^{m_l}_{(l,\tilde{\underline{l}})}\Big\rangle_{\mathbb{S}^{d-1}}\;.\end{aligned} \tag{A.36}$$

The upper line induces

$$\overline{\Big(D^{l}_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\Big)_{m_l' m_l}} \;=\; \Big(D^{l}_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\Big)_{-m_l',-m_l}\;. \tag{A.37}$$

The last line in (A.36) denotes the inner product of two functions on $\mathbb{S}^{d-1}$, and in between we have used

$$\langle\Omega\,|\,\Omega'\rangle \;=\; \frac{\delta^{(d-1)}(\Omega,\Omega')}{\sqrt{g_{\mathbb{S}^{d-1}}}}\;.$$

Again the crucial point is that the total spatial angular momentum l on $\mathbb{S}^{d-1}$ is conserved under SO(d) rotations around the center of this sphere. Thus again we can write the rotated spherical harmonics as a linear combination of unrotated ones, with the coefficients provided by elements of Wigner's D-matrix:

$$\begin{aligned}Y^{m_l'}_{(l,\tilde{\underline{l}}')}(\hat{R}(\underline{\alpha})\Omega) \;&=\; \sum_{\tilde{\underline{l}},\,m_l} Y^{m_l}_{(l,\tilde{\underline{l}})}(\Omega)\;\Big\langle Y^{m_l}_{(l,\tilde{\underline{l}})},\;Y^{m_l'}_{(l,\tilde{\underline{l}}')}\circ\hat{R}(\underline{\alpha})\Big\rangle\\ &=\; \sum_{\tilde{\underline{l}},\,m_l} Y^{m_l}_{(l,\tilde{\underline{l}})}(\Omega)\;\overline{\Big(D^{l}_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\Big)_{m_l' m_l}}\;.\end{aligned} \tag{A.38}$$

By complex conjugation we obtain from this relation

$$\begin{aligned}\overline{Y^{m_l'}_{(l,\tilde{\underline{l}}')}(\hat{R}(\underline{\alpha})\Omega)} \;&=\; \sum_{\tilde{\underline{l}},\,m_l} \overline{Y^{m_l}_{(l,\tilde{\underline{l}})}(\Omega)}\;\Big(D^{l}_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\Big)_{m_l' m_l}\\ =\; Y_l^{-m_l'}(\hat{R}(\underline{\alpha})\Omega) \;&=\; \sum_{\tilde{\underline{l}},\,m_l} Y^{m_l}_{(l,\tilde{\underline{l}})}(\Omega)\;\Big(D^{l}_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\Big)_{m_l',-m_l}\;.\end{aligned} \tag{A.39}$$

Since also for SO(d)-rotations we have $(\hat{R}(\underline{\alpha}))^{-1} \;=\; (\hat{R}(\underline{\alpha}))^{\top} \;=\; (\hat{R}(\underline{\alpha}))^{\dagger}$, we can also write

$$\begin{aligned}\Big(D^{l}_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\Big)_{m_l' m_l} \;&=\; \int \mathrm{d}^{d-1}\,\Omega\;\overline{Y^{m_l'}_{(l,\tilde{\underline{l}}')}(\Omega)}\;Y^{m_l}_{(l,\tilde{\underline{l}})}((\hat{R}(\underline{\alpha}))^{-1}\Omega)\\ &=\; \Big\langle Y^{m_l'}_{(l,\tilde{\underline{l}}')},\;Y^{m_l}_{(l,\tilde{\underline{l}})}\circ(\hat{R}(\underline{\alpha}))^{-1}\Big\rangle_{\mathbb{S}^{d-1}}\;,\end{aligned} \tag{A.40}$$

and (mind the position of the prime!)

$$Y^{m_l'}_{(l,\underline{\tilde{l}}')}((\hat{R}(\underline{\alpha}))^{-1}\Omega) \;=\; \sum_{\underline{\tilde{l}},\,m_l} Y^{m_l}_{(l,\underline{\tilde{l}})}(\Omega) \left\langle Y^{m_l}_{(l,\underline{\tilde{l}})},\; Y^{m_l'}_{(l,\underline{\tilde{l}}')} \circ (\hat{R}(\underline{\alpha}))^{-1} \right\rangle$$

$$= \sum_{\underline{\tilde{l}},\,m_l} Y^{m_l}_{(l,\underline{\tilde{l}})}(\Omega) \left(D^l_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha}) \right)_{m_l m_l'} \; . \tag{A.41}$$

By complex conjugation we obtain from this relation

$$\overline{Y^{m_l'}_{(l,\underline{\tilde{l}}')}(\hat{R}(\underline{\alpha})^{-1}\Omega)} \;=\; \sum_{\underline{\tilde{l}},\,m_l} \overline{Y^{m_l}_{(l,\underline{\tilde{l}})}(\Omega)}\; \overline{\left(D^l_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha}) \right)_{m_l m_l'}}$$

$$= Y^{-m_l'}_{(l,\underline{\tilde{l}}')}(\hat{R}(\underline{\alpha})^{-1}\Omega) \;=\; \sum_{\underline{\tilde{l}},\,m_l} Y^{m_l}_{(l,\underline{\tilde{l}})}(\Omega)\; \overline{\left(D^l_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha}) \right)_{-m_l, m_l'}} \; . \tag{A.42}$$

What will be most important for us is the following completeness relation fulfilled by Wigner's D-matrix:

$$\sum_{\underline{\tilde{l}},\,m_l} (D^l_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha}))_{m_l' m_l}\; \overline{(D^l_{\underline{\tilde{l}}'',\underline{\tilde{l}}}(\underline{\alpha}))_{m_l'' m_l}} \;=\; \delta_{\underline{\tilde{l}}',\underline{\tilde{l}}''}\, \delta_{m_l' m_l''} \; . \tag{A.43}$$

It can be obtained again via

$$\sum_{\underline{\tilde{l}},\,m_l} (D^l_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha}))_{m_l' m_l}\; \overline{(D^l_{\underline{\tilde{l}}'',\underline{\tilde{l}}}(\underline{\alpha}))_{m_l'' m_l}} = \sum_{\underline{\tilde{l}},m_l} \left\langle l\,\underline{\tilde{l}}'\,m_l' \middle| \hat{R}(\underline{\alpha}) \middle| l\,\underline{\tilde{l}}\,m_l \right\rangle \left\langle l\,\underline{\tilde{l}}\,m_l \middle| (\hat{R}(\underline{\alpha}))^\dagger \middle| l\,\underline{\tilde{l}}''\,m_l'' \right\rangle$$

$$= \left\langle l\,\underline{\tilde{l}}'\,m_l' \middle| \hat{R}(\underline{\alpha})(\hat{R}(\underline{\alpha}))^{-1} \middle| l\,\underline{\tilde{l}}''\,m_l'' \right\rangle = \delta_{\underline{\tilde{l}}',\underline{\tilde{l}}''}\, \delta_{m_l' m_l''}$$

$$= \int \mathrm{d}^{d-1}\Omega \left\langle l\,\underline{\tilde{l}}'\,m_l' \middle| \Omega \right\rangle \left\langle \Omega \middle| l\,\underline{\tilde{l}}''\,m_l'' \right\rangle$$

$$= \int \mathrm{d}^{d-1}\Omega\; \overline{Y^{m_l'}_{l,\underline{\tilde{l}}'}(\Omega)}\; Y^{m_l''}_{l,\underline{\tilde{l}}''}(\Omega)$$

$$= \delta_{\underline{\tilde{l}}',\underline{\tilde{l}}''}\, \delta_{m_l' m_l''} \; .$$

Appendix B

Minkowski spacetime

B.1 Minkowski basics

B.1.1 Prelude: Killing vector fields on $\mathbb{R}^{(p,q)}$

In this section we list the Killing vector fields on $\mathbb{R}^{(p,q)}$, with diagonal metric η and cartesian coordinates X^A, where the index $A = 1, ..., (p+q)$. These are well known, see e.g. example (7.10) about Minkowski spacetime in [48]. Nevertheless, we shall give some details here for the sake of completeness. The reason for doing this is that we need the Killing vectors for Minkowski spacetime which is $\mathbb{R}^{(1,3)}$. Moreover, some Killing vectors on $\mathbb{R}^{(2,d)}$ leave a radial coordinate $R := \sqrt{\eta(X,X)}$ invariant (it is well defined only where $\eta(X,X) \geq 0$). Therefore, these are also Killing vectors on $\mathrm{AdS}_{1,d}$, which can be seen as a submanifold of $\mathbb{R}^{(2,d)}$ with $R = \mathrm{const.}$

In this section we *shall not* use Einstein's sum convention of summing over repeated indices, but write summations explicitly. Latin uppercase indices range as in $A = 1, \ldots, p, (p+1), \ldots, (p+q)$. Since the embedding space metric η is constant, we already have the $(p+q)$ Killing vector fields corresponding to translations: $T_A = \partial_A$ with $A = 1, \ldots, (p+q)$. These do not leave R invariant, however, and thus are of lesser interest for us.

Further, we can define the following linear combinations of translations, denoting them by K for Killing:

$$\begin{aligned} K^A_B(X) &:= X^A\,\partial_B - \eta_{BB}\eta^{AA}\,X^B\,\partial_A \\ K_{AB}(X) &:= X_A\,\partial_B - X_B\,\partial_A \;=\; \eta_{AA}\,K^A_B\,. \end{aligned} \tag{B.1}$$

(We define both versions, since a priori it is not clear which one will be more useful later.) Because the embedding space metric is diagonal we have $X_A = \sum_Q \eta_{AQ}\,X^Q = \eta_{AA}\,X^A = \pm X^A$, depending on the sign of η_{AA}. Note that the A and B attached to the Killing vectors K here are mere labels, not coordinate indices. The components of these vector fields are given by

$$\begin{aligned} (K^A_B)^Q(X) &= X^A\,\delta^Q_B - \eta_{BB}\eta^{AA}X^B\,\delta^Q_A \\ (K_{AB})^Q(X) &= X_A\,\delta^Q_B - X_B\,\delta^Q_A\,. \end{aligned} \tag{B.2}$$

wherein δ^Q_A has the same value as the Kronecker delta δ_{QA}. For $A = B$ both Killing vector fields vanish: $K^A_B \equiv 0 \equiv K_{AB}$. Since therefore $A \neq B$, we have $[(p+q)^2 - (p+q)]/2 = (p+q)(p+q-1)/2$ Killing vector fields K^A_B (respectively K_{AB}).

While $K_{AB} = -K_{BA}$ is always antisymmetric, K^A_B is antisymmetric for $\eta_{AA} = \eta_{BB}$ (rotations in embedding space) but symmetric for $\eta_{AA} = -\eta_{BB}$ (boosts in embedding space). Moreover, we can make the usual identification of $\mathbb{R}^{(p,q)}$ with its tangent space, and define the vector field X at a point X to be the vector from the origin to the point X. Then the Killing vector fields at point X

is (pseudo)orthogonal to the vector field X at point X:

$$0 \;=\; \eta(K^A_B, X) \;=\; \eta(K_{AB}, X)\;.$$

Next let us check that our Killing vectors indeed fulfill the Killing equation (see e.g. (7.121) in [48])

$$\begin{aligned} 0 &\overset{!}{=} (\nabla_M\, K^A_B)_N + (\nabla_N\, K^A_B)_M \\ 0 &\overset{!}{=} (\nabla_M\, K_{AB})_N + (\nabla_N\, K_{AB})_M\;. \end{aligned} \tag{B.3}$$

Since the metric η is constant, all Christoffel symbols vanish and the covariant derivatives reduce to partial ones:

$$\begin{aligned} 0 &\overset{!}{=} \eta_{NN}(\partial_M\, K^A_B)^N + \eta_{MM}(\partial\, K^A_B)^M \\ 0 &\overset{!}{=} \eta_{NN}(\partial_M\, K_{AB})^N + \eta_{MM}(\partial_N\, K_{AB})^M\;. \end{aligned}$$

We can plug in the respective components according to (B.2), and then evaluate for several different cases (remember that always $A \neq B$, else the Killing vectors vanish trivially): (1) $A = M$ and $B = N$, (2) $A = M$ and $B \neq N \;\Rightarrow\; B \neq M$, (3) $B = N$ and $A \neq M \;\Rightarrow\; A \neq N$, (4) $A = N$ and $B = M$, (5) $A = N$ and $B \neq M \;\Rightarrow\; B \neq N$, (6) $B = M$ and $A \neq N \;\Rightarrow\; A \neq M$, (7) $M \neq A \neq N$ and $M \neq B \neq N$. Using $\partial_M\, X^A = \delta^A_M$ and $\partial_M\, X_A = \eta_{AA}\, \delta^A_M$, for all cases we obtain zero, confirming that K^A_B and K_{AB} are indeed Killing vectors.

Calculating the Lie bracket $[K_{AB}, K_{CD}]$ (using again $\partial_M\, X_A = \eta_{AA}\, \delta^A_M$), we can verify that the Killing vectors K_{AB} form a representation of the Lie algebra $\mathbf{so}\,(p, q)$:

$$[K_{AB}, K_{CD}] \;=\; -\eta_{AC}\, K_{BD} + \eta_{BC}\, K_{AD} - \eta_{BD}\, K_{AC} + \eta_{AD}\, K_{BC}\;. \tag{B.4}$$

(This is the same algebra as (4.21) in [26] up to an overall sign.) We note that both sides actually give zero for many combinations of labels. For $A = B$ or $C = D$ one Killing vector itself vanishes, giving zero. For $A = C$ with $B = D$ the Killing vectors coincide, giving zero. Ditto for $A = D$ with $B = C$. For no label agreeing with another label ($A \neq B, C, D$ with $B \neq C, D$ with $C \neq D$) both sides give zero. That is, we only get a nonvanishing contribution whenever one of the left labels coincides with one of the right labels with all other labels differing from each other (for example $A = C \neq B, D$ with $B \neq D$). Then, only one term on the right hand side survives and coincides with the left hand side.

An m-dimensional manifold admitting $m(m+1)/2$ Killing vector fields is called maximally symmetric space, see Section 7.7.1 in [48]. Since embedding space $\mathbb{R}^{(p,q)}$ of dimension $m = (p+q)$ has $(p+q)$ translations T_A and $(p+q)(p+q-1)/2$ Killing vectors K_{AB}, it has in total $(p+q)(p+q+1)/2$ Killing vectors and is thus maximally symmetric.

B.1.2 Killing vectors for Minkowski spacetime $\mathbb{R}^{(1,3)}$

Now we apply the result of the previous section to find the Killing vectors of Minkowski spacetime in spherical coordinates. Later we will let them act on Klein-Gordon solutions, and see what kind of action results and how it transcribes from the position representation to the momentum representation of the solutions. In this section lowercase Greek letters range over $\mu = 0, 1, 2, 3$ and lowercase Latin letters range over $k = 1, 2, 3$. On Minkowski spacetime we use the metric $\eta = \mathrm{diag}\,(-, +, +, +)$, and have the usual cartesian $x = (x_0, \underline{x})$ with $\underline{x} = (x_1, x_2, x_3)$ and spherical coordinates (t, r, Ω) with $\Omega = (\theta, \varphi)$ related through

$$\begin{aligned} x^0 &= -x_0 = t & & \\ x_1 &= +x^1 = r\,\xi_1(\Omega) & \xi_1(\Omega) &= \sin\theta\, \cos\varphi \\ x_2 &= +x^2 = r\,\xi_2(\Omega) & \xi_2(\Omega) &= \sin\theta\, \sin\varphi \\ x_3 &= +x^3 = r\,\xi_3(\Omega) & \xi_3(\Omega) &= \cos\theta\;, \end{aligned} \tag{B.5}$$

and

$$
\begin{aligned}
r &= \sqrt{\underline{x}^2} & \partial_{x^j} r &= \xi_j \\
\xi_k &= \frac{x_k}{\sqrt{\underline{x}^2}} & \partial_{x^j} \xi_k &= \frac{\delta_{jk} - \xi_j \xi_k}{r} \ .
\end{aligned} \tag{B.6}
$$

Therein, δ_{jk} is the Kronecker delta. We thus have as Killing vectors 4 translations T_μ, 3 rotations K_{jk} and 3 boosts K_{0k}. The translations are given by

$$
T_0(x) = \partial_{x^0} = \partial_t \tag{B.7}
$$

$$
T_k(x) = \partial_{x^k} = \xi_k \, \partial_r + \frac{1}{r} \left(\partial_{\xi_k} - \xi_k \xi_i \, \partial_{\xi_i} \right) . \tag{B.8}
$$

The rotations are given by

$$
\begin{aligned}
K_{jk}(x) &= x_j \, \partial_{x^k} - x_k \, \partial_{x^j} \\
&= \xi_j \, \partial_{\xi_k} - \xi_k \, \partial_{\xi_j} \ ,
\end{aligned} \tag{B.9}
$$

and the boosts by

$$
\begin{aligned}
K_{0k}(x) &= x_0 \, \partial_{x^k} - x_k \, \partial_{x^0} \\
&= -r\xi_k \, \partial_t - t\xi_k \, \partial_r - \frac{t}{r} \left(\partial_{\xi_k} - \xi_k \xi_i \, \partial_{\xi_i} \right) .
\end{aligned} \tag{B.10}
$$

The translations commute among themselves:

$$
[T_\mu, T_\nu] = 0, \tag{B.11}
$$

and the boosts and rotations form the Lorentz algebra $\mathbf{so}\,(1,3)$ as in (B.4):

$$
[K_{\alpha\beta}, K_{\mu\nu}] = -\eta_{\alpha\mu} \, K_{\beta\nu} + \eta_{\beta\mu} \, K_{\alpha\nu} - \eta_{\beta\nu} \, K_{\alpha\mu} + \eta_{\alpha\nu} \, K_{\beta\mu} \ . \tag{B.12}
$$

The commutation relations between translations and the generators of the Lorentz algebra are:

$$
[T_\alpha, K_{\mu\nu}] = \eta_{\alpha\mu} T_\nu - \eta_{\alpha\nu} T_\mu. \tag{B.13}
$$

Hence, the translations T_μ together with the rotations K_{jk} and boosts K_{0j} generate the Poincaré algebra.

Actions of rotations and boosts on coordinates

Next we calculate the action of the rotators K_{jk} on the angular coordinates (θ, φ). Using

$$
\begin{aligned}
\xi_1(\Omega) &= \sin\theta \, \cos\varphi & \frac{\partial\theta}{\partial\xi_1} &= \frac{\cos\varphi}{\cos\theta} & \frac{\partial\varphi}{\partial\xi_1} &= -\frac{\sin\varphi}{\sin\theta} \\
\xi_2(\Omega) &= \sin\theta \, \sin\varphi & \frac{\partial\theta}{\partial\xi_2} &= \frac{\sin\varphi}{\cos\theta} & \frac{\partial\varphi}{\partial\xi_2} &= +\frac{\cos\varphi}{\sin\theta} \\
\xi_3(\Omega) &= \cos\theta & \frac{\partial\theta}{\partial\xi_3} &= \frac{-1}{\sin\theta} & \frac{\partial\varphi}{\partial\xi_3} &= 0 \ ,
\end{aligned} \tag{B.14}
$$

we find

$$
K_{12} = \xi_1 \, \partial_{\xi_2} - \xi_2 \, \partial_{\xi_1} = \partial_\varphi \tag{B.15}
$$

$$
K_{23} = \xi_2 \, \partial_{\xi_3} - \xi_3 \, \partial_{\xi_2} = -2 \sin\varphi \, \partial_\theta - \frac{\cos\varphi}{\tan\theta} \, \partial_\varphi \tag{B.16}
$$

$$
K_{31} = \xi_3 \, \partial_{\xi_1} - \xi_1 \, \partial_{\xi_3} = +2 \cos\varphi \, \partial_\theta - \frac{\sin\varphi}{\tan\theta} \, \partial_\varphi \ . \tag{B.17}
$$

Thus K_{12} generates rotations in φ-direction. Evaluating K_{23} in the (x_2, x_3)-plane for $(x_2 > 0) \Leftrightarrow (\varphi = \pi/2)$ and for $(x_2 < 0) \Leftrightarrow (\varphi = -\pi/2)$, we see that for the former case $K_{23} = -2\partial_\theta$ while for the latter $K_{23} = +2\partial_\theta$. That is, K_{23} rotates the x_2-axis towards the x_3-axis. In a similar way we can see that K_{31} rotates the x_3-axis towards the x_1-axis.

In order to illustrate the action of the boosts K_{0j}, we consider that within our inertial system I with coordinates $(t, \underline{x})$ there is another inertial system I' with coordinates $(t', \underline{x}')$, which is moving with a 3-velocity $\underline{v}$ within I. Without restriction of generality, we let the x^μ-axes of I be parallel to the respective $x^{\mu\prime}$-axes of I', and we let I' move in x^3-direction with speed $v \ll 1$, in order to have an infinitesimal boost. Fixing the origins to coincide at $t = t' = 0$, we always have $x^{1\prime} = x^1$ and $x^{2\prime} = x^2$, and moreover

$$t' = \gamma(v)\,(t - vx^3) \qquad\qquad x^{3\prime} = \gamma(v)\,(x^3 - vt)\ ,$$

wherein $\gamma(v) = (1-v^2)^{-1/2}$. For $v = \varepsilon \ll 1$ we have in leading order $\gamma(\varepsilon) \approx 1 + v^2/2$. Up to linear order we thus get

$$t' = t - \varepsilon x^3 \qquad\qquad x^{3\prime} = x^3 - \varepsilon t\ .$$

Now this is just the infinitesimal action of K_{03}, and thus an infinitesimal boost $\mathbb{1} + \varepsilon K_{0j}$ gives us the coordinates in a system that moves with speed ε in x^j-direction:

$$(\mathbb{1} + \varepsilon K_{03}) \begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = (\mathbb{1} - \varepsilon t\partial_3 - \varepsilon x_3 \partial_t) \begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t - \varepsilon x^3 \\ x^1 \\ x^2 \\ x^3 - \varepsilon t \end{pmatrix} .$$

In summary, our conventions lead to a natural and convenient action of the rotators K_{jk} and boosts K_{0j}. This was achieved by working with the K_{AB} Killing vectors (and not the K^A_B), choosing the metric's overall sign to give $\eta = \mathrm{diag}\,(-,+,+,+)$, and hence setting $x_0 = -x^0 = -t$ (and not $x_0 = t$).

B.1.3 Penrose diagram of Minkowski spacetime

In this section we briefly sum up how to find the Penrose diagram for Minkowski spacetime, and then examine the behaviour of the Killing vectors on its conformal boundary. The former is well known, see e.g. Section 3.1 in [11] Departing from spherical coordinates (t, r, Ω), we first define the new coordinates

$$\begin{aligned} v_\pm &:= t \pm r & v_\pm &\in (-\infty, +\infty) \\ t &= \tfrac{1}{2}(v_+ + v_-) & t &\in (-\infty, +\infty) \\ r &= \tfrac{1}{2}(v_+ - v_-) & r &\in [0, +\infty)\ . \end{aligned}$$

Since $r \geq 0$ the $v_\pm$-coordinates have the restriction

$$v_+ \geq v_-\ .$$

The metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\ , \tag{B.18}$$

with $\mathrm{d}\Omega_2^2$ the area element on $\mathbb{S}^2$. Next we define "compactified" versions of these coordinates:

$$\tilde{v}_\pm := \arctan v_\pm \qquad v_\pm := \tan \tilde{v}_\pm \qquad \tilde{v}_\pm \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)\ . \tag{B.19}$$

Since $v_+ \geq v_-$ and the arctangent is monotonically increasing, we have in addition

$$\tilde{v}_+ \geq \tilde{v}_-\ .$$

Finally, we can define "compactified" time and radial coordinates as

$$\tilde{t} := \tfrac{1}{2}(\tilde{v}_+ + \tilde{v}_-) \qquad \tilde{t} \in (-\frac{\pi}{2}, +\frac{\pi}{2})$$
$$\tilde{r} := \tfrac{1}{2}(\tilde{v}_+ - \tilde{v}_-) \qquad \tilde{r} \in [0, +\frac{\pi}{2})$$
$$\tilde{v}_\pm = \tilde{t} \pm \tilde{r} \,.$$

We have $\tilde{r} \geq 0$ because of $\tilde{v}_+ \geq \tilde{v}_-$. Further, since $\tilde{v}_\pm \in (-\frac{\pi}{2}, +\frac{\pi}{2})$ we have $(\tilde{t} \pm \tilde{r}) \in (-\frac{\pi}{2}, +\frac{\pi}{2})$, that is, in particular

$$\tilde{t} \leq \frac{\pi}{2} - \tilde{r} \qquad \tilde{t} \geq \tilde{r} - \frac{\pi}{2} \,.$$

After applying a conformal transformation

$$d\tilde{s}^2 := \cos^2\tilde{v}_+ \, \cos^2\tilde{v}_- \, ds^2 \,, \tag{B.20}$$

we get the conformally rescaled metric

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \sin^2\tilde{r} \, \cos^2\tilde{r} \, d\Omega_2^2 \,. \tag{B.21}$$

We can now draw Minkowski spacetime in a conformal diagram called Penrose diagram, see Figure B.22.

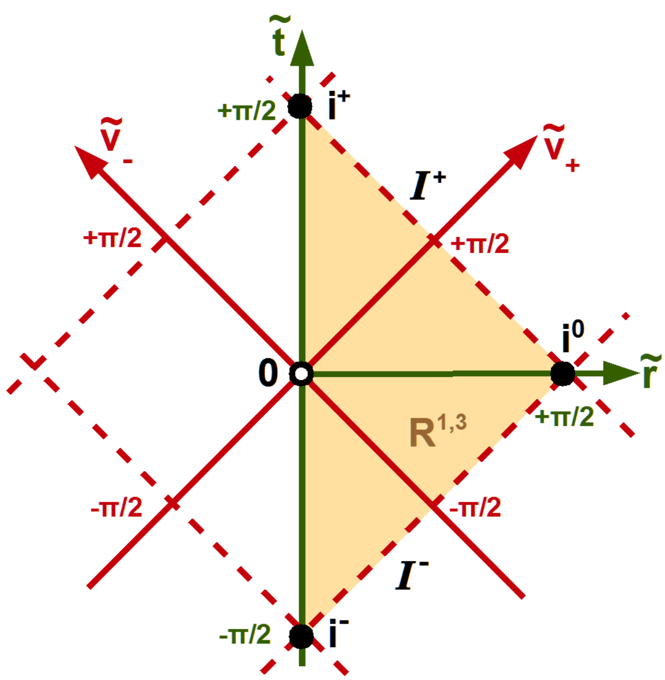


Figure B.22: Penrose diagram of Minkowski spacetime: $(\tilde{t}, \tilde{r})$ and $\tilde{v}_\pm$-coordinates.

The $\tilde{t}$-coordinate is on the vertical axis and $\tilde{r}$ on the horizontal one. The $\tilde{v}_\pm$-coordinates are on the diagonal axes. Minkowski spacetime is the shaded triangular region where we have $\tilde{r} \in [0, \pi/2)$ and $\tilde{t} \in (-\pi/2, +\pi/2)$ with $\tilde{t} \leq \frac{\pi}{2} - \tilde{r}$ and $\tilde{t} \geq \tilde{r} - \frac{\pi}{2}$, or equivalently $\tilde{v}_\pm \in (-\pi/2, +\pi/2)$ with $\tilde{v}_+ \geq \tilde{v}_-$. The point i^- is called past timelike infinity, i^+ is future timelike infinity, and i^0 is spacelike infinity.

Every point in the diagram corresponds to a two-sphere in Minkowski space, except for the points on the time axis $r = \tilde{r} = 0$ and i^-, i^+ and i^0. For these exceptions the prefactor $\sin^2\tilde{r} \, \cos^2\tilde{r}$ of the spherical part of the metric $d\tilde{s}^2$ vanishes, and thus their corresponding two-spheres degenerate to points.

The conformal boundary of Minkowski spacetime consists of the "lines" (whose points correspond to two-spheres) $\mathcal{I}^-$ (past null infinity) where $\tilde{v}_- = -\pi/2$ and $\mathcal{I}^+$ (future null infinity) where $\tilde{v}_+ = +\pi/2$, plus the points i^-, i^+ and i^0. The conformal boundary is not part of Minkowski spacetime.

The direct relation between original and compactified coordinates is given by

$$\begin{aligned} \tilde{t} &= \frac{1}{2}\Big(\arctan(t{+}r) + \arctan(t{-}r)\Big) & t &= \frac{1}{2}\Big(\tan(\tilde{t}{+}\tilde{r}) + \tan(\tilde{t}{-}\tilde{r})\Big) \\ \tilde{r} &= \frac{1}{2}\Big(\arctan(t{+}r) - \arctan(t{-}r)\Big) & r &= \frac{1}{2}\Big(\tan(\tilde{t}{+}\tilde{r}) - \tan(\tilde{t}{-}\tilde{r})\Big)\,. \end{aligned}$$

This permits us to draw the lines of constant t and r in the Penrose diagram.

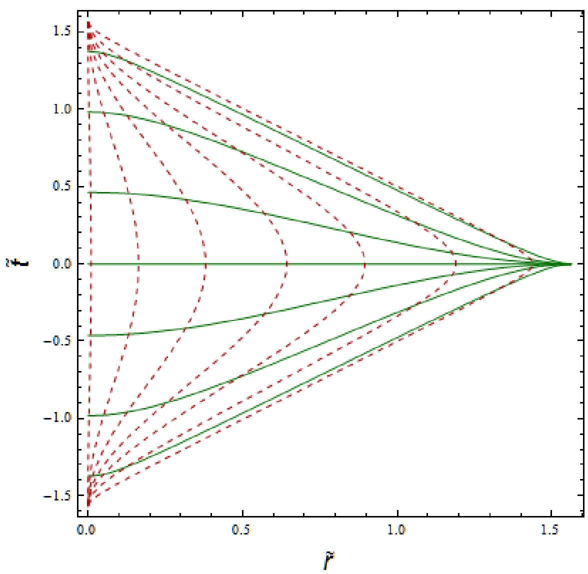


Figure B.23: Penrose diagram of Minkowski spacetime: lines of constant t and r coordinates.

Analyzing the Killing vectors in the Penrose diagram is done most easily with the $\tilde{v}_\pm$-coordinates, while for drawing them we need the $(\tilde{t}, \tilde{r})$-coordinates. Since the rotations only move points on a two-sphere, and thus leave (t, r), $(\tilde{v}_-, \tilde{v}_+)$ and $(\tilde{t}, \tilde{r})$ invariant, we shall only deal with translations and boosts here. Using

$$\tilde{v}_\pm = \arctan(t \pm r) \qquad t = \frac{1}{2}\Big(\tan\tilde{v}_+ + \tan\tilde{v}_-\Big) \qquad r = \frac{1}{2}\Big(\tan\tilde{v}_+ - \tan\tilde{v}_-\Big)\,,$$

and thus

$$\begin{aligned} \partial_t &= \tan^{-2}\tilde{v}_+\,\partial_{\tilde{v}_+} + \tan^{-2}\tilde{v}_-\,\partial_{\tilde{v}_-} = \frac{1}{2}\big(\cos^2(\tilde{t}{+}\tilde{r}) + \cos^2(\tilde{t}{-}\tilde{r})\big)\partial_{\tilde{t}} + \frac{1}{2}\Big(\cos^2(\tilde{t}{+}\tilde{r}) - \cos^2(\tilde{t}{-}\tilde{r})\Big)\partial_{\tilde{r}} \\ \partial_r &= \tan^{-2}\tilde{v}_+\,\partial_{\tilde{v}_+} - \tan^{-2}\tilde{v}_-\,\partial_{\tilde{v}_-} = \frac{1}{2}\big(\cos^2(\tilde{t}{+}\tilde{r}) - \cos^2(\tilde{t}{-}\tilde{r})\big)\partial_{\tilde{t}} + \frac{1}{2}\Big(\cos^2(\tilde{t}{+}\tilde{r}) + \cos^2(\tilde{t}{-}\tilde{r})\Big)\partial_{\tilde{r}}\,, \end{aligned}$$

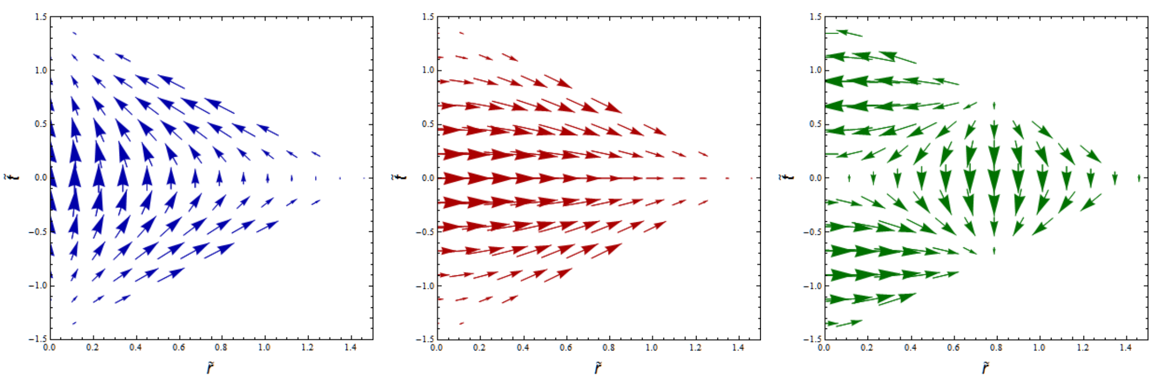
we can write the translations as

$$
\begin{aligned}
T_0(\tilde v_-,\tilde v_+,\underline{\xi}) &= \tan^{-2}\tilde v_+\ \partial_{\tilde v_+} + \tan^{-2}\tilde v_-\ \partial_{\tilde v_-} && \text{(B.24)}\\
&= \frac{1}{2}\big(\cos^2(\tilde t+\tilde r)+\cos^2(\tilde t-\tilde r)\big)\partial_{\tilde t} + \frac{1}{2}\Big(\cos^2(\tilde t+\tilde r)-\cos^2(\tilde t-\tilde r)\Big)\partial_{\tilde r} && \text{(B.25)}\\
T_j(\tilde v_-,\tilde v_+,\underline{\xi}) &= \frac{\xi_j}{\tan^2\tilde v_+}\,\partial_{\tilde v_+} - \frac{\xi_j}{\tan^2\tilde v_-}\,\partial_{\tilde v_-} + \underbrace{\frac{2}{\tan\tilde v_+ - \tan\tilde v_-}}_{1/r}\big(\partial_{\xi_j} - \xi_j\xi_i\,\partial_{\xi_i}\big) && \text{(B.26)}\\
&= \xi_j\frac{1}{2}\big(\cos^2(\tilde t+\tilde r)-\cos^2(\tilde t-\tilde r)\big)\partial_{\tilde t} + \frac{1}{2}\Big(\cos^2(\tilde t+\tilde r)+\cos^2(\tilde t-\tilde r)\Big)\partial_{\tilde r}\\
&\quad + \frac{2}{\tan(\tilde t+\tilde r)-\tan(\tilde t-\tilde r)}\big(\partial_{\xi_j} - \xi_j\xi_i\partial_{\xi_i}\big)\,, && \text{(B.27)}
\end{aligned}
$$

and the boosts as

$$
\begin{aligned}
K_{0j}(\tilde v_-,\tilde v_+,\underline{\xi}) &= -\frac{\xi_j}{\tan\tilde v_+}\,\partial_{\tilde v_+} + \frac{\xi_j}{\tan\tilde v_-}\,\partial_{\tilde v_-} - \underbrace{\frac{\tan\tilde v_+ + \tan\tilde v_-}{\tan\tilde v_+ - \tan\tilde v_-}}_{t/r}\big(\partial_{\xi_j} - \xi_j\xi_i\,\partial_{\xi_i}\big)\,.\\
&= -\frac{\xi_j}{2}\cos(2\tilde t)\,\sin(2\tilde r)\,\partial_{\tilde t} - \frac{\xi_j}{2}\sin(2\tilde t)\,\cos(2\tilde r)\,\partial_{\tilde r}\\
&\quad - \frac{\tan(\tilde t+\tilde r)+\tan(\tilde t-\tilde r)}{\tan(\tilde t+\tilde r)-\tan(\tilde t-\tilde r)}\big(\partial_{\xi_j} - \xi_j\xi_i\partial_{\xi_i}\big)\,.
\end{aligned}
$$

This is sketched in the following Figure B.28, wherein the compactified (t, x_1)-plane of Minkowski space is drawn.



(a) Time translation T_0 (b) x^j-Translation T_j (c) x^j-Boost K_{0j}

Figure B.28: Penrose diagram of Minkowski spacetime: Translation and Boost Killing vectors.

Now we can study the behaviour of the Killing vectors on the conformal boundary of Minkowski space. The rotations K_{jk} simply move points on the boundary on their respective two-sphere. On $\mathcal{I}^-$, where $\tilde v_- = -\pi/2$, we have

$$
\begin{aligned}
T_0(-\pi/2,\tilde v_+,\underline{\xi}) &= \tan^{-2}\tilde v_+\ \partial_{\tilde v_+}\\
T_j(-\pi/2,\tilde v_+,\underline{\xi}) &= \frac{\xi_j}{\tan^2\tilde v_+}\,\partial_{\tilde v_+}\\
K_{0j}(-\pi/2,\tilde v_+,\underline{\xi}) &= -\frac{\xi_j}{\tan\tilde v_+}\,\partial_{\tilde v_+} + \big(\partial_{\xi_j} - \xi_j\xi_i\,\partial_{\xi_i}\big)\,,
\end{aligned}
$$

while on $\mathcal{I}^+$, where $\tilde{v}_+ = +\pi/2$, we have

$$
\begin{aligned}
T_0(\tilde{v}_-, +\pi/2, \underline{\xi}) &= \tan^{-2}\tilde{v}_- \, \partial_{\tilde{v}_-} \\
T_j(\tilde{v}_-, +\pi/2, \underline{\xi}) &= -\frac{\xi_j}{\tan^2\tilde{v}_-}\, \partial_{\tilde{v}_-} \\
K_{0j}(\tilde{v}_-, +\pi/2, \underline{\xi}) &= +\frac{\xi_j}{\tan\tilde{v}_-}\, \partial_{\tilde{v}_-} - \left(\partial_{\xi_j} - \xi_j \xi_i \, \partial_{\xi_i}\right) .
\end{aligned}
$$

This means, that time translations move points along the boundary: on $\mathcal{I}^-$ towards i^0 and on $\mathcal{I}^+$ towards i^+. Since on $\mathcal{I}^\pm$ we have $t = \pm\infty$, this *is not* a movement in future direction. It is a result of the limiting process when approaching null infinity. Any finite translation starting at $t = \pm\infty$ remains at $t = \pm\infty$. On the interior of Minkoswki spacetime the time translation is always future directed.

Spatial translations $T_j = \partial_{x^j}$ increase the radius r on the interior. On $\mathcal{I}^-$ they move points along the boundary towards i^0, and on $\mathcal{I}^-$, too. As before, this *is not* a movement increasing the radius but a result of the limiting process when approaching null infinity. Any finite translation starting at $r = +\infty$ remains at $r = +\infty$.

The boosts behave basically like the spatial translations, their $\underline{\xi}$-components are of lesser interest here. Note that when approaching any of $i^\pm, i^0$, the translations vanish, and only the $\underline{\xi}$-part of the boosts survives.

At timelike infinity $i^\pm$ we have $(\tilde{v}_-, \tilde{v}_+) = (\pm\pi/2, \pm\pi, 2)$, which corresponds to $t = \pm\infty$ with r not being well defined. Here, T_0 seems to vanish in $(\tilde{v}_-, \tilde{v}_+)$ coordinates in (B.24), but using the original expression $T_0(i^\pm) = \partial_t$ reveals this an artifact of the coordinate transformation. The boosts and spatial translations are not well defined here.

At spatial infinity i^0 we have $(\tilde{v}_-, \tilde{v}_+) = (-\pi/2, +\pi, 2)$, and $r = +\infty$, but the time t is not well defined here. Thus T_0 and the boosts are not well defined either. The spatial translations in $(\tilde{v}_-, \tilde{v}_+)$ coordinates seem to vanish as well in (B.26), but using the original expression (B.8) reveals that this too is an artifact of the coordinate transformation: $T_j(i^0) = \xi_j \, \partial_r$.

Hence we can see that the Minkowski Killing vector fields map the conformal boundary to itself.

B.2 Minkowski isometry actions on solution spaces

B.2.1 Minkowski rod region

In this section we compute the action of the isometries: first on the modes, and then on a general solution written as a mode expansion. From (B.8) we know the form of the generator T_3 of translations in x^3-direction. However, as in the derivation of (C.172) for AdS, we shall use here the following form derived directly from $x^3 = r\cos\theta$:

$$
\begin{aligned}
T_3 \;=\; \partial_{x^3} &= \frac{\partial r}{\partial x^3}\, \partial_r + \frac{\partial \cos\theta}{\partial x^3}\, \partial_{\cos\theta} \\
&= \cos\theta\, \partial_r + \tfrac{(1-\cos^2\theta)}{r}\, \partial_{\cos\theta} .
\end{aligned} \tag{B.29}
$$

For the boost generator K_{03} we use a similar form as for T_3:

$$
\begin{aligned}
K_{03} &= x_0\, \partial_{x^3} - x_3\, \partial_{x^0} \;=\; -t\, \partial_{x^3} - x^3\, \partial_t \\
&= -t\cos\theta\, \partial_r - \tfrac{t}{r}\, (1-\cos^2\theta) \partial_{\cos\theta} - r\cos\theta\, \partial_t .
\end{aligned} \tag{B.30}
$$

In the calculation we shall need the recurrence relations DLMF [10.51.1+2] for spherical Bessel and Neumann functions:

$$\partial_z j_l(z) \;=\; \frac{l}{2l+1}\, j_{l-1}(z) - \frac{l+1}{2l+1}\, j_{l+1}(z) \qquad \partial_z n_l(z) \;=\; \frac{l}{2l+1}\, n_{l-1}(z) - \frac{l+1}{2l+1}\, n_{l+1}(z) \tag{B.31}$$

$$\frac{1}{z}\, j_l(z) \;=\; \frac{1}{2l+1}\, j_{l-1}(z) + \frac{1}{2l+1}\, j_{l+1}(z) \qquad \frac{1}{z}\, n_l(z) \;=\; \frac{1}{2l+1}\, n_{l-1}(z) + \frac{1}{2l+1}\, n_{l+1}(z) \tag{B.32}$$

$$\partial_z j_l(z) \;=\; +j_{l-1}(z) - \frac{l+1}{z}\, j_l(z) \qquad \partial_z n_l(z) \;=\; +n_{l-1}(z) - \frac{l+1}{z}\, n_l(z) \tag{B.33}$$

$$\partial_z j_l(z) \;=\; -j_{l+1}(z) + \frac{l}{z}\, j_l(z) \qquad \partial_z n_l(z) \;=\; -n_{l+1}(z) + \frac{l}{z}\, n_l(z). \tag{B.34}$$

We also recall the recurrence relations (A.15)

$$\begin{aligned} \cos\theta\, Y_l^{m_l}(\Omega) \;&=\; \chi_-^{(2)}(l,m_l)\, Y_{l-1}^{m_l}(\Omega) + \chi_+^{(2)}(l,m_l)\, Y_{l+1}^{m_l}(\Omega) \\ (1-\cos^2\theta)\, \frac{\mathrm{d}}{\mathrm{d}\cos\theta}\, Y_l^{m_l}(\Omega) \;&=\; (l+1)\chi_-^{(2)}(l,m_l)\, Y_{l-1}^{m_l}(\Omega) - l\chi_+^{(2)}(l,m_l)\, Y_{l+1}^{m_l}(\Omega). \end{aligned}$$

Further, for $p_E^{\mathbb{R}} := \sqrt{|E^2 - m^2|}$, which sometimes we abbreviate by p, we will need the derivative

$$\partial_E p_E^{\mathbb{R}} \;=\; \pm E/p_E^{\mathbb{R}} \qquad E^2 \gtrless m^2. \tag{B.35}$$

We recall the Minkowski mode definitions (2.109)

$$\begin{aligned} \mu^{(a)}_{Elm_l}(t,r,\Omega) \;&:=\; \frac{p_E^{\mathbb{R}}}{4\pi}\, \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{\jmath}_{El}(r) \\ \mu^{(b)}_{Elm_l}(t,r,\Omega) \;&:=\; \frac{p_E^{\mathbb{R}}}{4\pi}\, \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{n}_{El}(r), \end{aligned} \tag{B.36}$$

wherein

$$\check{\jmath}_{El}(r) \;=\; \begin{cases} j_l(p_E^{\mathbb{R}} r) & E^2 > m^2 \\ \mathrm{i}^{-l}\, j_l(\mathrm{i}p_E^{\mathbb{R}} r) & E^2 < m^2 \end{cases} \qquad \check{n}_{El}(r) \;=\; \begin{cases} n_l(p_E^{\mathbb{R}} r) & E^2 > m^2 \\ \mathrm{i}^{l+1}\, n_l(\mathrm{i}p_E^{\mathbb{R}} r) & E^2 < m^2 \end{cases}. \tag{B.37}$$

First, we let the translation generator act on the modes:

$$\big(T_3 \triangleright \mu^{(a)}_{Elm_l}\big)(t,r,\Omega) \;=\; -T_3\, \mu^{(a)}_{Elm_l}(t,r,\Omega). \tag{B.38}$$

On the left is the action $\triangleright$ of the generator T_3 as an operator in solution space, while on the right T_3 acts as a differential operator. Using the above ingredients, it is straightforward to let T_3 as given by (B.29) act on the modes. It turns out that the terms containing $Y_{l-1}^{m_l}(\Omega)\, \check{\jmath}_{E,l+1}(r)$ add up to zero, ditto for $Y_{l+1}^{m_l}(\Omega)\, \check{\jmath}_{E,l-1}(r)$, ditto for $Y_{l\pm1}^{m_l}(\Omega)\, \check{n}_{E,l\mp1}(r)$. Grouping the remaining terms, we find that the action of T_3 on the modes in solution space can be written rather nicely as (with the χ-factors (A.17))

$$\begin{aligned} T_3 \triangleright \mu^{(a)}_{Elm_l} \;&=\; -p_E^{\mathbb{R}}\, \chi_-^{(2)}(l,m_l)\, \mu^{(a)}_{E,l-1,m_l} \pm p_E^{\mathbb{R}}\, \chi_+^{(2)}(l,m_l)\, \mu^{(a)}_{E,l+1,m_l} \qquad E^2 \gtrless m^2 \\ T_3 \triangleright \mu^{(b)}_{Elm_l} \;&=\; \mp p_E^{\mathbb{R}}\, \chi_-^{(2)}(l,m_l)\, \mu^{(b)}_{E,l-1,m_l} + p_E^{\mathbb{R}}\, \chi_+^{(2)}(l,m_l)\, \mu^{(b)}_{E,l+1,m_l} \qquad E^2 \gtrless m^2. \end{aligned} \tag{B.39}$$

The next step is to transform the action of T_3 on the Klein-Gordon modes into an action within the momentum representation. Letting T_3 act as above on a solution expanded as in (2.108):

$$\phi(t,r,\Omega) \;=\; \int \mathrm{d}E \sum_{l,m_l} \Big\{ \phi^a_{Elm_l} \mu^{(a)}_{Elm_l}(t,r,\Omega) + \phi^b_{Elm_l} \mu^{(b)}_{Elm_l}(t,r,\Omega) \Big\}, \tag{B.40}$$

after shifting $l \to l \pm 1$ yields the actions of T_3 in the momentum representation for $E^2 \gtrless m^2$:

$$\begin{aligned} (T_3 \triangleright \phi)^a_{Elm_l} \;&=\; \pm p_E^{\mathbb{R}}\, \chi_-^{(2)}(l,m_l)\, \phi^a_{E,l-1,m_l} - p_E^{\mathbb{R}}\, \chi_+^{(2)}(l,m_l)\, \phi^a_{E,l+1,m_l} \\ (T_3 \triangleright \phi)^b_{Elm_l} \;&=\; +p_E^{\mathbb{R}}\, \chi_-^{(2)}(l,m_l)\, \phi^b_{E,l-1,m_l} \mp p_E^{\mathbb{R}}\, \chi_+^{(2)}(l,m_l)\, \phi^b_{E,l+1,m_l}. \end{aligned} \tag{B.41}$$

Second, we let the boost generator K_{03} act on the modes. As for T_3, the terms containing $Y^{m_l}_{l\pm1}(\Omega)\,\check{\jmath}_{E,l\mp1}(r)$ and $Y^{m_l}_{l\pm1}(\Omega)\,\check{n}_{E,l\mp1}(r)$ add up to zero respectively, and we obtain for $E^2 \gtrless m^2$:

$$\begin{aligned} K_{03}\triangleright\mu^{(a)}_{Elm_l}(t,r,\Omega) &= -K_{03}\,\mu^{(a)}_{Elm_l}(t,r,\Omega) \\ &= +tp^{\mathbb{R}}_E\chi^{(2)}_-(l,m_l)\mu^{(a)}_{E,l-1,m_l}(t,r,\Omega) \mp tp^{\mathbb{R}}_E\chi^{(2)}_+(l,m_l)\mu^{(a)}_{E,l+1,m_l}(t,r,\Omega) \\ &\quad - \mathrm{i}rE\,\frac{p^{\mathbb{R}}_E}{4\pi}\,\mathrm{e}^{-\mathrm{i}Et}\check{\jmath}_{El}(r)\Big(\chi^{(2)}_-(l,m_l)Y^{m_l}_{l-1}(\Omega) + \chi^{(2)}_+(l,m_l)Y^{m_l}_{l+1}(\Omega)\Big) \end{aligned} \tag{B.42}$$

$$\begin{aligned} K_{03}\triangleright\mu^{(b)}_{Elm_l}(t,r,\Omega) &= -K_{03}\,\mu^{(a)}_{Elm_l}(t,r,\Omega) \\ &= \pm tp^{\mathbb{R}}_E\chi^{(2)}_-(l,m_l)\mu^{(b)}_{E,l-1,m_l}(t,r,\Omega) - tp^{\mathbb{R}}_E\chi^{(2)}_+(l,m_l)\mu^{(b)}_{E,l+1,m_l}(t,r,\Omega) \\ &\quad - \mathrm{i}rE\,\frac{p^{\mathbb{R}}_E}{4\pi}\,\mathrm{e}^{-\mathrm{i}Et}\check{n}_{El}(r)\Big(\chi^{(2)}_-(l,m_l)Y^{m_l}_{l-1}(\Omega) + \chi^{(2)}_+(l,m_l)Y^{m_l}_{l+1}(\Omega)\Big) \end{aligned} \tag{B.43}$$

Since the right hand sides of (B.42) and (B.43) are not easily recognizable as Klein-Gordon solutions, we shall check explicitly that they are such. The only ingredients that we need to this end are the last two lines of (B.31) in the form

$$\begin{aligned} j_{l-1}(z) &= \frac{l+1}{z}\,j_l(z) + \partial_z j_l(z) & n_{l-1}(z) &= \frac{l+1}{z}\,n_l(z) + \partial_z n_l(z) \\ j_{l+1}(z) &= \frac{l}{z}\,j_l(z) - \partial_z j_l(z) & n_{l+1}(z) &= \frac{l}{z}\,n_l(z) - \partial_z n_l(z), \end{aligned}$$

plus the fact that the modes $\mu^{a,b}$ are homogeneous Klein-Gordon solutions. With these and the Laplace-Beltrami $\Box = -\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r + r^{-2}\Box_{\mathbb{S}^2}$, we find that for $E^2 \gtrless m^2$

$$\begin{aligned} \Big(\big(\Box - m^2\big)K_{03}\,\mu^{(a)}_{Elm_l}\Big)(t,r,\Omega) &= -\mathrm{i}E(p^{\mathbb{R}}_E)^2\,r\,\frac{p^{\mathbb{R}}_E}{4\pi}\,\mathrm{e}^{-\mathrm{i}Et}\Big(\chi^{(2)}_-(l,m_l)Y^{m_l}_{l-1}(\Omega) + \chi^{(2)}_+(l,m_l)Y^{m_l}_{l+1}(\Omega)\Big)\cdot \\ &\qquad \cdot\Big(\partial^2_{pr} + \frac{2}{p^{\mathbb{R}}_E r}\,\partial_{pr} \pm 1 - \frac{l(l+1)}{(p^{\mathbb{R}}_E r)^2}\Big)\,\check{\jmath}_{E,l}(r) \\ \Big(\big(\Box - m^2\big)K_{03}\,\mu^{(b)}_{Elm_l}\Big)(t,r,\Omega) &= -\mathrm{i}E(p^{\mathbb{R}}_E)^2\,r\,\frac{p^{\mathbb{R}}_E}{4\pi}\,\mathrm{e}^{-\mathrm{i}Et}\Big(\chi^{(2)}_-(l,m_l)Y^{m_l}_{l-1}(\Omega) + \chi^{(2)}_+(l,m_l)Y^{m_l}_{l+1}(\Omega)\Big)\cdot \\ &\qquad \cdot\Big(\partial^2_{pr} + \frac{2}{p^{\mathbb{R}}_E r}\,\partial_{pr} \pm 1 - \frac{l(l+1)}{(p^{\mathbb{R}}_E r)^2}\Big)\,\check{n}_{E,l}(r). \end{aligned}$$

For both $E^2 > m^2$ (with $z = p^{\mathbb{R}}_E r$) and $E^2 < m^2$ (with $z = \mathrm{i}p^{\mathbb{R}}_E r$) the terms involving spherical Bessel functions (respectively spherical Neumann functions) thus turn into the spherical Bessel differential equation, and therefore these terms add up to zero. That is, letting act K_{03} on the $\mu^{a,b}$-modes indeed yields Klein-Gordon solutions. The spherical Bessel DEQ can be found in DLMF [10.51.2] to be:

$$0 = \Big(\partial_z^2 + \frac{2}{z}\,\partial_z + 1 - \frac{l(l+1)}{z^2}\Big)f(z).$$

As done for cartesian coordinates in Section 2.5.5, we now want to find the momentum space representation of the boost generator (B.30). That is, we want to find an operator consisting only of energy and angular momenta which reproduces the actions (B.42) and (B.43). As a first step in this direction we note that

$$p^{\mathbb{R}}_E\partial_E\check{\jmath}_{E,l}(r) = \pm Er\,\check{\jmath}_{E,l-1}(r) \mp (l+1)(E/p^{\mathbb{R}}_E)\,\check{\jmath}_{E,l}(r) \tag{B.44}$$

$$= -Er\,\check{\jmath}_{E,l+1}(r) \pm (l)(E/p^{\mathbb{R}}_E)\,\check{\jmath}_{E,l}(r) \tag{B.45}$$

$$p^{\mathbb{R}}_E\partial_E\check{n}_{E,l}(r) = +Er\,\check{n}_{E,l-1}(r) \mp (l+1)(E/p^{\mathbb{R}}_E)\,\check{n}_{E,l}(r) \tag{B.46}$$

$$= \mp Er\,\check{n}_{E,l+1}(r) \pm (l)(E/p^{\mathbb{R}}_E)\,\check{n}_{E,l}(r). \tag{B.47}$$

Next we introduce the shifting operators $L^\pm$, which we define to increase/decrease the angular momentum l by one: $L^\pm f_l := f_{l\pm1}$. We note that $L^\pm$ and ∂_E commute when acting on $\mathrm{e}^{-\mathrm{i}Et}Y^{m_l}_l(\Omega)\check{\jmath}_{E,l}(r)$ and $\mathrm{e}^{-\mathrm{i}Et}Y^{m_l}_l(\Omega)\check{n}_{E,l}(r)$. We also recall that the lowering coefficient $\chi^{(2)}_-(l,m_l)$ vanishes for $l = 0$ and also for $|m_l| = l$. Moreover, $\chi_-^{(2)}(l+1, m_l) = \chi_+^{(2)}(l, m_l)$. When for short we write only $\chi_\pm^{(2)}$, then the argument is understood to be always (l, m_l). As well, p is understood to be $p_E^{\mathbb{R}}$. Combining L^+ with (B.44) respectively (B.46), and L^- with (B.45) respectively (B.47), it is somewhat lengthy but straightforward to check that

$$-\Big[(\chi_-^{(2)} L^- \mp \chi_+^{(2)} L^+)(-\mathrm{i} p_E^{\mathbb{R}} \partial_E) \pm \mathrm{i}(l-1) \tfrac{E}{p} \chi_-^{(2)} L^- + \mathrm{i}(l+2) \tfrac{E}{p} \chi_+^{(2)} L^+\Big]\ \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega) \check{\jmath}_{E,l}(r) \qquad (B.48)$$
$$= -K_{03}\ \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega) \check{\jmath}_{E,l}(r)$$
$$-\Big[\pm(\chi_-^{(2)} L^- - \chi_+^{(2)} L^+)(-\mathrm{i} p_E^{\mathbb{R}} \partial_E) + \mathrm{i}(l-1) \tfrac{E}{p} \chi_-^{(2)} L^- \pm \mathrm{i}(l+2) \tfrac{E}{p} \chi_+^{(2)} L^+\Big]\ \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega) \check{n}_{E,l}(r) \qquad (B.49)$$
$$= -K_{03}\ \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega) \check{n}_{E,l}(r).$$

That is, setting for the action on $\mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega) \check{\jmath}_{E,l}(r)$

$$K_{03} = \Big[(\chi_-^{(2)} L^- \mp \chi_+^{(2)} L^+)(-\mathrm{i} p_E^{\mathbb{R}} \partial_E) \pm \mathrm{i}(l-1) \tfrac{E}{p} \chi_-^{(2)} L^- + \mathrm{i}(l+2) \tfrac{E}{p} \chi_+^{(2)} L^+\Big] \qquad (B.50)$$

and for the action on $\mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega) \check{n}_{E,l}(r)$

$$K_{03} = \Big[\pm(\chi_-^{(2)} L^- - \chi_+^{(2)} L^+)(-\mathrm{i} p_E^{\mathbb{R}} \partial_E) + \mathrm{i}(l-1) \tfrac{E}{p} \chi_-^{(2)} L^- \pm \mathrm{i}(l+2) \tfrac{E}{p} \chi_+^{(2)} L^+\Big] \qquad (B.51)$$

reproduces the actions (B.42) and (B.43) of K_{03} in its configuration representation. That is, (B.50) and (B.51) are the momentum representation of K_{03}. Compared to the form (2.137) on the equal-time plane in cartesian coordinates, $K_{03} = E_{\underline{k}} \partial_{k^3}$, we note that the above are considerably less simple. Moreover, they have different signs for the two classes of modes (Bessel and Neumann modes), while in cartesian coordinates $K_{03} = E_{\underline{k}} \partial_{k^3}$ holds for both classes of modes (positive and negative frequencies).

The next step is to transform the action of K_{03} on the Klein-Gordon modes into an action within the momentum representation. Letting K_{03} act as above on a solution expanded as in (2.108):

$$\phi(t, r, \Omega) = \int \mathrm{d}E \sum_{l, m_l} \Big\{ \phi^a_{Elm_l} \mu^{(a)}_{Elm_l}(t, r, \Omega) + \phi^b_{Elm_l} \mu^{(b)}_{Elm_l}(t, r, \Omega) \Big\}, \qquad (B.52)$$

we obtain a somewhat long expression. In half of the terms of it, the derivative operator ∂_E acts to the right on the functions $\mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega) \check{\jmath}_{E,l}(r)$ and $\mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega) \check{n}_{E,l}(r)$. As for $E_{\underline{k}} \partial_{k^3}$ in cartesian coordinates, we now need to find its adjoint operator (with respect to the integral we use here) which acts to the left. Using partial integration and sufficient decay properties of the involved functions, in (2.140) for $E_{\underline{k}} \partial_{k^3}$, we find the adjoint $-E_{\underline{k}} \partial_{k^3}$. We note that the complete derivative operator appearing above is $\mathrm{i}p^2\, \partial_E$ due to the factor $p/(4\pi)$ in the mode definitions. For sufficiently decaying $f(E)$ and bounded $g(E)$, we then find

$$\int_{\mathbb{R}} \mathrm{d}E\ p^2 f\, (\partial_E g) = p^2 f g|_{-\infty}^{+\infty} - \int_{\mathbb{R}} \mathrm{d}E\ (\partial_E\, p^2 f)\, g = -\int_{\mathbb{R}} \mathrm{d}E\ p^2\, (\partial_E f)\, g - \int_{\mathbb{R}} \mathrm{d}E\ (\partial_E p^2)\, f g$$
$$= -\int_{\mathbb{R}} \mathrm{d}E\ p^2\, (\partial_E f)\, g \mp \int_{\mathbb{R}} \mathrm{d}E\ 2E\, f g.$$

That is, the adjoint of $\mathrm{i}p^2\, \partial_E$ contains one derivative to the left and one multiplicative term. This writes as the following replacement rule:

$$\overrightarrow{\mathrm{i}p^2\, \partial_E} \ \to\ -\overleftarrow{\mathrm{i}p^2\, \partial_E} \mp 2\mathrm{i}E. \qquad (B.53)$$

Applying this rule after letting K_{03} act on expansion (2.108) as in (B.50) and (B.51), shifting again $l \to l \pm 1$ and cleaning up, we obtain the following actions of K_{03} in the momentum representation of the solutions:

$$\begin{aligned}(K_{03}\triangleright\phi)^a_{Elm_l} &= \pm\chi^{(2)}_-(l,m_l)\big(\mathrm{i}p^{\mathbb{R}}_E\partial_E\phi^a_{E,l-1,m_l}\big)-\chi^{(2)}_+(l,m_l)\big(\mathrm{i}p^{\mathbb{R}}_E\partial_E\phi^a_{E,l+1,m_l}\big) \\ &\quad -\mathrm{i}(E/p^{\mathbb{R}}_E)(l-1)\chi^{(2)}_-(l,m_l)\,\phi^a_{E,l-1,m_l}\mp\mathrm{i}(E/p^{\mathbb{R}}_E)(l+2)\chi^{(2)}_+(l,m_l)\,\phi^a_{E,l+1,m_l}\end{aligned}\tag{B.54}$$

$$\begin{aligned}(K_{03}\triangleright\phi)^b_{Elm_l} &= +\chi^{(2)}_-(l,m_l)\big(\mathrm{i}p^{\mathbb{R}}_E\partial_E\phi^b_{E,l-1,m_l}\big)\mp\chi^{(2)}_+(l,m_l)\big(\mathrm{i}p^{\mathbb{R}}_E\partial_E\phi^b_{E,l+1,m_l}\big) \\ &\quad \mp\mathrm{i}(E/p^{\mathbb{R}}_E)(l-1)\chi^{(2)}_-(l,m_l)\,\phi^b_{E,l-1,m_l}-\mathrm{i}(E/p^{\mathbb{R}}_E)(l+2)\chi^{(2)}_+(l,m_l)\,\phi^b_{E,l+1,m_l}.\end{aligned}\tag{B.55}$$

B.2.2 Minkowski time-interval region

Here we use essentially the same modes as near the hypercylinder, except for that now the modes are parametrized by the momentum p instead of the energy E. With always $p>0$ and $E>m$ we define the $\mu^{(j)}$-modes as in (2.92):

$$\begin{aligned}\mu^{(j)}_{plm_l}(t,r,\Omega) &:= \frac{2p}{\sqrt{2\pi}}\,\mathrm{e}^{-\mathrm{i}E_pt}Y^{m_l}_l(\Omega)\,j_l(pr) \qquad\qquad E_p := \sqrt{p^2+m^2} \\ &= \frac{8\pi}{\sqrt{2\pi}}\,\mu^{(a)}_{E_plm_l}(t,r,\Omega).\end{aligned}\tag{B.56}$$

From the results of the hypercylinder, we can read off the actions for the $\mu^{(j)}$-modes:

$$\begin{aligned}T_3\triangleright\mu^{(j)}_{plm_l} &= -p\,\chi^{(2)}_-(l,m_l)\,\mu^{(j)}_{p,l-1,m_l}+p\,\chi^{(2)}_+(l,m_l)\,\mu^{(j)}_{p,l+1,m_l} \\ T_3\triangleright\overline{\mu^{(j)}_{plm_l}} &= -p\,\chi^{(2)}_-(l,m_l)\,\overline{\mu^{(j)}_{p,l-1,m_l}}+p\,\chi^{(2)}_+(l,m_l)\,\overline{\mu^{(j)}_{p,l+1,m_l}}.\end{aligned}\tag{B.57}$$

Using expansion (2.91)

$$\phi(t,r,\Omega) = \int_0^\infty\!\mathrm{d}p\sum_{l,m_l}\Big\{\phi^+_{plm_l}\mu^{(j)}_{plm_l}(t,r,\Omega)+\overline{\phi^-_{plm_l}}\;\overline{\mu^{(j)}_{plm_l}(t,r,\Omega)}\Big\},\tag{B.58}$$

we thus find the following actions in the momentum representation:

$$\begin{aligned}\big(T_3\triangleright\phi\big)^+_{plm_l} &= +p\,\chi^{(2)}_-(l,m_l)\,\phi^+_{p,l-1,m_l}-p\,\chi^{(2)}_+(l,m_l)\,\phi^+_{p,l+1,m_l} \\ \overline{\big(T_3\triangleright\phi\big)^-_{plm_l}} &= +p\,\chi^{(2)}_-(l,m_l)\,\overline{\phi^-_{p,l-1,m_l}}-p\,\chi^{(2)}_+(l,m_l)\,\overline{\phi^-_{p,l+1,m_l}}.\end{aligned}\tag{B.59}$$

For the boost generator K_{03} we recall (B.48), which here writes as

$$\begin{aligned}-\Big[(\chi^{(2)}_-L^- - \chi^{(2)}_+L^+)(-\mathrm{i}p\partial_{E_p})+\mathrm{i}(l-1)\tfrac{E_p}{p}\chi^{(2)}_-L^- + \mathrm{i}(l+2)\tfrac{E_p}{p}\chi^{(2)}_+L^+\Big]\;\mathrm{e}^{-\mathrm{i}E_pt}Y^{m_l}_l(\Omega)j_l(pr) \\ = -K_{03}\;\mathrm{e}^{-\mathrm{i}E_pt}Y^{m_l}_l(\Omega)j_l(pr).\end{aligned}\tag{B.60}$$

With $p\partial_{E_p}=E_p\partial_p$ this becomes

$$\begin{aligned}-\Big[(\chi^{(2)}_-L^- - \chi^{(2)}_+L^+)(-\mathrm{i}E_p\partial_p)+\mathrm{i}(l-1)\tfrac{E_p}{p}\chi^{(2)}_-L^- + \mathrm{i}(l+2)\tfrac{E_p}{p}\chi^{(2)}_+L^+\Big]\;\mathrm{e}^{-\mathrm{i}E_pt}Y^{m_l}_l(\Omega)j_l(pr) \\ = -K_{03}\;\mathrm{e}^{-\mathrm{i}E_pt}Y^{m_l}_l(\Omega)j_l(pr),\end{aligned}\tag{B.61}$$

which induces the following action on the conjugate functions:

$$\begin{aligned}-\Big[(\chi^{(2)}_-L^- - \chi^{(2)}_+L^+)(+\mathrm{i}E_p\partial_p)-\mathrm{i}(l-1)\tfrac{E_p}{p}\chi^{(2)}_-L^- - \mathrm{i}(l+2)\tfrac{E_p}{p}\chi^{(2)}_+L^+\Big]\;\mathrm{e}^{+\mathrm{i}E_pt}\overline{Y^{m_l}_l(\Omega)}j_l(pr) \\ = -K_{03}\;\mathrm{e}^{+\mathrm{i}E_pt}\overline{Y^{m_l}_l(\Omega)}j_l(pr).\end{aligned}\tag{B.62}$$

We can apply this action to expansion (2.91), and then in half of the terms we have the derivative operator $pE_p\partial_p$ acting to the right on the functions $\mathrm{e}^{-\mathrm{i}E_pt}Y^{m_l}_l(\Omega)j_l(pr)$ and $\mathrm{e}^{+\mathrm{i}E_pt}\overline{Y^{m_l}_l(\Omega)}j_l(pr)$. As before, we want it to act to the left. We can use the condition that $\phi^{\pm}_{plm_l}$ vanishes for $p=0$ and for $p \to \infty$. The first option to this is replacing

$$\overrightarrow{pE_p\partial_p} \;\to\; -\overleftarrow{pE_p\partial_p} - (E_p + \tfrac{p}{E_p}).$$

However, for later application in the symplectic structure it turns out that the following replacement is of advantage:

$$\overrightarrow{pE_p\partial_p} \;\to\; -\overleftarrow{pE_p\partial_p E_p} - E_p. \tag{B.63}$$

(That is: $f\,pE_p\,\partial_p g \;\to\; -g\,p\,\partial_p E f - E\,fg$.) Applying it, we find the following actions of the boost K_{03} on the momentum representation of the solutions:

$$\begin{aligned}\big(K_{03}\triangleright\phi\big)^{+}_{plm_l} &= +\mathrm{i}\chi^{(2)}_{-}(l,m_l)\,\big(\partial_p E_p \phi^{+}_{p,l-1,m_l}\big) - \mathrm{i}(l)\,\tfrac{E_p}{p}\,\chi^{(2)}_{-}(l,m_l)\,\phi^{+}_{p,l-1,m_l} \\ &\quad - \mathrm{i}\chi^{(2)}_{+}(l,m_l)\,\big(\partial_p E_p \phi^{+}_{p,l+1,m_l}\big) - \mathrm{i}(l+1)\,\tfrac{E_p}{p}\,\chi^{(2)}_{+}(l,m_l)\,\phi^{+}_{p,l+1,m_l}\end{aligned} \tag{B.64}$$

$$\begin{aligned}\overline{\big(K_{03}\triangleright\phi\big)^{-}_{plm_l}} &= -\mathrm{i}\chi^{(2)}_{-}(l,m_l)\,\big(\partial_p E_p \overline{\phi^{-}_{p,l-1,m_l}}\big) + \mathrm{i}(l)\,\tfrac{E_p}{p}\,\chi^{(2)}_{-}(l,m_l)\,\overline{\phi^{-}_{p,l-1,m_l}} \\ &\quad + \mathrm{i}\chi^{(2)}_{+}(l,m_l)\,\big(\partial_p E_p \overline{\phi^{-}_{p,l+1,m_l}}\big) + \mathrm{i}(l+1)\,\tfrac{E_p}{p}\,\chi^{(2)}_{+}(l,m_l)\,\overline{\phi^{-}_{p,l+1,m_l}}.\end{aligned} \tag{B.65}$$

B.3 Commutation of complex structure and isometries

B.3.1 Minkowski rod region

Having found the isometries' actions, we can study how to make them commute with the complex structure J_r on the hypercylinder. As for AdS, we can make the most general ansatz for a linear J_r, and then impose commutation with the actions of time translation and spatial rotations. Then, we impose the essential properties of the complex structure and obtain (C.358)

$$\begin{aligned}\big(J_r\phi\big)^{a}_{Elm_l} &= j^{aa}_{El}\,\phi^{a}_{Elm_l} + j^{ab}_{El}\,\phi^{b}_{Elm_l} \\ \big(J_r\phi\big)^{b}_{Elm_l} &= j^{ba}_{El}\,\phi^{a}_{Elm_l} - j^{aa}_{El}\,\phi^{b}_{Elm_l} \\ \big(j^{aa}_{El}\big)^2 &= -j^{ab}_{El}\,j^{ba}_{El} - 1 \geq 0.\end{aligned} \tag{B.66}$$

For a moment we leave aside the issue of positive-definiteness of the induced real g-product. First, the action (B.41) of the spatial translation T_3 induces the following actions for $E^2 \gtrless m^2$:

$$\begin{aligned}\big(J_r(T_3\triangleright\phi)\big)^{a}_{Elm_l} &= j^{aa}_{El}(T_3\triangleright\phi)^{a}_{Elm_l} + j^{ab}_{El}(T_3\triangleright\phi)^{b}_{Elm_l} \\ &= j^{aa}_{El}\Big(\pm p^{\mathbb{R}}_E\,\chi^{(2)}_{-}(l,m_l)\,\phi^{a}_{E,l-1,m_l} - p^{\mathbb{R}}_E\,\chi^{(2)}_{+}(l,m_l)\,\phi^{a}_{E,l+1,m_l}\Big) \\ &\quad + j^{ab}_{El}\Big(p^{\mathbb{R}}_E\,\chi^{(2)}_{-}(l,m_l)\,\phi^{b}_{E,l-1,m_l} \mp p^{\mathbb{R}}_E\,\chi^{(2)}_{+}(l,m_l)\,\phi^{b}_{E,l+1,m_l}\Big) \\ \big(T_3\triangleright(J_r\phi)\big)^{a}_{Elm_l} &= \pm p^{\mathbb{R}}_E\,\chi^{(2)}_{-}(l,m_l)\,(J_r\phi)^{a}_{E,l-1,m_l} - p^{\mathbb{R}}_E\,\chi^{(2)}_{+}(l,m_l)\,(J_r\phi)^{a}_{E,l+1,m_l} \\ &= \pm p^{\mathbb{R}}_E\,\chi^{(2)}_{-}(l,m_l)\,\Big(j^{aa}_{E,l-1}\,\phi^{a}_{E,l-1,m_l} + j^{ab}_{E,l-1}\,\phi^{b}_{E,l-1,m_l}\Big) \\ &\quad - p^{\mathbb{R}}_E\,\chi^{(2)}_{+}(l,m_l)\,\Big(j^{aa}_{E,l+1}\,\phi^{a}_{E,l+1,m_l} + j^{ab}_{E,l+1}\,\phi^{b}_{E,l+1,m_l}\Big).\end{aligned}$$

Comparing for both actions first the factors in front of $\phi^{a}_{E,l-1,m_l}$, then those in front of $\phi^{a}_{E,l+1,m_l}$ and so on, we can read off that both actions coincide precisely if

$$j^{aa}_{E,l+1} \;=\; j^{aa}_{E,l} \qquad \forall\, E \in \mathbb{R}$$

whereas

$$j^{ab}_{E,l+1} \;=\; \pm j^{ab}_{E,l} \qquad E^2 \gtrless m^2.$$

Doing the same for the actions of $\big(J_r(T_3\triangleright\phi)\big)^b_{Elm_l}$ and $\big(T_3\triangleright(J_r\phi)\big)^b_{Elm_l}$ yields in addition

$$j^{ba}_{E,l+1} \;=\; \pm j^{ba}_{E,l} \qquad E^2 \gtrless m^2.$$

(We note that these three relations are consistently linked through $(j^{aa}_{E,l\pm1})^2 = -j^{ab}_{E,l\pm1}\, j^{ba}_{E,l\pm1} - 1 = -(\pm)^2 j^{ab}_{El}\, j^{ba}_{El} - 1 = (j^{aa}_{El})^2$.) This means that for all energies j^{aa}_{El} must be independent of l, such that we can write it as j^{aa}_E from now on. $j^{ab}_{E,l}$ and $j^{ba}_{E,l}$ are also l-independent, but only for $E^2 > m^2$. For $E^2 < m^2$ they are only l-independent up to sign:

$$j^{ab}_{E,l+1} \;=\; -j^{ab}_{E,l} \qquad\qquad j^{ba}_{E,l+1} \;=\; -j^{ba}_{E,l} \qquad\qquad E^2 < m^2. \tag{B.67}$$

Thus for $E^2 > m^2$ we write now $j^{ab}_{El} = j^{ab}_E$, and for $E^2 < m^2$ we write now $j^{ab}_{El} = (-1)^l j^{ab}_E$, ditto for j^{ba}. We now put to use these results considering the action (B.54) of K_{03}. For $E^2 > m^2$ we get:

$$\begin{aligned}
\big(J_r\,(K_{03}\triangleright\phi)\big)^a_{Elm_l} \;=\;& j^{aa}_E\,(K_{03}\triangleright\phi)^a_{Elm_l} + j^{ab}_E\,(K_{03}\triangleright\phi)^b_{Elm_l}\\
\;=\;& +j^{aa}_E\Big\{+\mathrm{i}\chi^{(2)}_-(l,m_l)\,p^{\mathbb{R}}_E(\partial_E\phi^a_{E,l-1,m_l}) - \mathrm{i}\chi^{(2)}_-(l,m_l)\,\frac{E}{p^{\mathbb{R}}_E}\,(l-1)\,\phi^a_{E,l-1,m_l}\\
&\qquad - \mathrm{i}\chi^{(2)}_+(l,m_l)\,p^{\mathbb{R}}_E(\partial_E\phi^a_{E,l+1,m_l}) - \mathrm{i}\chi^{(2)}_+(l,m_l)\,\frac{E}{p^{\mathbb{R}}_E}\,(l+2)\,\phi^a_{E,l+1,m_l}\Big\}\\
&+ j^{ab}_E\Big\{+\mathrm{i}\chi^{(2)}_-(l,m_l)\,p^{\mathbb{R}}_E(\partial_E\phi^b_{E,l-1,m_l}) - \mathrm{i}\chi^{(2)}_-(l,m_l)\,\frac{E}{p^{\mathbb{R}}_E}\,(l-1)\,\phi^b_{E,l-1,m_l}\\
&\qquad - \mathrm{i}\chi^{(2)}_+(l,m_l)\,p^{\mathbb{R}}_E(\partial_E\phi^b_{E,l+1,m_l}) - \mathrm{i}\chi^{(2)}_+(l,m_l)\,\frac{E}{p^{\mathbb{R}}_E}\,(l+2)\,\phi^b_{E,l+1,m_l}\Big\}.
\end{aligned}$$

Reversing the order of the operators yields:

$$\begin{aligned}
\big(K_{03}\triangleright(J_r\phi)\big)^a_{Elm_l} \;=\;& +\mathrm{i}\chi^{(2)}_-(l,m_l)\,p^{\mathbb{R}}_E\partial_E\Big(j^{aa}_E\,\phi^a_{E,l-1,m_l} + j^{ab}_E\,\phi^b_{E,l-1,m_l}\Big)\\
&- \mathrm{i}\chi^{(2)}_-(l,m_l)\,\frac{E}{p^{\mathbb{R}}_E}(l-1)\Big(j^{aa}_E\,\phi^a_{E,l-1,m_l} + j^{ab}_E\,\phi^b_{E,l-1,m_l}\Big)\\
&- \mathrm{i}\chi^{(2)}_+(l,m_l)\,p^{\mathbb{R}}_E\partial_E\Big(j^{aa}_E\,\phi^a_{E,l+1,m_l} + j^{ab}_E\,\phi^b_{E,l+1,m_l}\Big)\\
&- \mathrm{i}\chi^{(2)}_+(l,m_l)\,\frac{E}{p^{\mathbb{R}}_E}(l+2)\Big(j^{aa}_E\,\phi^a_{E,l+1,m_l} + j^{ab}_E\,\phi^b_{E,l+1,m_l}\Big).
\end{aligned}$$

Comparing again the respective factors, we now find that for $E^2 > m^2$ both actions coincide iff

$$\partial_E j^{aa}_E \;=\; 0 \qquad\qquad \partial_E j^{ab}_E \;=\; 0.$$

Doing the same for the actions of $\big(J_r\,(K_{03}\triangleright\phi)\big)^b_{Elm_l}$ and $\big(K_{03}\triangleright(J_r\phi)\big)^b_{Elm_l}$ yields

$$\partial_E j^{ba}_E \;=\; 0.$$

Despite the additonal signs of $(-1)^l$, we obtain the same conditions for $E^2 < m^2$. These conditions mean that neither of the j-factors depends on E. Thus the fact that the action of K_{03} involves derivatives ∂_E turns out rather useful for our purposes, since it severely restricts the possible forms of the complex structure. Together with the previous results this says that all j-factors are constants (up to sign). For $E^2 < m^2$, we denote these constant factors by a tilde as in $\tilde{\jmath}^{aa}$. Thus we have found the following form for the complex structure for $E^2 > m^2$:

$$\begin{aligned}
\big(J_r\phi\big)^a_{Elm_l} \;&=\; j^{aa}_>\,\phi^a_{Elm_l} + j^{ab}_>\,\phi^b_{Elm_l}\\
\big(J_r\phi\big)^b_{Elm_l} \;&=\; j^{ba}_>\,\phi^a_{Elm_l} - j^{aa}_>\,\phi^b_{Elm_l}\\
0 \le \big(j^{aa}_>\big)^2 \;&=\; -j^{ab}_>\,j^{ba}_> - 1,
\end{aligned} \tag{B.68}$$

while for $E^2 < m^2$ the complex structure with $j^{ab}_{<,l} = (-1)^l \tilde{\jmath}^{ab}_<$ and $j^{ba}_{<,l} = (-1)^l \tilde{\jmath}^{ba}_<$ is given by

$$\begin{aligned} \left(J_r\phi\right)^a_{Elm_l} &= j^{aa}_< \,\phi^a_{Elm_l} + j^{ab}_{<,l}\,\phi^b_{Elm_l} \\ \left(J_r\phi\right)^b_{Elm_l} &= j^{ba}_{<,l}\,\phi^a_{Elm_l} - j^{aa}_<\,\phi^b_{Elm_l} \\ 0 \le \left(j^{aa}_<\right)^2 &= -\tilde{\jmath}^{ab}_<\,\tilde{\jmath}^{ba}_< - 1. \end{aligned} \tag{B.69}$$

Let us compare this form of J_r to the form given in [59]. Therein, the frequency momentum expansion (2.112) for solutions is used. The frequency momentum representation and our momentum representation (2.107) are related through (2.111)

$$\begin{aligned} \phi^a_{Elm_l} &= \phi^+_{Elm_l} + \overline{\phi^-_{-E,l,-m_l}} & \qquad \phi^+_{Elm_l} &= \tfrac{1}{2}\,\phi^a_{Elm_l} - \tfrac{\mathrm{i}}{2}\,\phi^b_{Elm_l} \\ \phi^b_{Elm_l} &= \mathrm{i}\phi^+_{Elm_l} - \mathrm{i}\,\overline{\phi^-_{-E,l,-m_l}} & \qquad \overline{\phi^-_{-E,l,-m_l}} &= \tfrac{1}{2}\,\phi^a_{Elm_l} + \tfrac{\mathrm{i}}{2}\,\phi^b_{Elm_l}\,. \end{aligned}$$

The choice (81) in [59] for the action of the complex structure in the frequency momentum representation

$$(J^{\mathrm{pos}}_r\phi)^{\pm}_{Elm_l} = \mathrm{i}\,\phi^{\pm}_{Elm_l} \tag{B.70}$$

induces a positive-definite real g-product, and induces the following action in our momentum representation for *all* energies E:

$$\begin{aligned} (J^{\mathrm{pos}}_r\phi)^a_{Elm_l} &= (J^{\mathrm{pos}}_r\phi)^+_{Elm_l} + \overline{(J^{\mathrm{pos}}_r\phi)^-_{-E,l,-m_l}} = \mathrm{i}\,\phi^+_{Elm_l} - \mathrm{i}\,\overline{\phi^-_{-E,l,-m_l}} = +\phi^b_{Elm_l} \\ (J^{\mathrm{pos}}_r\phi)^b_{Elm_l} &= \mathrm{i}(J^{\mathrm{pos}}_r\phi)^+_{Elm_l} - \mathrm{i}\,\overline{(J^{\mathrm{pos}}_r\phi)^-_{-E,l,-m_l}} = -\phi^+_{Elm_l} - \overline{\phi^-_{-E,l,-m_l}} = -\phi^a_{Elm_l}. \end{aligned} \tag{B.71}$$

This corresponds to $j^{aa}_{El} = 0$ with $j^{ab}_{El} = 1$ and $j^{ba}_{El} = -1$ in (B.66). As we have seen above, this choice for the complex structure is compatible with the isometry actions *only* for the modes with $E^2 > m^2$, while for the modes with $E^2 < m^2$ this choice *does not* let the complex structure commute with the actions of the translation T_3 and the boost K_{03}. The complex structure most similar to this choice while commuting with all isometry actions is given by $j^{aa}_> = j^{aa}_< = 0$ with $j^{ab}_> = -1$ and $\tilde{\jmath}^{ab}_< = \mp 1$ while $j^{ba}_> = 1$ and $\tilde{\jmath}^{ba}_< = \pm 1$ in (B.68). We are free to choose whether the constant $\tilde{\jmath}^{ab}_<$ is either 1 or -1, and $\tilde{\jmath}^{ba}_<$ then is the opposite. We call the resulting complex structure J^{iso}_r:

$$\begin{aligned} \left(J^{\mathrm{iso}}_r\phi\right)^a_{Elm_l} &= -\phi^b_{Elm_l} & \qquad \left(J^{\mathrm{iso}}_r\phi\right)^b_{Elm_l} &= +\phi^a_{Elm_l} & \qquad E^2 > m^2 \\ \left(J^{\mathrm{iso}}_r\phi\right)^a_{Elm_l} &= \mp(-1)^l\,\phi^b_{Elm_l} & \qquad \left(J^{\mathrm{iso}}_r\phi\right)^b_{Elm_l} &= \pm(-1)^l\,\phi^a_{Elm_l} & \qquad E^2 < m^2 \end{aligned} \tag{B.72}$$

In order to transcribe this into the frequency representation, we write

$$\begin{aligned} \begin{pmatrix} (J^{\mathrm{iso}}_r\phi)^+_{Elm_l} \\ \overline{(J^{\mathrm{iso}}_r\phi)^-_{-E,l,-m_l}} \end{pmatrix} &= \tfrac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}\begin{pmatrix} (J_r\phi)^a_{Elm_l} \\ (J_r\phi)^b_{Elm_l} \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}\begin{pmatrix} j^{aa} & j^{ab} \\ j^{ba} & -j^{aa} \end{pmatrix}\begin{pmatrix} \phi^a_{Elm_l} \\ \phi^b_{Elm_l} \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}\begin{pmatrix} j^{aa} & j^{ab} \\ j^{ba} & -j^{aa} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}\begin{pmatrix} \phi^+_{Elm_l} \\ \overline{\phi^-_{-E,l,-m_l}} \end{pmatrix} \\ &= \begin{pmatrix} \tfrac{\mathrm{i}}{2}\,(j^{ab} - j^{ba}) & j^{aa} - \tfrac{\mathrm{i}}{2}\,(j^{ab} + j^{ba}) \\ j^{aa} + \tfrac{\mathrm{i}}{2}\,(j^{ab} + j^{ba}) & -\tfrac{\mathrm{i}}{2}\,(j^{ab} - j^{ba}) \end{pmatrix}\begin{pmatrix} \phi^+_{Elm_l} \\ \overline{\phi^-_{-E,l,-m_l}} \end{pmatrix}. \end{aligned}$$

It is easy to check that the square of the matrix in the final line is $-\mathbb{1}$ (using the condition $\left(j^{aa}\right)^2 = -j^{ab}\,j^{ba} - 1$). Instead of (B.70) we thus obtain:

$$\begin{aligned} (J^{\mathrm{iso}}_r\phi)^+_{Elm_l} &= \mathrm{i}\,\phi^+_{Elm_l} & \qquad (J^{\mathrm{iso}}_r\phi)^-_{Elm_l} &= \mathrm{i}\,\phi^-_{Elm_l} & \qquad E^2 > m^2 \\ (J^{\mathrm{iso}}_r\phi)^+_{Elm_l} &= \pm(-1)^l\,\mathrm{i}\,\phi^+_{Elm_l} & \qquad (J^{\mathrm{iso}}_r\phi)^-_{Elm_l} &= \pm(-1)^l\,\mathrm{i}\,\phi^-_{Elm_l} & \qquad E^2 < m^2. \end{aligned} \tag{B.73}$$

B.3.2 Minkowski time-interval region

Now let us check that the complex structure (2.97) commutes with the action (B.59) of T_3 and the action (B.64) of K_{03}:

$$\big(J_t\,\phi\big)^{\pm}_{p,l,m_l} \;=\; -\mathrm{i}\,\phi^{\pm}_{plm_l}\;. \tag{B.74}$$

First, we can quickly convince us that J_t commutes with T_3:

$$\begin{aligned}\Big(J_t\big(T_3\triangleright\phi\big)\Big)^{+}_{plm_l} &= -\mathrm{i}\Big(+p\,\chi^{(2)}_{-}(l,m_l)\,\phi^{+}_{p,l-1,m_l} - p\,\chi^{(2)}_{+}(l,m_l)\,\phi^{+}j_{p,l+1,m_l}\Big)\\ = \Big(T_3\triangleright\big(J_t\phi\big)\Big)^{+}_{plm_l} &= +p\,\chi^{(2)}_{-}(l,m_l)\,(-\mathrm{i})\phi^{+}_{p,l-1,m_l} - p\,\chi^{(2)}_{+}(l,m_l)\,(-\mathrm{i})\phi^{+}j_{p,l+1,m_l}\end{aligned}$$

and

$$\begin{aligned}\overline{\Big(J_t\big(T_3\triangleright\phi\big)\Big)^{-}_{plm_l}} &= \mathrm{i}\Big(+p\,\chi^{(2)}_{-}(l,m_l)\,\overline{\phi^{-}_{p,l-1,m_l}} - p\,\chi^{(2)}_{+}(l,m_l)\,\overline{\phi^{-}j_{p,l+1,m_l}}\Big)\\ = \overline{\Big(T_3\triangleright\big(J_t\phi\big)\Big)^{-}_{plm_l}} &= +p\,\chi^{(2)}_{-}(l,m_l)\,\mathrm{i}\,\overline{\phi^{-}_{p,l-1,m_l}} - p\,\chi^{(2)}_{+}(l,m_l)\,\mathrm{i}\,\overline{\phi^{-}j_{p,l+1,m_l}}.\end{aligned}$$

The same happens with K_{03}:

$$\begin{aligned}\Big(J_t\big(K_{03}\triangleright\phi\big)\Big)^{+}_{plm_l} &= \Big(K_{03}\triangleright\big(J_t\phi\big)\Big)^{+}_{plm_l} = -\mathrm{i}\big(K_{03}\triangleright\phi\big)^{+}_{plm_l}\\ \overline{\Big(J_t\big(K_{03}\triangleright\phi\big)\Big)^{-}_{plm_l}} &= \overline{\Big(K_{03}\triangleright\big(J_t\phi\big)\Big)^{-}_{plm_l}} = \mathrm{i}\,\overline{\big(K_{03}\triangleright\phi\big)^{-}_{plm_l}}\,.\end{aligned}$$

That is, the standard complex structure commutes nicely with the actions of all isometries.

B.4 Making amplitudes coincide for rod and time-interval

So far we have fixed the complex structure J_t associated to equal-time planes Σ_t to be the standard one: $\big(J_t\phi\big)^{\pm}_{plm_l} = -\mathrm{i}\,\phi^{\pm}_{plm_l}$, while for the complex structure J^{iso}_r on hypercylinders Σ_r we have found conditions (B.68) for $E^2 > m^2$, wherein all j-factors are constants:

$$\begin{aligned}\big(J^{\mathrm{iso}}_r\phi\big)^{a}_{Elm_l} &= j^{aa}_{>}\,\phi^{a}_{Elm_l} + j^{ab}_{>}\,\phi^{b}_{Elm_l}\\ \big(J^{\mathrm{iso}}_r\phi\big)^{b}_{Elm_l} &= j^{ba}_{>}\,\phi^{a}_{Elm_l} - j^{aa}_{>}\,\phi^{b}_{Elm_l}\\ 0 \le \big(j^{aa}_{>}\big)^2 &= -j^{ab}_{>}\,j^{ba}_{>} - 1,\end{aligned} \tag{B.75}$$

and conditions (B.69) for $E^2 < m^2$:

$$\begin{aligned}\big(J^{\mathrm{iso}}_r\phi\big)^{a}_{Elm_l} &= j^{aa}_{<}\,\phi^{a}_{Elm_l} + (-1)^l\,\tilde{j}^{ab}_{<}\,\phi^{b}_{Elm_l}\\ \big(J^{\mathrm{iso}}_r\phi\big)^{b}_{Elm_l} &= (-1)^l\,\tilde{j}^{ba}_{<}\,\phi^{a}_{Elm_l} - j^{aa}_{<}\,\phi^{b}_{Elm_l}\\ 0 \le \big(j^{aa}_{<}\big)^2 &= -\tilde{j}^{ab}_{<}\,\tilde{j}^{ba}_{<} - 1.\end{aligned} \tag{B.76}$$

These conditions make J^{iso}_r commute with all Minkowski isometries. Then in (B.72) we found that the J^{iso}_r most similar to the choice J^{pos}_r of (81) in [59] is given by

$$\begin{aligned}j^{aa}_{>} &= 0 \qquad & j^{ab}_{>} &= -1 \qquad & j^{ba}_{>} &= +1\\ j^{aa}_{<} &= 0 \qquad & \tilde{j}^{ab}_{<} &= \mp 1 \qquad & \tilde{j}^{ba}_{<} &= \pm 1.\end{aligned} \tag{B.77}$$

(The signs of $\tilde{\jmath}^{ab}_{<}$ and $\tilde{\jmath}^{ba}_{<}$ are not fixed yet.) That is, the complex structure acts as

$$\begin{aligned}
\left(J_r^{\rm iso}\phi\right)^a_{Elm_l} &= -\phi^b_{Elm_l} & \left(J_r^{\rm iso}\phi\right)^b_{Elm_l} &= +\phi^a_{Elm_l} & E^2 &> m^2 \\
\left(J_r^{\rm iso}\phi\right)^a_{Elm_l} &= \mp(-1)^l\,\phi^b_{Elm_l} & \left(J_r^{\rm iso}\phi\right)^b_{Elm_l} &= \pm(-1)^l\,\phi^a_{Elm_l} & E^2 &< m^2
\end{aligned} \tag{B.78}$$

We now want to justify this choice further by showing that it makes the two complex structures J_t and $J_r^{\rm iso}$ amplitude-equivalent as in (3.71). That is, J_t and $J_r^{\rm iso}$ make the following amplitudes coincide for all global solutions η_1 and ζ_2:

$$\begin{aligned}
\rho_{[t_1,t_2]}\Big(K^{\eta_1}_{\Sigma_1}\otimes\overline{K^{\zeta_2}_{\Sigma_2}}\Big) &= \exp\Big(-{\rm g}_t(\xi^{\rm I}_{12},\,\xi^{\rm I}_{12}) - {\rm i}{\rm g}_t(\xi^{\rm R}_{12},\,\xi^{\rm I}_{12})\Big) \\
\stackrel{!}{=}\ \rho_{r_0}\big(\overline{K^{\xi_0}_{\Sigma_0}}\big) &= \exp\Big(-\tfrac{1}{2}{\rm g}_r(\xi^{\rm I}_{12},\,\xi^{\rm I}_{12}) - \tfrac{\rm i}{2}{\rm g}_r(\xi^{\rm R}_{12},\,\xi^{\rm I}_{12})\Big),
\end{aligned} \tag{B.79}$$

wherein

$$\xi^{\rm R}_{12} \,=\, \tfrac{1}{2}(\eta_1+\zeta_2) \qquad \xi^{\rm I}_{12} \,=\, \tfrac{1}{2}(-J_t\eta_1+J_t\zeta_2). \qquad \xi_0 \,=\, \xi^{\rm R}_{12} - J_r^{\rm iso}\xi^{\rm I}_{12}. \tag{B.80}$$

As discussed above (3.73), J_t and $J_r^{\rm iso}$ are amplitude-equivalent for global solutions, precisely if for all global solutions η_1,ζ_2 they induce

$${\rm g}_t(\eta_1,\zeta_2) = \tfrac{1}{2}\,{\rm g}_r^{\rm iso}(\eta_1,\zeta_2). \tag{B.81}$$

We now check whether our $J_r^{\rm iso}$ fulfills this requirement. To this end we first compare expansions of global solutions near equal-time hypersurfaces and hypercylinders. We can expand any global solution ξ as done for solutions near equal-time hypersurfaces in (2.91)

$$\xi(t,r,\Omega) \,=\, \int_0^\infty\!{\rm d}p\sum_{l,m_l}\Big\{\xi^+_{plm_l}\mu^{(j)}_{plm_l}(t,r,\Omega) + \overline{\xi^-_{plm_l}}\;\overline{\mu^{(j)}_{plm_l}(t,r,\Omega)}\Big\} \tag{B.82}$$

wherein the $\mu^{(j)}$-modes are defined as

$$\mu^{(j)}_{plm_l}(t,r,\Omega) \,:=\, \frac{2p}{\sqrt{2\pi}}\,{\rm e}^{-{\rm i}E_p t}Y_l^{m_l}(\Omega)\,j_l(pr) \qquad E_p \,:=\, \sqrt{E^2-m^2}. \tag{B.83}$$

As usual for solutions near equal-time hypersurfaces, here we have only modes with $p>0$, that is: $E_p>m$. We can expand ξ also using the more general expansion (2.108) for solutions near hypercylinders

$$\xi(t,r,\Omega) \,=\, \int\!{\rm d}E\sum_{l,m_l}\Big\{\xi^a_{Elm_l}\mu^{(a)}_{Elm_l}(t,r,\Omega) + \xi^b_{Elm_l}\mu^{(b)}_{Elm_l}(t,r,\Omega)\Big\} \tag{B.84}$$

with

$$\begin{aligned}
\mu^{(a)}_{Elm_l}(t,r,\Omega) &:= \frac{p_E^{\mathbb{R}}}{4\pi}\,{\rm e}^{-{\rm i}Et}Y_l^{m_l}(\Omega)\,\check{\jmath}_{El}(r) && \forall\, E\in\mathbb{R} \\
\mu^{(b)}_{Elm_l}(t,r,\Omega) &:= \frac{p_E^{\mathbb{R}}}{4\pi}\,{\rm e}^{-{\rm i}Et}Y_l^{m_l}(\Omega)\,\check{n}_{El}(r). && \forall\, E\in\mathbb{R} \\
\mu^{(a)}_{Elm_l}(t,r,\Omega) &= \tfrac{\sqrt{2\pi}}{8\pi}\,\mu^{(j)}_{p_E lm_l}(t,r,\Omega) && \forall\, E>m \\
\mu^{(a)}_{Elm_l}(t,r,\Omega) &= \tfrac{\sqrt{2\pi}}{8\pi}\,\overline{\mu^{(j)}_{p_E,l,-m_l}(t,r,\Omega)} && \forall\, E<-m.
\end{aligned}$$

However, then in order for ξ to be a global solution it must have $\xi^b_{Elm_l}\equiv 0$, and further for all $E^2<m^2$ it must have $\xi^a_{Elm_l}=0$. Then, both expansions are equivalent, and we can transcribe the

momentum representations into each other:

$$\begin{aligned}\xi^a_{Elm_l} &= +\xi^+_{p_E lm_l}\,\tfrac{E}{p_E}\,\tfrac{8\pi}{\sqrt{2\pi}} && E>m\\ \xi^a_{Elm_l} &= -\overline{\xi^-_{p_E,l,-m_l}}\,\tfrac{E}{p_E}\,\tfrac{8\pi}{\sqrt{2\pi}} && E<m\\ \xi^a_{-E,l,-m_l} &= +\overline{\xi^-_{p_E lm_l}}\,\tfrac{E}{p_E}\,\tfrac{8\pi}{\sqrt{2\pi}} && E>m.\end{aligned} \tag{B.85}$$

We recall that the real g-product g_t for the equal-time hypersurface is given by (2.98) to be

$$\mathrm{g}_t\big(\eta,\,\zeta\big) \;=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 2E_p\,\Big\{\overline{\eta^-_{plm_l}}\,\zeta^+_{plm_l} \;+\eta^+_{plm_l}\,\overline{\zeta^-_{plm_l}}\Big\}\;. \tag{B.86}$$

We proceed by reproducing this result from the real g-product $\mathrm{g}^{\mathrm{iso}}_r$ of the hypercylinder which is given by (2.64):

$$\mathrm{g}^{\mathrm{iso}}_r(\eta,\zeta) \;:=\; 2\omega_r(\eta,\,J^{\mathrm{iso}}_r\zeta).$$

Plugging the complex structure (B.75) into the symplectic form (2.116) for global solutions η,ζ results in

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{g}^{\mathrm{iso}}_r\big(\eta,\,\zeta\big) \;&:=\; \omega_r(\eta,\,J^{\mathrm{iso}}_r\zeta)\\ &= \int_{E^2>m^2}\mathrm{d}E\sum_{l,m_l}\frac{p_E}{16\pi}\Big\{\eta^a_{Elm_l}\,(J^{\mathrm{iso}}_r\zeta)^b_{-E,l,-m_l} - \eta^b_{Elm_l}\,(J^{\mathrm{iso}}_r\zeta)^a_{-E,l,-m_l}\Big\}\\ &= \int_{E^2>m^2}\mathrm{d}E\sum_{l,m_l}\frac{p_E}{16\pi}\Big\{\eta^a_{Elm_l}\Big[j^{ba}_>\zeta^a_{-E,l,-m_l} - j^{aa}_>\underbrace{\zeta^b_{-E,l,-m_l}}_{0}\Big] - \underbrace{\eta^b_{Elm_l}}_{0}\Big[j^{aa}_>\zeta^a_{-E,l,-m_l} + j^{ab}_>\zeta^b_{-E,l,-m_l}\Big]\Big\}\\ &= \int_{E^2>m^2}\mathrm{d}E\sum_{l,m_l}\frac{p_E}{16\pi}\,\eta^a_{Elm_l}\,j^{ba}_>\,\zeta^a_{-E,l,-m_l}\\ &= \int_{E>m}\mathrm{d}E\sum_{l,m_l}\frac{p_E}{16\pi}\,j^{ba}_>\,\Big\{\eta^a_{Elm_l}\zeta^a_{-E,l,-m_l} + \eta^a_{-E,l,-m_l}\zeta^a_{Elm_l}\Big\}.\end{aligned}$$

Plugging transcriptions (B.85) into this expression we obtain

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{g}^{\mathrm{iso}}_r\big(\eta,\,\zeta\big) \;&=\; \int_{E>m}\mathrm{d}E\sum_{l,m_l} j^{ba}_>\,\frac{p_E\,E^2\,(8\pi)^2}{p_E^2\,16\pi\sqrt{2\pi}^{\,2}}\,\Big\{\eta^+_{p_E lm_l}\overline{\zeta^-_{p_E lm_l}} + \overline{\eta^-_{p_E lm_l}}\,\zeta^+_{p_E lm_l}\Big\}\\ &= \int_{E>m}\mathrm{d}E\sum_{l,m_l} j^{ba}_>\,\frac{E^2}{p_E}\,2\,\Big\{\eta^+_{p_E lm_l}\overline{\zeta^-_{p_E lm_l}} + \overline{\eta^-_{p_E lm_l}}\,\zeta^+_{p_E lm_l}\Big\}\\ &= \int_0^\infty \mathrm{d}p\sum_{l,m_l} j^{ba}_>\,2E_p\,\Big\{\eta^+_{plm_l}\overline{\zeta^-_{plm_l}} + \overline{\eta^-_{plm_l}}\,\zeta^+_{plm_l}\Big\}.\end{aligned}$$

We can now read off directly that this agress with

$$\mathrm{g}_t\big(\eta,\,\zeta\big) \;=\; \int_0^\infty \mathrm{d}p \sum_{l,m_l} 2E_p\,\Big\{\overline{\eta^-_{plm_l}}\,\zeta^+_{plm_l} \;+\eta^+_{plm_l}\,\overline{\zeta^-_{plm_l}}\Big\} \tag{B.87}$$

precisely if

$$j^{ba}_> = 1. \tag{B.88}$$

This coincides with our above choice (B.77). However, we see that requiring amplitude equivalence between J_t and J_r^{iso} fixes only the coefficient $j_>^{ba}$ while leaving $j_>^{ab}$, $j_>^{aa}$ and all $j_<^{\cdot\cdot}$ undetermined, because we can only consider a-modes with energy $E^2 > m^2$ here. We can fix $j_\gtrless^{aa} = 0$ by requiring that J_r^{iso} maps a-modes to b-modes and vice versa. This is the most natural implementation of the property that J_r^{iso} maps solutions well defined on the whole interior of the rod region to solutions that are well defined only near the boundary hypercylinder (and vice versa). With $j_\gtrless^{aa} = 0$ we have $j_\gtrless^{ab} = -1/j_\gtrless^{ba}$. The only remaining ambiguity is thus the overall sign of $\tilde{\jmath}_<^{ba}$. We choose to fix $\tilde{\jmath}_<^{ba} = j_>^{ba}$, such that we get a positive-definite real g-product $\mathrm{g}_r^{\text{iso}}$ for all modes with $E^2 > m^2$ and also for all modes with $E^2 < m^2$ which have even l. The real g-product then becomes negative-definite only for modes with $E^2 < m^2$ which have odd l. Our final choice is thus:

$$\begin{aligned} j_>^{aa} &= 0 & j_>^{ab} &= -1 & j_>^{ba} &= +1 \\ j_<^{aa} &= 0 & \tilde{\jmath}_<^{ab} &= -1 & \tilde{\jmath}_<^{ba} &= +1. \end{aligned} \tag{B.89}$$

that is

$$\begin{aligned} \left(J_r^{\text{iso}}\phi\right)^a_{Elm_l} &= -\,\phi^b_{Elm_l} & \left(J_r^{\text{iso}}\phi\right)^b_{Elm_l} &= +\,\phi^a_{Elm_l} & E^2 &> m^2 \\ \left(J_r^{\text{iso}}\phi\right)^a_{Elm_l} &= -(-1)^l\,\phi^b_{Elm_l} & \left(J_r^{\text{iso}}\phi\right)^b_{Elm_l} &= +(-1)^l\,\phi^a_{Elm_l} & E^2 &< m^2 \end{aligned} \tag{B.90}$$

For two solutions η, ζ near a Minkowski hypercylinder, this makes the real g-product into (wherein again in ± 1 the upper sign holds for $E^2 > m^2$ and the lower sign for $E^2 < m^2$)

$$\begin{aligned} \mathrm{g}_r^{\text{iso}}(\eta,\zeta) &:= 2\omega_r(\eta,\, J_r^{\text{iso}}\zeta) \\ &= 2\int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{16\pi} \left\{ \eta^a_{Elm_l}\,(J_r^{\text{iso}}\zeta)^b_{-E,l,-m_l} - \eta^b_{Elm_l}\,(J_r^{\text{iso}}\zeta)^a_{-E,l,-m_l} \right\} \\ &= \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{8\pi} \left\{ \eta^a_{Elm_l}\,(\pm 1)^l \zeta^a_{-E,l,-m_l} - -\eta^b_{Elm_l}\,(\pm 1)^l \zeta^b_{-E,l,-m_l} \right\} \\ &= \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{8\pi}\,(\pm 1)^l \left\{ \eta^a_{Elm_l}\zeta^a_{-E,l,-m_l} + \eta^b_{Elm_l}\zeta^b_{-E,l,-m_l} \right\}. \end{aligned} \tag{B.91}$$

Finally, we observe that J_r^{pos} induces the same amplitude equivalence, since only modes with $E^2 > m^2$ matter and J_r^{pos} coincides with J_r^{iso} for these energies.

Appendix C

Anti de Sitter spacetime

C.1 AdS basics

We use global coordinates with the time coordinate $t \in (-\infty, +\infty)$, a radial coordinate $\widetilde{\rho} \in [0, +\infty)$ or its compactified version $\rho \in [0, \frac{\pi}{2})$, and denote the $(d-1)$ angular coordinates on $\mathbb{S}^{d-1}$ collectively by $\Omega := (\theta_1, \ldots, \theta_{d-1})$. (The finite range of the ρ does not mean that AdS is spatially compact. It roughly corresponds to introducing spherical coordinates (r, θ, φ) on Minkowski spacetime and rescaling r by $\tilde{r} = \arctan r \in [0, \frac{\pi}{2})$.) The radial coordinates are related to each other by

$$\sinh \widetilde{\rho} = \tan \rho \qquad \cosh \widetilde{\rho} = \frac{1}{\cos \rho} \qquad \tanh \widetilde{\rho} = \sin \rho \tag{C.1}$$

With $\mathrm{d}\rho = \mathrm{d}\widetilde{\rho}/\cosh \widetilde{\rho}$ and R_{AdS} denoting the curvature radius of AdS, the metric writes

$$\mathrm{d}s^2_{\mathrm{AdS}} = \frac{R^2_{\mathrm{AdS}}}{\cos^2 \rho} \left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2 \rho \, \mathrm{d}s^2_{\mathbb{S}^{d-1}} \right) \tag{C.2}$$

$$= R^2_{\mathrm{AdS}} \left(-\cosh^2 \widetilde{\rho} \, \mathrm{d}t^2 + \mathrm{d}\widetilde{\rho}^2 + \sinh^2 \widetilde{\rho} \, \mathrm{d}s^2_{\mathbb{S}^{d-1}} \right) . \tag{C.3}$$

For the precise form of coordinates and metric on the unit sphere $\mathbb{S}^{d-1}$ see Appendix A.3. From the second line we can read off that $\tilde{\rho}$ is the metric distance of the point $(t, \tilde{\rho}, \Omega)$ respectively $(t, \rho(\tilde{\rho}), \Omega)$ from the time axis.

With $\partial_\rho = \cosh \tilde{\rho} \, \partial_{\tilde{\rho}}$ the Laplace-Beltrami operator is given by

$$\Box_{\mathrm{AdS}} := \frac{1}{\sqrt{|g|}} \, \partial_\mu \sqrt{|g|} \, g^{\mu\nu} \partial_\nu$$

$$= R^{-2}_{\mathrm{AdS}} \left\{ -\cos^2 \rho \, \partial_t^2 + \cos^2 \rho \, \tan^{1-d} \rho \, \partial_\rho \tan^{d-1} \rho \, \partial_\rho + \tan^{-2} \rho \, \Box_{\mathbb{S}^{d-1}} \right\} \tag{C.4}$$

$$= R^{-2}_{\mathrm{AdS}} \left\{ -\cos^2 \rho \, \partial_t^2 + \cos^2 \rho \, \partial_\rho^2 + \frac{(d-1)}{\tan \rho} \, \partial_\rho + \tan^{-2} \rho \, \Box_{\mathbb{S}^{d-1}} \right\} \tag{C.5}$$

$$= R^{-2}_{\mathrm{AdS}} \left\{ -\cosh^{-2} \tilde{\rho} \, \partial_t^2 + \cosh^{-2} \tilde{\rho} \, \partial_{\tilde{\rho}}^2 + (d-1) \coth \tilde{\rho} \, \partial_{\tilde{\rho}} + \tanh \tilde{\rho} \, \partial_{\tilde{\rho}} + \sinh^{-2} \tilde{\rho} \, \Box_{\mathbb{S}^{d-1}} \right\} . \tag{C.6}$$

Although this point of view a priori is not necessary, it is sometimes useful to regard $\mathrm{AdS}_{1,d}$ as embedded in $\mathbb{R}^{2,d}$, that is, $\mathbb{R}^{d+2}$ with metric of signature $(2, d)$: $\eta = \mathrm{diag}\,(-,+,...,+,-)$. The (covariant) cartesian coordinates in this embedding space are usually denoted by $X = (X_0, \underline{X}, X_{d+1})$ with $\underline{X} = (X_1, ..., X_d)$. The embedding space metric is thus

$$\mathrm{d}s^2_{(2,d)} = -\mathrm{d}X_0^2 + \mathrm{d}\underline{X}^2 - \mathrm{d}X_{d+1}^2 . \tag{C.7}$$

We can now introduce (see p.17 in [72]) so-called orispherical coordinates (R, t, ρ, Ω) and hyperbolical coordinates $(R, t, \tilde{\rho}, \Omega)$ on the part of embedding space on which $R^2 = -X^2 = (X_0)^2 + (X_{d+1})^2 - \underline{X}^{\,2} > 0$:

$$
\begin{aligned}
X_0 &= -R\,\sin t\,\cos^{-1}\rho && = -R\,\sin t\,\cosh\tilde{\rho} && \text{(C.8)}\\
X_{d+1} &= +R\,\cos t\,\cos^{-1}\rho && = +R\,\cos t\,\cosh\tilde{\rho} && \text{(C.9)}\\
X_k &= +R\,\xi_k(\Omega)\,\tan\rho && = +R\,\xi_k(\Omega)\,\sinh\tilde{\rho} && \text{(C.10)}
\end{aligned}
$$

The $\xi_k(\Omega)$ are the standard cartesian coordinates on the unit sphere given by (A.9) in Appendix A.3. The hyperboloid obtained by fixing some $R = R_{\text{AdS}}$ then is AdS with curvature radius R_{AdS} and time coordinate $t \in [0, 2\pi)$. This version of AdS is called hyperboloidal AdS and contains closed timelike curves. The version of AdS which we use is the universal covering space of this hyperboloid and obtained by "unwrapping" it. That is, by extending the range of time to $t \in (-\infty, +\infty)$ and unidentifying the points in embedding space obtained by $t \to t + 2\pi$. The AdS metric (C.2) is then induced by the embedding space metric (C.7).

C.1.1 The flat limit $R_{\text{AdS}} \to \infty$

In this section we consider the limit of large curvature radius R_{AdS}. We introduce the new global coordinates

$$
\begin{aligned}
r &:= R_{\text{AdS}}\,\rho && r \in [0, \tfrac{\pi}{2}\,R_{\text{AdS}}) && \text{(C.11)}\\
\tau &:= R_{\text{AdS}}\,t && \tau \in (-\infty, +\infty).
\end{aligned}
$$

The AdS metric now writes as given in [36]

$$
\begin{aligned}
\mathrm{d}s^2_{\text{AdS}} &= \frac{R^2_{\text{AdS}}}{\cos^2\rho}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}s^2_{\mathbb{S}^{d-1}}\right)\\
&= \left(\cos\frac{r}{R_{\text{AdS}}}\right)^{-2}\left[-\mathrm{d}\tau^2 + \mathrm{d}r^2 + R^2_{\text{AdS}}\left(\sin\frac{r}{R_{\text{AdS}}}\right)^2\mathrm{d}s^2_{\mathbb{S}^{d-1}}\right],
\end{aligned}
$$

and for $\rho = r/R_{\text{AdS}} \ll 1$ it approximates Minkowski spacetime:

$$
\mathrm{d}s^2_{\text{AdS}} \approx -\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}s^2_{\mathbb{S}^{d-1}} = \mathrm{d}s^2_{\text{Mink}}\,. \qquad \text{(C.12)}
$$

Therefore the large-R_{AdS} limit is also called flat limit. This means, that with increasing curvature radius the AdS region where $\rho \ll 1$ corresponds to an increasing part of Minkowski spacetime, covering all of it in the flat limit $R_{\text{AdS}} \to \infty$. If we wish to consider some fixed value of r, then for some sufficiently large R_{AdS} we have $r \ll R_{\text{AdS}}$ and the flat approximation holds. Hence in the flat limit $R_{\text{AdS}} \to \infty$ it holds for all r.

This can also be seen in the following way[1]. Let us start by considering AdS with some fixed curvature radius $R_{\text{AdS}} = R_1$, for example let $R_1 = 1$. The AdS-metric then writes

$$
\mathrm{d}s^2_{R_1} = \frac{R_1^2}{\cos^2\rho}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}s^2_{\mathbb{S}^{d-1}}\right).
$$

If we consider a neighborhood U of some point, for example the origin, which is "small enough", then we can neglect the curvature, and we can regard this small neighborhood as flat. For simplicity, let U a (finite) rod hypercylinder: $U = [t_1, t_2] \times [0, \rho_0] \times \mathbb{S}^{d-1}$ with $\rho_0 \ll 1$. Considering next AdS with curvature radius $R_{\text{AdS}} = R_2 > R_1$, we have

$$
\mathrm{d}s^2_{R_2} = \frac{R_2^2}{\cos^2\rho}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}s^2_{\mathbb{S}^{d-1}}\right) = \frac{R_2^2}{R_1^2}\,\mathrm{d}s^2_{R_1}.
$$

[1] We thank José Antonio Zapata (CCM-UNAM) for pointing us to this isometry.

That is, AdS spacetimes with different curvature radii are isometric to each other up to a constant conformal factor R_1^2/R_2^2. On AdS with R_2 our neighborhood U has the same internal coordiantes $U = [t_1, t_2] \times [0, \rho_0] \times \mathbb{S}^{d-1}$, but due to $R_2 > R_1$ it is now bigger. That is, our approximately flat U has grown. In the flat limit $R_2 \to \infty$ our approximately flat neighborhood U becomes infinitely large, and thus we can accomodate all of Minkowski spacetime within it. This is sketched in figure C.1.1

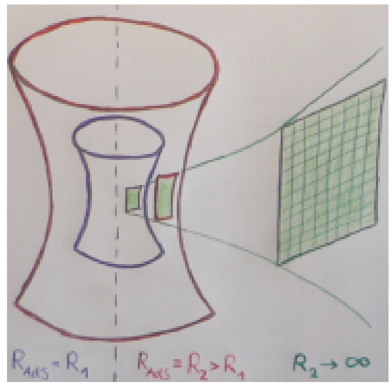


Figure C.13: Flat limit of AdS spacetime: $R_{\text{AdS}} \to \infty$.

In the flat limit the AdS Laplace-Beltrami approximates its Minkowski version:

$$\Box_{\text{AdS}} = R_{\text{AdS}}^{-2} \left\{ -\cos^2\rho\, \partial_t^2 + \cos^2\rho\, \partial_\rho^2 + \frac{(d-1)}{\tan\rho}\, \partial_\rho + \tan^{-2}\rho\, \Box_{\mathbb{S}^{d-1}} \right\} \tag{C.14}$$

$$\approx -\partial_\tau^2 + \partial_r^2 + \frac{(d-1)}{r}\, \partial_r + r^{-2}\, \Box_{\mathbb{S}^{d-1}} = \Box_{\text{Mink}}\,. \qquad \forall\, r \ll R_{\text{AdS}} \tag{C.15}$$

Thus the AdS Klein-Gordon equation with mass parameter m^2 in the flat limit approximates the Minkowski Klein-Gordon equation with the same m^2. Hence we define the flat limit, or large-R_{AdS} limit, by letting R_{AdS} grow larger and larger while keeping fixed the coordinates $\tau = t\, R_{\text{AdS}}$ and $r = \rho\, R_{\text{AdS}}$ and the parameters m^2, $\tilde{\omega}$ and $\tilde{p}_{\tilde{\omega}}$. We denote the flat limit of an expression by the symbol

$$\xrightarrow[\text{lim.}]{\text{flat}} \,.$$

C.1.2 Killing vector fields on $\text{AdS}_{1,d}$

We can now write the embedding space Killing vector fields

$$K_{AB}(X) := X_A\, \partial_B - X_B\, \partial_A \tag{C.16}$$

in orispherical coordinates $(R, t, \rho, \underline{\xi}(\Omega))$. We use Einstein's sum convention of summing over indices that appear twice. Latin uppercase indices have the range like $A = 0, \ldots, (d{+}1)$, while Latin lowercase indices range like $k = 1, \ldots, d$. Using

$$R = \left(X_0^2 + X_{d-1}^2 - \underline{X}^{\,2} \right)^{1/2} \qquad t = \arctan \frac{-X_0}{X_{d+1}}$$

$$\rho = \arctan \sqrt{\frac{\underline{X}^{\,2}}{X_0^2 + X_{d-1}^2 - \underline{X}^{\,2}}} \qquad \xi_k = \frac{X_k}{(\underline{X}\,)^{1/2}}\,,$$

and thus

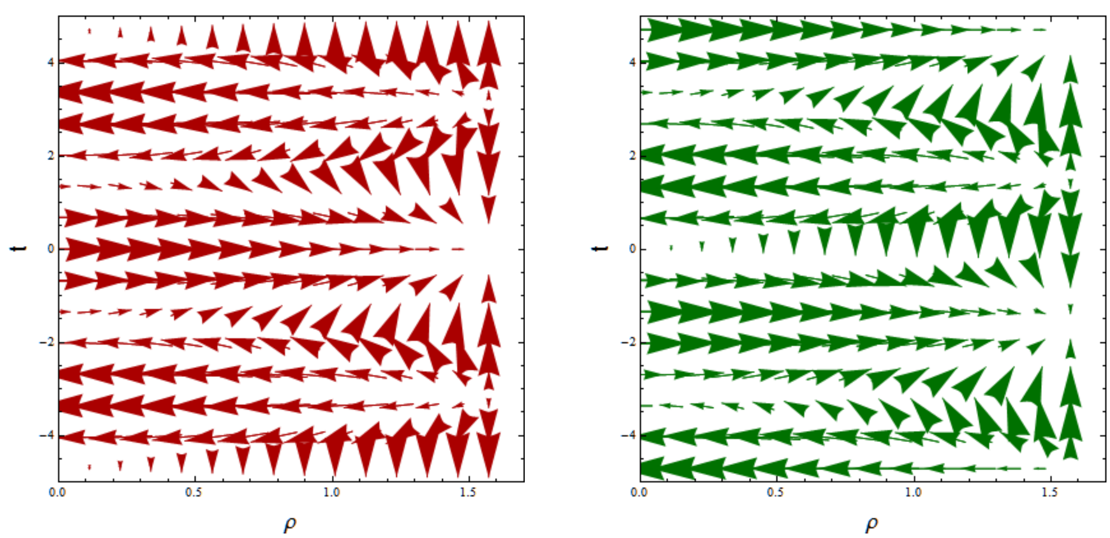
$$
\begin{aligned}
\frac{\partial R}{\partial X^0} &= +\frac{\sin t}{\cos\rho} & \frac{\partial t}{\partial X^0} &= R^{-1}\cos t\,\cos\rho & \frac{\partial \rho}{\partial X^0} &= -R^{-1}\sin t\,\sin\rho \\
\frac{\partial R}{\partial X^{d+1}} &= -\frac{\cos t}{\cos\rho} & \frac{\partial t}{\partial X^{d+1}} &= R^{-1}\sin t\,\cos\rho & \frac{\partial \rho}{\partial X^{d+1}} &= +R^{-1}\cos t\,\sin\rho \\
\frac{\partial R}{\partial X^j} &= -\xi_j\,\tan\rho & \frac{\partial \xi_k}{\partial X^j} &= \frac{\delta_{jk}-\xi_j\xi_k}{R\,\tan\rho} & \frac{\partial \rho}{\partial X^j} &= R^{-1}\,\xi_j\;,
\end{aligned}
\tag{C.17}
$$

wherein δ_{jk} is the Kronecker delta, we find for the Killing vectors

$$
\begin{aligned}
K_{d+1,0} &= X_{d+1}\,\partial_0 - X_0\,\partial_{d+1} \\
&= \partial_t \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (C.18) \\
K_{jk} &= X_j\,\partial_k - X_k\,\partial_j \\
&= \xi_j\partial_{\xi_k} - \xi_k\partial_{\xi_j} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (C.19) \\
K_{0j} &= X_0\,\partial_j - X_j\,\partial_0 \\
&= -\xi_j\cos t\,\sin\rho\,\partial_t - \xi_j\sin t\,\cos\rho\,\partial_\rho - \frac{\sin t}{\sin\rho}\left(\partial_{\xi_j} - \xi_j\xi_i\,\partial_{\xi_i}\right) \qquad (C.20) \\
K_{d+1,j} &= X_{d+1}\,\partial_j + X_j\,\partial_{d+1} \\
&= -\xi_j\sin t\,\sin\rho\,\partial_t + \xi_j\cos t\,\cos\rho\,\partial_\rho + \frac{\cos t}{\sin\rho}\left(\partial_{\xi_j} - \xi_j\xi_i\,\partial_{\xi_i}\right)\,. \qquad (C.21)
\end{aligned}
$$

Note that none of these Killing vectors has a ∂_R component, and that they therefore leave the R-coordinate invariant. Since $\mathrm{AdS}_{1,d}$ is nothing but a submanifold of $\mathbb{R}^{(2,d)}$ with fixed $R = R_{\mathrm{AdS}}$ and induced metric, the $(d+1)(d+2)/2$ embedding space Killing vectors K_{AB} are Killing vectors on $\mathrm{AdS}_{1,d}$ as well, thereby making it a maximally symmetric space(time). On AdS we thus have the following types of Killing vectors: only one translation $K_{d+1,0}$, $d(d-1)/2$ spatial rotations K_{jk}, and $(2d)$ boosts K_{0j} and $K_{d+1,j}$.

The AdS-Killing vectors for the boosts are sketched in the following Figure C.22. The time translation $K_{d+1,0} = \partial_t$ is constant and the rotations only move points on their $(d-1)$-sphere, hence both are less interesting for drawing.



(a) $K_{d+1,j}$-Boost (b) K_{0j}-Boost

Figure C.22: Penrose diagram of Minkowski spacetime: Boost Killing vectors.

Recalling from Section C.1.1 that the flat limit of AdS is related to the region near the origin $(t = 0, \rho = 0)$, the correspondence of Table 2.161 is nicely visible in Figure C.22. Comparing Subfigure C.22(a) to Subfigure B.28(b), we see that near the origin the former looks like the latter, that is, the AdS $K_{d+1,j}$-boosts look like Minkowski x^j-translations. Comparing Subfigure C.22(b) near the origin to Subfigure B.28(c), we see that AdS K_{0j}-boosts look like Minkowski x^j-boosts.

C.2 Classical Klein-Gordon solutions on AdS

C.2.1 Ingredients for the radial solutions on AdS

We will use the three special functions ψ, $\widetilde{\psi}$ and F. ψ is the Digamma function defined in DLMF [5.5.2] as the relative change of the Gamma function and $\widetilde{\psi}$ is a customized shorthand for our purposes:

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}, \qquad \forall\, x \in \mathbb{C} \setminus (-\mathbb{N}_0), \tag{C.23}$$

$$\widetilde{\psi}_k(a,b,c) := \psi(a) + \psi(b) - \psi(c) - \psi(k+1)\ . \tag{C.24}$$

$\widetilde{\psi}$ is invariant under exchange of a and b. F is Gauss' hypergeometric series/function defined in DLMF [15.2.1] as

$$F(a,\, b;\, c;\, x) \equiv {}_2F_1(a,\, b;\, c;\, x) := \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\, k!}\, x^k\ . \tag{C.25}$$

Note that it is invariant under exchange of a and b as well. The Pochhammer symbols are defined as

$$(a)_k := a \cdot (a+1) \cdot \ldots \cdot (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}\ , \tag{C.26}$$

wherein the second equality only holds where the Gamma function is well defined. Usually $a \in \mathbb{C}$ and $k \in \mathbb{N}_0$. However, the definition via the Euler Gammas holds for all $k \in \mathbb{C}$, as long as neither a nor $a+k$ become nonpositive integers.

Generically $F(a,\, b;\, c;\, x)$ does not exist if $c \in \mathbb{Z}^{\leq}(0)$. However, if also $a \in \mathbb{Z}^{\leq}(0)$ and $c \leq a$, then it can be interpreted in the sense DLMF [15.2.5] to be DLMF [15.2.4]

$$F(a,\, b;\, c;\, x) = \sum_{k=0}^{-a} \frac{(a)_k\,(b)_k}{(c)_k\, k!}\, x^k = \sum_{k=0}^{-a} (-1)^k \binom{-a}{k} \frac{(b)_k}{(c)_k}\, x^k\ . \tag{C.27}$$

which is just the usual definition of the hypergeometric series, except that it terminates after finitely many terms because $(a)_k$ becomes zero for all $k \geq -a+1$.

We also need what we call the double Pochhammer symbol: for all $a \in \mathbb{C}$ and $n \in \mathbb{N}_0$ we define it to be

$$(\!(a)\!)_n := a\,(a+2)\,(a+4) \cdot \ldots \cdot (a+2n-2)\ . \tag{C.28}$$

The special cases are $(\!(a)\!)_0 := 1$ and $(\!(a)\!)_1 = a$. Although we did not find this notation in the literature, its definition is quite natural and probably has been used before.

In order to relate the double to the original Pochhammer symbol, we recall that the definition of the latter is $(a)_n := a \cdot (a+1) \cdot (a+2) \cdot \ldots \cdot (a+n-1)$. We thus have the relation $(\!(2a)\!)_n = 2^n\,(a)_n$. (There is a similar relation between factorial and double factorial $(2n)!! = 2^n\, n!$, which is not much surprising since for natural subindex values the Pochhammer symbol is just a truncated factorial: $(a)_n = (a+n-1)!/(a-1)!$.) Since we can also write the Pochhammer symbol as the quotient

$(a)_n = \Gamma(a+n)/\Gamma(a)$ of Euler Gamma functions, we can write the double Pochhammer symbol in this way as well. This actually extends its definition from natural to complex subindex values:

$$(\!(a)\!)_z := 2^z \frac{\Gamma(\frac{a}{2}+z)}{\Gamma(\frac{a}{2})} \qquad \begin{smallmatrix}\forall\, a \in \mathbb{C}\\ \forall\, z \in \mathbb{C}\end{smallmatrix} . \tag{C.29}$$

For $z \in \mathbb{N}_0$ this reduces to the relation (C.28). Care has to still be taken of values for a and z that let the Euler Gamma become ill defined, that is, values for which either $\frac{a}{2}$ or $\frac{a}{2}+z$ or both take nonpositive integer values.

C.2.2 From Klein-Gordon to hypergeometric DEQ on AdS

The action for a free, real scalar field $\phi(x)$ living in AdS is

$$S[\phi] = \int \mathrm{d}^{d+1}x \underbrace{\sqrt{|g|}\ \tfrac{1}{2}\left[-g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2\right]}_{\mathcal{L}} \tag{C.30}$$

whose Euler-Lagrange equation

$$0 = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi}$$

is the free Klein-Gordon equation

$$0 = (-\Box_{\text{AdS}} + m^2)\,\phi . \tag{C.31}$$

In order to solve it we follow Mezincescu and Townsend in [46], and Breitenlohner and Freedman in [13], [14]. The separation ansatz $\phi(t,\rho,\Omega) = T(t)\,f(\rho)\,Y(\Omega)$ leads directly to the complex product

$$\phi(t,\rho,\Omega) = \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, f(\rho) . \tag{C.32}$$

$Y^{m_l}_{\underline{l}}$ denotes the hyperspherical harmonics, see Appendix A.4. The realness of the field is assured by expanding it over the complex modes and their complex conjugates. ω is the frequency of the solution and a priori allowed to be complex, although later it turns out that only real frequencies are needed. For the time-interval regions a complete ONS on an equal-time hypersurface is given by a discrete set of modes, with each mode a product of hyperspherical harmonic and a Jacobi polynomial. The Klein-Gordon equation then implies for each such mode one associated frequency ω. These frequencies are discrete and real. The hypercylinder regions contain temporal infinity, thus for these regions ω must be real, too, or the field will diverge for large times.

The remaining radial differential equation to be obeyed by the function $f(\rho)$ then is

$$0 = \left(\omega^2\cos^2\!\rho - \frac{l(l+d-2)}{\tan^2\!\rho} - m^2 R^2_{\text{AdS}}\right) f(\rho) + \frac{d-1}{\tan\rho}\,\partial_\rho f(\rho) + \cos^2\!\rho\,\partial^2_\rho f(\rho) .$$

We shall call the solutions f of this equation simply radial solutions, and the solutions ϕ of (C.31) Klein-Gordon solutions. As shown by Balasubramanian et al. in [9], we can make two ansätze for $f(\rho)$, which we shall call the sine respectively cosine ansatz:

$$f(\rho) \to S(\rho) = \sin^{\tilde{l}}\!\rho\, \cos^{\widetilde{m}}\!\rho\, \widetilde{S}(\sin^2\rho) \tag{C.33}$$

$$f(\rho) \to C(\rho) = \sin^{\tilde{l}}\!\rho\, \cos^{\widetilde{m}}\!\rho\, \widetilde{C}(\cos^2\rho) . \tag{C.34}$$

We denote $\tilde{l}$ and $\widetilde{m}$ like that simply because $\tilde{l}$ will depend on l and $\widetilde{m}$ on m. In [9] $\tilde{l}$ is denoted as $2b$ and $\widetilde{m}$ as $2h$. In [13] $\widetilde{m}$ is denoted as μ and in [46] as λ.

If we impose the following two conditions

$$0 \overset{!}{=} \widetilde{m}^2 - \widetilde{m}d - m^2 R^2_{\text{AdS}} \qquad\qquad 0 \overset{!}{=} \tilde{l}^2 + \tilde{l}(d-2) - l(l+d-2) \tag{C.35}$$

$$\widetilde{m}_\pm = \frac{d}{2} \pm \nu \qquad\qquad \tilde{l}_+ = l \tag{C.36}$$

$$\nu := \sqrt{\frac{d^2}{4} + m^2 R^2_{\text{AdS}}} \qquad\qquad \tilde{l}_- = -(l+d-2)\ ,$$

then plugging the sine or cosine ansatz into the radial equation yields the hypergeometric differential equation DLMF[15.10.1] (wherein the primes denote derivation with respect to $\sin^2\rho$ respectively $\cos^2\rho$)

$$0 = \sin^2\rho\left(1-\sin^2\rho\right)\widetilde{S}''(\sin^2\rho) + \left[\gamma^{\text{S}} - \sin^2\rho\,(\alpha+\beta+1)\right]\widetilde{S}'(\sin^2\rho) - \alpha\beta\,\widetilde{S}(\sin^2\rho)\ , \tag{C.37}$$

$$0 = \cos^2\rho\left(1-\cos^2\rho\right)\widetilde{C}''(\cos^2\rho) + \left[\gamma^{\text{C}} - \cos^2\rho\,(\alpha+\beta+1)\right]\widetilde{C}'(\cos^2\rho) - \alpha\beta\,\widetilde{C}(\cos^2\rho)\ . \tag{C.38}$$

The parameters therein depend on $\tilde{l}, \widetilde{m}$ and ω:

$$\begin{aligned} \alpha &= (\tilde{l} + \widetilde{m} - \omega)/2 \\ \beta &= (\tilde{l} + \widetilde{m} + \omega)/2 \\ \gamma^{\text{S}} &= \tilde{l} + d/2 \\ \gamma^{\text{C}} &= \widetilde{m} - d/2 + 1\ . \end{aligned} \tag{C.39}$$

We remark that the conditions imposed on $\widetilde{m}$ and $\tilde{l}$ are not motivated from physical principles here, and hence must be seen as forming part of the ansatz. However, they will be justified a posteriori by the fact that with these conditions holding, we get systems of solutions that are complete on the time-interval regions respectively on rod and tube regions. Moreover Breitenlohner and Freedman show after equation (11) of [13] that the $\widetilde{m}$-condition leads to a positive energy. The mass squared value where ν vanishes is called Breitenlohner-Freedman mass:

$$m^2_{\text{BF}} = \frac{-d^2}{4R^2_{\text{AdS}}}\ . \tag{C.40}$$

C.2.3 Sets of radial solutions for AdS

In this section we will give an overview of the Klein-Gordon solutions obtained via the two ansätze. We use the following notation for the parameters:

$$\begin{aligned} \alpha_\pm &= (l + \widetilde{m}_\pm - \omega)/2 \\ \beta_\pm &= (l + \widetilde{m}_\pm + \omega)/2 \\ \gamma^{\text{S}} &= l + d/2 \\ \gamma^{\text{C}}_\pm &= \widetilde{m}_\pm - d/2 + 1 = \pm\nu + 1\ . \end{aligned} \tag{C.41}$$

We shall set $\tilde{l} = \tilde{l}_+ = l$ and $\widetilde{m} = \widetilde{m}_+$. This is without loss of generality because the other possible choices yield the same solutions. Note however, that although having fixed this choice, in some radial solutions we encounter linear combinations of the parameters α_+,β_+,γ^{S}, γ^{C}_+ and $\widetilde{m}_+$ adding up to α_-,β_-,γ^{S}, γ^{C}_- and $\widetilde{m}_-$. The appearances of the plus and minus versions of these parameters do not depend on our choice, but are an intrinsic property of the respective solutions.

In order to give a complete list of solutions we will rely heavily on DLMF §15.10.(i). We will use several special functions given in Appendix C.2.1. Each of the hypergeometric equations (C.37) and (C.38) has two linear independent solutions which we shall denote by ${}^1\tilde{S}_{\omega l}$ and ${}^2\tilde{S}_{\omega l}$, respectively ${}^1\tilde{C}_{\omega l}$ and ${}^2\tilde{C}_{\omega l}$. Via the sine ansatz (C.33) respectively cosine ansatz (C.34) this provides us the solutions of the radial part of the Klein-Gordon equation, ${}^1S_{\omega l}$ and ${}^2S_{\omega l}$, respectively ${}^1C_{\omega l}$ and ${}^2C_{\omega l}$.

The form of the solutions of (C.37) depends on wether $\gamma^{\rm S}$ is integer or not, and of (C.38) on wether $\gamma^{\rm C}$ is integer or not. $\gamma^{\rm S}$ is integer precisely if the spatial dimension d is even, and noninteger if not. $\gamma^{\rm C}$ is integer precisely if ν is integer, else noninteger. Where necessary we equip our solutions with labels distinguishing between these cases. Table C.42 shows of which radial solutions we dispose in which case. The S-solutions and the C-solutions generically are not linear independent, their relations are examined in Appendix C.2.4. Since the parameters α,β,$\gamma^{\rm S}$ and $\gamma^{\rm C}$ depend on ω and l, each radial solution is labeled by these parameters.

	d odd		d even	
ν noninteger	${}^1S_{\omega l}$ ${}^2S^{\rm odd}_{\omega l}$	${}^1C_{\omega l}$ ${}^2C^{\rm non}_{\omega l}$	${}^1S_{\omega l}$ ${}^2S^{\rm eve}_{\omega l}$	${}^1C_{\omega l}$ ${}^2C^{\rm non}_{\omega l}$
ν integer	${}^1S_{\omega l}$ ${}^2S^{\rm odd}_{\omega l}$	${}^1C_{\omega l}$ ${}^2C^{\rm int}_{\omega l}$	${}^1S_{\omega l}$ ${}^2S^{\rm eve}_{\omega l}$	${}^1C_{\omega l}$ ${}^2C^{\rm int}_{\omega l}$

Table C.42: Cases of radial solutions

Further below we list the radial solutions appearing in Table C.42. We also give the corresponding DLMF equations, remarking that the solutions appearing there need to be multiplied by the factor $\sin^l\rho\ \cos^{\tilde m_\pm}\rho$ stemming from the sine and cosine ansatzes C.33. The following parameter relations have been used to give the solutions the form we consider most useful.

$$\begin{aligned} 2-\gamma^{\rm C}_\pm &= \gamma^{\rm C}_\mp & 1-\gamma^{\rm C}_\pm &= \gamma^{\rm C}_\mp - 1 \\ \alpha_\pm+\beta_\pm-\gamma^{\rm S}+1 &= \gamma^{\rm C}_\pm \\ \alpha_\pm-\gamma^{\rm C}_\pm+1 &= \alpha_\mp & \beta_\pm-\gamma^{\rm C}_\pm+1 &= \beta_\mp \\ 1-\alpha_\mp &= \gamma^{\rm C}_\pm-\alpha_\pm & 1-\beta_\mp &= \gamma^{\rm C}_\pm-\beta_\pm\ . \end{aligned} \tag{C.43}$$

We shall see later that for certain discrete frequency sets the solutions show special behaviour. The magic frequencies are defined as

$$\omega^\pm_{nl} := 2n+l+\tilde m_\pm = 2n+\gamma^{\rm S}\pm\nu \qquad \begin{matrix}\forall\, l\in\mathbb{N}_0\\ \forall\, n\in\mathbb{N}_0\end{matrix} \tag{C.44}$$

and the submagic frequencies as

$$\sigma^{\rm S}_{nl} := 2n+l+\tilde m_\pm = 2n+\gamma^{\rm S}+\gamma^{\rm C}_+-1 \qquad \begin{matrix}\forall\, l\in\mathbb{N}_0\\ \forall\, n\in\{-(\gamma^{\rm S}-1),\dots,-1\}\end{matrix} \tag{C.45}$$

$$\sigma^{\rm C}_{nl} := 2n+l+\tilde m_\pm = 2n+\gamma^{\rm S}+\gamma^{\rm C}_+-1 \qquad \begin{matrix}\forall\, l\in\mathbb{N}_0\\ \forall\, n\in\{-(\gamma^{\rm C}-1),\dots,-1\}\end{matrix}\ . \tag{C.46}$$

The magic frequencies ω^+_{nl} are allways positive, while the sign of ω^-_{nl} depends on n, l, ν and d. However, we will see later that only for $\nu<1$ we can make use of the ω^-_{nl}, and for this case they are positive, too. The sign of $\sigma^{\rm S/C}_{nl}$ depends on n, l, ν and d. The following solutions come from DLMF [15.10.2] for odd d and noninteger ν (which is the only case we shall study in detail in this work):

$$\begin{aligned} {}^1S_{\omega l}(\rho) &= \sin^l\rho\ \cos^{\tilde m_+}\rho\ F(\alpha_+,\,\beta_+;\,\gamma^{\rm S};\,\sin^2\rho) && \text{(C.47)}\\ {}^2S^{\rm odd}_{\omega l}(\rho) &= -(\sin\rho)^{2-l-d}\ \cos^{\tilde m_+}\rho\ F(\alpha_+-\gamma^{\rm S}+1,\,\beta_+-\gamma^{\rm S}+1;\,2-\gamma^{\rm S};\,\sin^2\rho) && \text{(C.48)}\\ {}^1C_{\omega l}(\rho) &= \sin^l\rho\ \cos^{\tilde m_+}\rho\ F(\alpha_+,\,\beta_+;\,\gamma^{\rm C}_+;\,\cos^2\rho) && \text{(C.49)}\\ {}^2C^{\rm non}_{\omega l}(\rho) &= \sin^l\rho\ \cos^{\tilde m_-}\rho\ F(\alpha_-,\,\beta_-;\,\gamma^{\rm C}_-;\,\cos^2\rho)\ . && \text{(C.50)} \end{aligned}$$

Moreover, DLMF 15.10.(a+b) tell us that the solutions ${}^1S_{\omega l}(\rho)$ and ${}^1C_{\omega l}(\rho)$ also hold for the other three cases where d is even and/or ν is integer. The functions ${}^2S^{\rm eve}$ and ${}^2C^{\rm int}$ are different however, and need to be defined via case-by-case distinction. We indicate the corresponding case by bestowing yet another label (bottom left) to the solutions. Case 1 corresponds to DLMF [15.10.8], case 2 to 15.10(i)(a) with interpretation (C.27), and case 3 to DLMF [15.10.9]. The case of DLMF [15.10.10], where both α_+ and β_+ are nonpositive, cannot occur for our situation since independently of positive or negative ω, either α_+ or β_+ is positive. For being precise, ior denotes "inclusive or" and xor denotes "exclusive or" below.

$$
{}^{2}S^{\text{eve}}_{\omega l} := \begin{cases} {}^{2}_{1}S^{\text{eve}}_{\omega l} & \text{neither } \alpha_+ \text{ nor } \beta_+ \notin \mathbb{Z}^{\leq}(\gamma^{\text{S}}-1) \\ {}^{2}_{2}S^{\text{eve}}_{\omega l} & \text{ior } \begin{array}{l} \alpha_+ = -n \in \{1,\dots,\gamma^{\text{C}}-1\} \Leftrightarrow \omega = +\sigma^{\text{S}}_{nl} \\ \beta_+ = -n' \in \{1,\dots,\gamma^{\text{C}}-1\} \Leftrightarrow \omega = -\sigma^{\text{S}}_{n'l} \end{array} \\ {}^{2}_{3}S^{\text{eve}}_{\omega l} & \text{xor } \begin{array}{l} (\alpha_+ = -n \in \mathbb{Z}^{\leq}(0) \Rightarrow \beta_+ \notin \mathbb{Z}^{\leq}(\gamma^{\text{S}}-1)) \Leftrightarrow \omega = +\omega^{+}_{nl} \\ (\beta_+ = -n \in \mathbb{Z}^{\leq}(0) \Rightarrow \alpha_+ \notin \mathbb{Z}^{\leq}(\gamma^{\text{S}}-1)) \Leftrightarrow \omega = -\omega^{+}_{nl} \end{array} \end{cases} \tag{C.51}
$$

$$
{}^{2}C^{\text{int}}_{\omega l} := \begin{cases} {}^{2}_{1}C^{\text{int}}_{\omega l} & \text{neither } \alpha_+ \text{ nor } \beta_+ \notin \mathbb{Z}^{\leq}(\gamma^{\text{C}}_{+}-1) \\ {}^{2}_{2}C^{\text{int}}_{\omega l} & \text{ior } \begin{array}{l} \alpha_+ = -n \in \{1,\dots,\gamma^{\text{C}}-1\} \Leftrightarrow \omega = +\sigma^{\text{C}}_{nl} \\ \beta_+ = -n' \in \{1,\dots,\gamma^{\text{C}}-1\} \Leftrightarrow \omega = -\sigma^{\text{S}}_{n'l} \end{array} \\ {}^{2}_{3}C^{\text{int}}_{\omega l} & \text{xor } \begin{array}{l} (\alpha_+ = -n \in \mathbb{Z}^{\leq}(0) \Rightarrow \beta_+ \notin \mathbb{Z}^{\leq}(\gamma^{\text{C}}_{+}-1)) \Leftrightarrow \omega = +\omega^{+}_{nl} \\ (\beta_+ = -n \in \mathbb{Z}^{\leq}(0) \Rightarrow \alpha_+ \notin \mathbb{Z}^{\leq}(\gamma^{\text{C}}_{+}-1)) \Leftrightarrow \omega = -\omega^{+}_{nl} \end{array} \end{cases} \tag{C.52}
$$

$$
\begin{aligned} {}^{2}_{1}S^{\text{eve}}_{\omega l}(\rho) = \sin^{l}\rho \cos^{\tilde{m}_+}\rho \Bigg\{ & \ln(\sin^2\rho)\, F(\alpha_+, \beta_+; \gamma^{\text{S}}; \sin^2\rho) \\ & - \sum_{k=1}^{\gamma^{\text{S}}-1} \frac{(k-1)!\,(1-\gamma^{\text{S}})_k}{(1-\alpha_+)_k\,(1-\beta_+)_k} (\sin^2\rho)^{-k} \\ & + \sum_{k=0}^{\infty} \frac{(\alpha_+)_k\,(\beta_+)_k}{(\gamma^{\text{S}})_k\,k!} (\sin^2\rho)^k\, \widetilde{\psi}_k(\alpha_++k, \beta_++k, \gamma^{\text{S}}+k) \Bigg\} \end{aligned} \tag{C.53}
$$

$$
{}^{2}_{2}S^{\text{eve}}_{\omega l}(\rho) = {}^{2}S^{\text{odd}}_{\omega l}(\rho)
$$

$$
\begin{aligned} {}^{2}_{3}S^{\text{eve}}_{\omega l}(\rho) = \sin^{l}\rho \cos^{\tilde{m}_+}\rho \Bigg\{ & \ln(\sin^2\rho)\, F(\alpha_+, \beta_+; \gamma^{\text{S}}; \sin^2\rho) \\ & - \sum_{k=1}^{\gamma^{\text{S}}-1} \frac{(k-1)!\,(1-\gamma^{\text{S}})_k}{(1-\alpha_+)_k\,(1-\beta_+)_k} (\sin^2\rho)^{-k} \\ & + \sum_{k=0}^{n} \frac{(\alpha_+)_k\,(\beta_+)_k}{(\gamma^{\text{S}})_k\,k!} (\sin^2\rho)^k\, \widetilde{\psi}_k(1+n-k, n+l+\tilde{m}_++k, \gamma^{\text{S}}+k) \\ & + (-1)^n\, n! \sum_{k=n+1}^{\infty} \frac{(k-n-1)!\,(n+l+\widetilde{m}_+)_k}{(\gamma^{\text{S}})_k\,k!} (\sin^2\rho)^k \Bigg\} \end{aligned} \tag{C.54}
$$

$$
\begin{aligned} {}^{2}_{1}C^{\text{int}}_{\omega l}(\rho) = \sin^{l}\rho \cos^{\tilde{m}_+}\rho \Bigg\{ & \ln(\cos^2\rho)\, F(\alpha_+, \beta_+; \gamma^{\text{C}}_{+}; \cos^2\rho) \\ & - \sum_{k=1}^{\gamma^{\text{C}}_{+}-1} \frac{(k-1)!\,(1-\gamma^{\text{C}}_{+})_k}{(1-\alpha_+)_k\,(1-\beta_+)_k} (\cos^2\rho)^{-k} \\ & + \sum_{k=0}^{\infty} \frac{(\alpha_+)_k\,(\beta_+)_k}{(\gamma^{\text{C}}_{+})_k\,k!} (\cos^2\rho)^k\, \widetilde{\psi}_k(\alpha_++k, \beta_++k, \gamma^{\text{C}}_{+}+k) \Bigg\} \end{aligned} \tag{C.55}
$$

$$
\begin{aligned}
{}^2_2C^{\rm int}_{\omega l}(\rho) \;&=\; {}^2C^{\rm non}_{\omega l}(\rho) \\
{}^2_3C^{\rm int}_{\omega l}(\rho) \;&=\; \sin^l\!\rho\,\cos^{\tilde m_+}\!\rho\, \Bigg\{ \ln(\cos^2\!\rho)\, F(\alpha_+,\,\beta_+;\,\gamma^{\rm C}_+;\,\cos^2\!\rho) \\
&\qquad - \sum_{k=1}^{\gamma^{\rm C}_+-1} \frac{(k-1)!\,(1-\gamma^{\rm C}_+)_k}{(1-\alpha_+)_k\;(1-\beta_+)_k}\,(\cos^2\!\rho)^{-k} \\
&\qquad + \sum_{k=0}^{n} \frac{(\alpha_+)_k\;(\beta_+)_k}{(\gamma^{\rm C}_+)_k\;k!}\,(\cos^2\!\rho)^k\, \widetilde{\psi}_k(1+n-k,\,n+l+\tilde m_++k,\,\gamma^{\rm C}_++k) \\
&\qquad + (-1)^n\,n!\sum_{k=n+1}^{\infty} \frac{(k-n-1)!\,(n+l+\widetilde m_+)_k}{(\gamma^{\rm C}_+)_k\;k!}\,(\cos^2\!\rho)^k \Bigg\}
\end{aligned}
\tag{C.56}
$$

The exceptional radial solutions ${}^2_2S^{\rm eve}$, ${}^2_3S^{\rm eve}$ and ${}^2_2C^{\rm int}$, ${}^2_3C^{\rm int}$ can be presented here thanks to the extended DLMF section on the hypergeometric DEQ, we have not found them in the existing literature on AdS. The other solutions agree (up to small typos) with those in e.g. [46] and [9].

Finally, for the magic frequencies $|\omega| = \omega^{\pm}_{nl}$ the solutions ${}^1S_{\omega l}(\rho)$ form a discrete set of special solutions which we denote by $J^{(\pm)}_{nl}(\rho)$ (see also Appendix C.2.4):

$$
\begin{aligned}
J^{(+)}_{nl}(\rho) \;&:=\; \tfrac{n!}{(\gamma^{\rm S})_n}\,\sin^l\!\rho\,\cos^{\tilde m_+}\!\rho\, P_n^{(\gamma^{\rm S}-1,+\nu)}(\cos 2\rho) &&\text{for } \omega \;=\; \pm\omega^+_{nl} \\
J^{(-)}_{nl}(\rho) \;&:=\; \tfrac{n!}{(\gamma^{\rm S})_n}\,\sin^l\!\rho\,\cos^{\tilde m_-}\!\rho\, P_n^{(\gamma^{\rm S}-1,-\nu)}(\cos 2\rho) &&\text{for } \omega \;=\; \pm\omega^-_{nl}\,.
\end{aligned}
\tag{C.57}
$$

$P_n^{(\cdot,\cdot)}$ are Jacobi polynomials. These are the only radial solutions leading to normalizable fields on time-interval regions $\mathbb{M}_{[t_1,t_2]}$, see Appendix C.2.5. They are always of this form, independently of wether d is even or odd and wether ν is integer or not.

We have discussed above why only real frequencies ω can be used. Since we also restrict to masses for which $m^2 \geq m^2_{\rm BF}$, all parameters entering the hypergeometric functions (respectively Jacobi polynomials) are real, and therefore all radial solutions given here are real.

C.2.4 Linear (in)dependence of radial solutions in AdS

The solutions for the case of odd d and noninteger ν are related by the following matrix equation, which can be obtained without too much effort using AS[15.3.3+6]. The label "on" stands for odd-noninteger, and "no" for noninteger-odd. Note that the matrices $M^{\rm on}_{\omega l}$ and $M^{\rm no}_{\omega l}$ can be obtained from each other by mutually replacing $\gamma^{\rm S}$ with $\gamma^{\rm C}_+$.

$$
\begin{pmatrix} {}^1S_{\omega l} \\ {}^2S^{\rm odd}_{\omega l} \end{pmatrix} = M^{\rm on}_{\omega l} \begin{pmatrix} {}^1C_{\omega l} \\ {}^2C^{\rm non}_{\omega l} \end{pmatrix}
\qquad\qquad
\begin{pmatrix} {}^1C_{\omega l} \\ {}^2C^{\rm non}_{\omega l} \end{pmatrix} = M^{\rm no}_{\omega l} \begin{pmatrix} {}^1S_{\omega l} \\ {}^2S^{\rm odd}_{\omega l} \end{pmatrix}
\tag{C.58}
$$

$$
\begin{aligned}
(M^{\rm on}_{\omega l})_{11} &= +\frac{\Gamma(\gamma^{\rm S})\,\Gamma(1-\gamma^{\rm C}_+)}{\Gamma(\gamma^{\rm S}-\alpha_+)\,\Gamma(\gamma^{\rm S}-\beta_+)}
& (M^{\rm no}_{\omega l})_{11} &= +\frac{\Gamma(\gamma^{\rm C}_+)\,\Gamma(1-\gamma^{\rm S})}{\Gamma(\gamma^{\rm C}_+-\alpha_+)\,\Gamma(\gamma^{\rm C}_+-\beta_+)} \\
&= +\frac{\Gamma(\gamma^{\rm S})\,\Gamma(1-\gamma^{\rm C}_+)}{\Gamma(1-\alpha^{S,b})\,\Gamma(1-\beta^{S,b})}
& &= +\frac{\Gamma(\gamma^{\rm C}_+)\,\Gamma(1-\gamma^{\rm S})}{\Gamma(\beta^{S,b})\,\Gamma(\alpha^{S,b})} \\
(M^{\rm on}_{\omega l})_{12} &= +\frac{\Gamma(\gamma^{\rm S})\,\Gamma(\gamma^{\rm C}_+-1)}{\Gamma(\alpha_+)\,\Gamma(\beta_+)}
& (M^{\rm no}_{\omega l})_{12} &= -\frac{\Gamma(\gamma^{\rm C}_+)\,\Gamma(\gamma^{\rm S}-1)}{\Gamma(\alpha_+)\,\Gamma(\beta_+)} \\
(M^{\rm on}_{\omega l})_{21} &= -\frac{\Gamma(2-\gamma^{\rm S})\,\Gamma(1-\gamma^{\rm C}_+)}{\Gamma(1-\alpha_+)\,\Gamma(1-\beta_+)}
& (M^{\rm no}_{\omega l})_{21} &= +\frac{\Gamma(2-\gamma^{\rm C}_+)\,\Gamma(1-\gamma^{\rm S})}{\Gamma(1-\alpha_+)\,\Gamma(1-\beta_+)} \\
(M^{\rm on}_{\omega l})_{22} &= -\frac{\Gamma(2-\gamma^{\rm S})\,\Gamma(\gamma^{\rm C}_+-1)}{\Gamma(\gamma^{\rm C}_+-\alpha_+)\,\Gamma(\gamma^{\rm C}_+-\beta_+)}
& (M^{\rm no}_{\omega l})_{22} &= -\frac{\Gamma(2-\gamma^{\rm C}_+)\,\Gamma(\gamma^{\rm S}-1)}{\Gamma(\gamma^{\rm S}-\alpha_+)\,\Gamma(\gamma^{\rm S}-\beta_+)} \\
&= -\frac{\Gamma(2-\gamma^{\rm S})\,\Gamma(\gamma^{\rm C}_+-1)}{\Gamma(\beta^{S,b})\,\Gamma(\alpha^{S,b})}
& &= -\frac{\Gamma(2-\gamma^{\rm C}_+)\,\Gamma(\gamma^{\rm S}-1)}{\Gamma(1-\alpha^{S,b})\,\Gamma(1-\beta^{S,b})}\,.
\end{aligned}
$$

Plugging in the definitions of the hypergeometric parameters this becomes

$$(M^{\text{on}}_{\omega l})_{11} = +\frac{\Gamma(l+\frac{d}{2})\,\Gamma(-\nu)}{\Gamma(-\frac{1}{2}(\tilde{m}_+-\omega-l-d))\,\Gamma(-\frac{1}{2}(\tilde{m}_++\omega-l-d))} \qquad (M^{\text{no}}_{\omega l})_{11} = +\frac{\Gamma(1+\nu)\,\Gamma(1-l-\frac{d}{2})}{\Gamma(\frac{1}{2}(\tilde{m}_++\omega-l-d+2))\Gamma(\frac{1}{2}(\tilde{m}_+-\omega-l-d+2))}$$

$$(M^{\text{on}}_{\omega l})_{12} = +\frac{\Gamma(l+\frac{d}{2})\,\Gamma(\nu)}{\Gamma(\frac{1}{2}(\tilde{m}_++\omega+l))\,\Gamma(\frac{1}{2}(\tilde{m}_+-\omega+l))} \qquad (M^{\text{no}}_{\omega l})_{12} = -\frac{\Gamma(1+\nu)\,\Gamma(l+\frac{d}{2}-1)}{\Gamma(\frac{1}{2}(\tilde{m}_++\omega+l))\,\Gamma(\frac{1}{2}(\tilde{m}_+-\omega+l))}$$

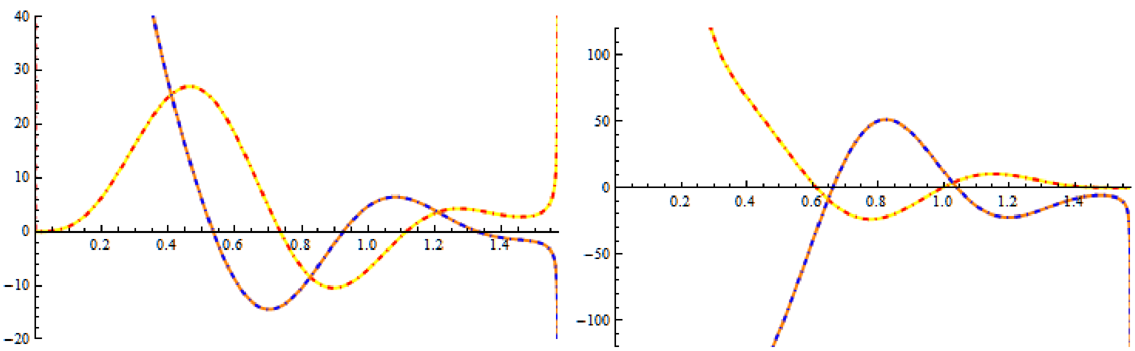
$$(M^{\text{on}}_{\omega l})_{21} = -\frac{\Gamma(2-l-\frac{d}{2})\,\Gamma(-\nu)}{\Gamma(-\frac{1}{2}(\tilde{m}_+-\omega+l-2))\,\Gamma(-\frac{1}{2}(\tilde{m}_++\omega+l-2))} \qquad (M^{\text{no}}_{\omega l})_{21} = +\frac{\Gamma(1-\nu)\,\Gamma(1-l-\frac{d}{2})}{\Gamma(-\frac{1}{2}(\tilde{m}_+-\omega+l-2))\,\Gamma(-\frac{1}{2}(\tilde{m}_++\omega+l-2))}$$

$$(M^{\text{on}}_{\omega l})_{22} = -\frac{\Gamma(2-l-\frac{d}{2})\,\Gamma(\nu)}{\Gamma(\frac{1}{2}(\tilde{m}_++\omega-l-d+2))\Gamma(\frac{1}{2}(\tilde{m}_+-\omega-l-d+2))} \qquad (M^{\text{no}}_{\omega l})_{22} = -\frac{\Gamma(1-\nu)\,\Gamma(l+\frac{d}{2}-1)}{\Gamma(-\frac{1}{2}(\tilde{m}_+-\omega-l-d))\,\Gamma(-\frac{1}{2}(\tilde{m}_++\omega-l-d))}\,.$$

The two matrices $M^{\text{no}}_{\omega l}$ and $M^{\text{on}}_{\omega l}$ must be the inverse of each other, which we verify at the end of this section. Further, we there calculate the determinants of the matrices, giving the following result (which again confirms that both matrices are inverse to each other, since their determinants multiply to unity):

$$\det\, M^{\text{on}}_{\omega l} \;=\; \frac{(2l+d-2)}{2\nu} \qquad\qquad \det\, M^{\text{no}}_{\omega l} \;=\; \frac{2\nu}{(2l+d-2)}\;. \tag{C.59}$$

In Figure C.60 we plot the original functions (e.g. ${}^1S_{\omega l}$) and their decomposition into the other two functions (e.g. $(M^{\text{on}}_{\omega l})_{11}\,{}^1C_{\omega l} + (M^{\text{on}}_{\omega l})_{12}\,{}^2C^{\text{non}}_{\omega l}$) in order to check that we calculated all matrix elements correctly.



(a) radial S-solutions (b) radial C-solutions

Figure C.60: Linear dependence of the radial solutions.

Herein, the horizontal axis is the ρ-axis going from 0 to $\pi/2$, and the vertical axis shows the value of the radial solutions. For both plots we set $d = 3$ while $m = 1.5$ and $R_{\text{AdS}} = 1$, and moreover $\omega = 10$ while $l = 3$. The plots illustrate that the left and right hand sides of the four linear dependencies of (C.58) agree. This indicates that our matrix elements of $M^{\text{on}}_{\omega l}$ and $M^{\text{no}}_{\omega l}$ indeed are correct. We found the same agreement for all sets of parameter values that we tested.

In C.60(a) we plot the equations in the left column of (C.58). The continuous yellow line is ${}^1S_{10,3}(\rho)$ and the dotdashed red line is the corresponding decomposition $(M^{\text{on}}_{10,3})_{11}{}^1C_{10,3}(\rho)+(M^{\text{on}}_{10,3})_{12}{}^2C^{\text{non}}_{10,3}(\rho)$ (both zoomed by a factor of 1000). The continuous orange line is ${}^2S^{\text{odd}}_{10,3}(\rho)$ and the dotdashed blue line is $(M^{\text{on}}_{10,3})_{21}{}^1C_{10,3}(\rho) + (M^{\text{on}}_{10,3})_{22}{}^2C^{\text{non}}_{10,3}(\rho)$ (both zoomed by a factor of -0.2).

In C.60(b) we plot the equations on the right column of (C.58). The continuous yellow line is ${}^1C_{10,3}(\rho)$ and the dotdashed red line is the corresponding decomposition (again zoomed by a factor of 1000) $(M^{\text{no}}_{10,3})_{11}{}^1S_{10,3}(\rho)+(M^{\text{no}}_{10,3})_{12}{}^2S^{\text{odd}}_{10,3}(\rho)$. The continuous orange line is ${}^2S^{\text{odd}}_{10,3}(\rho)$ and the dotdashed blue line is $(M^{\text{no}}_{10,3})_{21}{}^1S_{10,3}(\rho) + (M^{\text{no}}_{10,3})_{22}{}^2S^{\text{odd}}_{10,3}(\rho)$ (both multiplied by -1 for visibility).

Next we have a look at exceptional cases of these dependencies. As stated in [9], ${}^1S$ is proportional to ${}^1C$ whenever M^{on}_{12} and M^{no}_{12} vanish, i.e., whenever α_+ or β_+ are nonpositive integers. This happens precisely if the frequency is one of the "magic" frequencies $\pm\omega^+_{nl}$ from (C.44).

${}^1S$ is proportional to ${}^2C^{\text{non}}$ whenever M^{on}_{11} and M^{no}_{22} vanish, i.e., whenever $(\gamma^{\text{S}}-\alpha_+)$ or $(\gamma^{\text{S}}-\beta_+)$ are nonpositive integers. This happens iff the frequency is one of the "magic" frequencies $\pm\omega^-_{nl}$ from (C.44). In these cases the functions are related through:

$$
\begin{aligned}
{}^1S_{\omega l} &= (-1)^n \tfrac{(\gamma^{\text{C}}_+)_n}{(\gamma^{\text{S}})_n}\, {}^1C_{\omega l} = \tfrac{n!}{(\gamma^{\text{S}})_n} \sin^l\!\rho \cos^{\tilde m_+}\!\rho\, P_n^{(\gamma^{\text{S}}-1,+\nu)}(\cos 2\rho) &&\Leftrightarrow \quad \omega = \pm\omega^+_{nl} \\
{}^1S_{\omega l} &= (-1)^n \tfrac{(\gamma^{\text{C}}_-)_n}{(\gamma^{\text{S}})_n}\, {}^1C_{\omega l} = \tfrac{n!}{(\gamma^{\text{S}})_n} \sin^l\!\rho \cos^{\tilde m_-}\!\rho\, P_n^{(\gamma^{\text{S}}-1,-\nu)}(\cos 2\rho) &&\Leftrightarrow \quad \omega = \pm\omega^-_{nl} \quad \nu<1
\end{aligned}
\tag{C.61}
$$

In both cases ${}^1S$ becomes regular at the boundary $\rho = \frac{\pi}{2}$. The Jacobi polynomials $P_n^{(\cdot,\cdot)}$ arise from the hypergeometric function if the frequency is magic via DLMF [15.9.1]:

$$
\tfrac{n!}{(a+1)_n}\, P_n^{(a,b)}(1-2x) = F(-n,\, a+b+n+1;\, a+1;\, x) \qquad a,b > -1\,. \tag{C.62}
$$

${}^2S^{\text{odd}}$ is proportional to ${}^1C$ whenever M^{no}_{11} and M^{on}_{22} vanish, i.e., whenever $(\gamma^{\text{C}}_+ - \alpha_+)$ or $(\gamma^{\text{C}}_+ - \beta_+)$ are nonpositive integers. This happens exactly if the frequency is one of the frequencies $\pm\omega = -(l+d-2) + \tilde m_+ + 2n$ with $n \in \mathbb{N}_0$.

As promised above, we now show that the matrices are indeed inverses of each other, and calculate their determinant. Writing shorter $\lambda := d/2+l-1$, they write as follows:

$$
\begin{aligned}
(M^{\text{on}}_{\omega l})_{11} &= +\frac{\Gamma(\lambda+1)\,\Gamma(-\nu)}{\Gamma((1+\lambda-\nu+\omega)/2)\,\Gamma((1+\lambda-\nu-\omega)/2)} &
(M^{\text{no}}_{\omega l})_{11} &= +\frac{\Gamma(1+\nu)\,\Gamma(-\lambda)}{\Gamma((1-\lambda+\nu+\omega)/2)\,\Gamma((1-\lambda+\nu-\omega)/2)} \\
(M^{\text{on}}_{\omega l})_{12} &= +\frac{\Gamma(\lambda+1)\,\Gamma(\nu)}{\Gamma((1+\lambda+\nu+\omega)/2)\,\Gamma((1+\lambda+\nu-\omega)/2)} &
(M^{\text{no}}_{\omega l})_{12} &= -\frac{\Gamma(1+\nu)\,\Gamma(\lambda)}{\Gamma((1+\lambda+\nu+\omega)/2)\,\Gamma((1+\lambda+\nu-\omega)/2)} \\
(M^{\text{on}}_{\omega l})_{21} &= -\frac{\Gamma(1-\lambda)\,\Gamma(-\nu)}{\Gamma((1-\lambda-\nu+\omega)/2)\,\Gamma((1-\lambda-\nu-\omega)/2)} &
(M^{\text{no}}_{\omega l})_{21} &= +\frac{\Gamma(1-\nu)\,\Gamma(-\lambda)}{\Gamma((1-\lambda-\nu+\omega)/2)\,\Gamma((1-\lambda-\nu-\omega)/2)} \\
(M^{\text{on}}_{\omega l})_{22} &= -\frac{\Gamma(1-\lambda)\,\Gamma(\nu)}{\Gamma((1-\lambda+\nu+\omega)/2)\,\Gamma((1-\lambda+\nu-\omega)/2)} &
(M^{\text{no}}_{\omega l})_{22} &= -\frac{\Gamma(1-\nu)\,\Gamma(\lambda)}{\Gamma((1+\lambda-\nu+\omega)/2)\,\Gamma((1+\lambda-\nu-\omega)/2)}\,.
\end{aligned}
$$

What we want to show is thus

$$
\begin{aligned}
M^{\text{on}}_{\omega l}\, M^{\text{no}}_{\omega l} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \det M^{\text{on}}_{\omega l} &= \frac{\lambda}{\nu} \\
M^{\text{no}}_{\omega l}\, M^{\text{on}}_{\omega l} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \det M^{\text{no}}_{\omega l} &= \frac{\nu}{\lambda}\,.
\end{aligned}
\tag{C.63}
$$

Plugging the matrix elements into the matrix multiplication, we can see that the zeros in the unity matrix are obtained quickly using only the recursion relation AS[6.1.15]=DLMF[5.5.3]

$$
z\,\Gamma(z) = \Gamma(1+z)\,. \tag{C.64}
$$

Further, generously applying the reflection property AS[6.1.17]=DLMF[5.5.3] of Euler's Gamma function

$$
\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}\,, \tag{C.65}
$$

(plus $\sin(x+\pi/2) = \cos x$) we find that obtaining the ones in the unity matrix of the matrix multiplication and obtaining the values of the determinant corresponds to showing the validity of

$$
\sin(\pi\lambda)\,\sin(\pi\nu) = \cos([\lambda-\nu+\omega]\pi/2)\,\cos([\lambda-\nu-\omega]\pi/2) - \cos([\lambda+\nu+\omega]\pi/2)\,\cos([\lambda-\nu-\omega]\pi/2)\,.
$$

This can be achieved using first AS[4.3.42] (eliminating the ω-terms) and then AS[4.3.41]:

$$
\begin{aligned}
\cos(z_1+z_2)\,\cos(z_1-z_2) &= \cos^2 z_1 - \sin^2 z_2 \\
\cos^2 z_2 - \cos^2 z_1 &= \sin(z_2+z_1)\,\sin(z_2-z_1)\,.
\end{aligned}
$$

This completes the demonstration of (C.63).

C.2.5 Normalizability on equal-time hypersurface in AdS

At a later stage we will need to integrate solutions over equal-time hypersurfaces Σ_t. The radial part of this integral for a radial solution $\chi(\rho)$ writes

$$\int_0^{\pi/2} \mathrm{d}\rho\ \tan^{d-1}\rho\ \chi^2(\rho)\ .$$

We need solutions giving us a regular integrand on the whole Σ_t. This happens exactly if the frequency is magic, because only then the solution inherits the regularity of ${}^1S$ at the origin plus the regularity of ${}^1C$ (and ${}^2C^{\mathrm{non}}$ for small ν) at $\rho = \frac{\pi}{2}$. Using (A.1) and (A.6) we get (with $\nu < 1$ for the minus case)

$$\int_0^{\pi/2} \mathrm{d}\rho\ \tan^{d-1}\rho\ \left(\sin^l\rho\ \cos^{\tilde{m}_\pm}\rho\ P_n^{(\gamma^{\mathrm{S}}-1,\pm\nu)}(\cos 2\rho)\right)^2 \;=\; \frac{\Gamma(n+\gamma^{\mathrm{S}})\,\Gamma(n\pm\nu+1)}{2\omega^\pm_{nl}\, n!\,\Gamma(n\pm\nu+\gamma^{\mathrm{S}})}\ , \tag{C.66}$$

and can thus define the following normalization constant:

$$\mathcal{N}^\pm_{nl} \;:=\; \int_0^{\pi/2} \mathrm{d}\rho\ \tan^{d-1}\rho\ \left(J^{(\pm)}_{nl}(\rho)\right)^2 \;=\; \frac{n!\ \Gamma(\gamma^{\mathrm{S}})^2\,\Gamma(n\pm\nu+1)}{2\omega^\pm_{nl}\ \Gamma(n+\gamma^{\mathrm{S}})\ \Gamma(n\pm\nu+\gamma^{\mathrm{S}})}\ . \tag{C.67}$$

For the $J^{(+)}_{nl}$ this holds for all $\nu \geq 0$, that is: all masses $m^2 \geq m^2_{\mathrm{BF}}$ above the Breitenlohner-Freedman mass, while for the $J^{(-)}_{nl}$ it holds only for $\nu \in [0,1)$, that is, for the negative values $m^2 \in [m^2_{\mathrm{BF}}, (4-d^2)/(4R^2_{\mathrm{AdS}})\ )$. Hence for the respective allowed values of ν we have positive $\omega^\pm_{nl}$ and the exceptional case (described in Section A.2) of $-1 = a+b = \gamma^{\mathrm{S}}-1\pm\nu$ cannot occur.

Finally, we note that the integrals with $n \neq n'$ vanish:

$$0 \;=\; \int_0^{\pi/2} \mathrm{d}\rho\ \tan^{d-1}\rho\ J^{(\pm)}_{nl}(\rho)\ J^{(\pm)}_{n'l}(\rho)\ , \tag{C.68}$$

due to the orthogonality of the Jacobi polynomials, see AS[22.1.1+2] and AS[22.2.1].

C.2.6 Wronskians for radial solutions on AdS

In this section we calculate the Wronskians ${}^1S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\mathrm{odd}}_{\omega l}(\rho)$ and ${}^1C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2C^{\mathrm{non}}_{\omega l}(\rho)$ of the radial solutions, which are needed for the symplectic structure. Starting with the former, for greater clarity we allow us to simplify our notation a bit for the moment: $\alpha_+ \;\to\; \alpha$, $\beta_+ \;\to\; \beta$, $\gamma^{\mathrm{S}} \;\to\; \gamma$ and

$\sin^2\rho \;\to\; x$. The calculation proceeds as follows:

$$\begin{aligned}
\tan^{d-1}\rho\,\big({}^1S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\rm odd}_{\omega l}(\rho)\big) \;&=\; -2(1-x)^{\alpha+\beta-\gamma+1}\Big(F(\alpha,\,\beta;\,\gamma;\,x)\,F(\alpha-\gamma+1,\,\beta-\gamma+1;\,2-\gamma;\,x)\,(1-\gamma)\\
&\qquad + F(\alpha,\,\beta;\,\gamma;\,x)\,F(\alpha-\gamma+2,\,\beta-\gamma+2;\,3-\gamma;\,x)\,x\;\frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)}\\
&\qquad - F(\alpha+1,\,\beta+1;\,\gamma+1;\,x)\,F(\alpha-\gamma+1,\,\beta-\gamma+1;\,2-\gamma;\,x)\,x\;\frac{\alpha\beta}{\gamma}\Big)\\
&=\; -2(1-x)^{\alpha+\beta-\gamma+1}\Big(F(\alpha,\,\beta;\,\gamma;\,x)\,F(\alpha-\gamma+2,\,\beta-\gamma+1;\,2-\gamma;\,x)\,(1+\alpha-\gamma)\\
&\qquad - F(\alpha+1,\,\beta;\,\gamma;\,x)\,F(\alpha-\gamma+1,\,\beta-\gamma+1;\,2-\gamma;\,x)\,a\Big)\\
&=\; -2F(\alpha,\,\beta;\,\gamma;\,x)\,F(-\alpha,\,1-\beta;\,2-\gamma;\,x)\,(1+\alpha-\gamma)\\
&\quad + 2F(\alpha+1,\,\beta;\,\gamma;\,x)\,F(1-\alpha,\,1-\beta;\,2-\gamma;\,x)\,(1-x)\,a\\
&=\; 2F(\alpha,\,\beta;\,\gamma;\,x)\,F(-\alpha,-\beta;-\gamma;\,x)\,(x-1)\\
&\quad - F(\alpha+1,\,1+\beta;\,2+\gamma;\,x)\,F(1-\alpha,\,1-\beta;\,2-\gamma;\,x)\,x^2\;\frac{\alpha\beta(\alpha-\gamma)(\beta-\gamma)}{\gamma^2\,(1+\gamma)}\\
&=\; 2(\gamma-1)\,. \qquad\qquad (C.69)
\end{aligned}$$

Since here $\gamma \;=\; \gamma^S \;=\; l+d/2$, we obtain the result

$$ {}^1S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\rm odd}_{\omega l}(\rho) \;=\; 2(\gamma^S-1)\,\tan^{1-d}\rho \;=\; (2l+d-2)\,\tan^{1-d}\rho\,. \qquad (C.70)$$

Thus the S-Wronskian times the metric factor $\tan^{d-1}\rho$ is actually independent of ρ (and of ω), and moreover has positive sign for all l. In the calculation we make use of the following three hypergeometric contiguous relations, on whose origin we comment further below.

$$\begin{aligned}
F(\alpha+1,\,\beta;\,\gamma;\,x)\,(1-x)\,\gamma^2(1+\gamma) \;&=\; F(\alpha,\,\beta;\,\gamma;\,x)\,\gamma(1+\gamma)[\gamma+x(\beta-\gamma)]\\
&\qquad - F(\alpha+1,\,\beta+1;\,\gamma+2;\,x)\,x^2\,\beta(\alpha-\gamma)(\beta-\gamma) \qquad (C.71)\\
F(\alpha,\,\beta+1;\,\gamma+2;\,x)\,\gamma(1+\gamma-\alpha) \;&=\; F(\alpha,\,\beta;\,\gamma;\,x)\,\gamma(1+\gamma) - F(\alpha+1,\,\beta+1;\,\gamma+2;\,x)\,\alpha[\gamma+x(\beta-\gamma)] \qquad (C.72)\\
F(\alpha+1,\,\beta+1;\,\gamma+1;\,x)\,x\,\beta \;&=\; -F(\alpha,\,\beta;\,\gamma;\,x)\,\gamma + F(\alpha+1,\,\beta;\,\gamma;\,x)\,\gamma\,. \qquad (C.73)
\end{aligned}$$

The first equality in the Wronskian calculation (C.70) comes directly from plugging in the radial functions and cleaning up a bit. The second then is achieved by using the third contiguous relation for both $F(\alpha-\gamma+2,\,\beta-\gamma+2;\,3-\gamma;\,x)$ and $F(\alpha+1,\,\beta+1;\,\gamma+1;\,x)$. The third follows by using AS[15.3.3] = DLMF[15.8.1]:

$$F(\alpha,\,\beta;\,\gamma;\,x) \;=\; (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\,\gamma-\beta;\,\gamma;\,x)$$

for both $F(\alpha-\gamma+2,\,\beta-\gamma+1;\,2-\gamma;\,x)$ and $F(\alpha-\gamma+1,\,\beta-\gamma+1;\,2-\gamma;\,x)$. The fourth results from using the second contiguous relation for $F(-\alpha,\,1-\beta;\,2-\gamma;\,x)$ and the first contiguous relation for $F(\alpha+1,\,\beta;\,\gamma;\,x)$. The last then can finally be read off from DLMF[15.16.4]:

$$1 \;=\; F(\alpha,\,\beta;\,\gamma;\,x)\,F(-\alpha,-\beta;-\gamma;\,x) \;+\; F(1+\alpha,\,1+\beta;\,2+\gamma;\,x)\,F(1-\alpha,\,1-\beta;\,2-\gamma;\,x)\,x^2\;\frac{\alpha\beta(\alpha-\gamma)(\beta-\gamma)}{c^2\,(1-c^2)}\;.$$

The three hypergeometric contiguous relations above were found using the Mathematica code file provided by A. Ibrahim and M. Rakha in [40], see also [70] by the same authors and A. Rathie. While greatly benefiting from their work we should remark the following: the code in [40] is more general then the theorems presented therein: it computes the coefficients c_i for the more general contiguous relation (3.4) in [70], making it a potentially powerful and practical tool.

However, it seems to us that in the proof of the essential Lemma 2 in [70], the transition from (2.5) to (2.6) looks clean at first while actually not well justified. In addition, in the given form the code does use the definitions of Section 2 until (2.3), but then seems to implement not the formulas (3.7) and (3.3) which are used in the proof of Theorem 3, but other formulas which we where unable to indentify. That said, their code works well for *some sets* of parameter shifts (which is why we tried to fully understand the magic at work therein). Checking the three relations above (and others generated by the code) with Mathematica's FullSimplify command (and where possible with the Handbook of Abramowitz and Stegun) results in complete agreement. They can also be checked by writing the hypergeometric function as a power series in x and comparing the coefficients for each power of x on left and right hand sides, using AS [15.1.1]:

$$F(\alpha,\,\beta;\,\gamma;\,x)\;=\;\sum_{n=0}^{\infty}\frac{(\alpha)_n\;(\beta)_n}{(\gamma)_n}\;\frac{x^n}{n!}\;\;. \tag{C.74}$$

However, for other sets of parameter shifts the code *fails to work*, e.g., some of the sets needed in Appendices C.3.4 and C.3.5. This can be guessed quickly from overly complex coefficients returned by the code for these cases, and is further confirmed by the code's own numerical test of the results delivered by its algorithm. The results for these cases (which we obtained by hand) can be verified again through Mathematica's FullSimplify command and power series expansion, but disagree with the results given by the code. We therefore recommend treating the results of this code with due care, and hope that it can be improved such that it may work for all sets of parameter shifts.

The Wronskian for the radial C-solutions $\mathcal{W}^{C}_{\omega l}(\rho)\;:=\;{}^{1}C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_{\rho}{}^{2}C^{\rm non}_{\omega l}(\rho)$ can be calculated in exactly the same way. To see this, we use the simplified notation $\alpha_+\;\to\;\alpha$, $\beta_+\;\to\;\beta$, $\gamma^{\rm C}_+\;\to\;\gamma$ and $\cos^2\rho\;\to\;y$. Then, using parameter relations (C.43), we can write the radial C-solutions in a similar form as the S-solutions

$$\begin{aligned}{}^{1}C_{\omega l}(\rho)\;&=\;\sin^{l}\!\rho\,\cos^{\tilde m_+}\!\rho\;F(\alpha,\,\beta;\,\gamma;\,\cos^2\!\rho)\\ {}^{2}C^{\rm non}_{\omega l}(\rho)\;&=\;\sin^{l}\!\rho\,\cos^{\tilde m_-}\!\rho\;F(\alpha\!-\!\gamma\!+\!1,\,\beta\!-\!\gamma\!+\!1;\,2\!-\!\gamma;\,\cos^2\!\rho)\;.\end{aligned} \tag{C.75}$$

Plugging this form of the C-solutions into the Wronskian, we obtain the first line of the S-calculation, and thus can directly jump to its last line:

$$\begin{aligned}\tan^{d-1}\!\rho\,\big({}^{1}C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_{\rho}{}^{2}C^{\rm non}_{\omega l}(\rho)\big)\;&=\;-2(1-y)^{\alpha+\beta-\gamma+1}\Big(F(\alpha,\,\beta;\,\gamma;\,y)\,F(\alpha\!-\!\gamma\!+\!1,\,\beta\!-\!\gamma\!+\!1;\,2\!-\!\gamma;\,y)\,(1\!-\!\gamma)\\ &\qquad+F(\alpha,\,\beta;\,\gamma;\,y)\,F(\alpha\!-\!\gamma\!+\!2,\,\beta\!-\!\gamma\!+\!2;\,3\!-\!\gamma;\,y)\,y\;\frac{(\alpha\!-\!\gamma\!+\!1)(\beta\!-\!\gamma\!+\!1)}{(2\!-\!\gamma)}\\ &\qquad-F(\alpha\!+\!1,\,\beta\!+\!1;\,\gamma\!+\!1;\,y)\,F(\alpha\!-\!\gamma\!+\!1,\,\beta\!-\!\gamma\!+\!1;\,2\!-\!\gamma;\,y)\,y\;\frac{\alpha\beta}{\gamma}\Big)\\ &=\;2(\gamma-1)\;.\end{aligned}$$

Since here $\gamma\;=\;\gamma^{C}_{+}\;=\;1\!+\!\nu$, we obtain the result

$${}^{1}C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_{\rho}{}^{2}C^{\rm non}_{\omega l}(\rho)\;=\;2(\gamma^{C}_{+}\!-\!1)\,\tan^{1-d}\!\rho\;=\;2\nu\,\tan^{1-d}\!\rho\;. \tag{C.76}$$

In Figure C.77 we plot the Wronskians (multiplied by $\tan^{d-1}\rho$) in order to illustrate that our results are correct.

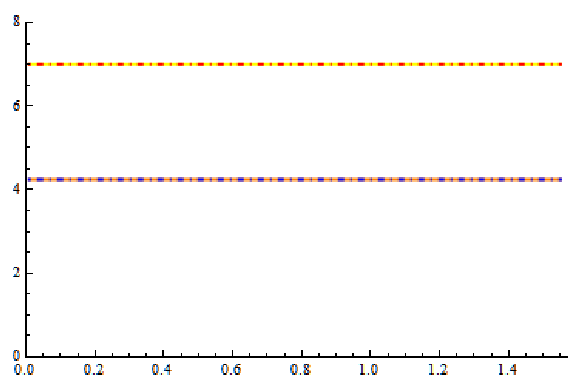


Figure C.77: Wronskians ${}^{1}S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_{\rho}{}^{2}S^{\rm odd}_{\omega l}(\rho)$ and ${}^{1}C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_{\rho}{}^{2}C^{\rm non}_{\omega l}(\rho)$.

Herein, the horizontal axis is the ρ-axis going from 0 to $\pi/2$, and the vertical axis shows the value of the Wronskians. Again we set $d = 3$ while $m = 1.5$ and $R_{\text{AdS}} = 1$, giving us $(2\nu) = \sqrt{18} \approx 4.24$, and moreover we set $\omega = 10$ while $l = 3$. The continuous yellow line is the numerical evaluation of ${}^1S_{10,3}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\text{odd}}_{10,3}(\rho)$ and the dotdashed red line is the value of $(2l{+}d{-}2)$. The continuous orange line is the numerical evaluation of ${}^1C_{10,3}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2C^{\text{non}}_{10,3}(\rho)$ and the dotdashed blue line is the value of (2ν). The plot illustrates that our results $(2l{+}d{-}2)$ and (2ν) agree perfectly with the numerical evaluation of the Wronskians. We found the same agreement for all sets of parameter values that we tested.

The two Wronskians ${}^1S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\text{odd}}_{\omega l}(\rho)$ and ${}^1C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2C^{\text{non}}_{\omega l}(\rho)$ are related in the following way. In the definition of the C-Wronskian we can write the C-solutions as linear combinations of the S-solutions using the matrix $M^{\text{no}}_{\omega l}$ of (C.58). The analogue can be done for the S-Wronskian with the matrix $M^{\text{on}}_{\omega l}$. It is then straightforward to check that

$$\begin{aligned}\left({}^1C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2C^{\text{non}}_{\omega l}(\rho)\right) &= \left({}^1S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\text{odd}}_{\omega l}(\rho)\right)\det M^{\text{no}}_{\omega l}\\ \left({}^1S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\text{odd}}_{\omega l}(\rho)\right) &= \left({}^1C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2C^{\text{non}}_{\omega l}(\rho)\right)\det M^{\text{on}}_{\omega l}\ ,\end{aligned} \tag{C.78}$$

which via inserting (C.70) and (C.76) implies that

$$\begin{aligned}\det M^{\text{no}}_{\omega l} &= \frac{\left({}^1C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2C^{\text{non}}_{\omega l}(\rho)\right)}{\left({}^1S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\text{odd}}_{\omega l}(\rho)\right)} = \frac{2\nu}{2l{+}d{-}2}\\ \det M^{\text{on}}_{\omega l} &= \frac{\left({}^1S_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2S^{\text{odd}}_{\omega l}(\rho)\right)}{\left({}^1C_{\omega l}(\rho)\,\overleftrightarrow{\partial}_\rho\,{}^2C^{\text{non}}_{\omega l}(\rho)\right)} = \frac{2l{+}d{-}2}{2\nu}\ .\end{aligned} \tag{C.79}$$

This agrees exactly with (C.59).

C.2.7 Flat limits of the radial functions

In this section we compare the flat limit of the S^a and S^b-modes with the Klein-Gordon solutions in (3+1)-dimensional Minkowski spacetime given in Section 5.3 of [59]. We do not use Einstein's sum convention in this section. We recall the notation of (C.11) for new global coordinates

$$\begin{aligned} r &:= R_{\text{AdS}}\,\rho & r &\in [0, \tfrac{\pi}{2}\,R_{\text{AdS}})\\ \tau &:= R_{\text{AdS}}\,t & \tau &\in (-\infty, +\infty)\end{aligned} \tag{C.80}$$

and introduce new parameters for frequency and momentum

$$\begin{aligned} \widetilde{\omega} &:= \omega/R_{\text{AdS}} & \widetilde{\omega} &\in (-\infty, +\infty)\\ p_\omega &:= \sqrt{\omega^2 - \tilde{m}_+^2} & |p_\omega| &\in [0,\infty)\\ p^{\mathbb{R}}_\omega &:= \sqrt{|\omega^2 - \tilde{m}_+^2|} & p^{\mathbb{R}}_\omega &\in [0,\infty)\\ \tilde{p}_{\tilde{\omega}} &= \sqrt{\tilde{\omega}^2 - m^2} & |\tilde{p}_{\tilde{\omega}}| &\in [0,\infty)\\ \tilde{p}^{\mathbb{R}}_{\tilde{\omega}} &= \sqrt{|\tilde{\omega}^2 - m^2|} & \tilde{p}^{\mathbb{R}}_{\tilde{\omega}} &\in [0,\infty).\end{aligned} \tag{C.81}$$

The following case distinctions thus arise:

$$p_\omega = \begin{cases} p^{\mathbb{R}}_\omega & \omega^2 > \tilde{m}_+^2\\ \mathrm{i}\,p^{\mathbb{R}}_\omega & \omega^2 < \tilde{m}_+^2\end{cases} \qquad \tilde{p}_{\tilde{\omega}} = \begin{cases} \tilde{p}^{\mathbb{R}}_{\tilde{\omega}} & \tilde{\omega}^2 > m^2\\ \mathrm{i}\,\tilde{p}^{\mathbb{R}}_{\tilde{\omega}} & \tilde{\omega}^2 < m^2\end{cases}\ . \tag{C.82}$$

For $R_{\text{AdS}} \to \infty$ we have the limits

$$p_\omega/R_{\text{AdS}} \xrightarrow[\text{lim.}]{\text{flat}} \tilde{p}_{\tilde{\omega}} \qquad p^{\mathbb{R}}_\omega/R_{\text{AdS}} \xrightarrow[\text{lim.}]{\text{flat}} \tilde{p}^{\mathbb{R}}_{\tilde{\omega}}. \tag{C.83}$$

The temporal part $e^{-i\omega t} = e^{-i\tilde{\omega}\tau}$ and the angular part $Y_l^{m_l}(\Omega)$ of the solutions remain unchanged, and we only need to consider the radial part. To this end, in the radial solutions we first replace ρ by r/R_{AdS}, and then taking the flat limit $R_{\mathrm{AdS}} \to \infty$ we replace

$$\widetilde{m}_\pm,\, \pm\nu \;\to\; \pm m R_{\mathrm{AdS}} \qquad \sin\tfrac{r}{R_{\mathrm{AdS}}} \;\to\; \tfrac{r}{R_{\mathrm{AdS}}} \qquad \cos^{\tilde{m}_+}\tfrac{r}{R_{\mathrm{AdS}}} \;\to\; 1\;.$$

The last replacement is justified by the fact that for arbitrary r a Taylor development in R_{AdS}^{-1} yields

$$\cos^{\tilde{m}_+}\tfrac{r}{R_{\mathrm{AdS}}} \;=\; 1 - \sum_{k=1}^{\infty} c_k(m,r)\, R_{\mathrm{AdS}}^{-k}\;.$$

Using (C.27) we can write for $d = 3$

$$S^a_{\omega l}(\rho) \approx \Big(\frac{r}{R_{\mathrm{AdS}}}\Big)^l \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_+{+}k)\,\Gamma(\beta_+{+}k)}{\Gamma(\alpha_+)\,\Gamma(\beta_+)}\;\frac{\Gamma(\gamma^{\mathrm{S}})}{\Gamma(\gamma^{\mathrm{S}}{+}k)}\;\frac{(r/R_{\mathrm{AdS}})^{2k}}{k!}$$

$$S^b_{\omega l}(\rho) \approx -\Big(\frac{r}{R_{\mathrm{AdS}}}\Big)^{2-d-l} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_+{-}\gamma^{\mathrm{S}}{+}1{+}k)\,\Gamma(\beta_+{-}\gamma^{\mathrm{S}}{+}1{+}k)}{\Gamma(\alpha_+{-}\gamma^{\mathrm{S}}{+}1)\,\Gamma(\beta_+{-}\gamma^{\mathrm{S}}{+}1)}\;\frac{\Gamma(2{-}\gamma^{\mathrm{S}})}{\Gamma(2{-}\gamma^{\mathrm{S}}{+}k)}\;\frac{(r/R_{\mathrm{AdS}})^{2k}}{k!}$$

We have

$$\frac{\Gamma(\gamma^{\mathrm{S}})}{\Gamma(\gamma^{\mathrm{S}}{+}k)} \;=\; 2^k\,\frac{(2l{+}1)!!}{(2l{+}2k{+}1)!!} \qquad \frac{\Gamma(2{-}\gamma^{\mathrm{S}})}{\Gamma(2{-}\gamma^{\mathrm{S}}{+}k)} \;=\; 2^k\Big/\prod_{j=1}^{k}(2j{-}2l{-}1)\;.$$

With each k, l being small compared to R_{AdS} in the flat limit, we get (note that all expressions are real numbers)

$$\frac{\Gamma(\alpha_+{+}k)\,\Gamma(\beta_+{+}k)}{\Gamma(\alpha_+)\,\Gamma(\beta_+)} \;\xrightarrow[\text{lim.}]{\text{flat}}\; (-1)^k\,2^{-2k}R_{\mathrm{AdS}}^{2k}\big(\widetilde{\omega}^2{-}m^2\big)^k \;=\; (-1)^k\,2^{-2k}R_{\mathrm{AdS}}^{2k}\begin{cases}(\tilde{p}^{\mathbb{R}}_{\tilde{\omega}})^{2k} & \widetilde{\omega}^2 \ge m^2\\ (\mathrm{i}\tilde{p}^{\mathbb{R}}_{\tilde{\omega}})^{2k} & \widetilde{\omega}^2 < m^2\end{cases}$$

$$\frac{\Gamma(\alpha_+{-}\gamma^{\mathrm{S}}{+}1{+}k)\,\Gamma(\beta_+{-}\gamma^{\mathrm{S}}{+}1{+}k)}{\Gamma(\alpha_+{-}\gamma^{\mathrm{S}}{+}1)\,\Gamma(\beta_+{-}\gamma^{\mathrm{S}}{+}1)} \;\xrightarrow[\text{lim.}]{\text{flat}}\; (-1)^k\,2^{-2k}R_{\mathrm{AdS}}^{2k}\big(\widetilde{\omega}^2{-}m^2\big)^k \;=\; (-1)^k\,2^{-2k}R_{\mathrm{AdS}}^{2k}\begin{cases}(\tilde{p}^{\mathbb{R}}_{\tilde{\omega}})^{2k} & \widetilde{\omega}^2 \ge m^2\\ (\mathrm{i}\tilde{p}^{\mathbb{R}}_{\tilde{\omega}})^{2k} & \widetilde{\omega}^2 < m^2\end{cases}.$$

Now we can put these parts together and compare with the power series DLMF [10.53.1,2] of the spherical Bessel and Neumann functions (with the usual convention $(-1)!! = 1$)

$$j_l(z) \;=\; z^l \sum_{k=0}^{\infty} \frac{(-1)^k\,2^{-k}\,z^{2k}}{k!\,(2l{+}2k{+}1)!!} \tag{C.84}$$

$$n_l(z) \;=\; \frac{-1}{z^{l+1}}\sum_{k=0}^{l}\frac{(2l{-}2k{-}1)!!}{k!\,2^k}\,z^{2k} - \frac{(-1)^l}{z^{l+1}}\sum_{k=l+1}^{\infty}\frac{(-1)^k\,2^{-k}}{k!\,(2k{-}2l{-}1)!!}\,z^{2k} \tag{C.85}$$

$$= -\,\frac{(2l{-}1)!!}{z^{l+1}}\sum_{k=0}^{\infty}\frac{(-1)^k\,z^{2k}}{2^k\,k!}\prod_{j=1}^{k}(2j{-}2l{-}1)^{-1}\;. \tag{C.86}$$

This shows that for $d = 3$ we have the flat limits

$$\tfrac{(p^{\mathbb{R}}_{\omega})^l}{(2l+d-2)!!}\,S^a_{\omega l}(\rho) \;\xrightarrow[\text{lim.}]{\text{flat}}\; \check{\jmath}_{\tilde{\omega}l}(r) \qquad \tfrac{(2l+d-4)!!}{(p^{\mathbb{R}}_{\omega})^{l+d-2}}\,S^b_{\omega l}(\rho) \;\xrightarrow[\text{lim.}]{\text{flat}}\; \check{n}_{\tilde{\omega}l}(r)\;. \tag{C.87}$$

Since $J^{(\pm)}_{nl}(\rho)$ is a special case of $S^a_{\omega l}$ we find the following flat limit for $d = 3$

$$\tfrac{(p^{\mathbb{R}}_{\omega})^l}{(2l+d-2)!!}\,J^{(+)}_{nl}(\rho) \;\xrightarrow[\text{lim.}]{\text{flat}}\; j_l(\tilde{p}_{\tilde{\omega}} r)\;. \tag{C.88}$$

C.2.8 Flat limits of the field expansions

In order to compare the field expansions with the corresponding ones in Minkowski spacetime, we need to switch to the AdS (τ, r, Ω) coordinates and parameter $\tilde{\omega}$, which correspond to the Minkowski coordinates (t, r, Ω) and energy E, see Section C.1.1. We shall label momentum representations referring to the "tilded" parameters like $\tilde{\omega}$ by a tilde as in $\tilde{\phi}_{\tilde{\omega}}$. For brevity we only consider real fields, with the complex case being completely analoguous. From the two field expansions (with f only abbreviating symbolically the true dependencies)

$$\begin{aligned} \phi(t,\rho,\Omega) \;&=\; \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi_{\omega\underline{l}m_l}\, f(t,\rho,\Omega,\omega,\underline{l},m_l) + \text{ c.c. } \Big\} \\ = \phi(\tau/R_{\text{AdS}}, r/R_{\text{AdS}},\Omega) \;&=\; \int\!\mathrm{d}\tilde{\omega} \sum_{\underline{l},m_l} \Big\{ \tilde{\phi}_{\tilde{\omega}\underline{l}m_l}\, f(\tau/R_{\text{AdS}},\, r/R_{\text{AdS}},\Omega,\tilde{\omega}R_{\text{AdS}},\underline{l},m_l) + \text{ c.c. } \Big\} \end{aligned}$$

we can deduce

$$\tilde{\phi}_{\tilde{\omega}\underline{l}m_l} \;=\; R_{\text{AdS}}\phi_{\omega\underline{l}m_l} \;=\; R_{\text{AdS}}\phi_{\tilde{\omega}R_{\text{AdS}},\underline{l},m_l}\;. \tag{C.89}$$

This is consistent with the reconstruction formula for the momentum representation on an equal-time plane

$$\begin{aligned} \phi_{\omega\underline{l}m_l} \;&=\; \int\!\mathrm{d}\rho\,\mathrm{d}^{d-1}\Omega\; f'(t,\rho,\Omega,\omega,\underline{l},m_l) \\ \tilde{\phi}_{\tilde{\omega},l,\underline{m}} \;&=\; \int\!\mathrm{d}r\,\mathrm{d}^{d-1}\Omega\; f'(\tau/R_{\text{AdS}},\, r/R_{\text{AdS}},\Omega,\tilde{\omega}R_{\text{AdS}},\underline{l},m_l) \end{aligned}$$

respectively equal-radius hypercylinder

$$\begin{aligned} \phi_{\omega\underline{l}m_l} \;&=\; \int\!\mathrm{d}t\,\mathrm{d}^{d-1}\Omega\; f''(t,\rho,\Omega,\omega,\underline{l},m_l) \\ \tilde{\phi}_{\tilde{\omega},l,\underline{m}} \;&=\; \int\!\mathrm{d}\tau\,\mathrm{d}^{d-1}\Omega\; f''(\tau/R_{\text{AdS}},\, r/R_{\text{AdS}},\Omega,\tilde{\omega}R_{\text{AdS}},\underline{l},m_l)\;. \end{aligned}$$

First we study the flat limit of our field expansion (2.201) on the equal-time hypersurface. The functions $J^{(-)}_{nl}(\rho)$ are allowed for negative masses only, which we do not consider for the Klein-Gordon field on Minkowski spacetime. Hence we shall only deal with the case of $\tilde{m} = \tilde{m}_+$ and the corresponding functions $J^{(+)}_{nl}(\rho)$. We can write the summation over n as a sum over ω^+_{nl} (with l fixed), which in the flat limit becomes an integral over $\tilde{\omega}$ (abbreviating $R = R_{\text{AdS}}$, and with $\tilde{\omega}_{\tilde{p}} = \sqrt{\tilde{p}^2 + m^2}$):

$$\begin{aligned} \sum_{n=0}^{\infty} f(\omega^+_{nl}) \;&=\; \tfrac{1}{2}\,\Delta\omega \sum_{\omega=\omega^+_{0l}}^{\Delta\omega=2} f(\omega) \;=\; \tfrac{R}{2}\,\frac{\Delta\omega}{R} \sum_{\omega/R=\omega^+_{0l}/R}^{\Delta\omega/R=2/R} f(R\omega/R) \;=\; \tfrac{R}{2}\,\Delta\tilde{\omega} \sum_{\tilde{\omega}=\omega^+_{0l}/R}^{\Delta\tilde{\omega}=2/R} f(R\tilde{\omega}) \\ &\xrightarrow[\text{lim.}]{\text{flat}}\; \tfrac{R}{2}\int_m^{\infty}\!\mathrm{d}\tilde{\omega}\; f(R\tilde{\omega}) \;=\; \tfrac{R}{2}\int_0^{\infty}\!\mathrm{d}\tilde{p}\;\frac{\tilde{p}}{\tilde{\omega}_{\tilde{p}}}\, f(R\tilde{\omega}_{\tilde{p}})\;. \end{aligned} \tag{C.90}$$

Therein we proceed as in the limiting process from a sum towards the Riemann integral: $\Delta\omega^+_{nl}$ times $f(\omega^+_{nl})$ is the area of the rectangle, whose width $\Delta\omega^+_{nl}$ becomes infinitesimal in the limit of large R_{AdS}. We can rescale the momentum representation

$$\phi^{\pm}_{n\underline{l}m_l} \;=\; \phi^{\pm}_{\omega^+_{nl}\underline{l}m_l} \;=\; \phi^{\mathrm{F},\pm}_{n\underline{l}m_l}\,\frac{4\tilde{\omega}^+_{nl}}{\sqrt{2\pi}}\,\frac{(p^{\mathbb{R}}_{nl})^l}{(2l+d-2)!!} \tag{C.91}$$

$$\phi^{\mathrm{F},\pm}_{n\underline{l}m_l} \;=\; R^{-1}_{\text{AdS}}\,\tilde{\phi}^{\mathrm{F},\pm}_{\tilde{p}\underline{l}m_l} \qquad \text{with here } \tilde{p} \text{ short for } \tilde{p}_{nl}, \tag{C.92}$$

wherein we call $\phi^{\mathrm{F},\pm}_{n\underline{l}m_l}$ the flat Jacobi representation. Plugging the flat Jacobi representation into the Jacobi expansion (2.201), we get what we call the flat Jacobi expansion. Using (C.88), for $d=3$ it is then straightforward to check that the flat limit of the flat expansion becomes the expansion of a solution near a Minkowski equal-time plane:

$$\phi(t,r,\Omega) \xrightarrow[\text{lim.}]{\text{flat}} \int_0^\infty \mathrm{d}\tilde{p} \sum_{l,m_l} 2\tilde{p}\,(2\pi)^{-1/2}\, j_l(\tilde{p}r) \Big\{ \tilde{\phi}^{\mathrm{F},+}_{\tilde{p}lm_l}\, \mathrm{e}^{-\mathrm{i}\tilde{\omega}_{\tilde{p}}\tau}\, Y_l^{m_l}(\Omega) + \overline{\tilde{\phi}^{\mathrm{F},-}_{\tilde{p}lm_l}}\, \mathrm{e}^{\mathrm{i}\tilde{\omega}_{\tilde{p}}\tau}\, \overline{Y_l^{m_l}(\Omega)} \Big\}\,. \tag{C.93}$$

For the tube region, with definitions (2.102) and rescaling the momentum representation

$$\begin{aligned} \phi^{S,a}_{\omega\underline{l}\,m_l} &= \phi^{\mathrm{F},a}_{\tilde{\omega}\underline{l}m_l}\, \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\, \frac{(p^{\mathbb{R}}_{\omega})^l}{(2l+d-2)!!} &= R^{-1}_{\mathrm{AdS}}\tilde{\phi}^{\mathrm{F},a}_{\tilde{\omega}\underline{l}m_l}\, \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\, \frac{(p^{\mathbb{R}}_{\omega})^l}{(2l+d-2)!!} \\ \phi^{S,b}_{\omega\underline{l}\,m_l} &= \phi^{\mathrm{F},b}_{\tilde{\omega}\underline{l}m_l}\, \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\, \frac{(2l+d-4)!!}{(p^{\mathbb{R}}_{\omega})^{l+d-2}} &= R^{-1}_{\mathrm{AdS}}\tilde{\phi}^{\mathrm{F},b}_{\tilde{\omega}\underline{l}m_l}\, \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\, \frac{(2l+d-4)!!}{(p^{\mathbb{R}}_{\omega})^{l+d-2}}\,. \end{aligned} \tag{C.94}$$

We call $\phi^{\mathrm{F},a/b}_{\tilde{\omega}\underline{l}m_l}$ the flat S-representation. Plugging it into the S-expansion (2.186), we obtain what we call the flat S-expansion. With (C.87) we get the following flat limit for $d=3$ of the flat S-expansion:

$$\phi(t,r,\Omega) \xrightarrow[\text{lim.}]{\text{flat}} \int \mathrm{d}\tilde{\omega} \sum_{l,m_l} \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{4\pi} \Big\{ \tilde{\phi}^{\mathrm{F},a}_{\tilde{\omega}lm_l} \mathrm{e}^{-\mathrm{i}\tilde{\omega}\tau} Y_l^{m_l}(\Omega)\, \check{\jmath}_{\tilde{\omega}l}(r) + \tilde{\phi}^{\mathrm{F},b}_{\tilde{\omega}lm_l} \mathrm{e}^{-\mathrm{i}\tilde{\omega}\tau} Y_l^{m_l}(\Omega)\, \check{n}_{\tilde{\omega}l}(r) \Big\}\,. \tag{C.95}$$

The discrete (respectively continuous) flat limit above fixes how we map solutions on AdS to solutions on Minkowski spacetime for neigborhoods of equal-time hypersurfaces (respectively hypercylinders).

As a consistency check, let us verify that the following diagram commutes for *global* Klein-Gordon solutions on AdS. Therein, we start at the top left corner with a global solution $\phi(t,\rho,\Omega)$ written as a solution in a neighborhood of an AdS hypercylinder. It is characterized by its S-representation $\phi^{S,a/b}_{\omega lm_l}$.

$$\begin{array}{ccc} \phi^{S,a/b}_{\omega lm_l} & \xrightarrow{\text{cont. flat lim.}} & \tilde{\phi}^{a/b}_{\tilde{\omega}lm_l} \\ {\scriptstyle \delta_{n\omega}}\big\downarrow & & \big\downarrow{\scriptstyle \delta_{n\tilde{\omega}}} \\ \phi^{\pm}_{nlm_l} & \xrightarrow{\text{disc. flat lim.}} & \tilde{\phi}^{\pm}_{\tilde{p}lm_l} \end{array}$$

Since the solution is global we have $\phi^{S,b}_{\omega lm_l} \equiv 0$ together with

$$\phi^{S,a}_{\omega lm_l} = \sum_{n=0}^{\infty} \Big(\delta(\omega-\omega^+_{nl})\, \phi^+_{nlm_l} + \delta(\omega+\omega^+_{nl})\, \overline{\phi^-_{n,l,-m_l}} \Big). \tag{C.96}$$

From this general solution we can obtain a solution well-defined near an AdS equal-time hypersurface, that is: a global solution, by integrating over ω, resulting in the Jacobi expansion (2.201). In the diagram this process is represented by the shorthand $\delta_{n\omega}$, and the global AdS solution is characterized by its Jacobi representation $\phi^{\pm}_{nlm_l}$. As shown above, we can then take a discrete flat limit of this global AdS solution which results in a solution well-defined near a Minkowski equal-time plane, represented in the diagram by $\tilde{\phi}^{\pm}_{\tilde{p}lm_l}$. The consistency check consists in taking first the continuous flat limit, turning $\delta(\omega-\omega^+_{nl})$ into $\delta(\tilde{\omega}-\tilde{\omega}^+_{nl})/R$, then integrating over $\tilde{\omega}$ (represented by $\delta_{n\tilde{\omega}}$ in the diagram) and converting the sum over n into an integral over $\tilde{p}$ as in (C.90). Unsurprisingly, this first-right-then-down way yields the same result as the first-down-then-right way. We have thus verified that for global AdS solutions the discrete and continuous flat limit are mutually consistent.

C.3 Action of isometries on solution space

In this section we consider the action of (a representation of) elements of the isometry group $\mathrm{SO}\,(2,d)$ of $\mathrm{AdS}_{1,d}$ on points of $\mathrm{AdS}_{1,d}$. We can do this for finite and infinitesimal transformations. We *do not use* Einstein's sum convention in this section, but write summations explicitly. Again we use uppercase Latin indices to range as in $A = 0, 1, ..., d, (d{+}1)$, and lowercase Latin indices to range as in $k = 1, ..., d$. $\mathrm{SO}\,(2,d)$ is a Lie group, and we coordinatize its representation by the coordinates λ^{AB}. (We think of AB as *one* index, not two.) The generators K_{AB} of the Lie algebra $\mathbf{so}\,(2,d)$ are the Killing vectors K_{AB} of Section C.1.2:

$$K_{AB} \;=\; \big(X_A\,\partial_B - X_B\,\partial_A\big)\;. \tag{C.97}$$

This choice is the same as in (4.18) in [26] up to a factor of $\pm\mathrm{i}$. The (representation of a) group element $g(\lambda)$ (that is continuously connected to the identity) with coordinates λ^{AB} can then be written as

$$g(\lambda) \;=\; \exp\Big(\sum_{AB}\lambda^{AB}\,K_{AB}\Big)\;. \tag{C.98}$$

An infinitesimal action in direction of K_{AB} arises through $\lambda^{AB} = \varepsilon \ll 1$ with $\lambda^{CD} = 0$ for all $C \neq A$ and $D \neq B$. It is represented by the Taylor series of (C.99) up to linear order, that is

$$g(\lambda^{AB}{=}\varepsilon) \;=\; \mathbb{1} + \varepsilon\,K_{AB}\;. \tag{C.99}$$

Because of our choice (C.97), the transformations have very natural meanings: $g(\lambda^{d+1,0} = \varepsilon)$ is an infinitesimal time translation $t \to t + \varepsilon$, and $g(\lambda^{jk} = \varepsilon)$ is an infinitesimal rotation of angle ε in the (ξ_j, ξ_k)-plane, see also the end of Section B.1.2. The generators can be derived from the finite transformations by

$$K_{AB} \;=\; \frac{\partial}{\partial\lambda^{AB}}\bigg|_{\lambda=0} g(\lambda)\;. \tag{C.100}$$

The Lie algebra $\mathbf{so}\,(2,d)$ is determined by the Lie bracket (B.4), which coincides with (4.21) in [26] up to an overall sign:

$$[K_{AB},\, K_{CD}] \;=\; -\eta_{AC}\,K_{BD} + \eta_{BC}\,K_{AD} - \eta_{BD}\,K_{AC} + \eta_{AD}\,K_{BC}\;. \tag{C.101}$$

Note that the algebra writes like that independently of the overall sign of the metric. For the various combinations of time translation, rotations and boosts this Lie bracket writes as

$$\begin{aligned}
&[K_{d+1,0},\, K_{jk}] = 0 && [K_{0j},\, K_{0k}] = \eta_{00}K_{kj} && [K_{0q},\, K_{jk}] = \eta_{jq}K_{0k} - \eta_{kq}K_{0j}\\
&[K_{d+1,j},\, K_{d+1,k}] = \eta_{d+1,d+1}K_{kj} && [K_{d+1,0},\, K_{0k}] = \eta_{00}K_{d+1,k} && [K_{d+1,q},\, K_{jk}] = \eta_{jq}K_{d+1,k} - \eta_{kq}K_{d+1,j}\\
&[K_{d+1,k},\, K_{d+1,0}] = \eta_{d+1,d+1}K_{0k} && [K_{0k},\, K_{d+1,j}] = \eta_{jk}K_{d+1,0} && [K_{jk},\, K_{pq}] = \eta_{kp}K_{jq} - \eta_{jp}K_{kq}\\
&&&&& \qquad + \eta_{jq}K_{kp} - \eta_{kq}K_{jp}.
\end{aligned} \tag{C.102}$$

It is a bit lengthy but straightforward to verify these relations both in the cartesian X-coordinates of embedding space $\mathbb{R}^{(2,d)}$ and in the orispherical $\mathrm{AdS}_{1,d}$-coordinates $(t, \rho, \underline{\xi}(\Omega))$. The infinitesimal action of a generator on the coordinates is simply

$$(g(\lambda^{AB} = \varepsilon)\,x)^\mu \;=\; x^\mu + \varepsilon(K_{AB})^\mu\;. \tag{C.103}$$

A finite action generically has a more involved expression. The inverse transformation is given by the negative of the generator:

$$(g(\lambda^{AB} = \varepsilon)^{-1}\,x)^\mu \;=\; x^\mu - \varepsilon(K_{AB})^\mu\;. \tag{C.104}$$

We denote the (infinitesimal) action of a generator K_{AB} respectively group element $g(\lambda)$ on a field $\phi(x)$ by

$$(\mathbb{1} + \varepsilon K_{AB}) \triangleright \phi \qquad\qquad g(\lambda) \triangleright \phi \; .$$

Requiring the transformed field at transformed coordinates to agree with the original field at original coordinates as in

$$\Big((\mathbb{1}+\varepsilon K_{AB}) \triangleright \phi\Big)((\mathbb{1}+\varepsilon K_{AB})x) \;\stackrel{!}{=}\; \phi(x) \qquad\qquad (g(\lambda) \triangleright \phi)(g(\lambda)\, x) \;\stackrel{!}{=}\; \phi(x) \;, \tag{C.105}$$

we get for the transformed field at the original coordinates:

$$\Big((\mathbb{1}+\varepsilon K_{AB}) \triangleright \phi\Big)(x) \;=\; \phi((\mathbb{1}-\varepsilon K_{AB})x) \qquad\qquad (g(\lambda) \triangleright \phi)(x) \;=\; \phi(g(\lambda)^{-1} x) \; . \tag{C.106}$$

This is the usual definition for the action of $\mathrm{SO}\,(d,2)$-elements on functions, see e.g. relation (2) on page 40 in Vilenkin and Klimyk's [72]. In the following sections we calculate these actions for the time translation, rotations and boosts for Klein-Gordon solutions on the time-interval and tube regions. Our goal will always be to transcribe the action from the coordinate representation to the momentum representation. That is, we start from a field expansion over modes $\mu^{(S,a,b)}_{\omega \underline{l} m_l}(x)$ of momentum $(\omega, \underline{l}, m_l)$ like

$$(g(\lambda) \triangleright \phi)(x) \;=\; \phi(g(\lambda)^{-1}x) \;=\; \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{\omega \underline{l} m_l}\, \mu^{(S,a)}_{\omega \underline{l} m_l}(g(\lambda)^{-1}x) + \phi^{S,b}_{\omega \underline{l} m_l}\, \mu^{(S,b)}_{\omega \underline{l} m_l}(g(\lambda)^{-1}x) \Big\} \;, \tag{C.107}$$

wherein $\phi^{S,a}_{\omega \underline{l} m_l}$ and $\phi^{S,a}_{\omega \underline{l} m_l}$ are the momentum representation of the Klein-Gordon solution. What we want to find is an explicit expression for the momentum representations $(g(\lambda) \triangleright \phi)^{S,a,b}_{\omega \underline{l} m_l}$ of the transformed field, such that we can directly write the transformed field in the original coordinates:

$$(g(\lambda) \triangleright \phi)(x) \;=\; \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ (g(\lambda) \triangleright \phi)^{S,a}_{\omega \underline{l} m_l}\, \mu^{(S,a)}_{\omega \underline{l} m_l}(x) + (g(\lambda) \triangleright \phi)^{S,b}_{\omega \underline{l} m_l}\, \mu^{(S,b)}_{\omega \underline{l} m_l}(x) \Big\} \; . \tag{C.108}$$

One motivation for this is that we want to plug the momentum representations of the transformed fields into (the momentum representation of) the symplectic form, in order to show that it remains invariant under the action of isometries.

C.3.1 Time translations' action on AdS solutions

AdS time-interval region

The action of the infinitesimal time translation $K_{d+1,0} \;=\; \partial_t$ on the Jacobi modes

$$\begin{aligned} \mu^{(+)}_{n \underline{l} m_l}(t,\rho,\Omega) \;&=\; \mathrm{e}^{-\mathrm{i}\omega^{+}_{nl} t}\; Y^{m_l}_{\underline{l}}(\Omega)\; J^{(+)}_{nl}(\rho) \\ \mu^{(-)}_{n \underline{l} m_l}(t,\rho,\Omega) \;&=\; \mathrm{e}^{-\mathrm{i}\omega^{-}_{nl} t}\; Y^{m_l}_{\underline{l}}(\Omega)\; J^{(-)}_{nl}(\rho) \end{aligned} \tag{C.109}$$

according to (C.106) is simply

$$\begin{aligned} \Big(K_{d+1,0} \triangleright \mu^{(\pm)}_{n \underline{l} m_l}\Big)(t,\rho,\Omega) \;&=\; +\mathrm{i}\omega^{\pm}_{nl}\; \mu^{(\pm)}_{n \underline{l} m_l}(t,\rho,\Omega) \\ \Big(K_{d+1,0} \triangleright \overline{\mu^{(\pm)}_{n \underline{l} m_l}}\Big)(t,\rho,\Omega) \;&=\; -\mathrm{i}\omega^{\pm}_{nl}\; \overline{\mu^{(\pm)}_{n \underline{l} m_l}(t,\rho,\Omega)} \; . \end{aligned} \tag{C.110}$$

Applying this to expansion (2.201) we obtain

$$\Big(K_{d+1,0} \triangleright \phi\Big)(t,\rho,\Omega) \;=\; \sum_{n \underline{l} m_l} \Big\{ \phi^{+}_{n \underline{l} m_l}\, (\mathrm{i}\omega^{\pm}_{nl})\, \mu^{(\pm)}_{n \underline{l} m_l}(t,\rho,\Omega) + \overline{\phi^{-}_{n \underline{l} m_l}}\, (-\mathrm{i}\omega^{\pm}_{nl})\, \overline{\mu^{(\pm)}_{n \underline{l} m_l}(t,\rho,\Omega)} \Big\} \;, \tag{C.111}$$

from which we can directly read off the corresponding actions for infinitesimal time translations in the momentum representation:

$$\Big(K_{d+1,0} \triangleright \phi\Big)^{+}_{n\underline{l}m_l} \;=\; (+\mathrm{i}\omega^{\pm}_{nl})\;\phi^{+}_{n\underline{l}m_l} \tag{C.112}$$

$$\overline{\Big(\mathrm{i}K_{d+1,0}) \triangleright \phi\Big)^{-}_{n\underline{l}m_l}} \;=\; (-\mathrm{i}\omega^{\pm}_{nl})\;\overline{\phi^{-}_{n\underline{l}m_l}}\,. \tag{C.113}$$

It is also straightforward to calculate the action of a finite time translation, denoting it by $k_{\Delta t}$: $t \;\rightarrow\; t+\Delta t$:

$$\begin{aligned}\big(k_{\Delta t} \triangleright \mu^{(\pm)}_{n\underline{l}m_l}\big)(t,\rho,\Omega) \;&=\; \mathrm{e}^{\mathrm{i}\omega^{\pm}_{nl}\Delta t}\;\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)\\ \big(k_{\Delta t} \triangleright \overline{\mu^{(\pm)}_{n\underline{l}m_l}}\big)(t,\rho,\Omega) \;&=\; \mathrm{e}^{-\mathrm{i}\omega^{\pm}_{nl}\Delta t}\;\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\,.\end{aligned} \tag{C.114}$$

Applying this to expansion (2.201) we obtain

$$\big(k_{\Delta t} \triangleright \phi\big)(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l}\Big\{\phi^{+}_{n\underline{l}m_l}\;\mathrm{e}^{\mathrm{i}\omega^{\pm}_{nl}\Delta t}\;\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)+\overline{\phi^{-}_{n\underline{l}m_l}}\;\mathrm{e}^{-\mathrm{i}\omega^{\pm}_{nl}\Delta t}\;\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\}\,, \tag{C.115}$$

from which we can directly read off the corresponding actions for finite time translations in the momentum representation:

$$\big(k_{\Delta t} \triangleright \phi\big)^{+}_{n\underline{l}m_l} \;=\; \mathrm{e}^{\mathrm{i}\omega^{\pm}_{nl}\Delta t}\;\phi^{+}_{n\underline{l}m_l} \qquad\qquad \overline{\big(k_{\Delta t} \triangleright \phi\big)^{-}_{n\underline{l}m_l}} \;=\; \mathrm{e}^{-\mathrm{i}\omega^{\pm}_{nl}\Delta t}\;\overline{\phi^{-}_{n\underline{l}m_l}}\,. \tag{C.116}$$

Comparing the respective formulas for infinitesimal and finite time translation, we see that the former arises nicely from the latter for $\Delta t = \varepsilon \ll 1$. We can use the momentum representation to write the transformed field using the original coordinates as

$$\big(k_{\Delta t} \triangleright \phi\big)(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l}\Big\{\big(k_{\Delta t} \triangleright \phi\big)^{+}_{n\underline{l}m_l}\;\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)+\overline{\big(k_{\Delta t} \triangleright \phi\big)^{-}_{n\underline{l}m_l}}\;\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\}\,. \tag{C.117}$$

AdS tube region

The action of the time translation $K_{0,d+1} \;=\; \partial_t$ on the hypergeometric S^a and S^b-modes

$$\begin{aligned}\mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \;&=\; \mathrm{e}^{-\mathrm{i}\omega t}\;Y^{m_l}_{\underline{l}}(\Omega)\;S^{a}_{\omega l}(\rho)\\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \;&=\; \mathrm{e}^{-\mathrm{i}\omega t}\;Y^{m_l}_{\underline{l}}(\Omega)\;S^{b}_{\omega l}(\rho)\end{aligned} \tag{C.118}$$

is basically the same as for the time-interval region. Since the infinitesimal action arises from the finite one through $\Delta t = \varepsilon \ll 1$, we only give the finite action here. Denoting it again by $k_{\Delta t}$: $t \;\rightarrow\; t+\Delta t$, the action is given by

$$\begin{aligned}\Big(k_{\Delta t} \triangleright \mu^{(S,a)}_{\omega\underline{l}m_l}\Big)(t,\rho,\Omega) \;&=\; \mathrm{e}^{\mathrm{i}\omega\Delta t}\;\mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \qquad \Big(k_{\Delta t} \triangleright \mu^{(S,a)}_{-\omega,\underline{l},-m_l}\Big)(t,\rho,\Omega) \;=\; \mathrm{e}^{-\mathrm{i}\omega\Delta t}\;\mu^{(S,a)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega)\\ \Big(k_{\Delta t} \triangleright \mu^{(S,b)}_{\omega\underline{l}m_l}\Big)(t,\rho,\Omega) \;&=\; \mathrm{e}^{\mathrm{i}\omega\Delta t}\;\mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \qquad \Big(k_{\Delta t} \triangleright \mu^{(S,b)}_{-\omega,\underline{l},-m_l}\Big)(t,\rho,\Omega) \;=\; \mathrm{e}^{-\mathrm{i}\omega\Delta t}\;\mu^{(S,b)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega)\,.\end{aligned}$$

Applying this to expansion (2.186) we obtain

$$\big(k_{\Delta t} \triangleright \phi\big)(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\phi^{S,a}_{\omega\underline{l}m_l}\;\mathrm{e}^{\mathrm{i}\omega\Delta t}\;\mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega)+\phi^{S,b}_{\omega\underline{l}m_l}\;\mathrm{e}^{\mathrm{i}\omega\Delta t}\;\mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega)\Big\}\,, \tag{C.119}$$

from which we can read off the corresponding actions for finite time translations in the momentum representation:

$$\begin{aligned}\big(k_{\Delta t} \triangleright \phi\big)^{S,a}_{\omega\underline{l}m_l} \;&=\; \mathrm{e}^{\mathrm{i}\omega\Delta t}\;\phi^{S,a}_{\omega\underline{l}m_l} \qquad\qquad \big(k_{\Delta t} \triangleright \phi\big)^{S,a}_{-\omega,\underline{l},-m_l} \;=\; \mathrm{e}^{-\mathrm{i}\omega\Delta t}\;\phi^{S,a}_{-\omega,\underline{l},-m_l}\\ \big(k_{\Delta t} \triangleright \phi\big)^{S,b}_{\omega\underline{l}m_l} \;&=\; \mathrm{e}^{\mathrm{i}\omega\Delta t}\;\phi^{S,b}_{\omega\underline{l}m_l} \qquad\qquad \big(k_{\Delta t} \triangleright \phi\big)^{S,b}_{-\omega,\underline{l},-m_l} \;=\; \mathrm{e}^{-\mathrm{i}\omega\Delta t}\;\phi^{S,b}_{-\omega,\underline{l},-m_l}\,.\end{aligned} \tag{C.120}$$

The action of the time translation $K_{0,d+1} = \partial_t$ on the hypergeometric C-modes

$$\begin{aligned} \mu^{(C,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) &= \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, C^a_{\omega l}(\rho) \\ \mu^{(C,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) &= \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, C^b_{\omega l}(\rho) \end{aligned} \tag{C.121}$$

is completely analoguous to the action on the S-modes:

$$\begin{aligned} \Big(k_{\Delta t} \triangleright \mu^{(C,a)}_{\omega\underline{l}m_l}\Big)(t,\rho,\Omega) &= \mathrm{e}^{\mathrm{i}\omega\Delta t}\, \mu^{(C,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \qquad \Big(k_{\Delta t} \triangleright \mu^{(C,a)}_{-\omega,\underline{l},-m_l}\Big)(t,\rho,\Omega) = \mathrm{e}^{-\mathrm{i}\omega\Delta t}\, \mu^{(C,a)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) \\ \Big(k_{\Delta t} \triangleright \mu^{(C,b)}_{\omega\underline{l}m_l}\Big)(t,\rho,\Omega) &= \mathrm{e}^{\mathrm{i}\omega\Delta t}\, \mu^{(C,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \qquad \Big(k_{\Delta t} \triangleright \mu^{(C,b)}_{-\omega,\underline{l},-m_l}\Big)(t,\rho,\Omega) = \mathrm{e}^{-\mathrm{i}\omega\Delta t}\, \mu^{(C,b)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega)\,. \end{aligned}$$

Thus the actions for finite time translations in the momentum representation write:

$$\begin{aligned} \big(k_{\Delta t} \triangleright \phi\big)^{C,a}_{\omega\underline{l}m_l} &= \mathrm{e}^{\mathrm{i}\omega\Delta t}\, \phi^{C,a}_{\omega\underline{l}m_l} \qquad \big(k_{\Delta t} \triangleright \phi\big)^{C,a}_{-\omega,\underline{l},-m_l} = \mathrm{e}^{-\mathrm{i}\omega\Delta t}\, \phi^{C,a}_{-\omega,\underline{l},-m_l} \\ \big(k_{\Delta t} \triangleright \phi\big)^{C,b}_{\omega\underline{l}m_l} &= \mathrm{e}^{\mathrm{i}\omega\Delta t}\, \phi^{C,b}_{\omega\underline{l}m_l} \qquad \big(k_{\Delta t} \triangleright \phi\big)^{C,b}_{-\omega,\underline{l},-m_l} = \mathrm{e}^{-\mathrm{i}\omega\Delta t}\, \phi^{C,b}_{-\omega,\underline{l},-m_l}\,. \end{aligned} \tag{C.122}$$

We can use the momentum representation to write the transformed field using the original coordinates as

$$\begin{aligned} \big(k_{\Delta t} \triangleright \phi\big)(t,r,\Omega) &= \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \big(k_{\Delta t} \triangleright \phi\big)^{S,a}_{\omega\underline{l}m_l} \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \big(k_{\Delta t} \triangleright \phi\big)^{S,b}_{\omega\underline{l}m_l} \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\} \\ &= \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \big(k_{\Delta t} \triangleright \phi\big)^{C,a}_{\omega\underline{l}m_l} \mu^{(C,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \big(k_{\Delta t} \triangleright \phi\big)^{C,b}_{\omega\underline{l}m_l} \mu^{(C,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\}\,. \end{aligned}$$

C.3.2 Rotations' action on AdS solutions

Rotations: AdS time-interval region

In this section we consider the action of finite rotations on Klein-Gordon solutions on time-interval regions. Although in principle there is no problem in working out the explicit expressions for the rotation generators K_{jk} (departing from (A.9), and then deriving expressions like (B.14) and below) and letting the rotators act on the hyperspherical harmonics (A.22) (making use of relations like (A.24)), this would become quite involved. Therefore we shall be content with the finite rotations.

For these we need Wigner's D-matrix, which is introduced in Appendix A.4. Using the notation of that appendix, let $\hat{R}(\underline{\alpha})$ denote our finite rotation, with $\underline{\alpha}$ denoting the rotation angles. Starting again with expansion (2.201)

$$\phi(t,\rho,\Omega) = \sum_{n\underline{l}m_l} \Big\{ \phi^{+}_{n\underline{l}m_l}\, \mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\phi^{-}_{n\underline{l}m_l}}\, \overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)} \Big\}\,, \tag{C.123}$$

according to (C.106) we get directly

$$\begin{aligned} (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)(t,\rho,\Omega) = \phi(t,\rho,\hat{R}(\underline{\alpha})\,\Omega) &= \sum_{n\underline{l}m_l} \Big\{ \phi^{+}_{n\underline{l}m_l}\, \mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) + \overline{\phi^{-}_{n\underline{l}m_l}}\, \overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega)} \Big\} \\ (\hat{R}(\underline{\alpha}) \triangleright \phi)(t,\rho,\Omega) = \phi(t,\rho,\hat{R}(\underline{\alpha})^{-1}\,\Omega) &= \sum_{n\underline{l}m_l} \Big\{ \phi^{+}_{n\underline{l}m_l}\, \mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) + \overline{\phi^{-}_{n\underline{l}m_l}}\, \overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega)} \Big\}\,, \end{aligned}$$

From (A.39) and (A.40) respectively (A.41) and (A.42)

$$Y^{m_l}_{(l,\tilde{\underline{l}})}(\hat{R}(\underline{\alpha})\Omega) = \sum_{\tilde{\underline{l}}', m_l'} Y^{m_l'}_{(l,\tilde{\underline{l}}')}(\Omega) \, \overline{\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'}} \tag{C.124}$$

$$\overline{Y^{m_l}_{(l,\tilde{\underline{l}})}(\hat{R}(\underline{\alpha})\Omega)} = \sum_{\tilde{\underline{l}}', m_l'} \overline{Y^{m_l'}_{(l,\tilde{\underline{l}}')}(\Omega)} \, \left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \tag{C.125}$$

$$Y^{m_l}_{(l,\tilde{\underline{l}})}(\hat{R}(\underline{\alpha})^{-1}\Omega) = \sum_{\tilde{\underline{l}}', m_l'} Y^{m_l'}_{(l,\tilde{\underline{l}}')}(\Omega) \, \left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l} \tag{C.126}$$

$$\overline{Y^{m_l}_{(l,\tilde{\underline{l}})}(\hat{R}(\underline{\alpha})^{-1}\Omega)} = \sum_{\tilde{\underline{l}}', m_l'} \overline{Y^{m_l'}_{(l,\tilde{\underline{l}}')}(\Omega)} \, \overline{\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l}} \,, \tag{C.127}$$

we can easily deduce

$$\mu^{(\pm)}_{nl\,\tilde{\underline{l}}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) = \sum_{\tilde{\underline{l}}', m_l'} \mu^{(\pm)}_{n\,l\tilde{\underline{l}}'\,m_l'}(t,\rho,\Omega) \, \overline{\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'}} \tag{C.128}$$

$$\overline{\mu^{(\pm)}_{nl\,\tilde{\underline{l}}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega)} = \sum_{\tilde{\underline{l}}', m_l'} \overline{\mu^{(\pm)}_{n\,l\tilde{\underline{l}}'\,m_l'}(t,\rho,\Omega)} \, \left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \tag{C.129}$$

$$\mu^{(\pm)}_{nl\,\tilde{\underline{l}}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) = \sum_{\tilde{\underline{l}}', m_l'} \mu^{(\pm)}_{n\,l\tilde{\underline{l}}'\,m_l'}(t,\rho,\Omega) \, \left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l} \tag{C.130}$$

$$\overline{\mu^{(\pm)}_{nl\,\tilde{\underline{l}}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega)} = \sum_{\tilde{\underline{l}}', m_l'} \overline{\mu^{(\pm)}_{n\,l\tilde{\underline{l}}'\,m_l'}(t,\rho,\Omega)} \, \overline{\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l}} \,. \tag{C.131}$$

This implies that the action of a finite rotation in the momentum representation is given by (mind the position of the primes!)

$$\begin{aligned}
\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{+}_{nl\,\tilde{\underline{l}}\,m_l} &= \sum_{\tilde{\underline{l}}', m_l'} \phi^{+}_{nl\,\tilde{\underline{l}}'\,m_l'} \, \overline{\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l}} \\
\overline{\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{-}_{nl\,\tilde{\underline{l}}\,m_l}} &= \sum_{\tilde{\underline{l}}', m_l'} \overline{\phi^{-}_{nl\,\tilde{\underline{l}}'\,m_l'}} \, \left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l}
\end{aligned} \tag{C.132}$$

$$\begin{aligned}
\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{+}_{nl\,\tilde{\underline{l}}\,m_l} &= \sum_{\tilde{\underline{l}}', m_l'} \phi^{+}_{nl\,\tilde{\underline{l}}'\,m_l'} \, \left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \\
\overline{\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{-}_{nl\,\tilde{\underline{l}}\,m_l}} &= \sum_{\tilde{\underline{l}}', m_l'} \overline{\phi^{-}_{nl\,\tilde{\underline{l}}'\,m_l'}} \, \overline{\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'}} \,,
\end{aligned} \tag{C.133}$$

so that we can now write

$$\begin{aligned}
(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)(t,\rho,\Omega) &= \sum_{n\underline{l}m_l} \Big\{ \left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{+}_{n\underline{l}m_l} \mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{-}_{n\underline{l}m_l}} \, \overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)} \Big\} \\
(\hat{R}(\underline{\alpha}) \triangleright \phi)(t,\rho,\Omega) &= \sum_{n\underline{l}m_l} \Big\{ \left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{+}_{n\underline{l}m_l} \mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{-}_{n\underline{l}m_l}} \, \overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)} \Big\} \,.
\end{aligned} \tag{C.134}$$

Rotations: AdS tube region

Next we consider the action of finite rotations on Klein-Gordon solutions on tube regions. This is completely analoguous to the previous section. Again we need Wigner's D-matrix from Appendix A.4 and let $\hat{R}(\underline{\alpha})$ denote our finite rotation. Starting with expansion (2.186)

$$\phi(t,r,\Omega) = \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{\omega\underline{l}m_l}\ \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^{S,b}_{\omega\underline{l}m_l}\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\}\,, \tag{C.135}$$

we obtain according to (C.106)

$$(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)(t,\rho,\Omega) = \phi(t,\rho,\hat{R}(\underline{\alpha})\Omega) = \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{\omega\underline{l}m_l}\ \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) + \phi^{S,b}_{\omega\underline{l}m_l}\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) \Big\}$$

$$(\hat{R}(\underline{\alpha}) \triangleright \phi)(t,\rho,\Omega) = \phi(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) = \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{\omega\underline{l}m_l}\ \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) + \phi^{S,b}_{\omega\underline{l}m_l}\ \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) \Big\}.$$

From (A.38) respectively (A.41)

$$Y^{m_l}_{(l,\underline{\tilde{l}})}(\hat{R}(\underline{\alpha})\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} Y^{m_l'}_{(l,\underline{\tilde{l}}')}(\Omega)\ \overline{\left(D^l_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\right)_{m_l m_l'}}\,. \tag{C.136}$$

$$Y^{m_l}_{(l,\underline{\tilde{l}})}(\hat{R}(\underline{\alpha})^{-1}\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} Y^{m_l'}_{(l,\underline{\tilde{l}}')}(\Omega)\ \left(D^l_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\right)_{m_l' m_l} \tag{C.137}$$

we deduce

$$\mu^{(S,a)}_{\omega l\,\underline{\tilde{l}}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} \mu^{(S,a)}_{\omega l\,\underline{\tilde{l}}'\,m_l'}(t,\rho,\Omega)\ \overline{\left(D^l_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\right)_{m_l m_l'}} \tag{C.138}$$

$$\mu^{(S,b)}_{\omega l\,\underline{\tilde{l}}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} \mu^{(S,b)}_{\omega l\,\underline{\tilde{l}}'\,m_l'}(t,\rho,\Omega)\ \overline{\left(D^l_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\right)_{m_l m_l'}} \tag{C.139}$$

$$\mu^{(S,a)}_{\omega l\,\underline{\tilde{l}}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} \mu^{(S,a)}_{\omega l\,\underline{\tilde{l}}'\,m_l'}(t,\rho,\Omega)\ \left(D^l_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\right)_{m_l' m_l} \tag{C.140}$$

$$\mu^{(S,b)}_{\omega l\,\underline{\tilde{l}}\,m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} \mu^{(S,b)}_{\omega l\,\underline{\tilde{l}}'\,m_l'}(t,\rho,\Omega)\ \left(D^l_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\right)_{m_l' m_l}\,. \tag{C.141}$$

And

$$\begin{aligned} \overline{\mu^{(S,a)}_{\omega l\,\underline{\tilde{l}}\,m_l}(t,\rho,\Omega)} &= \mu^{(S,a)}_{-\omega,l,\underline{\tilde{l}},-m_l}(t,\rho,\Omega) \\ \overline{\mu^{(S,b)}_{\omega l\,\underline{\tilde{l}}\,m_l}(t,\rho,\Omega)} &= \mu^{(S,b)}_{-\omega,l,\underline{\tilde{l}},-m_l}(t,\rho,\Omega) \end{aligned} \tag{C.142}$$

leads to

$$\mu^{(S,a)}_{-\omega,l,\underline{\tilde{l}},-m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} \mu^{(S,a)}_{-\omega,l,\underline{\tilde{l}}',-m_l'}(t,\rho,\Omega)\ \left(D^l_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \tag{C.143}$$

$$\mu^{(S,b)}_{-\omega,l,\underline{\tilde{l}},-m_l}(t,\rho,\hat{R}(\underline{\alpha})\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} \mu^{(S,b)}_{-\omega,l,\underline{\tilde{l}}',-m_l'}(t,\rho,\Omega)\ \left(D^l_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \tag{C.144}$$

$$\mu^{(S,a)}_{-\omega,l,\underline{\tilde{l}},-m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} \mu^{(S,a)}_{-\omega,l,\underline{\tilde{l}}',-m_l'}(t,\rho,\Omega)\ \overline{\left(D^l_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\right)_{m_l' m_l}} \tag{C.145}$$

$$\mu^{(S,b)}_{-\omega,l,\underline{\tilde{l}},-m_l}(t,\rho,\hat{R}(\underline{\alpha})^{-1}\Omega) = \sum_{\underline{\tilde{l}}',\,m_l'} \mu^{(S,b)}_{-\omega,l,\underline{\tilde{l}}',-m_l'}(t,\rho,\Omega)\ \overline{\left(D^l_{\underline{\tilde{l}}',\underline{\tilde{l}}}(\underline{\alpha})\right)_{m_l' m_l}}\,. \tag{C.146}$$

This implies that the action of the finite rotation in the momentum representation is given by (mind the position of the primes!)

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{S,a}_{\omega l\,\tilde{\underline{l}}\,m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{S,a}_{\omega l\,\tilde{\underline{l}}'\,m_l'}\,\overline{\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l}} \tag{C.147}$$

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{S,b}_{\omega l\,\tilde{\underline{l}}\,m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{S,b}_{\omega l\,\tilde{\underline{l}}'\,m_l'}\,\overline{\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l}} \tag{C.148}$$

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{S,a}_{-\omega,l,\tilde{\underline{l}},-m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{S,a}_{-\omega,l,\tilde{\underline{l}}',-m_l'}\,\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l} \tag{C.149}$$

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{S,b}_{-\omega,l,\tilde{\underline{l}},-m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{S,b}_{-\omega,l,\tilde{\underline{l}}',-m_l'}\,\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l} \tag{C.150}$$

$$\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{S,a}_{\omega l\,\tilde{\underline{l}}\,m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{S,a}_{\omega l\,\tilde{\underline{l}}'\,m_l'}\,\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \tag{C.151}$$

$$\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{S,b}_{\omega l\,\tilde{\underline{l}}\,m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{S,b}_{\omega l\,\tilde{\underline{l}}'\,m_l'}\,\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \tag{C.152}$$

$$\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{S,a}_{-\omega,l,\tilde{\underline{l}},-m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{S,a}_{-\omega,l,\tilde{\underline{l}}',-m_l'}\,\overline{\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'}} \tag{C.153}$$

$$\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{S,b}_{-\omega,l,\tilde{\underline{l}},-m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{S,b}_{-\omega,l,\tilde{\underline{l}}',-m_l'}\,\overline{\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'}}\,, \tag{C.154}$$

The action on the C-modes is completely analoguous:

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{C,a}_{\omega l\,\tilde{\underline{l}}\,m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{C,a}_{\omega l\,\tilde{\underline{l}}'\,m_l'}\,\overline{\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l}} \tag{C.155}$$

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{C,b}_{\omega l\,\tilde{\underline{l}}\,m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{C,b}_{\omega l\,\tilde{\underline{l}}'\,m_l'}\,\overline{\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l}} \tag{C.156}$$

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{C,a}_{-\omega,l,\tilde{\underline{l}},-m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{C,a}_{-\omega,l,\tilde{\underline{l}}',-m_l'}\,\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l} \tag{C.157}$$

$$\left(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi\right)^{C,b}_{-\omega,l,\tilde{\underline{l}},-m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{C,b}_{-\omega,l,\tilde{\underline{l}}',-m_l'}\,\left(D^l_{\tilde{\underline{l}}',\tilde{\underline{l}}}(\underline{\alpha})\right)_{m_l' m_l} \tag{C.158}$$

$$\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{C,a}_{\omega l\,\tilde{\underline{l}}\,m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{C,a}_{\omega l\,\tilde{\underline{l}}'\,m_l'}\,\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \tag{C.159}$$

$$\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{C,b}_{\omega l\,\tilde{\underline{l}}\,m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{C,b}_{\omega l\,\tilde{\underline{l}}'\,m_l'}\,\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'} \tag{C.160}$$

$$\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{C,a}_{-\omega,l,\tilde{\underline{l}},-m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{C,a}_{-\omega,l,\tilde{\underline{l}}',-m_l'}\,\overline{\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'}} \tag{C.161}$$

$$\left(\hat{R}(\underline{\alpha}) \triangleright \phi\right)^{C,b}_{-\omega,l,\tilde{\underline{l}},-m_l} = \sum_{\tilde{\underline{l}}',\,m_l'} \phi^{C,b}_{-\omega,l,\tilde{\underline{l}}',-m_l'}\,\overline{\left(D^l_{\tilde{\underline{l}},\tilde{\underline{l}}'}(\underline{\alpha})\right)_{m_l m_l'}}\,. \tag{C.162}$$

Thus we can now write the rotated solution in the unrotated coordinates as

$$(\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)(t,\rho,\Omega)$$
$$= \int d\omega \sum_{\underline{l},m_l} \Big\{ (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)^{S,a}_{\omega \underline{l} m_l} \; \mu^{(S,a)}_{\omega \underline{l} m_l}(t,\rho,\Omega) + (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)^{S,b}_{\omega \underline{l} m_l} \; \mu^{(S,b)}_{\omega \underline{l} m_l}(t,\rho,\Omega) \Big\} \tag{C.163}$$
$$= \int d\omega \sum_{\underline{l},m_l} \Big\{ (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)^{S,a}_{-\omega,\underline{l},-m_l} \; \mu^{(S,a)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) + (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)^{S,b}_{-\omega,\underline{l},-m_l} \; \mu^{(S,b)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) \Big\} \tag{C.164}$$
$$= \int d\omega \sum_{\underline{l},m_l} \Big\{ (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)^{C,a}_{\omega \underline{l} m_l} \; \mu^{(C,a)}_{\omega \underline{l} m_l}(t,\rho,\Omega) + (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)^{C,b}_{\omega \underline{l} m_l} \; \mu^{(C,b)}_{\omega \underline{l} m_l}(t,\rho,\Omega) \Big\} \tag{C.165}$$
$$= \int d\omega \sum_{\underline{l},m_l} \Big\{ (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)^{C,a}_{-\omega,\underline{l},-m_l} \; \mu^{(C,a)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) + (\hat{R}(\underline{\alpha})^{-1} \triangleright \phi)^{C,b}_{-\omega,\underline{l},-m_l} \; \mu^{(C,b)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) \Big\} \tag{C.166}$$

$$(\hat{R}(\underline{\alpha}) \triangleright \phi)(t,\rho,\Omega)$$
$$= \int d\omega \sum_{\underline{l},m_l} \Big\{ (\hat{R}(\underline{\alpha}) \triangleright \phi)^{S,a}_{\omega \underline{l} m_l} \; \mu^{(S,a)}_{\omega \underline{l} m_l}(t,\rho,\Omega) + (\hat{R}(\underline{\alpha}) \triangleright \phi)^{S,b}_{\omega \underline{l} m_l} \; \mu^{(S,b)}_{\omega \underline{l} m_l}(t,\rho,\Omega) \Big\} \tag{C.167}$$
$$= \int d\omega \sum_{\underline{l},m_l} \Big\{ (\hat{R}(\underline{\alpha}) \triangleright \phi)^{S,a}_{-\omega,\underline{l},-m_l} \; \mu^{(S,a)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) + (\hat{R}(\underline{\alpha}) \triangleright \phi)^{S,b}_{-\omega,\underline{l},-m_l} \; \mu^{(S,b)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) \Big\} \tag{C.168}$$
$$= \int d\omega \sum_{\underline{l},m_l} \Big\{ (\hat{R}(\underline{\alpha}) \triangleright \phi)^{C,a}_{\omega \underline{l} m_l} \; \mu^{(C,a)}_{\omega \underline{l} m_l}(t,\rho,\Omega) + (\hat{R}(\underline{\alpha}) \triangleright \phi)^{C,b}_{\omega \underline{l} m_l} \; \mu^{(C,b)}_{\omega \underline{l} m_l}(t,\rho,\Omega) \Big\} \tag{C.169}$$
$$= \int d\omega \sum_{\underline{l},m_l} \Big\{ (\hat{R}(\underline{\alpha}) \triangleright \phi)^{C,a}_{-\omega,\underline{l},-m_l} \; \mu^{(C,a)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) + (\hat{R}(\underline{\alpha}) \triangleright \phi)^{C,b}_{-\omega,\underline{l},-m_l} \; \mu^{(C,b)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) \Big\} . \tag{C.170}$$

C.3.3 Boosts' action on AdS solutions

In this section the goal is to calculate the action of the AdS boost generators $K_{d+1,j}$ and K_{0j} on (the momentum representation of) Klein-Gordon solutions on time-interval and tube regions.

We only consider the infinitesimal version of the boosts,because our efforts to obtain explicit expressions for the finite ones were in vain. To this end we calculated the effect of a finite boost on the cartesian embedding space coordinates $X(t,\rho,\Omega) = (X^0, \underline{X}\,, X^{d+1})(t,\rho,\Omega)$, which gave us the new cartesian coordinates $X'(X)$ as a function of the original orispherical coordinates (t,ρ,Ω). However, trying to further untangle the equality $X'(t',\rho',\Omega') = X'(X(t,\rho,\Omega))$ into explicit expressions $t' = t'(t,\rho,\Omega)$, $\rho' = \rho'(t,\rho,\Omega)$ and $\Omega' = \Omega'(t,\rho,\Omega)$ was not successful.

The expressions (C.20) and (C.21) for the boosts tend to become rather complicated when converting the $\xi_k(\Omega)$ into trigonometric functions of the θ-angles of the orispherical coordinates. We are rescued by the fact that for the two d-boosts $K_{d+1,d}$ and K_{0d} this can be done without too much effort. Instead of converting $\partial_{\xi_d} - \xi_d \xi_i \,\partial_{\xi_i}$, we derive the explicit expressions for these boosts from their original definition in cartesian embedding space coordinates:

$$K_{d+1,d} \;=\; X_{d+1}\partial_d - X_d\partial_{d+1} \qquad\qquad K_{0d} \;=\; X_0\partial_d - X_d\partial_0 \,. \tag{C.171}$$

Using

$$X_0 \;=\; -R\,\sin t\;\cos^{-1}\rho \qquad X_{d+1} \;=\; R\,\cos t\;\cos^{-1}\rho \qquad X_d \;=\; R\,\cos\theta_{d-1}\;\tan\rho\,,$$

and thus

$$\cos\theta_{d-1} \;=\; \frac{X_d}{\sqrt{\underline{X}^{\,2}}} \qquad\qquad \frac{\partial\cos\theta_{d-1}}{\partial X^d} \;=\; \frac{\sin^2\theta_{d-1}}{R\,\tan\rho}\;,$$

together with (C.17) we find that

$$K_{d+1,d} \;=\; -\sin t\,\sin\rho\,\cos\theta_{d-1}\,\partial_t + \cos t\,\cos\rho\,\cos\theta_{d-1}\,\partial_\rho + \frac{\cos t}{\sin\rho}\,\sin^2\theta_{d-1}\,\partial_{\cos\theta_{d-1}} \tag{C.172}$$

$$K_{0d} \;=\; -\cos t\,\sin\rho\,\cos\theta_{d-1}\,\partial_t - \sin t\,\cos\rho\,\cos\theta_{d-1}\,\partial_\rho - \frac{\sin t}{\sin\rho}\,\sin^2\theta_{d-1}\,\partial_{\cos\theta_{d-1}}\,. \tag{C.173}$$

But letting these d-boosts act directly on Jacobi or hypergeometric modes still results in too many terms for us to make sense of. However, in [32] Dorn et al. note below equation (3.18) that a complex linear combination of these boosts increases the frequency ω by one. Inspired by their equation (2.7) we therefore define the Z-vectors

$$\begin{aligned} Z_d \;&:=\; K_{0d} + \mathrm{i}K_{d+1,d} \\ &=\; -\mathrm{e}^{\mathrm{i}t}\,\sin\rho\,\cos\theta_{d-1}\,\partial_t + \mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,\cos\rho\,\cos\theta_{d-1}\,\partial_\rho + \mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,\sin^{-1}\!\rho\,\sin^2\theta_{d-1}\,\partial_{\cos\theta_{d-1}} \qquad \text{(C.174)} \\ \overline{Z_d} \;&:=\; K_{0d} - \mathrm{i}K_{d+1,d} \\ &=\; -\mathrm{e}^{-\mathrm{i}t}\,\sin\rho\,\cos\theta_{d-1}\,\partial_t - \mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\,\cos\rho\,\cos\theta_{d-1}\,\partial_\rho - \mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\,\sin^{-1}\!\rho\,\sin^2\theta_{d-1}\,\partial_{\cos\theta_{d-1}}\,. \qquad \text{(C.175)} \end{aligned}$$

The reverse of this is

$$K_{0d} \;=\; \frac{1}{2}\Big(Z_d + \overline{Z_d}\Big) \qquad\qquad K_{d+1,d} \;=\; \frac{1}{2\mathrm{i}}\Big(Z_d - \overline{Z_d}\Big)\,. \tag{C.176}$$

The strategy is now the following: first calculate the action of $\overline{Z_d}$ and Z_d on the modes, which can be done with somewhat large but finite effort. Second, to calculate the action of the d-boosts $K_{d+1,d}$ and K_{0d} on the modes using (C.176). In Section 2.6.7 we then show that it is enough to know the action of these two boosts on the symplectic structure, because the action of the other boosts can be obtained from the Lie bracket/commutator of these two boosts with some rotators: the Lie brackets (C.102) give us (no summation over repeated indices) $K_{0,j-1} = \eta_{jj}\,[K_{0,j}\,,\,K_{j,j-1}]$ and $K_{d+1,j-1} = \eta_{jj}\,[K_{d+1,j}\,,\,K_{j,j-1}]$.

Boosts: AdS time-interval region

In Appendix C.3.4 we work out the actions of Z_d and $\overline{Z_d}$, and then those of $K_{d+1,d}$ and K_{0d} on the Jacobi modes. Applying actions (C.232) -(C.235) to expansion (2.201)

$$\phi(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l} \Big\{ \phi^+_{n\underline{l}m_l}\;\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\phi^-_{n\underline{l}m_l}}\;\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)} \Big\}\,, \tag{C.177}$$

and shifting n and l by ± 1 depending on the respective term, we can work out the corresponding actions in the momentum representation in the same way as for time translation and rotations. Since for time-interval regions we have the ranges $n, l \in \mathbb{N}_0$, for notational convenience we set to zero all quantities where n or l take values outside this range:

$$0 \;=\; \omega^\pm_{n,-1} \;=\; \omega^\pm_{-1,l} \tag{C.178}$$

$$\begin{aligned} 0 \;&=\; z^{(\pm)-+}_{n,-1,\tilde{l}} \;=\; z^{(\pm)-+}_{-1,l,\tilde{l}} \;=\; z^{(\pm)0-}_{n,-1,\tilde{l}} \;=\; z^{(\pm)0-}_{-1,l,\tilde{l}} \\ 0 \;&=\; \tilde{z}^{(\pm)+-}_{n,-1,\tilde{l}} \;=\; \tilde{z}^{(\pm)+-}_{-1,l,\tilde{l}} \;=\; \tilde{z}^{(\pm)0+}_{n,-1,\tilde{l}} \;=\; \tilde{z}^{(\pm)0+}_{-1,l,\tilde{l}} \qquad \text{(C.179)} \\ 0 \;&=\; \phi^+_{n,-1,\underline{\tilde{l}},m_l} = \phi^+_{-1,\underline{l},m_l} = \overline{\phi^-_{n,-1,\underline{\tilde{l}},m_l}} = \overline{\phi^-_{-1,\underline{l},m_l}}\,. \end{aligned}$$

Then the actions for infinitesimal d-boosts in the momentum representation write

$$\big(K_{0d} \triangleright \phi\big)^+_{n\underline{l}m_l} = \tfrac{\mathrm{i}}{2} z^{(+)0-}_{n,l+1,\tilde{l}}\,\phi^+_{n,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} z^{(+)-+}_{n+1,l-1,\tilde{l}}\,\phi^+_{n+1,l-1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} \tilde{z}^{(+)+-}_{n-1,l+1,\tilde{l}}\,\phi^+_{n-1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} \tilde{z}^{(+)0+}_{n,l-1,\tilde{l}}\,\phi^+_{n,l-1,\underline{\tilde{l}},m_l} \tag{C.180}$$

$$\overline{\big(K_{0d} \triangleright \phi\big)^-_{n\underline{l}m_l}} = -\tfrac{\mathrm{i}}{2} z^{(+)0-}_{n,l+1,\tilde{l}}\,\overline{\phi^-_{n,l+1,\underline{\tilde{l}},m_l}} - \tfrac{\mathrm{i}}{2} z^{(+)-+}_{n+1,l-1,\tilde{l}}\,\overline{\phi^-_{n+1,l-1,\underline{\tilde{l}},m_l}} - \tfrac{\mathrm{i}}{2} \tilde{z}^{(+)+-}_{n-1,l+1,\tilde{l}}\,\overline{\phi^-_{n-1,l+1,\underline{\tilde{l}},m_l}} - \tfrac{\mathrm{i}}{2} \tilde{z}^{(+)0+}_{n,l-1,\tilde{l}}\,\overline{\phi^-_{n,l-1,\underline{\tilde{l}},m_l}} \tag{C.181}$$

$$\big(K_{d+1,d} \triangleright \phi\big)^{+}_{n\underline{l}m_l} = \tfrac{1}{2} z^{(+)0-}_{n,l+1,\tilde{l}}\, \phi^{+}_{n,l+1,\underline{\tilde{l}},m_l} + \tfrac{1}{2} z^{(+)-+}_{n+1,l-1,\tilde{l}}\, \phi^{+}_{n+1,l-1,\underline{\tilde{l}},m_l} - \tfrac{1}{2} \tilde{z}^{(+)+-}_{n-1,l+1,\tilde{l}}\, \phi^{+}_{n-1,l+1,\underline{\tilde{l}},m_l} - \tfrac{1}{2} \tilde{z}^{(+)0+}_{n,l-1,\tilde{l}}\, \phi^{+}_{n,l-1,\underline{\tilde{l}},m_l} \tag{C.182}$$

$$\overline{\big(K_{d+1,d} \triangleright \phi\big)^{-}_{n\underline{l}m_l}} = \tfrac{1}{2} z^{(+)0-}_{n,l+1,\tilde{l}}\, \overline{\phi^{-}_{n,l+1,\underline{\tilde{l}},m_l}} + \tfrac{1}{2} z^{(+)-+}_{n+1,l-1,\tilde{l}}\, \overline{\phi^{-}_{n+1,l-1,\underline{\tilde{l}},m_l}} - \tfrac{1}{2} \tilde{z}^{(+)+-}_{n-1,l+1,\tilde{l}}\, \overline{\phi^{-}_{n-1,l+1,\underline{\tilde{l}},m_l}} - \tfrac{1}{2} \tilde{z}^{(+)0+}_{n,l-1,\tilde{l}}\, \overline{\phi^{-}_{n,l-1,\underline{\tilde{l}},m_l}} \tag{C.183}$$

Boosts: AdS tube region

In Appendix C.3.5 we work out the actions of Z_d and $\overline{Z_d}$, and then those of $K_{d+1,d}$ and K_{0d} on the hypergeometric S-modes. Applying actions (C.270)- (C.273) to expansion (2.186)

$$\phi(t,r,\Omega) = \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{\omega\underline{l}m_l}\; \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^{S,b}_{\omega\underline{l}m_l}\; \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\} \tag{C.184}$$

and shifting ω and l by ± 1 depending on the respective term, we can work out the corresponding actions in the momentum representation. Since for tube regions we have the ranges $\omega \in \mathbb{R}$ and $l \in \mathbb{N}_0$, for notational convenience for negative values of l we set to zero all quantities with values outside this range:

$$0 = \phi^{S,a}_{\omega,-1,\underline{\tilde{l}},m_l} = \phi^{S,b}_{\omega,-1,\underline{\tilde{l}},m_l} \tag{C.185}$$

$$\begin{aligned} 0 &= z^{(S,a)--}_{\omega,-1,\tilde{l}} = z^{(S,a)-+}_{\omega,-1,\tilde{l}} = z^{(S,b)--}_{\omega,-1,\tilde{l}} = z^{(S,b)-+}_{\omega,-1,\tilde{l}} \\ 0 &= \tilde{z}^{(S,a)+-}_{\omega,-1,\tilde{l}} = \tilde{z}^{(S,a)++}_{\omega,-1,\tilde{l}} = \tilde{z}^{(S,b)+-}_{\omega,-1,\tilde{l}} = \tilde{z}^{(S,b)++}_{\omega,-1,\tilde{l}} . \end{aligned} \tag{C.186}$$

Then the actions for the infinitesimal d-boosts in the momentum representation write

$$\begin{aligned} \big(K_{0d} \triangleright \phi\big)^{S,a}_{\omega\underline{l}m_l} &= \tfrac{\mathrm{i}}{2} \tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\, \phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} \tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\, \phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\, \phi^{S,a}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\, \phi^{S,a}_{\omega+1,l-1,\underline{\tilde{l}},m_l} \\ \big(K_{0d} \triangleright \phi\big)^{S,b}_{\omega\underline{l}m_l} &= \tfrac{\mathrm{i}}{2} \tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}}\, \phi^{S,b}_{\omega-1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} \tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\, \phi^{S,b}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} z^{(S,b)--}_{\omega+1,l+1,\tilde{l}}\, \phi^{S,b}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2} z^{(S,b)-+}_{\omega+1,l-1,\tilde{l}}\, \phi^{S,b}_{\omega+1,l-1,\underline{\tilde{l}},m_l} \end{aligned} \tag{C.187}$$

$$\begin{aligned} \big(K_{d+1,d} \triangleright \phi\big)^{S,a}_{\omega\underline{l}m_l} &= -\tfrac{1}{2} \tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\, \phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l} - \tfrac{1}{2} \tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\, \phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \tfrac{1}{2} z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\, \phi^{S,a}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \tfrac{1}{2} z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\, \phi^{S,a}_{\omega+1,l-1,\underline{\tilde{l}},m_l} \\ \big(K_{d+1,d} \triangleright \phi\big)^{S,b}_{\omega\underline{l}m_l} &= -\tfrac{1}{2} \tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}}\, \phi^{S,b}_{\omega-1,l+1,\underline{\tilde{l}},m_l} - \tfrac{1}{2} \tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\, \phi^{S,b}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \tfrac{1}{2} z^{(S,b)--}_{\omega+1,l+1,\tilde{l}}\, \phi^{S,b}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \tfrac{1}{2} z^{(S,b)-+}_{\omega+1,l-1,\tilde{l}}\, \phi^{S,b}_{\omega+1,l-1,\underline{\tilde{l}},m_l} . \end{aligned} \tag{C.188}$$

Using now (C.274)

$$\mu^{(S,a,b)}_{-\omega,\underline{l},-m_l} = \overline{\mu^{(S,a,b)}_{\omega\underline{l}m_l}} ,$$

we can equivalently write expansion (2.186) substituting $\omega \to -\omega$ and $m_l \to -m_l$ as

$$\phi(t,r,\Omega) = \int \mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^{S,a}_{-\omega,\underline{l},-m_l}\; \mu^{(S,a)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) + \phi^{S,b}_{-\omega,\underline{l},-m_l}\; \mu^{(S,b)}_{-\omega,\underline{l},-m_l}(t,\rho,\Omega) \Big\} . \tag{C.189}$$

With (C.275)

$$\begin{aligned} K_{d+1,d} \triangleright \mu^{(S,a,b)}_{-\omega,\underline{l},-m_l} &= K_{d+1,d} \triangleright \overline{\mu^{(S,a,b)}_{\omega,\underline{l},m_l}} = \overline{K_{d+1,d} \triangleright \mu^{(S,a,b)}_{\omega,\underline{l},m_l}} \\ K_{0d} \triangleright \mu^{(S,a,b)}_{-\omega,\underline{l},-m_l} &= \quad K_{0d} \triangleright \overline{\mu^{(S,a,b)}_{\omega,\underline{l},m_l}} = \overline{K_{0d} \triangleright \mu^{(S,a,b)}_{\omega,\underline{l},m_l}} . \end{aligned} \tag{C.190}$$

we then find these equivalent actions for the infinitesimal d-boosts in the momentum representation

$$\begin{aligned} \big(K_{0d} \triangleright \phi\big)^{S,a}_{-\omega,\underline{l},-m_l} &= -\tfrac{\mathrm{i}}{2} \tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\, \phi^{S,a}_{-(\omega-1),l+1,\underline{\tilde{l}},-m_l} - \tfrac{\mathrm{i}}{2} \tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\, \phi^{S,a}_{-(\omega-1),l-1,\underline{\tilde{l}},-m_l} - \tfrac{\mathrm{i}}{2} z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\, \phi^{S,a}_{-(\omega+1),l+1,\underline{\tilde{l}},-m_l} - \tfrac{\mathrm{i}}{2} z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\, \phi^{S,a}_{-(\omega+1),l-1,\underline{\tilde{l}},-m_l} \\ \big(K_{0d} \triangleright \phi\big)^{S,b}_{-\omega,\underline{l},-m_l} &= -\tfrac{\mathrm{i}}{2} \tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}}\, \phi^{S,b}_{-(\omega-1),l+1,\underline{\tilde{l}},-m_l} - \tfrac{\mathrm{i}}{2} \tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\, \phi^{S,b}_{-(\omega-1),l-1,\underline{\tilde{l}},-m_l} - \tfrac{\mathrm{i}}{2} z^{(S,b)--}_{\omega+1,l+1,\tilde{l}}\, \phi^{S,b}_{-(\omega+1),l+1,\underline{\tilde{l}},-m_l} - \tfrac{\mathrm{i}}{2} z^{(S,b)-+}_{\omega+1,l-1,\tilde{l}}\, \phi^{S,b}_{-(\omega+1),l-1,\underline{\tilde{l}},-m_l} \end{aligned} \tag{C.191}$$

$$\big(K_{d+1,d}\triangleright\phi\big)^{S,a}_{-\omega,\underline{l},-m_l}=-\tfrac{1}{2}\,\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,a}_{-(\omega-1),l+1,\tilde{\underline{l}},-m_l}-\tfrac{1}{2}\,\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,a}_{-(\omega-1),l-1,\tilde{\underline{l}},-m_l}+\tfrac{1}{2}\,z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,a}_{-(\omega+1),l+1,\tilde{\underline{l}},-m_l}+\tfrac{1}{2}\,z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,a}_{-(\omega+1),l-1,\tilde{\underline{l}},-m_l} \tag{C.192}$$

$$\big(K_{d+1,d}\triangleright\phi\big)^{S,b}_{-\omega,\underline{l},-m_l}=-\tfrac{1}{2}\,\tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,b}_{-(\omega-1),l+1,\tilde{\underline{l}},-m_l}-\tfrac{1}{2}\,\tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,b}_{-(\omega-1),l-1,\tilde{\underline{l}},-m_l}+\tfrac{1}{2}\,z^{(S,b)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,b}_{-(\omega+1),l+1,\tilde{\underline{l}},-m_l}+\tfrac{1}{2}\,z^{(S,b)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,b}_{-(\omega+1),l-1,\tilde{\underline{l}},-m_l}\,.$$

For the C-modes, by the same procedure using the results of Appendix C.3.6, we obtain the actions

$$\big(K_{0d}\triangleright\phi\big)^{C,a}_{\omega\underline{l}m_l}=\tfrac{\mathrm{i}}{2}\,\tilde{z}^{(C,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{C,a}_{\omega-1,l+1,\tilde{\underline{l}},m_l}+\tfrac{\mathrm{i}}{2}\,\tilde{z}^{(C,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{C,a}_{\omega-1,l-1,\tilde{\underline{l}},m_l}+\tfrac{\mathrm{i}}{2}\,z^{(C,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{C,a}_{\omega+1,l+1,\tilde{\underline{l}},m_l}+\tfrac{\mathrm{i}}{2}\,z^{(C,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{C,a}_{\omega+1,l-1,\tilde{\underline{l}},m_l} \tag{C.193}$$

$$\big(K_{0d}\triangleright\phi\big)^{C,b}_{\omega\underline{l}m_l}=\tfrac{\mathrm{i}}{2}\,\tilde{z}^{(C,b)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{C,b}_{\omega-1,l+1,\tilde{\underline{l}},m_l}+\tfrac{\mathrm{i}}{2}\,\tilde{z}^{(C,b)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{C,b}_{\omega-1,l-1,\tilde{\underline{l}},m_l}+\tfrac{\mathrm{i}}{2}\,z^{(C,b)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{C,b}_{\omega+1,l+1,\tilde{\underline{l}},m_l}+\tfrac{\mathrm{i}}{2}\,z^{(C,b)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{C,b}_{\omega+1,l-1,\tilde{\underline{l}},m_l}$$

$$\big(K_{d+1,d}\triangleright\phi\big)^{C,a}_{\omega\underline{l}m_l}=-\tfrac{1}{2}\,\tilde{z}^{(C,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{C,a}_{\omega-1,l+1,\tilde{\underline{l}},m_l}-\tfrac{1}{2}\,\tilde{z}^{(C,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{C,a}_{\omega-1,l-1,\tilde{\underline{l}},m_l}+\tfrac{1}{2}\,z^{(C,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{C,a}_{\omega+1,l+1,\tilde{\underline{l}},m_l}+\tfrac{1}{2}\,z^{(C,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{C,a}_{\omega+1,l-1,\tilde{\underline{l}},m_l} \tag{C.194}$$

$$\big(K_{d+1,d}\triangleright\phi\big)^{C,b}_{\omega\underline{l}m_l}=-\tfrac{1}{2}\,\tilde{z}^{(C,b)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{C,b}_{\omega-1,l+1,\tilde{\underline{l}},m_l}-\tfrac{1}{2}\,\tilde{z}^{(C,b)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{C,b}_{\omega-1,l-1,\tilde{\underline{l}},m_l}+\tfrac{1}{2}\,z^{(C,b)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{C,b}_{\omega+1,l+1,\tilde{\underline{l}},m_l}+\tfrac{1}{2}\,z^{(C,b)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{C,b}_{\omega+1,l-1,\tilde{\underline{l}},m_l}\,.$$

Using

$$\mu^{(C,a,b)}_{-\omega,\underline{l},-m_l}\;=\;\overline{\mu^{(C,a,b)}_{\omega\underline{l}m_l}}\;,$$

we then find these equivalent actions for the infinitesimal d-boosts in the momentum representation

$$\big(K_{0d}\triangleright\phi\big)^{C,a}_{-\omega,\underline{l},-m_l}=-\tfrac{\mathrm{i}}{2}\,\tilde{z}^{(C,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{C,a}_{-(\omega-1),l+1,\tilde{\underline{l}},-m_l}-\tfrac{\mathrm{i}}{2}\,\tilde{z}^{(C,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{C,a}_{-(\omega-1),l-1,\tilde{\underline{l}},-m_l}-\tfrac{\mathrm{i}}{2}\,z^{(C,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{C,a}_{-(\omega+1),l+1,\tilde{\underline{l}},-m_l}-\tfrac{\mathrm{i}}{2}\,z^{(C,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{C,a}_{-(\omega+1),l-1,\tilde{\underline{l}},-m_l} \tag{C.195}$$

$$\big(K_{0d}\triangleright\phi\big)^{C,b}_{-\omega,\underline{l},-m_l}=-\tfrac{\mathrm{i}}{2}\,\tilde{z}^{(C,b)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{C,b}_{-(\omega-1),l+1,\tilde{\underline{l}},-m_l}-\tfrac{\mathrm{i}}{2}\,\tilde{z}^{(C,b)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{C,b}_{-(\omega-1),l-1,\tilde{\underline{l}},-m_l}-\tfrac{\mathrm{i}}{2}\,z^{(C,b)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{C,b}_{-(\omega+1),l+1,\tilde{\underline{l}},-m_l}-\tfrac{\mathrm{i}}{2}\,z^{(C,b)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{C,b}_{-(\omega+1),l-1,\tilde{\underline{l}},-m_l}$$

$$\big(K_{d+1,d}\triangleright\phi\big)^{C,a}_{-\omega,\underline{l},-m_l}=-\tfrac{1}{2}\,\tilde{z}^{(C,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{C,a}_{-(\omega-1),l+1,\tilde{\underline{l}},-m_l}-\tfrac{1}{2}\,\tilde{z}^{(C,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{C,a}_{-(\omega-1),l-1,\tilde{\underline{l}},-m_l}+\tfrac{1}{2}\,z^{(C,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{C,a}_{-(\omega+1),l+1,\tilde{\underline{l}},-m_l}+\tfrac{1}{2}\,z^{(C,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{C,a}_{-(\omega+1),l-1,\tilde{\underline{l}},-m_l} \tag{C.196}$$

$$\big(K_{d+1,d}\triangleright\phi\big)^{C,b}_{-\omega,\underline{l},-m_l}=-\tfrac{1}{2}\,\tilde{z}^{(C,b)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{C,b}_{-(\omega-1),l+1,\tilde{\underline{l}},-m_l}-\tfrac{1}{2}\,\tilde{z}^{(C,b)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{C,b}_{-(\omega-1),l-1,\tilde{\underline{l}},-m_l}+\tfrac{1}{2}\,z^{(C,b)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{C,b}_{-(\omega+1),l+1,\tilde{\underline{l}},-m_l}+\tfrac{1}{2}\,z^{(C,b)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{C,b}_{-(\omega+1),l-1,\tilde{\underline{l}},-m_l}\,.$$

C.3.4 Jacobi recurrence relations for AdS

In this section we derive the ingredients necessary for calculating the action of Z_d and $\overline{Z_d}$ on the AdS-KG solutions of Jacobi type. We recall the notation $\underline{l}\;=\;(l\equiv l_{d-1},\,\tilde{\underline{l}})$ with $\tilde{\underline{l}}\;=\;(l_{d-2},...,l_2)$.

Action of $\overline{Z_d}$ on the AdS-Jacobi modes

The first thing we note after applying $\overline{Z_d}$ to the Klein-Gordon mode $\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)$ according to (C.106) and (2.228), is that there only appear terms with (magic) frequency $\omega^+_{nl}+1$. Since $\omega^+_{nl}\;=\;2n+l+\widetilde{m}_+$, we have only two possibilities to realize an increase by 1 through adjusting n and l:

$$\omega^+_{nl}+1\;=\;\omega^+_{n+1,l-1}\;=\;\omega^+_{n,l+1}\;.$$

This hints to try and find out wether we can decompose $\overline{Z_d}\triangleright\mu^{(+)}_{n\underline{l}m_l}$ as some linear combination of $\mu^{(+)}_{n+1,l-1,\tilde{\underline{l}},m_l}$ and $\mu^{(+)}_{n,l+1,\tilde{\underline{l}},m_l}$. For the hyperspherical harmonics we already dispose of the necessary relations, see Section A.4. The remaing task is thus to decompose the Jacobi polynomials $P^{(\alpha,\beta)}_n(x)$. In our case we have $\alpha\;=\;\gamma^{\mathrm{S}}-1\;=\;l+d/2-1$, while $\beta\;=\;\nu$ and $x\;=\;\cos 2\rho$. Thus $l\;\to\;l\pm1$ induces $\alpha\;\to\;\alpha\pm1$, and β and x remain unaffected. The Jacobi polynomials appear directly as $P^{(\alpha,\beta)}_n(x)$ and as derivative given by DLMF [18.9.15]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,P^{(\alpha,\beta)}_n(x)\;=\;\tfrac{1}{2}\,(n+\alpha+\beta+1)\,P^{(\alpha+1,\beta+1)}_{n-1}(x)\;. \tag{C.197}$$

(By definition, $P_n^{(\alpha,\beta)}(x) \equiv 0$ for all negative n.) We thus need to find the coefficients for the relations

$$P_n^{(\alpha,\beta)}(x) = a\, P_{n+1}^{(\alpha-1,\beta)}(x) + b\, P_n^{(\alpha+1,\beta)}(x) \tag{C.198}$$

$$P_{n-1}^{(\alpha+1,\beta+1)}(x) = c\, P_{n+1}^{(\alpha-1,\beta)}(x) + d\, P_n^{(\alpha+1,\beta)}(x)\ . \tag{C.199}$$

Instead of trying to puzzle together the various recurrence relations for Jacobi polynomials given in AS [22.7] and DLMF [18.9], we shall apply the following procedure: first write the Jacobi polynomials as hypergeometric functions using AS [22.5.42]:

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} F(-n,\, n+\alpha+\beta+1;\, \alpha+1;\, \tfrac{1-x}{2})\ . \tag{C.200}$$

We do this for all Jacobi polynomials in (C.198) and (C.199). Second, we use the program of Rakha et al. (see Appendix C.2.6) to determine the coefficients for the corresponding relations between hypergeometric functions. Third, we convert these relations back to Jacobi polynomials using again (C.200). We start writing (C.198) using hypergeometric functions:

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} F(-n,\, n+\alpha+\beta+1;\, \alpha+1;\, \tfrac{1-x}{2}) = \binom{n+\alpha}{n} F(A+1,\, B;\, C+1;\, y) \tag{C.201}$$

$$P_{n+1}^{(\alpha-1,\beta)}(x) = \binom{n+\alpha}{n+1} F(-n-1,\, n+\alpha+\beta+1;\, \alpha;\, \tfrac{1-x}{2}) = \binom{n+\alpha}{n+1} F(A,\, B;\, C;\, y) \tag{C.202}$$

$$P_n^{(\alpha+1,\beta)}(x) = \binom{n+\alpha+1}{n} F(-n,\, n+\alpha+\beta+2;\, \alpha+2;\, \tfrac{1-x}{2}) = \binom{n+\alpha+1}{n} F(A+1,\, B+1;\, C+2;\, y) \tag{C.203}$$

(The placeholders A, B, C, y will be used only to write the relations in a shorter way, their relation to α, β, n, x may be different in each calculation!) The program by Rakha et al. gives us the contiguous relation for the hypergeometric functions on the right hand side:

$$0 = C(1+C)\, F(A+1,\, B;\, C+1;\, y) + (-C)(C+1)\, F(A,\, B;\, C;\, y) + y\, B(A-C)\, F(A+1,\, B+1;\, C+2;\, y)\ . \tag{C.204}$$

It can be checked with Mathematica's FullSimplify command. Using it, in equation (C.201) we can replace $F(A+1,\, B;\, C+1;\, y)$ by a linear combination of $F(A,\, B;\, C;\, y)$ and $F(A+1,\, B+1;\, C+2;\, y)$. Then converting the hypergeometric functions back to Jacobi polynomials we obtain the simple recurrence relation

$$P_n^{(\alpha,\beta)}(x) = \frac{n+1}{\alpha}\, P_{n+1}^{(\alpha-1,\beta)}(x) + \frac{1-x}{2}\, \frac{n+\alpha+\beta+1}{\alpha}\, P_n^{(\alpha+1,\beta)}(x)\ . \tag{C.205}$$

It can be verified with Mathematica's FullSimplify command. Next we write (C.199) using hypergeometric functions:

$$P_{n-1}^{(\alpha+1,\beta+1)}(x) = \binom{n+\alpha}{n-1} F(-n+1,\, n+\alpha+\beta+2;\, \alpha+2;\, \tfrac{1-x}{2}) = \binom{n+\alpha}{n-1} F(A+2,\, B+1;\, C+2;\, y) \tag{C.206}$$

$$P_{n+1}^{(\alpha-1,\beta)}(x) = \binom{n+\alpha}{n+1} F(-n-1,\, n+\alpha+\beta+1;\, \alpha;\, \tfrac{1-x}{2}) = \binom{n+\alpha}{n+1} F(A,\, B;\, C;\, y) \tag{C.207}$$

$$P_n^{(\alpha+1,\beta)}(x) = \binom{n+\alpha+1}{n} F(-n,\, n+\alpha+\beta+2;\, \alpha+2;\, \tfrac{1-x}{2}) = \binom{n+\alpha+1}{n} F(A+1,\, B+1;\, C+2;\, y) \tag{C.208}$$

The program by Rakha et al. fails to return correct coefficients relating the three hypergeometric functions on the right hand side. However, using (C.72) with AS [15.2.10]

$$0 = (C-A)\, F(A-1,\, B;\, C;\, y) + (2A-C+y[B-A])\, F(A,\, B;\, C;\, y) + A\,(y-1)\, F(A+1,\, B;\, C;\, y) \tag{C.209}$$

with the shifts $A \to A+1$, while $B \to B+1$ and $C \to C+2$ we obtain the contiguous relation

$$\begin{aligned} 0 = (y-1)\, C(A+1)\, F(A+2,\, B+1;\, C+2;\, y) + C(C+1)\, F(A,\, B;\, C;\, y) \\ + \big(C[A-C] + y\, B[C-A]\big)\, F(A+1,\, B+1;\, C+2;\, y)\ . \end{aligned} \tag{C.210}$$

It can be checked with FullSimplify. Using it, in (C.206) we can replace $F(A+2,\, B+1;\, C+2;\, y)$ by a linear combination of $F(A,\, B;\, C;\, y)$ and $F(A+1,\, B+1;\, C+2;\, y)$. Then converting the hypergeometric functions back to Jacobi polynomials we obtain the recurrence relation

$$P_{n-1}^{(\alpha+1,\beta+1)}(x) = \frac{-2}{x+1}\, \frac{n+1}{\alpha}\, P_{n+1}^{(\alpha-1,\beta)}(x) + \frac{2/\alpha}{x+1}\, \big(\alpha + (x-1)(n+\alpha+\beta+1)/2\big)\, P_n^{(\alpha+1,\beta)}(x)\ . \tag{C.211}$$

It can be verified with Mathematica's FullSimplify command, too. We remark that $\alpha = \gamma^{\mathrm{S}} - 1 = l + d/2 - 1$ and thus $\alpha \geq 1/2$. Thus $\alpha - 1 \geq -1/2$ and the Jacobi polynomials are well-defined for these α because they are bigger than -1. Since $\beta = \nu \geq 0$ the same holds for this parameter. Now we can let $\overline{Z_d}$ act on the Klein-Gordon mode $\mu^{(+)}_{n\underline{l}m_l}(t, \rho, \Omega)$, and put to use relations (C.205), (C.211), (A.19) and (A.25). Cleaning up the large expressions resulting from this, we see that the terms containing $Y^{m_l}_{l+1} P^{(\alpha-1,\beta)}_n$ sum up to zero, and the same happens for those containing $Y^{m_l}_{l-1} P^{(\alpha+1,\beta)}_n$. Therefore, we only encounter terms containing either $Y^{m_l}_{l-1} P^{(\alpha-1,\beta)}_{n+1}$ (as needed for $\mu^{(+)}_{n+1,l-1,\underline{\tilde{l}},m_l}$) or $Y^{m_l}_{l+1} P^{(\alpha+1,\beta)}_n$ (as needed for $\mu^{(+)}_{n,l+1,\underline{\tilde{l}},m_l}$). This means that our hopes become fulfilled and we can indeed decompose $\overline{Z_d} \triangleright \mu^{(+)}_{n\underline{l}m_l}$ as a linear combination of $\mu^{(+)}_{n+1,l-1,\underline{\tilde{l}},m_l}$ and $\mu^{(+)}_{n,l+1,\underline{\tilde{l}},m_l}$ (recall the abbreviation $\tilde{l} := l_{d-2}$):

$$\begin{aligned} \overline{Z_d} \triangleright \mu^{(+)}_{n,l,\underline{\tilde{l}},m_l} &= +\mathrm{i}\tilde{z}^{(+)+-}_{nl\tilde{l}}\ \mu^{(+)}_{n+1,l-1,\underline{\tilde{l}},m_l} + \mathrm{i}\tilde{z}^{(+)0+}_{nl\tilde{l}}\ \mu^{(+)}_{n,l+1,\underline{\tilde{l}},m_l} \\ \tilde{z}^{(+)+-}_{nl\tilde{l}} &= +(2l+d-2)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\ \tilde{z}^{(+)0+}_{nl\tilde{l}} &= -2(n+l+\widetilde{m}_+)\,\frac{(n+l+\frac{d}{2})}{(l+\frac{d}{2})}\ \chi^{(d-1)}_{+}(l,\tilde{l})\ . \end{aligned} \tag{C.212}$$

Since $\chi^{(d-1)}_{-}(l,\tilde{l})$ vanishes for $l = 0$, so does $\tilde{z}^{(+)+-}_{nl\tilde{l}}$, and we don't need to worry about defining modes with negative l. Negative n cannot appear in this formula. Letting $\overline{Z_d}$ act on the "vacuum mode" $\mu^{(+)}_{0,\underline{0},0}$ thus creates a mode with $l = 1$:

$$\overline{Z_d} \triangleright \mu^{(+)}_{0,\underline{0},0} = -2\mathrm{i}\widetilde{m}_+ d^{-1/2} \mu^{(+)}_{0,1,\underline{\tilde{0}},0}$$

(because $\chi^{(d-1)}_{+}(0,0) = d^{-1/2}$). When considering the modes $\mu^{(-)}_{n,l,\underline{\tilde{l}},m_l}$, we realize that the whole calculation is exactly the same here, up to changing $\nu \to -\nu$ and thus $\tilde{m}_+ \to \tilde{m}_-$ and $\omega^+_{nl} \to \omega^-_{nl}$. Hence

$$\begin{aligned} \overline{Z_d} \triangleright \mu^{(-)}_{n,l,\underline{\tilde{l}},m_l} &= +\mathrm{i}\tilde{z}^{(-)+-}_{nl\tilde{l}}\ \mu^{(-)}_{n+1,l-1,\underline{\tilde{l}},m_l} + \mathrm{i}\tilde{z}^{(-)0+}_{nl\tilde{l}}\ \mu^{(-)}_{n,l+1,\underline{\tilde{l}},m_l} \\ \tilde{z}^{(-)+-}_{nl\tilde{l}} &= +(2l+d-2)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\ \tilde{z}^{(-)0+}_{nl\tilde{l}} &= -2(n+l+\widetilde{m}_-)\,\frac{(n+l+\frac{d}{2})}{(l+\frac{d}{2})}\ \chi^{(d-1)}_{+}(l,\tilde{l})\ . \end{aligned} \tag{C.213}$$

Letting $\overline{Z_d}$ act on the "vacuum mode" $\mu^{(-)}_{0,\underline{0},0}$ again creates a mode with $l = 1$:

$$\overline{Z_d} \triangleright \mu^{(-)}_{0,\underline{0},0} = -2\mathrm{i}\widetilde{m}_- d^{-1/2} \mu^{(-)}_{0,1,\underline{\tilde{0}},0}$$

Action of Z_d on the AdS-Jacobi modes

This calculation is completely analoguous to the previous, we give the details here for the readers eager enough to reproduce our results. After applying Z_d to the Klein-Gordon mode $\mu^{(+)}_{nl}$ now there only appear terms with (magic) frequency $\omega^+_{nl} - 1$. We have only two possibilities to realize a decrease by 1 through adjusting n and l:

$$\omega^+_{nl} - 1 = \omega^+_{n-1,l+1} = \omega^+_{n,l-1}\ .$$

This hints to decompose $\overline{Z_d} \triangleright \mu^{(+)}_{nl}$ as some linear combination of $\mu^{(+)}_{n-1,l+1}$ and $\mu^{(+)}_{n,l-1}$. Again, for the hyperspherical harmonics we already dispose of the necessary relations. The remaing task is once more to decompose the Jacobi polynomials $P^{(\alpha,\beta)}_n(x)$ and their derivatives. This time we need to find the coefficients for the relations

$$P^{(\alpha,\beta)}_n(x) = a\,P^{(\alpha-1,\beta)}_n(x) + b\,P^{(\alpha+1,\beta)}_{n-1}(x) \tag{C.214}$$

$$P^{(\alpha+1,\beta+1)}_{n-1}(x) = c\,P^{(\alpha-1,\beta)}_n(x) + d\,P^{(\alpha+1,\beta)}_{n-1}(x)\ . \tag{C.215}$$

Writing (C.214) using hypergeometric functions:

$$P_n^{(\alpha,\beta)}(x) \;=\; \binom{n+\alpha}{n}\, F(-n,\, n+\alpha+\beta+1;\; \alpha+1;\; \tfrac{1-x}{2}) \qquad =\; \binom{n+\alpha}{n}\, F(A,\, B+1;\; C+1;\; y) \tag{C.216}$$

$$P_n^{(\alpha-1,\beta)}(x) \;=\; \binom{n+\alpha-1}{n}\, F(-n,\, n+\alpha+\beta;\; \alpha;\; \tfrac{1-x}{2}) \qquad =\; \binom{n+\alpha-1}{n}\, F(A,\, B;\; C;\; y) \tag{C.217}$$

$$P_{n-1}^{(\alpha+1,\beta)}(x) \;=\; \binom{n+\alpha}{n-1}\, F(-n+1,\, n+\alpha+\beta+1;\; \alpha+2;\; \tfrac{1-x}{2}) \qquad =\; \binom{n+\alpha}{n-1}\, F(A+1,\, B+1;\; C+2;\; y) \tag{C.218}$$

The program by Rakha et al. gives us the contiguous relation for the hypergeometric functions on the right hand side:

$$0 \;=\; C(1+C)\, F(A,\, B+1;\, C+1;\; y) + (-C)(C+1)\, F(A,\, B;\, C;\; y) + y\, A(B-C)\, F(A+1,\, B+1;\, C+2;\; y)\;. \tag{C.219}$$

This is not too surprising: the hypergeometric functions on the right hand sides of (C.201) and below are related to those in (C.216) and below by interchanging A with B. Therefore also (C.204) and (C.219) are related by interchanging A with B.

Using (C.219), in (C.216) we can replace $F(A,\, B+1;\; C+1;\; y)$ by a linear combination of $F(A,\, B;\; C;\; y)$ and $F(A+1,\, B+1;\; C+2;\; y)$. Then converting the hypergeometric functions back to Jacobi polynomials we obtain the simple recurrence relation

$$P_n^{(\alpha,\beta)}(x) \;=\; \frac{n+\alpha}{\alpha}\, P_n^{(\alpha-1,\beta)}(x) + \frac{1-x}{2}\;\frac{n+\beta}{\alpha}\, P_{n-1}^{(\alpha+1,\beta)}(x)\;. \tag{C.220}$$

It can be verified with Mathematica's FullSimplify command. Note that in the case $n = 0$ it reduces to $1 = 1$, because of $P_{-1}^{(\alpha,\beta)} \equiv 0$ and $P_0^{(\alpha,\beta)} \equiv 1$. Next we write (C.215) using hypergeometric functions:

$$P_{n-1}^{(\alpha+1,\beta+1)}(x) \;=\; \binom{n+\alpha}{n-1}\, F(-n+1,\, n+\alpha+\beta+2;\; \alpha+2;\; \tfrac{1-x}{2}) \qquad =\; \binom{n+\alpha}{n-1}\, F(A+1,\, B+2;\; C+2;\; y) \tag{C.221}$$

$$P_n^{(\alpha-1,\beta)}(x) \;=\; \binom{n+\alpha-1}{n}\, F(-n,\, n+\alpha+\beta;\; \alpha;\; \tfrac{1-x}{2}) \qquad =\; \binom{n+\alpha-1}{n}\, F(A,\, B;\; C;\; y) \tag{C.222}$$

$$P_{n-1}^{(\alpha+1,\beta)}(x) \;=\; \binom{n+\alpha}{n-1}\, F(-n+1,\, n+\alpha+\beta+1;\; \alpha+2;\; \tfrac{1-x}{2}) \qquad =\; \binom{n+\alpha}{n-1}\, F(A+1,\, B+1;\; C+2;\; y) \tag{C.223}$$

The program by Rakha et al. fails to return correct coefficients relating the three hypergeometric functions on the right hand side. However, using (C.72) with AS [15.2.18]

$$0 \;=\; (C-A-B)\, F(A,\, B;\, C;\; y) + (A-C)\, F(A-1,\, B;\, C;\; y) + B\,(1-y)\, F(A,\, B+1;\, C;\; y) \tag{C.224}$$

with the shifts $A \;\to\; A+1$, while $B \;\to\; B+1$ and $C \;\to\; C+2$ we obtain the contiguous relation

$$\begin{aligned} 0 \;=\; (y-1)\, C(B+1)\, F(A+1,\, B+2;\, C+2;\; x) + C(1+C)\, F(A,\, B;\, C;\; x) \\ + \big(C[B-C] + y\, A[C-B]\big)\, F(A+1,\, B+1;\, C+2;\; x)\;. \end{aligned} \tag{C.225}$$

Again, equations (C.206) and below relate to (C.221) and below by interchanging A and B. Thus also (C.210) and (C.225) relate in this way. Using (C.225), in (C.221) we can replace $F(A+1,\, B+2;\, C+2;\; y)$ by a linear combination of $F(A,\, B;\, C;\; y)$ and $F(A+1,\, B+1;\, C+2;\; y)$. Then converting the hypergeometric functions back to Jacobi polynomials we obtain the recurrence relation

$$P_{n-1}^{(\alpha+1,\beta+1)}(x) \;=\; \frac{2/\alpha}{x+1}\;\frac{n(n+\alpha)}{(n+\alpha+\beta+1)}\; P_n^{(\alpha-1,\beta)}(x) + \frac{2/\alpha}{x+1}\;\frac{(n+\beta) + (1-x)n(n+\beta)/2}{\alpha(n+\alpha+\beta+1)}\; P_{n-1}^{(\alpha+1,\beta)}(x)\;. \tag{C.226}$$

It can be verified with Mathematica's FullSimplify command, too. The Jacobi polynomials are well-defined again since $\alpha, \beta > -1$. Now letting act Z_d act on $\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)$, and using (C.220), (C.226), (A.19) and (A.25), we see that again the terms containing $Y_{l+1}^{m_l} P_n^{(\alpha-1,\beta)}$ sum up to zero, and the same happens for those containing $Y_{l-1}^{m_l} P_n^{(\alpha+1,\beta)}$. This means that again we can decompose $Z_d \triangleright \mu^{(+)}_{n\underline{l}m_l}$ as a linear combination of $\mu^{(+)}_{n-1,l+1,\underline{\tilde{l}},m_l}$ and $\mu^{(+)}_{n,l-1,\underline{\tilde{l}},m_l}$:

$$\begin{aligned} Z_d \triangleright \mu^{(+)}_{n,l,\underline{\tilde{l}},m_l} \;&=\; +\mathrm{i} z^{(+)0-}_{nl\tilde{l}}\; \mu^{(+)}_{n,l-1,\underline{\tilde{l}},m_l} + \mathrm{i} z^{(+)-+}_{nl\tilde{l}}\; \mu^{(+)}_{n-1,l+1,\underline{\tilde{l}},m_l} \\ z^{(+)0-}_{nl\tilde{l}} \;&=\; -(2l+d-2)\, \chi_-^{(d-1)}(l,\tilde{l}) \;=\; -\tilde{z}^{(+)+-}_{nl\tilde{l}} \\ z^{(+)-+}_{nl\tilde{l}} \;&=\; +2n\, \frac{(n+\nu)}{(l+\frac{d}{2})}\; \chi_+^{(d-1)}(l,\tilde{l})\;. \end{aligned} \tag{C.227}$$

Since $\chi_-^{(d-1)}(l,\tilde{l})$ vanishes for $l=0$, so does $z_{nl\tilde{l}}^{(+)0-}$, and we don't need to worry about defining modes with negative l. And $z_{nl\tilde{l}}^{(+)-+}$ vanishes for $n=0$ so that we don't need to worry about defining modes with negative n either. Combining these two properties we see that the "vacuum mode" $\mu^{(+)}_{0,\underline{0},0}$ with magic frequency $\omega^+_{00} = \widetilde{m}_+$ is annihilated by the action of Z_d:

$$\Big(Z_d \triangleright \mu^{(+)}_{0,\underline{0},0}\Big)(t,\rho,\Omega) \;=\; 0\;.$$

The action of Z_d and $\overline{Z_d}$ on the complex-conjugated modes can be found by noting that

$$\begin{aligned} Z_d \triangleright \overline{\mu^{(+)}_{n,\underline{l},m_l}} &= \overline{\overline{Z_d} \triangleright \mu^{(+)}_{n,\underline{l},m_l}} \\ \overline{Z_d} \triangleright \overline{\mu^{(+)}_{n,\underline{l},m_l}} &= \overline{Z_d \triangleright \mu^{(+)}_{n,\underline{l},m_l}}\;, \end{aligned} \tag{C.228}$$

and thus results in

$$\begin{aligned} \overline{Z_d} \triangleright \overline{\mu^{(+)}_{n,l,\underline{\tilde{l}},m_l}} &= -\mathrm{i} z^{(+)0-}_{nl\tilde{l}}\; \overline{\mu^{(+)}_{n,l-1,\underline{\tilde{l}},m_l}} - \mathrm{i} z^{(+)-+}_{nl\tilde{l}}\; \overline{\mu^{(+)}_{n-1,l+1,\underline{\tilde{l}},m_l}} \\ Z_d \triangleright \overline{\mu^{(+)}_{n,l,\underline{\tilde{l}},m_l}} &= -\mathrm{i} \tilde{z}^{(+)+-}_{nl\tilde{l}}\; \overline{\mu^{(+)}_{n+1,l-1,\underline{\tilde{l}},m_l}} - \mathrm{i} \tilde{z}^{(+)0+}_{nl\tilde{l}}\; \overline{\mu^{(+)}_{n,l+1,\underline{\tilde{l}},m_l}}\;. \end{aligned} \tag{C.229}$$

Thus the "vacuum mode" $\overline{\mu^{(+)}_{0,\underline{0},0}}$ with magic frequency $\omega^+_{00} = \widetilde{m}_+$ is annihilated by the action of $\overline{Z_d}$:

$$\Big(\overline{Z_d} \triangleright \overline{\mu^{(+)}_{0,\underline{0},0}}\Big)(t,\rho,\Omega) \;=\; 0\;.$$

Considering the modes $\mu^{(-)}_{n,l,\underline{\tilde{l}},m_l}$, the whole calculation is the same again up to $\nu \to -\nu$ and thus $\tilde{m}_+ \to \tilde{m}_-$ and $\omega^+_{nl} \to \omega^-_{nl}$. Hence

$$\begin{aligned} Z_d \triangleright \mu^{(-)}_{n,l,\underline{\tilde{l}},m_l} &= +\mathrm{i} z^{(-)0-}_{nl\tilde{l}}\; \mu^{(-)}_{n,l-1,\underline{\tilde{l}},m_l} + \mathrm{i} z^{(-)-+}_{nl\tilde{l}}\; \mu^{(-)}_{n-1,l+1,\underline{\tilde{l}},m_l} \\ z^{(-)0-}_{nl\tilde{l}} &= -(2l+d-2)\,\chi_-^{(d-1)}(l,\tilde{l}) \;=\; -\tilde{z}^{(-)+-}_{nl\tilde{l}} \\ z^{(-)-+}_{nl\tilde{l}} &= +2n\,\frac{(n-\nu)}{(l+\frac{d}{2})}\;\chi_+^{(d-1)}(l,\tilde{l}) \end{aligned} \tag{C.230}$$

and

$$\begin{aligned} \overline{Z_d} \triangleright \overline{\mu^{(-)}_{n,l,\underline{\tilde{l}},m_l}} &= -\mathrm{i} z^{(-)0-}_{nl\tilde{l}}\; \overline{\mu^{(-)}_{n,l-1,\underline{\tilde{l}},m_l}} - \mathrm{i} z^{(-)-+}_{nl\tilde{l}}\; \overline{\mu^{(-)}_{n-1,l+1,\underline{\tilde{l}},m_l}} \\ Z_d \triangleright \overline{\mu^{(-)}_{n,l,\underline{\tilde{l}},m_l}} &= -\mathrm{i} \tilde{z}^{(-)+-}_{nl\tilde{l}}\; \overline{\mu^{(-)}_{n+1,l-1,\underline{\tilde{l}},m_l}} - \mathrm{i} \tilde{z}^{(-)0+}_{nl\tilde{l}}\; \overline{\mu^{(-)}_{n,l+1,\underline{\tilde{l}},m_l}}\;. \end{aligned} \tag{C.231}$$

Here the "vacuum modes" with magic frequency $\omega^-_{00} = \widetilde{m}_-$ are annihilated:

$$\begin{aligned} \Big(Z_d \triangleright \mu^{(-)}_{0,\underline{0},0}\Big)(t,\rho,\Omega) &= 0 \\ \Big(\overline{Z_d} \triangleright \overline{\mu^{(-)}_{0,\underline{0},0}}\Big)(t,\rho,\Omega) &= 0\;. \end{aligned}$$

Action of $K_{0,d}$ and $K_{d+1,d}$ on the AdS-Jacobi modes

With equations (2.228)

$$K_{0d} \;=\; \tfrac{1}{2}\left(Z_d + \overline{Z_d}\right) \qquad\qquad K_{d+1,d} \;=\; \tfrac{1}{2\mathrm{i}}\left(\mathrm{i}Z_d - \overline{Z_d}\right)\Big)\;.$$

it is now easy to wite down the action of $K_{0,d}$ and $K_{d+1,d}$ on the Jacobi modes:

$$\begin{aligned} K_{0d} \triangleright \mu^{(\pm)}_{n,l,\tilde{\underline{l}},m_l} &= +\frac{\mathrm{i}}{2}\tilde{z}^{(\pm)+-}_{nl\tilde{l}}\,\mu^{(\pm)}_{n+1,l-1,\tilde{\underline{l}},m_l} + \frac{\mathrm{i}}{2}\tilde{z}^{(\pm)0+}_{nl\tilde{l}}\,\mu^{(\pm)}_{n,l+1,\tilde{\underline{l}},m_l} \\ &\quad + \frac{\mathrm{i}}{2}z^{(\pm)0-}_{nl\tilde{l}}\,\mu^{(\pm)}_{n,l-1,\tilde{\underline{l}},m_l} + \frac{\mathrm{i}}{2}z^{(\pm)-+}_{nl\tilde{l}}\,\mu^{(\pm)}_{n-1,l+1,\tilde{\underline{l}},m_l} &\text{(C.232)} \\ K_{d+1,d} \triangleright \mu^{(\pm)}_{n,l,\tilde{\underline{l}},m_l} &= -\frac{1}{2}\tilde{z}^{(\pm)+-}_{nl\tilde{l}}\,\mu^{(\pm)}_{n+1,l-1,\tilde{\underline{l}},m_l} - \frac{1}{2}\tilde{z}^{(\pm)0+}_{nl\tilde{l}}\,\mu^{(\pm)}_{n,l+1,\tilde{\underline{l}},m_l} \\ &\quad + \frac{1}{2}z^{(\pm)0-}_{nl\tilde{l}}\,\mu^{(\pm)}_{n,l-1,\tilde{\underline{l}},m_l} + \frac{1}{2}z^{(\pm)-+}_{nl\tilde{l}}\,\mu^{(\pm)}_{n-1,l+1,\tilde{\underline{l}},m_l} &\text{(C.233)} \\ K_{0d} \triangleright \overline{\mu^{(\pm)}_{n,l,\tilde{\underline{l}},m_l}} &= -\frac{\mathrm{i}}{2}\tilde{z}^{(\pm)+-}_{nl\tilde{l}}\,\overline{\mu^{(\pm)}_{n+1,l-1,\tilde{\underline{l}},m_l}} - \frac{\mathrm{i}}{2}\tilde{z}^{(\pm)0+}_{nl\tilde{l}}\,\overline{\mu^{(\pm)}_{n,l+1,\tilde{\underline{l}},m_l}} \\ &\quad - \frac{\mathrm{i}}{2}z^{(\pm)0-}_{nl\tilde{l}}\,\overline{\mu^{(\pm)}_{n,l-1,\tilde{\underline{l}},m_l}} - \frac{\mathrm{i}}{2}z^{(\pm)-+}_{nl\tilde{l}}\,\overline{\mu^{(\pm)}_{n-1,l+1,\tilde{\underline{l}},m_l}} &\text{(C.234)} \\ K_{d+1,d} \triangleright \overline{\mu^{(\pm)}_{n,l,\tilde{\underline{l}},m_l}} &= -\frac{1}{2}\tilde{z}^{(\pm)+-}_{nl\tilde{l}}\,\overline{\mu^{(\pm)}_{n+1,l-1,\tilde{\underline{l}},m_l}} - \frac{1}{2}\tilde{z}^{(\pm)0+}_{nl\tilde{l}}\,\overline{\mu^{(\pm)}_{n,l+1,\tilde{\underline{l}},m_l}} \\ &\quad + \frac{1}{2}z^{(\pm)0-}_{nl\tilde{l}}\,\overline{\mu^{(\pm)}_{n,l-1,\tilde{\underline{l}},m_l}} + \frac{1}{2}z^{(\pm)-+}_{nl\tilde{l}}\,\overline{\mu^{(\pm)}_{n-1,l+1,\tilde{\underline{l}},m_l}}\,. &\text{(C.235)} \end{aligned}$$

C.3.5 Hypergeometric recurrence relations for AdS: S-modes

In this section we derive the ingredients necessary for calculating the action of Z_d and $\overline{Z_d}$ on the hypergeometric AdS S-modes. The course of the calculations is quite similar to the one for the Jacobi solutions. We recall the notation $\underline{l} = (l \equiv l_{d-1}, \tilde{\underline{l}})$ with $\tilde{\underline{l}} = (l_{d-2}, ..., l_2)$. The modes are given by

$$\begin{aligned} \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) &= \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, S^a_{\omega l}(\rho) &\text{(C.236)} \\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) &= \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, S^b_{\omega l}(\rho) &\text{(C.237)} \end{aligned}$$

with the real radial functions

$$\begin{aligned} S^a_{\omega l}(\rho) &= \sin^l\rho\, \cos^{\tilde{m}_+}\rho\, F(\alpha^{S,a}, \beta^{S,a}; \gamma^{S,a}; \sin^2\rho) &\text{(C.238)} \\ S^b_{\omega l}(\rho) &= -(\sin\rho)^{2-l-d}\, \cos^{\tilde{m}_+}\rho\, F(\alpha^{S,b}, \beta^{S,b}; \gamma^{S,b}; \sin^2\rho)\,. &\text{(C.239)} \end{aligned}$$

The parameters of the hypergeometric functions are

$$\begin{aligned} \alpha^{S,a} &= \tfrac{1}{2}(l+\widetilde{m}_+-\omega) & \alpha^{S,b} &= \alpha^{S,a} - \gamma^{S,a} + 1 \\ \beta^{S,a} &= \tfrac{1}{2}(l+\widetilde{m}_++\omega) & \beta^{S,b} &= \beta^{S,a} - \gamma^{S,a} + 1 &\text{(C.240)} \\ \gamma^{S,a} &= l + \tfrac{d}{2} & \gamma^{S,b} &= 2 - \gamma^{S,a}\,. \end{aligned}$$

We recall that

$$\overline{\mu^{(S,a)}_{\omega\underline{l}m_l}} = \mu^{(S,a)}_{-\omega,\underline{l},-m_l} \qquad \overline{\mu^{(S,b)}_{\omega\underline{l}m_l}} = \mu^{(S,b)}_{-\omega,\underline{l},-m_l}\,.$$

Action of $\overline{Z_d}$ on the hypergeometric AdS S-modes

After applying $\overline{Z_d}$ to the Klein-Gordon modes $\mu^{(S,a)}_{\omega\underline{l}m_l}$ and $\mu^{(S,b)}_{\omega\underline{l}m_l}$ according to (C.106) and (2.228), we note that there only appear terms with frequency $\omega+1$. Fueled by the success with the Jacobi solutions, we will try again to find a decomposition of $\overline{Z_d} \triangleright \mu^{(S,a)}_{\omega\underline{l}m_l}$ as some linear combination of $\mu^{(S,a)}_{\omega+1,l-1,\tilde{\underline{l}},m_l}$ and $\mu^{(S,a)}_{\omega+1,l+1,\tilde{\underline{l}},m_l}$ (and the same for the S^b-modes) and hope that the S^a and S^b-modes do not mix. For the hyperspherical harmonics we already dispose of the necessary relations, see Section A.4. The remaing task is thus to decompose the hypergeometric functions $F(\alpha^{S,a}, \beta^{S,a}; \gamma^{S,a}; \sin^2\rho)$.

Changing ω and l induces the following changes of the hypergeometric parameters:

$$\begin{matrix} \omega \to \omega{+}1 \\ l \to l{-}1 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \alpha^{S,a} \to \alpha^{S,a}{-}1 & \alpha^{S,b} \to \alpha^{S,b} \\ \beta^{S,a} \to \beta^{S,a} & \beta^{S,b} \to \beta^{S,b}{+}1 \\ \gamma^{S,a} \to \gamma^{S,a}{-}1 & \gamma^{S,b} \to \gamma^{S,b}{+}1 \end{matrix} \tag{C.241}$$

$$\begin{matrix} \omega \to \omega{+}1 \\ l \to l{+}1 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \alpha^{S,a} \to \alpha^{S,a} & \alpha^{S,b} \to \alpha^{S,b}{-}1 \\ \beta^{S,a} \to \beta^{S,a}{+}1 & \beta^{S,b} \to \beta^{S,b} \\ \gamma^{S,a} \to \gamma^{S,a}{+}1 & \gamma^{S,b} \to \gamma^{S,b}{-}1 \end{matrix} \; . \tag{C.242}$$

Thus $l \to l{+}1$ affects $(\alpha^{S,a}, \beta^{S,a}, \gamma^{S,a})$ like $l \to l{-}1$ affects $(\alpha^{S,b}, \beta^{S,b}, \gamma^{S,b})$ and vice versa. The hypergeometric functions appear directly as $F(A,\, B;\, C;\, x)$ and as derivative given by AS [15.2.1] = DLMF [15.5.1]:

$$\frac{\mathrm{d}}{\mathrm{d}x} F(A,\, B;\, C;\, x) \;=\; \frac{AB}{C}\, F(A{+}1,\, B{+}1;\, C{+}1;\, x)\;. \tag{C.243}$$

We thus need to find the coefficients for the relations

$$F(A,\, B;\, C;\, x) \;=\; a\, F(A{-}1,\, B;\, C{-}1;\, x) + b\, F(A,\, B{+}1;\, C{+}1;\, x) \tag{C.244}$$

$$F(A{+}1,\, B{+}1;\, C{+}1;\, x) \;=\; c\, F(A{-}1,\, B;\, C{-}1;\, x) + d\, F(A,\, B{+}1;\, C{+}1;\, x)\;. \tag{C.245}$$

Again we would like to use the program of Rakha et al. (see Appendix C.2.6) to determine these coefficients. Alas it doesn't work for these shifts of hypergeometric parameters, so we have to derive them by hand. For (C.244) we first shift $A \to A{+}1$ and $C \to C{+}1$ resulting in

$$F(A{+}1,\, B;\, C{+}1;\, x) \;=\; a\, F(A,\, B;\, C;\, x) + b\, F(A{+}1,\, B{+}1;\, C{+}2;\, x)\;. \tag{C.246}$$

Now we recall that (C.71) gives us a relation

$$e\, F(A{+}1,\, B;\, C;\, x) \;=\; f\, F(A,\, B;\, C;\, x) + g F(A{+}1,\, B{+}1;\, C{+}2;\, x)\;. \tag{C.247}$$

Thus, if we can find a relation

$$h\, F(A{+}1,\, B;\, C;\, x) \;=\; i\, F(A,\, B;\, C;\, x) + j F(A{+}1,\, B;\, C{+}1;\, x)\;, \tag{C.248}$$

then replacing $F(A{+}1,\, B;\, C;\, x)$ in (C.247) with the right hand side of (C.248) gives us the coefficients for (C.246), and reversing the shifts $A \to A{-}1$ and $C \to C{-}1$ we obtain the coefficients for (C.244). The good news is that relation (C.248) indeed can be found: shifting $A \to A{-}1$ it turns into

$$h\, F(A,\, B;\, C;\, x) \;=\; i\, F(A{-}1,\, B;\, C;\, x) + j F(A,\, B;\, C{+}1;\, x)\;. \tag{C.249}$$

This relation is provided by AS [15.2.20]:

$$(1{-}x)\, C\, F(A,\, B;\, C;\, x) \;=\; C\, F(A{-}1,\, B;\, C;\, x) + x\,(B{-}C)\, F(A,\, B;\, C{+}1;\, x)\;, \tag{C.250}$$

and shifting back $A \to A{+}1$ gives us the coefficients for (C.249). By carrying out in detail the sketched path of calculation we obtain for (C.244):

$$F(A,\, B;\, C;\, x) \;=\; F(A{-}1,\, B;\, C{-}1;\, x) + x\, \frac{B}{C}\, \frac{C{-}A}{C{-}1}\, F(A,\, B{+}1;\, C{+}1;\, x)\;. \tag{C.251}$$

This contiguous relation can be verified with Mathematica's FullSimplify command. For (C.245) we also shift $A \to A{+}1$ and $C \to C{+}1$ resulting in

$$F(A{+}2,\, B{+}1;\, C{+}2;\, x) \;=\; c\, F(A,\, B;\, C;\, x) + d\, F(A{+}1,\, B{+}1;\, C{+}2;\, x)\;. \tag{C.252}$$

(The placeholders $e, f, \ldots$ are now different from those of the previous calculations.) Now we recall that (C.72) gives us a relation

$$e\,F(A,\, B{+}1;\, C{+}2;\, x) \;=\; f\,F(A,\, B;\, C;\, x) + gF(A{+}1,\, B{+}1;\, C{+}2;\, x)\;. \tag{C.253}$$

Thus, if we can find a relation

$$h\,F(A,\, B{+}1;\, C{+}2;\, x) \;=\; i\,F(A{+}2,\, B{+}1;\, C{+}2;\, x) + jF(A{+}1,\, B{+}1;\, C{+}2;\, x)\;, \tag{C.254}$$

then replacing $F(A,\, B{+}1;\, C{+}2;\, x)$ in (C.253) with the right hand side of (C.254) gives us the coefficients for (C.252), and reversing the shifts $A \;\to\; A{-}1$ and $C \;\to\; C{-}1$ we obtain the coefficients for (C.245). Again relation (C.254) indeed can be found: shifting $A \;\to\; A{-}1$ while $B \;\to\; B{-}1$ and $C \;\to\; C{-}2$ it turns into

$$h\,F(A{-}1,\, B;\, C;\, x) \;=\; i\,F(A{+}1,\, B;\, C;\, x) + jF(A,\, B;\, C{+}1;\, x)\;. \tag{C.255}$$

This relation is provided by AS [15.2.10]:

$$(A{-}C)\,F(A{-}1,\, B;\, C;\, x) \;=\; (2A{-}C + x[B{-}A])\,F(A,\, B;\, C;\, x) + (x{-}1)\,A\,F(A{+}1,\, B;\, C;\, x)\;, \tag{C.256}$$

and shifting back $A \;\to\; A{+}1$ while $B \;\to\; B{+}1$ and $C \;\to\; C{+}2$ gives us the coefficients for (C.255). Now carrying out in detail the sketched path of calculation we obtain for (C.245):

$$F(A{+}1,\, B{+}1;\, C{+}1;\, x) = \tfrac{C/A}{1-x}F(A{-}1,\, B;\, C{-}1;\, x) + \tfrac{(C-1)(A-C)+xB(C-A)}{(1-x)\,A(C-1)}F(A,\, B{+}1;\, C{+}1;\, x)\,. \tag{C.257}$$

This contiguous relation can be verified with Mathematica's FullSimplify command. Now we can let $\overline{Z_d}$ act on the Klein-Gordon modes $\mu^{(S,a,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega)$, and put to use relations (C.251), (C.257), (A.19) and (A.25). Cleaning up the large expressions resulting from this, we see that the terms containing both $Y^{m_l}_{l-1}$ and the hypergeometric function obtained from $\begin{smallmatrix}\omega\,\to\,\omega+1\\ l\,\to\,l+1\end{smallmatrix}$ through (C.242) sum up to zero. The same happens for those containing both $Y^{m_l}_{l+1}$ and the hypergeometric function obtained from $\begin{smallmatrix}\omega\,\to\,\omega+1\\ l\,\to\,l-1\end{smallmatrix}$ through (C.241). This means that we can indeed decompose $\overline{Z_d}\triangleright\mu^{(S,a,b)}_{\omega\underline{l}m_l}$ as a linear combination of $\mu^{(S,a,b)}_{\omega+1,l-1,\tilde{\underline{l}},m_l}$ and $\mu^{(S,a,b)}_{\omega+1,l+1,\tilde{\underline{l}},m_l}$. For the S^a-modes this results in

$$\begin{aligned}
\overline{Z_d}\triangleright\mu^{(S,a)}_{\omega,l,\tilde{\underline{l}},m_l} \;&=\; +\mathrm{i}\tilde{z}^{(S,a)+-}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega+1,l-1,\tilde{\underline{l}},m_l} + \mathrm{i}\tilde{z}^{(S,a)++}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega+1,l+1,\tilde{\underline{l}},m_l}\\
\tilde{z}^{(S,a)+-}_{\omega l\tilde{l}} \;&=\; +2(\gamma^{\mathrm{S}}-1)\,\chi^{(d-1)}_{-}(l,\tilde{l})\\
&=\; (2l{+}d{-}2)\,\chi^{(d-1)}_{-}(l,\tilde{l})\\
\tilde{z}^{(S,a)++}_{\omega l\tilde{l}} \;&=\; +2(\alpha^{S,a}-\gamma^{S,a})\,\frac{2\beta^{S,a}}{2\gamma^{S,a}}\,\chi^{(d-1)}_{+}(l,\tilde{l})\;,\\
&=\; (\tilde{m}_+-\omega-l-d)\,\frac{(\tilde{m}_+ + \omega + l)}{(2l{+}d)}\,\chi^{(d-1)}_{+}(l,\tilde{l})\;,
\end{aligned} \tag{C.258}$$

and for the S^b-modes the result is

$$\begin{aligned}
\overline{Z_d}\triangleright\mu^{(S,b)}_{\omega,l,\tilde{\underline{l}},m_l} \;&=\; +\mathrm{i}\tilde{z}^{(S,b)+-}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega+1,l-1,\tilde{\underline{l}},m_l} + \mathrm{i}\tilde{z}^{(S,b)++}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega+1,l+1,\tilde{\underline{l}},m_l}\\
\tilde{z}^{(S,b)+-}_{\omega l\tilde{l}} \;&=\; +2(\alpha^{S,b}-\gamma^{S,b})\,\frac{2\beta^{S,b}}{2\gamma^{S,b}}\,\chi^{(d-1)}_{-}(l,\tilde{l})\\
&=\; -(\tilde{m}_+-\omega+l-2)\,\frac{(\tilde{m}_+ + \omega - l - d + 2)}{(2l{+}d{-}4)}\,\chi^{(d-1)}_{-}(l,\tilde{l})\\
\tilde{z}^{(S,b)++}_{\omega l\tilde{l}} \;&=\; -2(\gamma^{\mathrm{S}}-1)\,\chi^{(d-1)}_{+}(l,\tilde{l})\\
&=\; -(2l{+}d{-}2)\,\chi^{(d-1)}_{+}(l,\tilde{l})\;.
\end{aligned} \tag{C.259}$$

Since $\chi_-^{(d-1)}(l,\tilde{l})$ vanishes for $l = |\tilde{l}|$, so do $\tilde{z}^{(S,a)+-}$ and $\tilde{z}^{(S,b)+-}$ for this case. First, we thus don't need to worry about defining modes with negative l: for $l = 0$ there appears no $(l-1)$-term at all. Second, neither can there appear modes with $|\tilde{l}| > l$. Except for $|\tilde{l}| = l$, the factors $\tilde{z}^{(S,a)+-}_{\omega l\tilde{l}}$ and $\tilde{z}^{(S,b)++}_{\omega l\tilde{l}}$ are always finite. $\tilde{z}^{(S,a)++}_{\omega l\tilde{l}}$ vanishes if either $\beta^{S,a} = 0$ or $(\gamma^{S,a} - \alpha^{S,a}) = 0$, which happen for the magic frequencies

$$-\omega^+_{0l} \;=\; -(l + \widetilde{m}_+) \qquad \text{and} \qquad -\,\omega^-_{0l} \;=\; -(l + \widetilde{m}_-)\;.$$

Thus all S^a-modes $\mu^{(S,a)}_{\omega\underline{l}\,m_l}$ with (magic) frequencies $\omega = -\omega^{\pm}_{0l}$ while $l = |\tilde{l}|$ are annihilated by the action of $\overline{Z_d}$. $\tilde{z}^{(S,b)+-}_{\omega l\tilde{l}}$ vanishes only for $l = |\tilde{l}|$, and also if either $\beta^{S,b} = 0$ or $(\alpha^{S,b} - \gamma^{S,b}) = 0$, which happen for

$$\omega \;=\; (l-2) + \widetilde{m}_+ \qquad \text{and} \qquad \omega \;=\; (l-2) + \widetilde{m}_-\;.$$

Since $\tilde{z}^{(S,b)++}$ never vanishes, there are no b-modes that are annihilated by the action of $\overline{Z_d}$.

Action of Z_d on the hypergeometric AdS S-modes

This calculation is completely analog to the previous one. After applying Z_d to the Klein-Gordon modes $\mu^{(S,a,b)}_{\omega\underline{l}m_l}$ there only appear terms with frequency $\omega-1$. Thus we look for a decomposition of $Z_d \triangleright \mu^{(S,a)}_{\omega\underline{l}m_l}$ as some linear combination of $\mu^{(S,a)}_{\omega-1,l-1,\underline{\tilde{l}},m_l}$ and $\mu^{(S,a)}_{\omega-1,l+1,\underline{\tilde{l}},m_l}$ plus the original mode (and the same for the S^b-modes). Already disposing of the necessary hyperspherical relations, it remains to decompose the hypergeometric functions $F(\alpha^{S,a},\, \beta^{S,a};\, \gamma^{S,a};\, \sin^2\rho)$. Changing ω and l induces the following changes of the hypergeometric parameters:

$$\begin{matrix} \omega \to \omega-1 \\ l \to l-1 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \alpha^{S,a} \to \alpha^{S,a} & \alpha^{S,b} \to \alpha^{S,b}+1 \\ \beta^{S,a} \to \beta^{S,a}-1 & \beta^{S,b} \to \beta^{S,b} \\ \gamma^{S,a} \to \gamma^{S,a}-1 & \gamma^{S,b} \to \gamma^{S,b}+1 \end{matrix} \tag{C.260}$$

$$\begin{matrix} \omega \to \omega-1 \\ l \to l+1 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \alpha^{S,a} \to \alpha^{S,a}+1 & \alpha^{S,b} \to \alpha^{S,b} \\ \beta^{S,a} \to \beta^{S,a} & \beta^{S,b} \to \beta^{S,b}-1\;. \\ \gamma^{S,a} \to \gamma^{S,a}+1 & \gamma^{S,b} \to \gamma^{S,b}-1 \end{matrix} \tag{C.261}$$

Thus (C.260) relates to (C.241) by interchanging $\alpha^{S,a} \leftrightarrow \beta^{S,a}$ and $\alpha^{S,b} \leftrightarrow \beta^{S,b}$. The same relates (C.261) to (C.242). The hypergeometric functions appear directly and as derivative, and thus we need to find the coefficients for the relations

$$F(A,\, B;\, C;\, x) \;=\; a\, F(A,\, B-1;\, C-1;\, x) + b\, F(A+1,\, B;\, C+1;\, x) \tag{C.262}$$

$$F(A+1,\, B+1;\, C+1;\, x) \;=\; c\, F(A,\, B-1;\, C-1;\, x) + d\, F(A+1,\, B;\, C+1;\, x)\;. \tag{C.263}$$

Since these relations are obtained via $A \leftrightarrow B$ from (C.262) and (C.263), we can directly obtain them via $A \leftrightarrow B$ in (C.251) and (C.257):

$$F(A,\, B;\, C;\, x) = F(A,\, B-1;\, C-1;\, x) + x\, \tfrac{A}{C}\, \tfrac{C-B}{C-1}\, F(A+1,\, B;\, C+1;\, x) \tag{C.264}$$

$$F(A+1,\, B+1;\, C+1;\, x) = \tfrac{C/B}{1-x}\, F(A,\, B-1;\, C-1;\, x) + \tfrac{(C-1)(B-C)+xA(C-B)}{(1-x)\, B(C-1)}\, F(A+1,\, B;\, C+1;\, x)\,. \tag{C.265}$$

These contiguous relations can be verified with Mathematica's FullSimplify command. Now we can let Z_d act on the Klein-Gordon modes $\mu^{(S,a,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega)$, and put to use relations (C.264), (C.265), (A.19) and (A.25). Then, the terms containing both $Y^{m_l}_{l-1}$ and the hypergeometric function obtained from $\begin{smallmatrix} \omega \to \omega-1 \\ l \to l+1 \end{smallmatrix}$ through (C.261) sum up to zero. The same happens for those containing both $Y^{m_l}_{l+1}$ and the hypergeometric function obtained from $\begin{smallmatrix} \omega \to \omega-1 \\ l \to l-1 \end{smallmatrix}$ through (C.260). Thus we can decompose $Z_d \triangleright \mu^{(S,a,b)}_{\omega\underline{l}m_l}$

as a linear combination of $\mu^{(S,a,b)}_{\omega-1,l-1,\underline{\tilde{l}},m_l}$ and $\mu^{(S,a,b)}_{\omega-1,l+1,\underline{\tilde{l}},m_l}$. For the S^a-modes this results in

$$\begin{aligned}
Z_d \triangleright \mu^{(S,a)}_{\omega,l,\underline{\tilde{l}},m_l} &= +\mathrm{i}z^{(S,a)--}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \mathrm{i}z^{(S,a)-+}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega-1,l+1,\underline{\tilde{l}},m_l} \\
z^{(S,a)--}_{\omega l\tilde{l}} &= -2(\gamma^{\mathrm{S}}-1)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\
&= -(2l+d-2)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\
z^{(S,a)-+}_{\omega l\tilde{l}} &= +2(\gamma^{S,a}-\beta^{S,a})\,\frac{2\alpha^{S,a}}{2\gamma^{S,a}}\,\chi^{(d-1)}_{+}(l,\tilde{l})\ . \\
&= -(\tilde{m}_+ +\omega-l-d)\,\frac{(\tilde{m}_+ -\omega+l)}{(2l+d)}\,\chi^{(d-1)}_{+}(l,\tilde{l})\ .
\end{aligned} \tag{C.266}$$

And for the S^b-modes the result is

$$\begin{aligned}
Z_d \triangleright \mu^{(S,b)}_{\omega,l,\underline{\tilde{l}},m_l} &= +\mathrm{i}z^{(S,b)--}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \mathrm{i}z^{(S,b)-+}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega-1,l+1,\underline{\tilde{l}},m_l} \\
z^{(S,b)--}_{\omega l\tilde{l}} &= +2(\gamma^{S,b}-\beta^{S,b})\,\frac{2\alpha^{S,b}}{2\gamma^{S,b}}\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\
&= (\tilde{m}_+ +\omega+l-2)\,\frac{(\tilde{m}_+ -\omega-l-d+2)}{(2l+d-4)}\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\
z^{(S,b)-+}_{\omega l\tilde{l}} &= +2(\gamma^{\mathrm{S}}-1)\,\chi^{(d-1)}_{+}(l,\tilde{l})\ . \\
&= (2l+d-2)\,\chi^{(d-1)}_{+}(l,\tilde{l})\ .
\end{aligned} \tag{C.267}$$

Comparing the values of the z-factors, we can read off the following relations for all $d \geq 3$:

$$\begin{aligned}
\tilde{z}^{(S,a)+-}_{\omega l\tilde{l}} &= -z^{(S,a)--}_{-\omega,l,\tilde{l}} & \qquad \tilde{z}^{(S,a)++}_{\omega l\tilde{l}} &= -z^{(S,a)-+}_{-\omega,l,\tilde{l}} \\
\tilde{z}^{(S,b)+-}_{\omega l\tilde{l}} &= -z^{(S,b)--}_{-\omega,l,\tilde{l}} & \qquad \tilde{z}^{(S,b)++}_{\omega l\tilde{l}} &= -z^{(S,b)-+}_{-\omega,l,\tilde{l}}\ .
\end{aligned} \tag{C.268}$$

For $d = 3$ we have $\tilde{l} := m_l$, and since all $\chi^{(2)}_{\pm}(l,m_l)$ are invariant under $m_l \to -m_l$, for $d = 3$ in addition to the above we have:

$$\begin{aligned}
\tilde{z}^{(S,a)+-}_{\omega l\tilde{l}} &= -z^{(S,a)--}_{-\omega,l,-\tilde{l}} & \qquad \tilde{z}^{(S,a)++}_{\omega l\tilde{l}} &= -z^{(S,a)-+}_{-\omega,l,-\tilde{l}} \\
\tilde{z}^{(S,b)+-}_{\omega l\tilde{l}} &= -z^{(S,b)--}_{-\omega,l,-\tilde{l}} & \qquad \tilde{z}^{(S,b)++}_{\omega l\tilde{l}} &= -z^{(S,b)-+}_{-\omega,l,-\tilde{l}}\ .
\end{aligned} \tag{C.269}$$

Since $\chi^{(d-1)}_{-}(l,\tilde{l})$ vanishes only for $l = |\tilde{l}|$, so does $\tilde{z}^{+-}_{\omega l\tilde{l}}$, and again we don't need to worry about defining modes with negative l or with $l < |\tilde{l}|$. While $z^{(S,a)--}_{\omega l\tilde{l}}$ and $z^{(S,b)-+}_{\omega l\tilde{l}}$ are always finite except for $l = |\tilde{l}|$. The factor $z^{(S,a)-+}_{\omega l\tilde{l}}$ vanishes if either $\alpha^{S,a} = 0$ or $(\gamma^{S,a}-\beta^{S,a}) = 0$, which happen for the magic frequencies

$$\omega^{+}_{0l} = l + \widetilde{m}_+ \qquad \text{and} \qquad \omega^{-}_{0l} = l + \widetilde{m}_-\ .$$

Thus all S^a-modes $\mu^{(S,a)}_{\omega\underline{l}\,m_l}$ with (magic) frequencies $\omega = \omega^{\pm}_{0l}$ while $l = |\tilde{l}|$ are annihilated by the action of $\overline{Z_d}$. $z^{(S,b)--}_{\omega l\tilde{l}}$ vanishes only if $l = |\tilde{l}|$, and also if either $\alpha^{S,b} = 0$ or $(\gamma^{S,b}-\beta^{S,b}) = 0$, which happen for the frequencies

$$\omega = -(l-2) - \widetilde{m}_+ \qquad \text{and} \qquad \omega = -(l-2) - \widetilde{m}_-\ .$$

Since $z^{(S,b)-+}$ never vanishes, again there are no b-modes that are annihilated by the action of Z_d.

Action of $K_{0,d}$ and $K_{d+1,d}$ on the hypergeometric AdS S-modes

With equations (2.228)

$$K_{0d} = \tfrac{1}{2}\left(Z_d + \overline{Z_d}\right) \qquad\qquad K_{d+1,d} = \tfrac{1}{2\mathrm{i}}\left(Z_d - \overline{Z_d}\right)\ .$$

it is now easy to write down the action of $K_{0,d}$ and $K_{d+1,d}$ on the hypergeometric S-modes:

$$\begin{aligned}
K_{0d} \triangleright \mu^{(S,a)}_{\omega,l,\underline{\tilde{l}},m_l} \;&=\; +\,\frac{\mathrm{i}}{2}\,\tilde{z}^{(S,a)+-}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega+1,l-1,\underline{\tilde{l}},m_l} \,+\, \frac{\mathrm{i}}{2}\,\tilde{z}^{(S,a)++}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega+1,l+1,\underline{\tilde{l}},m_l} \\
&\qquad +\, \frac{\mathrm{i}}{2}\,z^{(S,a)--}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega-1,l-1,\underline{\tilde{l}},m_l} \,+\, \frac{\mathrm{i}}{2}\,z^{(S,a)-+}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega-1,l+1,\underline{\tilde{l}},m_l} && \text{(C.270)} \\
K_{d+1,d} \triangleright \mu^{(S,a)}_{\omega,l,\underline{\tilde{l}},m_l} \;&-\; \frac{1}{2}\,\tilde{z}^{(S,a)+-}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega+1,l-1,\underline{\tilde{l}},m_l} \,-\, \frac{1}{2}\,\tilde{z}^{(S,a)++}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega+1,l+1,\underline{\tilde{l}},m_l} \\
&\qquad +\, \frac{1}{2}\,z^{(S,a)--}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega-1,l-1,\underline{\tilde{l}},m_l} \,+\, \frac{1}{2}\,z^{(S,a)-+}_{\omega l\tilde{l}}\,\mu^{(S,a)}_{\omega-1,l+1,\underline{\tilde{l}},m_l} && \text{(C.271)} \\
K_{0d} \triangleright \mu^{(S,b)}_{\omega,l,\underline{\tilde{l}},m_l} \;&=\; +\,\frac{\mathrm{i}}{2}\,\tilde{z}^{(S,b)+-}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega+1,l-1,\underline{\tilde{l}},m_l} \,+\, \frac{\mathrm{i}}{2}\,\tilde{z}^{(S,b)++}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega+1,l+1,\underline{\tilde{l}},m_l} \\
&\qquad +\, \frac{\mathrm{i}}{2}\,z^{(S,b)--}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega-1,l-1,\underline{\tilde{l}},m_l} \,+\, \frac{\mathrm{i}}{2}\,z^{(S,b)-+}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega-1,l+1,\underline{\tilde{l}},m_l} && \text{(C.272)} \\
K_{d+1,d} \triangleright \mu^{(S,b)}_{\omega,l,\underline{\tilde{l}},m_l} \;&=\; -\,\frac{1}{2}\,\tilde{z}^{(S,b)+-}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega+1,l-1,\underline{\tilde{l}},m_l} \,-\, \frac{1}{2}\,\tilde{z}^{(S,b)++}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega+1,l+1,\underline{\tilde{l}},m_l} \\
&\qquad +\, \frac{1}{2}\,z^{(S,b)--}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega-1,l-1,\underline{\tilde{l}},m_l} \,+\, \frac{1}{2}\,z^{(S,b)-+}_{\omega l\tilde{l}}\,\mu^{(S,b)}_{\omega-1,l+1,\underline{\tilde{l}},m_l}\,. && \text{(C.273)}
\end{aligned}$$

Since for the S-modes we have the relation

$$\mu^{(S,a,b)}_{-\omega,\underline{l},-m_l} \;=\; \overline{\mu^{(S,a,b)}_{\omega\underline{l}m_l}}\;, \tag{C.274}$$

and both $K_{d+1,d}$ and K_{0d} are real, we can write

$$\begin{aligned}
K_{d+1,d} \triangleright \mu^{(S,a,b)}_{-\omega,\underline{l},-m_l} \;&=\; K_{d+1,d} \triangleright \overline{\mu^{(S,a,b)}_{\omega,\underline{l},m_l}} \;=\; \overline{K_{d+1,d} \triangleright \mu^{(S,a,b)}_{\omega,\underline{l},m_l}} \\
K_{0d} \triangleright \mu^{(S,a,b)}_{-\omega,\underline{l},-m_l} \;&=\; \quad K_{0d} \triangleright \overline{\mu^{(S,a,b)}_{\omega,\underline{l},m_l}} \;=\; \overline{K_{0d} \triangleright \mu^{(S,a,b)}_{\omega,\underline{l},m_l}}\;.
\end{aligned} \tag{C.275}$$

This is a quicker way of obtaining these actions compared to evaluating all coefficients manually with $-\omega$ instead of ω and $-m_l$ instead of m_l, and then using relations (C.268). The results are the same anyway. As a last remark, for $d = 3$ the raising and lowering coefficients are invariant under sign change of m_l, and for $d \geq 4$ they are independent of m_l, and $\tilde{l} := l_{d-2}$ is always nonnegative. Thus all $z, \tilde{z}$ are invariant under the sign change $m_l \to -m_l$.

C.3.6 Hypergeometric recurrence relations for AdS: C-modes

In this section we derive the ingredients necessary for calculating the action of Z_d and $\overline{Z_d}$ on the hypergeometric C-modes. They are given by

$$\mu^{(C,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \;=\; \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, C^a_{\omega l}(\rho) \tag{C.276}$$

$$\mu^{(C,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \;=\; \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_{\underline{l}}(\Omega)\, C^b_{\omega l}(\rho) \tag{C.277}$$

with the real radial functions

$$C^a_{\omega l}(\rho) \;=\; \sin^l\!\rho\, \cos^{\tilde{m}_+}\!\rho\, F(\alpha^{C,a},\, \beta^{C,a};\, \gamma^{C,a};\, \cos^2\!\rho) \tag{C.278}$$

$$C^b_{\omega l}(\rho) \;=\; \sin^l\!\rho\, \cos^{\tilde{m}_-}\!\rho\, F(\alpha^{C,b},\, \beta^{C,b};\, \gamma^{C,b};\, \cos^2\!\rho)\;. \tag{C.279}$$

The parameters of the hypergeometric functions are

$$\begin{aligned}
\alpha^{C,a} \;&=\; \tfrac{1}{2}(\widetilde{m}_+ - \omega + l) \qquad & \alpha^{C,b} \;&=\; \alpha^{C,a} - \gamma^{C,a} + 1 \;=\; \tfrac{1}{2}(\widetilde{m}_- - \omega + l) \;=\; -\tfrac{1}{2}(\tilde{m}_+ + \omega - l - d) \\
\beta^{C,a} \;&=\; \tfrac{1}{2}(\widetilde{m}_+ + \omega + l) \qquad & \beta^{C,b} \;&=\; \beta^{C,a} - \gamma^{C,a} + 1 \;=\; \tfrac{1}{2}(\widetilde{m}_- + \omega + l) \;=\; -\tfrac{1}{2}(\tilde{m}_+ - \omega - l - d) \\
\gamma^{C,a} \;&=\; 1 + \nu \qquad & \gamma^{C,b} \;&=\; \quad 2 - \gamma^{C,a} \quad \;=\; 1 - \nu\;,
\end{aligned} \tag{C.280}$$

and thus we have the relations

$$\begin{aligned}
\gamma^{C,a} - \alpha^{C,a} &= \tfrac{1}{2}(\tilde{m}_+ + \omega - l - d + 2) \qquad & \gamma^{C,b} - \alpha^{C,b} &= \tfrac{1}{2}(\tilde{m}_- + \omega - l - d + 2) = -\tfrac{1}{2}(\tilde{m}_+ - \omega + l - 2) \\
\gamma^{C,a} - \beta^{C,a} &= \tfrac{1}{2}(\tilde{m}_+ - \omega - l - d + 2) \qquad & \gamma^{C,b} - \beta^{C,b} &= \tfrac{1}{2}(\tilde{m}_- - \omega - l - d + 2) = -\tfrac{1}{2}(\tilde{m}_+ + \omega + l - 2) \\
\gamma^{C,a} - \alpha^{C,a} - \beta^{C,a} &= -\tfrac{1}{2}(2l + d - 2) \qquad & \gamma^{C,b} - \alpha^{C,b} - \beta^{C,b} &= \quad -\tfrac{1}{2}\,(2l + d - 2) \quad = \gamma^{C,a} - \alpha^{C,a} - \beta^{C,a}.
\end{aligned}$$

We also recall that

$$\overline{\mu^{(C,a)}_{\omega\underline{l}m_l}} = \mu^{(C,a)}_{-\omega,\underline{l},-m_l} \qquad\qquad \overline{\mu^{(C,b)}_{\omega\underline{l}m_l}} = \mu^{(C,b)}_{-\omega,\underline{l},-m_l} \,.$$

Action of $\overline{Z_d}$ on the hypergeometric AdS C-modes

After applying $\overline{Z_d}$ to the Klein-Gordon modes $\mu^{(C,a)}_{\omega\underline{l}m_l}$ and $\mu^{(C,b)}_{\omega\underline{l}m_l}$ according to (C.106) and (2.228), we note that (as for the S-modes) there only appear terms with frequency $\omega+1$. Thus we try again to find a decomposition of $\overline{Z_d} \triangleright \mu^{(C,a)}_{\omega\underline{l}m_l}$ as some linear combination of $\mu^{(C,a)}_{\omega+1,l-1,\tilde{\underline{l}},m_l}$ and $\mu^{(C,a)}_{\omega+1,l+1,\tilde{\underline{l}},m_l}$ (and the same for the C^b-modes). To this end we need to decompose the hypergeometric function $F(\alpha^{C,a},\, \beta^{C,a};\, \gamma^{C,a};\, \sin^2\rho)$ and its derivative. Changing ω and l induces the following changes of the hypergeometric parameters:

$$\begin{matrix} \omega \to \omega+1 \\ l \to l-1 \end{matrix} \quad\Longrightarrow\quad \begin{matrix} \alpha^{C,a} \to \alpha^{C,a}-1 \\ \beta^{C,a} \to \beta^{C,a} \\ \gamma^{C,a} \to \gamma^{C,a} \end{matrix} \qquad \begin{matrix} \alpha^{C,b} \to \alpha^{C,b}-1 \\ \beta^{C,b} \to \beta^{C,b} \\ \gamma^{C,b} \to \gamma^{C,b} \end{matrix} \tag{C.281}$$

$$\begin{matrix} \omega \to \omega+1 \\ l \to l+1 \end{matrix} \quad\Longrightarrow\quad \begin{matrix} \alpha^{C,a} \to \alpha^{C,a} \\ \beta^{C,a} \to \beta^{C,a}+1 \\ \gamma^{C,a} \to \gamma^{C,a} \end{matrix} \qquad \begin{matrix} \alpha^{C,b} \to \alpha^{C,b} \\ \beta^{C,b} \to \beta^{C,b}+1 \\ \gamma^{C,b} \to \gamma^{C,b} \end{matrix} \,. \tag{C.282}$$

Thus the hypergeometric parameters with superscripts a and b are affected in the same way, and moreover here the γ^C remain completely unchanged, which simplifies the following calculations a lot. The hypergeometric functions again appear directly as $F(A,\, B;\, C;\, x)$ and as derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x} F(A,\, B;\, C;\, x) \;=\; \frac{AB}{C}\, F(A{+}1,\, B{+}1;\, C{+}1;\, x)\,. \tag{C.283}$$

We thus need to find the coefficients for the relations

$$F(A,\, B;\, C;\, x) \;=\; a\, F(A{-}1,\, B;\, C;\, x) + b\, F(A,\, B{+}1;\, C;\, x) \tag{C.284}$$

$$F(A{+}1,\, B{+}1;\, C{+}1;\, x) \;=\; c\, F(A{-}1,\, B;\, C;\, x) + d\, F(A,\, B{+}1;\, C;\, x)\,. \tag{C.285}$$

For (C.284) we can read them off directly from DLMF [15.5.3], and plugging this result into DLMF [15.5.20] we find the coefficients for (C.285):

$$F(A,\, B;\, C;\, x) \;=\; \frac{C-A}{C-A-B}\, F(A{-}1,\, B;\, C;\, x) + \frac{B\,(x-1)}{C-A-B}\, F(A,\, B{+}1;\, C;\, x)\,. \tag{C.286}$$

$$F(A{+}1,\, B{+}1;\, C{+}1;\, x) \;=\; \frac{C}{xA}\,\frac{A-C}{C-A-B}\, F(A{-}1,\, B;\, C;\, x) + \frac{C}{xA}\,\frac{C-A-xB}{C-A-B}\, F(A,\, B{+}1;\, C;\, x)\,. \tag{C.287}$$

These contiguous relation can be verified with Mathematica's FullSimplify command. Now we can let $\overline{Z_d}$ act on the Klein-Gordon modes $\mu^{(C,a,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega)$, and put to use relations (C.286), (C.287), (A.19) and (A.25). Again, the terms containing both $Y^{m_l}_{l-1}$ and the hypergeometric function obtained from $\substack{\omega\,\to\,\omega+1\\ l\,\to\, l+1}$ through (C.242) sum up to zero. The same happens for those containing both $Y^{m_l}_{l+1}$ and the hypergeometric function obtained from $\substack{\omega\,\to\,\omega+1\\ l\,\to\, l-1}$ through (C.241). Thus again we can decompose $\overline{Z_d} \triangleright \mu^{(C,a,b)}_{\omega\underline{l}m_l}$ as a linear combination of $\mu^{(C,a,b)}_{\omega+1,l-1,\tilde{\underline{l}},m_l}$ and $\mu^{(C,a,b)}_{\omega+1,l+1,\tilde{\underline{l}},m_l}$. For the C^a-modes this results in

$$\begin{aligned} \overline{Z_d} \triangleright \mu^{(C,a)}_{\omega\underline{l}m_l} &= \mathrm{i}\tilde{z}^{(C,a)+-}_{\omega l\tilde{l}}\; \mu^{(C,a)}_{\omega+1,l-1,\tilde{\underline{l}},m_l} + \mathrm{i}\tilde{z}^{(C,a)++}_{\omega l\tilde{l}}\; \mu^{(C,a)}_{\omega+1,l+1,\tilde{\underline{l}},m_l} \\ \tilde{z}^{(C,a)+-}_{\omega l\tilde{l}} &= -2(\gamma^{C,a}-\alpha^{C,a})\, \chi^{(d-1)}_{-}(l,\tilde{l}) \\ &= -(\tilde{m}_{+}+\omega-l-d+2)\, \chi^{(d-1)}_{-}(l,\tilde{l}) \\ \tilde{z}^{(C,a)++}_{\omega l\tilde{l}} &= -2\beta^{C,a}\, \chi^{(d-1)}_{+}(l,\tilde{l}) \\ &= -(\tilde{m}_{+}+\omega+l)\, \chi^{(d-1)}_{+}(l,\tilde{l}). \end{aligned} \tag{C.288}$$

and for the C^b-modes the result is

$$\begin{aligned}
\overline{Z_d} \triangleright \mu^{(C,b)}_{\omega \underline{l} m_l} &= \mathrm{i}\tilde{z}^{(C,b)+-}_{\omega l\tilde{l}}\, \mu^{(C,b)}_{\omega+1,l-1,\underline{\tilde{l}},m_l} + \mathrm{i}\tilde{z}^{(C,b)++}_{\omega l\tilde{l}}\, \mu^{(C,b)}_{\omega+1,l+1,\underline{\tilde{l}},m_l} \\
\tilde{z}^{(C,b)+-}_{\omega l\tilde{l}} &= -2(\gamma^{C,b}-\alpha^{C,b})\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\
&= +(\tilde{m}_{+}-\omega+l-2)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\
&= -(\tilde{m}_{-}+\omega-l-d+2)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\
\tilde{z}^{(C,b)++}_{\omega l\tilde{l}} &= -2\beta^{C,b}\,\chi^{(d-1)}_{+}(l,\tilde{l}) \\
&= +(\tilde{m}_{+}-\omega-l-d)\,\chi^{(d-1)}_{+}(l,\tilde{l}) \\
&= -(\tilde{m}_{-}+\omega+l)\,\chi^{(d-1)}_{+}(l,\tilde{l}).
\end{aligned} \tag{C.289}$$

As for the S-modes, no modes with negative l or $l < |\tilde{l}|$ can appear. $\tilde{z}^{(C,a)+-}_{\omega l\tilde{l}}$ is always finite, except for $l = |\tilde{l}|$ and also for $\omega = -\tilde{m}_{+} + (l+d-2)$, then it vanishes. The factor $\tilde{z}^{(C,a)++}_{\omega l\tilde{l}}$ vanishes for the magic frequencies $\omega = -\omega^{+}_{0l} := -\tilde{m}_{+} - l$, for all other frequencies it is finite. Thus all C^a-modes $\mu^{(C,a)}_{\omega \underline{l} m_l}$ with (magic) frequencies $\omega = -\omega^{+}_{0l}$ while $l = |\tilde{l}|$ are annihilated by the action of $\overline{Z_d}$.

$\tilde{z}^{(C,b)+-}_{\omega l\tilde{l}}$ vanishes for $l = |\tilde{l}|$ and also for $\omega = -\tilde{m}_{-} + (l+d-2)$, for all other cases it is finite. The factor $\tilde{z}^{(C,b)++}_{\omega l\tilde{l}}$ vanishes for the magic frequencies $\omega = -\omega^{-}_{0l} := -\tilde{m}_{-} - l$, for all other frequencies it is finite. Thus all C^b-modes $\mu^{(C,b)}_{\omega \underline{l} m_l}$ with (magic) frequencies $\omega = -\omega^{-}_{0l}$ while $l = |\tilde{l}|$ are annihilated by the action of $\overline{Z_d}$.

Action of Z_d on the hypergeometric AdS C-modes

This calculation is completely analog to the previous one. After applying Z_d to the Klein-Gordon modes $\mu^{(C,a,b)}_{\omega \underline{l} m_l}$ there only appear terms with frequency $\omega-1$. Thus we look for a decomposition of $Z_d \triangleright \mu^{(C,a)}_{\omega \underline{l} m_l}$ as some linear combination of $\mu^{(C,a)}_{\omega-1,l-1,\underline{\tilde{l}},m_l}$ and $\mu^{(C,a)}_{\omega-1,l+1,\underline{\tilde{l}},m_l}$ plus the original mode (and the same for the C^b-modes).

Already disposing of the necessary hyperspherical relations, it remains to decompose the hypergeometric functions $F(\alpha^{C,a},\, \beta^{C,a};\, \gamma^{C,a};\, \sin^2\rho)$. Changing ω and l induces the following changes of the hypergeometric parameters:

$$\begin{matrix} \omega \to \omega-1 \\ l \to l-1 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \alpha^{C,a} \to \alpha^{C,a} \\ \beta^{C,a} \to \beta^{C,a}-1 \\ \gamma^{C,a} \to \gamma^{C,a} \end{matrix} \qquad \begin{matrix} \alpha^{C,b} \to \alpha^{C,b} \\ \beta^{C,b} \to \beta^{C,b}-1 \\ \gamma^{C,b} \to \gamma^{C,b} \end{matrix} \tag{C.290}$$

$$\begin{matrix} \omega \to \omega-1 \\ l \to l+1 \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \alpha^{C,a} \to \alpha^{C,a}+1 \\ \beta^{C,a} \to \beta^{C,a} \\ \gamma^{C,a} \to \gamma^{C,a} \end{matrix} \qquad \begin{matrix} \alpha^{C,b} \to \alpha^{C,b}+1 \\ \beta^{C,b} \to \beta^{C,b} \\ \gamma^{C,b} \to \gamma^{C,b} \end{matrix} \quad . \tag{C.291}$$

Thus the hypergeometric parameters with superscripts a and b are affected in the same way. The hypergeometric functions appear directly and as derivative, and thus we need to find the coefficients for the relations

$$F(A,\, B;\, C;\, x) = a\, F(A,\, B-1;\, C;\, x) + b\, F(A+1,\, B;\, C;\, x) \tag{C.292}$$

$$F(A+1,\, B+1;\, C+1;\, x) = c\, F(A,\, B-1;\, C;\, x) + d\, F(A+1,\, B;\, C;\, x)\ . \tag{C.293}$$

Since these relations are obtained via $A \leftrightarrow B$ from (C.262) and (C.263), we can directly obtain them via $A \leftrightarrow B$ in (C.251) and (C.257):

$$F(A,\, B;\, C;\, x) = \frac{C-B}{C-A-B}\, F(A,\, B-1;\, C;\, x) + \frac{A\,(x-1)}{C-A-B}\, F(A+1,\, B;\, C;\, x)\ . \tag{C.294}$$

$$F(A+1,\, B+1;\, C+1;\, x) = \frac{C}{xB}\,\frac{B-C}{C-A-B}\, F(A,\, B-1;\, C;\, x) + \frac{C}{xB}\,\frac{C-B-xA}{C-A-B}\, F(A+1,\, B;\, C;\, x)\,. \tag{C.295}$$

These contiguous relations can be verified with Mathematica's FullSimplify command. Now we can let Z_d act on the Klein-Gordon modes $\mu^{(C,a,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega)$, and put to use relations (C.264), (C.265), (A.19) and (A.25). Then, the terms containing both $Y^{m_l}_{l-1}$ and the hypergeometric function obtained from ${}^{\omega\,\to\,\omega-1}_{l\,\to\,l+1}$ through (C.261) sum up to zero. The same happens for those containing both $Y^{m_l}_{l+1}$ and the hypergeometric function obtained from ${}^{\omega\,\to\,\omega-1}_{l\,\to\,l-1}$ through (C.260). Thus we can decompose $Z_d \triangleright \mu^{(C,a,b)}_{\omega\underline{l}m_l}$ as a linear combination of $\mu^{(C,a,b)}_{\omega-1,l-1,\tilde{\underline{l}},m_l}$ and $\mu^{(C,a,b)}_{\omega-1,l+1,\tilde{\underline{l}},m_l}$. For the C^a-modes this results in

$$\begin{aligned} Z_d \triangleright \mu^{(C,a)}_{\omega\underline{l}m_l} &= \mathrm{i} z^{(C,a)--}_{\omega l\tilde{l}}\, \mu^{(C,a)}_{\omega-1,l-1,\tilde{\underline{l}},m_l} + \mathrm{i} z^{(C,a)-+}_{\omega l\tilde{l}}\, \mu^{(C,a)}_{\omega-1,l+1,\tilde{\underline{l}},m_l} \\ z^{(C,a)--}_{\omega l\tilde{l}} &= 2(\gamma^{C,a}-\beta^{C,a})\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\ &= +(\tilde{m}_+-\omega-l-d+2)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\ z^{(C,a)-+}_{\omega l\tilde{l}} &= 2\alpha^{C,a}\,\chi^{(d-1)}_{+}(l,\tilde{l}) \\ &= +(\tilde{m}_+-\omega+l)\,\chi^{(d-1)}_{+}(l,\tilde{l}). \end{aligned} \tag{C.296}$$

And for the C^b-modes the result is

$$\begin{aligned} Z_d \triangleright \mu^{(C,b)}_{\omega\underline{l}m_l} &= \mathrm{i} z^{(C,b)--}_{\omega l\tilde{l}}\, \mu^{(C,b)}_{\omega-1,l-1,\tilde{\underline{l}},m_l} + \mathrm{i} z^{(C,b)-+}_{\omega l\tilde{l}}\, \mu^{(C,b)}_{\omega-1,l+1,\tilde{\underline{l}},m_l} \\ z^{(C,b)--}_{\omega l\tilde{l}} &= 2(\gamma^{C,b}-\beta^{C,b})\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\ &= -(\tilde{m}_++\omega+l-2)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\ &= +(\tilde{m}_--\omega-l-d+2)\,\chi^{(d-1)}_{-}(l,\tilde{l}) \\ z^{(C,b)-+}_{\omega l\tilde{l}} &= 2\alpha^{C,b}\,\chi^{(d-1)}_{+}(l,\tilde{l}) \\ &= -(\tilde{m}_++\omega-l-d)\,\chi^{(d-1)}_{+}(l,\tilde{l}) \\ &= +(\widetilde{m}_--\omega+l)\,\chi^{(d-1)}_{+}(l,\tilde{l}). \end{aligned} \tag{C.297}$$

Comparing the values of the z-factors, we can read off the same relations for all $d \geq 3$ that we found for the z-factors of the S-modes:

$$\begin{aligned} \tilde{z}^{(C,a)+-}_{\omega l\tilde{l}} &= -z^{(C,a)--}_{-\omega,l,\tilde{l}} & \qquad \tilde{z}^{(C,a)++}_{\omega l\tilde{l}} &= -z^{(C,a)-+}_{-\omega,l,\tilde{l}} \\ \tilde{z}^{(C,b)+-}_{\omega l\tilde{l}} &= -z^{(C,b)--}_{-\omega,l,\tilde{l}} & \qquad \tilde{z}^{(C,b)++}_{\omega l\tilde{l}} &= -z^{(C,b)-+}_{-\omega,l,\tilde{l}}\,. \end{aligned} \tag{C.298}$$

For $d=3$ we have $\tilde{l} := m_l$, and since all $\chi^{(2)}_{\pm}(l,m_l)$ are invariant under $m_l \to -m_l$, for $d=3$ in addition to the above we have:

$$\begin{aligned} \tilde{z}^{(C,a)+-}_{\omega l\tilde{l}} &= -z^{(C,a)--}_{-\omega,l,-\tilde{l}} & \qquad \tilde{z}^{(C,a)++}_{\omega l\tilde{l}} &= -z^{(C,a)-+}_{-\omega,l,-\tilde{l}} \\ \tilde{z}^{(C,b)+-}_{\omega l\tilde{l}} &= -z^{(C,b)--}_{-\omega,l,-\tilde{l}} & \qquad \tilde{z}^{(C,b)++}_{\omega l\tilde{l}} &= -z^{(C,b)-+}_{-\omega,l,-\tilde{l}}\,. \end{aligned} \tag{C.299}$$

Again, no modes with negative l or $l < |\tilde{l}|$ can appear. $z^{(C,a)--}_{\omega l\tilde{l}}$ is always finite except for $l = |\tilde{l}|$ and also for $\omega = \tilde{m}_+ - (l+d-2)$, where it vanishes. The factor $z^{(C,a)-+}_{\omega l\tilde{l}}$ vanishes for the magic frequencies $\omega = \omega^+_{0l} := \tilde{m}_+ + l$, for all other cases it remains finite. Thus all C^a-modes $\mu^{(C,a)}_{\omega\underline{l}\,m_l}$ with (magic) frequencies $\omega = \omega^+_{0l}$ while $l = |\tilde{l}|$ are annihilated by the action of Z_d.

$z^{(C,b)--}_{\omega l\tilde{l}}$ vanishes for $l = |\tilde{l}|$ and also for $\omega = -\tilde{m}_- - (l+d-2)$, for all other cases it is finite. $z^{(C,b)-+}_{\omega l\tilde{l}}$ vanishes for the magic frequencies $\omega = \omega^-_{0l} := \tilde{m}_- - l$, for all other cases it remains finite. Thus all C^b-modes $\mu^{(C,a)}_{\omega\underline{l}\,m_l}$ with (magic) frequencies $\omega = \omega^-_{0l}$ while $l = |\tilde{l}|$ are annihilated by the action of Z_d.

Action of $K_{0,d}$ and $K_{d+1,d}$ on the hypergeometric AdS C-modes

With equations (2.228)

$$K_{0d} = \tfrac{1}{2}\left(Z_d+\overline{Z_d}\right) \qquad\qquad K_{d+1,d} = \tfrac{1}{2\mathrm{i}}\left(Z_d-\overline{Z_d}\right).$$

it is now easy to write down the action of $K_{0,d}$ and $K_{d+1,d}$ on the hypergeometric C-modes:

$$\begin{aligned}
K_{0d} \triangleright \mu^{(C,a)}_{\omega,l,\underline{l},m_l} &= +\frac{\mathrm{i}}{2}\,\tilde{z}^{(C,a)+-}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega+1,l-1,\underline{l},m_l} + \frac{\mathrm{i}}{2}\,\tilde{z}^{(C,a)++}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega+1,l+1,\underline{l},m_l} \\
&\quad + \frac{\mathrm{i}}{2}\,z^{(C,a)--}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega-1,l-1,\underline{l},m_l} + \frac{\mathrm{i}}{2}\,z^{(C,a)-+}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega-1,l+1,\underline{l},m_l} && \text{(C.300)}\\
K_{d+1,d} \triangleright \mu^{(C,a)}_{\omega,l,\underline{l},m_l} &= -\frac{1}{2}\,\tilde{z}^{(C,a)+-}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega+1,l-1,\underline{l},m_l} - \frac{1}{2}\,\tilde{z}^{(C,a)++}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega+1,l+1,\underline{l},m_l} \\
&\quad + \frac{1}{2}\,z^{(C,a)--}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega-1,l-1,\underline{l},m_l} + \frac{1}{2}\,z^{(C,a)-+}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega-1,l+1,\underline{l},m_l} && \text{(C.301)}\\
K_{0d} \triangleright \mu^{(C,b)}_{\omega,l,\underline{l},m_l} &= +\frac{\mathrm{i}}{2}\,\tilde{z}^{(C,b)+-}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega+1,l-1,\underline{l},m_l} + \frac{\mathrm{i}}{2}\,\tilde{z}^{(C,b)++}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega+1,l+1,\underline{l},m_l} \\
&\quad + \frac{\mathrm{i}}{2}\,z^{(C,b)--}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega-1,l-1,\underline{l},m_l} + \frac{\mathrm{i}}{2}\,z^{(C,b)-+}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega-1,l+1,\underline{l},m_l} && \text{(C.302)}\\
K_{d+1,d} \triangleright \mu^{(C,b)}_{\omega,l,\underline{l},m_l} &= -\frac{1}{2}\,\tilde{z}^{(C,b)+-}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega+1,l-1,\underline{l},m_l} - \frac{1}{2}\,\tilde{z}^{(C,b)++}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega+1,l+1,\underline{l},m_l} \\
&\quad + \frac{1}{2}\,z^{(C,b)--}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega-1,l-1,\underline{l},m_l} + \frac{1}{2}\,z^{(C,b)-+}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega-1,l+1,\underline{l},m_l}\,. && \text{(C.303)}
\end{aligned}$$

Since for the C-modes we also have

$$\mu^{(C,a,b)}_{-\omega,\underline{l},-m_l} = \overline{\mu^{(C,a,b)}_{\omega\underline{l}m_l}}\,, \tag{C.304}$$

and both $K_{d+1,d}$ and K_{0d} are real, again we can write

$$\begin{aligned}
K_{d+1,d} \triangleright \mu^{(C,a,b)}_{-\omega,\underline{l},-m_l} &= K_{d+1,d} \triangleright \overline{\mu^{(C,a,b)}_{\omega,\underline{l},m_l}} = \overline{K_{d+1,d} \triangleright \mu^{(C,a,b)}_{\omega,\underline{l},m_l}} \\
K_{0d} \triangleright \mu^{(C,a,b)}_{-\omega,\underline{l},-m_l} &= K_{0d} \triangleright \overline{\mu^{(C,a,b)}_{\omega,\underline{l},m_l}} = \overline{K_{0d} \triangleright \mu^{(C,a,b)}_{\omega,\underline{l},m_l}}\,.
\end{aligned} \tag{C.305}$$

C.3.7 Consistency checks of AdS recurrence relations

Action of $\overline{Z_d}$ and Z_d on hypergeometric modes and Jacobi modes

In Section C.3.4 we calculate the actions of $\overline{Z_d}$ and Z_d on the AdS Jacobi modes. We find that these actions map a Jacobi mode to a linear combination of two other Jacobi modes. We recall that the ordinary Jacobi modes arise as special cases of the C^a-modes when the frequency is magic: $|\omega| = \omega^+_{nl} := 2n + \tilde{m}_+ + l$. The exceptional Jacobi modes arise as special cases of the C^b-modes when $\nu < 1$ and the frequency is magic: $|\omega| = \omega^-_{nl} := 2n + \tilde{m}_- + l$. Both arise also from the S^a-modes if $|\omega| = \omega^\pm_{nl}$. We now check whether our results for S and C-modes reproduce this behaviour, that is, if the actions on magic modes (modes with magic frequencies) yield a linear combination of magic modes.

To see this, the special form of the $z^{S,C}$-factors is actually not relevant. The only important fact is that the actions map a mode with momenta $(\omega, l, \underline{\tilde{l}}, m_l)$ to a linear combination of contiguous modes with $\omega' = \omega \pm 1$ and $l' = l \pm 1$. It is easy to verify that if ω is a magic frequency $\pm\omega^\pm_{nl}$, then $\omega' = \omega \pm 1$ is again a magic frequency for both $l' = l + 1$ and $l' = l - 1$, with $n' \in \{n, n \pm 1\}$. Thus the actions of the boosts $\overline{Z_d}$ and Z_d on the hypergeometric modes respect the Jacobi modes as a subset of the hypergeometric modes.

Action of $\overline{Z_d}$ and Z_d on AdS S-modes and C-modes

In this subsection we calculate the action of $\overline{Z_d}$ and Z_d on the C-modes that is induced by their actions (C.258), (C.259), (C.266) and (C.267) on the S-modes via the linear dependence (C.58) of the S and C-modes. For clarity again we often suppress the indices $\underline{\tilde{l}}$ and m_l, since they remain unchanged in all expressions. Using linear dependence (C.58) and actions (C.258) and (C.259), we can first write the action of $\overline{Z_d}$ on $\mu^{(C,a)}_{\omega\underline{l}m_l}$ as

$$\begin{aligned}
\overline{Z_d}\triangleright\mu^{(C,a)}_{\omega l} &= (M^{\rm no}_{11})_{\omega l}\left(\mathrm{i}\tilde z^{(S,a)+-}_{\omega l}\,\mu^{(S,a)}_{\omega+1,l-1}+\mathrm{i}\tilde z^{(S,a)++}_{\omega l}\,\mu^{(S,a)}_{\omega+1,l+1}\right)+(M^{\rm no}_{12})_{\omega l}\left(\mathrm{i}\tilde z^{(S,b)+-}_{\omega l}\,\mu^{(S,b)}_{\omega+1,l-1}+\mathrm{i}\tilde z^{(S,b)++}_{\omega l}\,\mu^{(S,b)}_{\omega+1,l+1}\right)\\
&= (M^{\rm no}_{11})_{\omega+1,l-1}\,\mu^{(S,a)}_{\omega+1,l-1}\left(\frac{(M^{\rm no}_{11})_{\omega l}}{(M^{\rm no}_{11})_{\omega+1,l-1}}\,\mathrm{i}\tilde z^{(S,a)+-}_{\omega l}\right)+(M^{\rm no}_{12})_{\omega+1,l-1}\,\mu^{(S,b)}_{\omega+1,l-1}\left(\frac{(M^{\rm no}_{12})_{\omega l}}{(M^{\rm no}_{12})_{\omega+1,l-1}}\,\mathrm{i}\tilde z^{(S,b)+-}_{\omega l}\right)\\
&\quad+(M^{\rm no}_{11})_{\omega+1,l+1}\,\mu^{(S,a)}_{\omega+1,l+1}\left(\frac{(M^{\rm no}_{11})_{\omega l}}{(M^{\rm no}_{11})_{\omega+1,l+1}}\,\mathrm{i}\tilde z^{(S,a)++}_{\omega l}\right)+(M^{\rm no}_{12})_{\omega+1,l+1}\,\mu^{(S,b)}_{\omega+1,l+1}\left(\frac{(M^{\rm no}_{12})_{\omega l}}{(M^{\rm no}_{12})_{\omega+1,l+1}}\,\mathrm{i}\tilde z^{(S,b)++}_{\omega l}\right)
\end{aligned}$$

Plugging in the respective definitions, it is rather straightforward to calculate

$$\begin{aligned}
\frac{(M^{\rm no}_{11})_{\omega l}}{(M^{\rm no}_{11})_{\omega+1,l-1}}\,\tilde z^{(S,a)+-}_{\omega l} &= \frac{(M^{\rm no}_{12})_{\omega l}}{(M^{\rm no}_{12})_{\omega+1,l-1}}\,\tilde z^{(S,b)+-}_{\omega l} = -(\tilde m_+ +\omega-l-d+2)\,\chi^{(d-1)}_-(l) =: \tilde z^{(C,a)+-}_{\omega l}\\
\frac{(M^{\rm no}_{11})_{\omega l}}{(M^{\rm no}_{11})_{\omega+1,l+1}}\,\tilde z^{(S,a)++}_{\omega l} &= \frac{(M^{\rm no}_{12})_{\omega l}}{(M^{\rm no}_{12})_{\omega+1,l+1}}\,\tilde z^{(S,b)++}_{\omega l} = -(\tilde m_+ +\omega+l)\,\chi^{(d-1)}_+(l) =: \tilde z^{(C,a)++}_{\omega l}.
\end{aligned}$$

Therefore we can write this action as

$$\begin{aligned}
\overline{Z_d}\triangleright\mu^{(C,a)}_{\omega\underline{l}m_l} &= \mathrm{i}\tilde z^{(C,a)+-}_{\omega l\tilde l}\,\mu^{(C,a)}_{\omega+1,l-1,\underline{\tilde l},m_l}+\mathrm{i}\tilde z^{(C,a)++}_{\omega l\tilde l}\,\mu^{(C,a)}_{\omega+1,l+1,\underline{\tilde l},m_l} \qquad \text{(C.306)}\\
\tilde z^{(C,a)+-}_{\omega l\tilde l} &= -(\tilde m_+ +\omega-l-d+2)\,\chi^{(d-1)}_-(l,\tilde l)\\
&= -2(\gamma^{C,a}-\alpha^{C,a})\,\chi^{(d-1)}_-(l,\tilde l)\\
\tilde z^{(C,a)++}_{\omega l\tilde l} &= -(\tilde m_+ +\omega+l)\,\chi^{(d-1)}_+(l,\tilde l)\\
&= -2\beta^{C,a}\,\chi^{(d-1)}_+(l,\tilde l).
\end{aligned}$$

This agrees exactly with (C.288). Now we repeat this calculation also for the other three actions. The second induced action is

$$\begin{aligned}
\overline{Z_d}\triangleright\mu^{(C,b)}_{\omega l} &= (M^{\rm no}_{21})_{\omega l}\left(\mathrm{i}\tilde z^{(S,a)+-}_{\omega l}\,\mu^{(S,a)}_{\omega+1,l-1}+\mathrm{i}\tilde z^{(S,a)++}_{\omega l}\,\mu^{(S,a)}_{\omega+1,l+1}\right)+(M^{\rm no}_{22})_{\omega l}\left(\mathrm{i}\tilde z^{(S,b)+-}_{\omega l}\,\mu^{(S,b)}_{\omega+1,l-1}+\mathrm{i}\tilde z^{(S,b)++}_{\omega l}\,\mu^{(S,b)}_{\omega+1,l+1}\right)\\
&= (M^{\rm no}_{21})_{\omega+1,l-1}\,\mu^{(S,a)}_{\omega+1,l-1}\left(\frac{(M^{\rm no}_{21})_{\omega l}}{(M^{\rm no}_{21})_{\omega+1,l-1}}\,\mathrm{i}\tilde z^{(S,a)+-}_{\omega l}\right)+(M^{\rm no}_{22})_{\omega+1,l-1}\,\mu^{(S,b)}_{\omega+1,l-1}\left(\frac{(M^{\rm no}_{22})_{\omega l}}{(M^{\rm no}_{22})_{\omega+1,l-1}}\,\mathrm{i}\tilde z^{(S,b)+-}_{\omega l}\right)\\
&\quad+(M^{\rm no}_{21})_{\omega+1,l+1}\,\mu^{(S,a)}_{\omega+1,l+1}\left(\frac{(M^{\rm no}_{21})_{\omega l}}{(M^{\rm no}_{21})_{\omega+1,l+1}}\,\mathrm{i}\tilde z^{(S,a)++}_{\omega l}\right)+(M^{\rm no}_{22})_{\omega+1,l+1}\,\mu^{(S,b)}_{\omega+1,l+1}\left(\frac{(M^{\rm no}_{22})_{\omega l}}{(M^{\rm no}_{22})_{\omega+1,l+1}}\,\mathrm{i}\tilde z^{(S,b)++}_{\omega l}\right)
\end{aligned}$$

Plugging in the definitions we find

$$\begin{aligned}
\frac{(M^{\rm no}_{21})_{\omega l}}{(M^{\rm no}_{21})_{\omega+1,l-1}}\,\tilde z^{(S,a)+-}_{\omega l} &= \frac{(M^{\rm no}_{22})_{\omega l}}{(M^{\rm no}_{22})_{\omega+1,l-1}}\,\tilde z^{(S,b)+-}_{\omega l} = +(\tilde m_+ -\omega+l-2)\,\chi^{(d-1)}_-(l) =: \tilde z^{(C,b)+-}_{\omega l}\\
\frac{(M^{\rm no}_{21})_{\omega l}}{(M^{\rm no}_{21})_{\omega+1,l+1}}\,\tilde z^{(S,a)++}_{\omega l} &= \frac{(M^{\rm no}_{22})_{\omega l}}{(M^{\rm no}_{22})_{\omega+1,l+1}}\,\tilde z^{(S,b)++}_{\omega l} = +(\tilde m_+ -\omega-l-d)\,\chi^{(d-1)}_+(l) =: \tilde z^{(C,b)++}_{\omega l},
\end{aligned}$$

and thus obtain

$$\begin{aligned}
\overline{Z_d}\triangleright\mu^{(C,b)}_{\omega\underline{l}m_l} &= \mathrm{i}\tilde z^{(C,b)+-}_{\omega l\tilde l}\,\mu^{(C,b)}_{\omega+1,l-1,\underline{\tilde l},m_l}+\mathrm{i}\tilde z^{(C,b)++}_{\omega l\tilde l}\,\mu^{(C,b)}_{\omega+1,l+1,\underline{\tilde l},m_l} \qquad \text{(C.307)}\\
\tilde z^{(C,b)+-}_{\omega l\tilde l} &= +(\tilde m_+ -\omega+l-2)\,\chi^{(d-1)}_-(l,\tilde l)\\
&= -2(\gamma^{C,b}-\alpha^{C,b})\,\chi^{(d-1)}_-(l,\tilde l)\\
\tilde z^{(C,b)++}_{\omega l\tilde l} &= +(\tilde m_+ -\omega-l-d)\,\chi^{(d-1)}_+(l,\tilde l)\\
&= -2\beta^{C,b}\,\chi^{(d-1)}_+(l,\tilde l).
\end{aligned}$$

This reproduces (C.289). The third induced action is

$$\begin{aligned}
Z_d \triangleright \mu^{(C,a)}_{\omega l} &= (M^{\text{no}}_{11})_{\omega l}\left(\mathrm{i}z^{(S,a)--}_{\omega l}\,\mu^{(S,a)}_{\omega-1,l-1}+\mathrm{i}z^{(S,a)-+}_{\omega l}\,\mu^{(S,a)}_{\omega-1,l+1}\right)+(M^{\text{no}}_{12})_{\omega l}\left(\mathrm{i}z^{(S,b)--}_{\omega l}\,\mu^{(S,b)}_{\omega-1,l-1}+\mathrm{i}z^{(S,b)-+}_{\omega l}\,\mu^{(S,b)}_{\omega-1,l+1}\right)\\
&= (M^{\text{no}}_{11})_{\omega-1,l-1}\,\mu^{(S,a)}_{\omega-1,l-1}\left(\frac{(M^{\text{no}}_{11})_{\omega l}}{(M^{\text{no}}_{11})_{\omega-1,l-1}}\,\mathrm{i}z^{(S,a)--}_{\omega l}\right)+(M^{\text{no}}_{12})_{\omega-1,l-1}\,\mu^{(S,b)}_{\omega-1,l-1}\left(\frac{(M^{\text{no}}_{12})_{\omega l}}{(M^{\text{no}}_{12})_{\omega-1,l-1}}\,\mathrm{i}z^{(S,b)--}_{\omega l}\right)\\
&\quad +(M^{\text{no}}_{11})_{\omega-1,l+1}\,\mu^{(S,a)}_{\omega-1,l+1}\left(\frac{(M^{\text{no}}_{11})_{\omega l}}{(M^{\text{no}}_{11})_{\omega-1,l+1}}\,\mathrm{i}z^{(S,a)-+}_{\omega l}\right)+(M^{\text{no}}_{12})_{\omega-1,l+1}\,\mu^{(S,b)}_{\omega-1,l+1}\left(\frac{(M^{\text{no}}_{12})_{\omega l}}{(M^{\text{no}}_{12})_{\omega-1,l+1}}\,\mathrm{i}z^{(S,b)-+}_{\omega l}\right)
\end{aligned}$$

Plugging in the definitions we find

$$\begin{aligned}
\frac{(M^{\text{no}}_{11})_{\omega l}}{(M^{\text{no}}_{11})_{\omega-1,l-1}}\,z^{(S,a)--}_{\omega l} &= \frac{(M^{\text{no}}_{12})_{\omega l}}{(M^{\text{no}}_{12})_{\omega-1,l-1}}\,z^{(S,b)--}_{\omega l} = +(\tilde{m}_+-\omega-l-d+2)\,\chi^{(d-1)}_-(l) =: z^{(C,a)--}_{\omega l}\\
\frac{(M^{\text{no}}_{11})_{\omega l}}{(M^{\text{no}}_{11})_{\omega-1,l+1}}\,z^{(S,a)-+}_{\omega l} &= \frac{(M^{\text{no}}_{12})_{\omega l}}{(M^{\text{no}}_{12})_{\omega-1,l+1}}\,z^{(S,b)-+}_{\omega l} = +(\tilde{m}_+-\omega+l)\,\chi^{(d-1)}_+(l) =: z^{(C,a)-+}_{\omega l},
\end{aligned}$$

and thus obtain

$$\begin{aligned}
Z_d \triangleright \mu^{(C,a)}_{\omega\underline{l}m_l} &= \mathrm{i}z^{(C,a)--}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega-1,l-1,\underline{\tilde{l}},m_l}+\mathrm{i}z^{(C,a)-+}_{\omega l\tilde{l}}\,\mu^{(C,a)}_{\omega-1,l+1,\underline{\tilde{l}},m_l}\\
z^{(C,a)--}_{\omega l\tilde{l}} &= +(\tilde{m}_+-\omega-l-d+2)\,\chi^{(d-1)}_-(l,\tilde{l})\\
&= 2(\gamma^{C,a}-\beta^{C,a})\,\chi^{(d-1)}_-(l,\tilde{l})\\
z^{(C,a)-+}_{\omega l\tilde{l}} &= +(\tilde{m}_+-\omega+l)\,\chi^{(d-1)}_+(l,\tilde{l})\\
&= 2\alpha^{C,a}\,\chi^{(d-1)}_+(l,\tilde{l}).
\end{aligned}\tag{C.308}$$

This confirms (C.296). Finally, the fourth induced action is

$$\begin{aligned}
Z_d \triangleright \mu^{(C,b)}_{\omega l} &= (M^{\text{no}}_{21})_{\omega l}\left(\mathrm{i}z^{(S,a)--}_{\omega l}\,\mu^{(S,a)}_{\omega-1,l-1}+\mathrm{i}z^{(S,a)-+}_{\omega l}\,\mu^{(S,a)}_{\omega-1,l+1}\right)+(M^{\text{no}}_{22})_{\omega l}\left(\mathrm{i}z^{(S,b)--}_{\omega l}\,\mu^{(S,b)}_{\omega-1,l-1}+\mathrm{i}z^{(S,b)-+}_{\omega l}\,\mu^{(S,b)}_{\omega-1,l+1}\right)\\
&= (M^{\text{no}}_{21})_{\omega-1,l-1}\,\mu^{(S,a)}_{\omega-1,l-1}\left(\frac{(M^{\text{no}}_{21})_{\omega l}}{(M^{\text{no}}_{21})_{\omega-1,l-1}}\,\mathrm{i}z^{(S,a)--}_{\omega l}\right)+(M^{\text{no}}_{22})_{\omega-1,l-1}\,\mu^{(S,b)}_{\omega-1,l-1}\left(\frac{(M^{\text{no}}_{22})_{\omega l}}{(M^{\text{no}}_{22})_{\omega-1,l-1}}\,\mathrm{i}z^{(S,b)--}_{\omega l}\right)\\
&\quad +(M^{\text{no}}_{21})_{\omega-1,l+1}\,\mu^{(S,a)}_{\omega-1,l+1}\left(\frac{(M^{\text{no}}_{21})_{\omega l}}{(M^{\text{no}}_{21})_{\omega-1,l+1}}\,\mathrm{i}z^{(S,a)-+}_{\omega l}\right)+(M^{\text{no}}_{22})_{\omega-1,l+1}\,\mu^{(S,b)}_{\omega-1,l+1}\left(\frac{(M^{\text{no}}_{22})_{\omega l}}{(M^{\text{no}}_{22})_{\omega-1,l+1}}\,\mathrm{i}z^{(S,b)-+}_{\omega l}\right)
\end{aligned}$$

Plugging in the definitions we find

$$\begin{aligned}
\frac{(M^{\text{no}}_{21})_{\omega l}}{(M^{\text{no}}_{21})_{\omega-1,l-1}}\,z^{(S,a)--}_{\omega l} &= \frac{(M^{\text{no}}_{22})_{\omega l}}{(M^{\text{no}}_{22})_{\omega-1,l-1}}\,z^{(S,b)--}_{\omega l} = -(\tilde{m}_++\omega+l-2)\,\chi^{(d-1)}_-(l) =: z^{(C,b)--}_{\omega l}\\
\frac{(M^{\text{no}}_{21})_{\omega l}}{(M^{\text{no}}_{21})_{\omega-1,l+1}}\,z^{(S,a)-+}_{\omega l} &= \frac{(M^{\text{no}}_{22})_{\omega l}}{(M^{\text{no}}_{22})_{\omega-1,l+1}}\,z^{(S,b)-+}_{\omega l} = -(\tilde{m}_++\omega-l-d)\,\chi^{(d-1)}_+(l) =: z^{(C,b)-+}_{\omega l},
\end{aligned}$$

and thus obtain

$$\begin{aligned}
Z_d \triangleright \mu^{(C,b)}_{\omega\underline{l}m_l} &= \mathrm{i}z^{(C,b)--}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega-1,l-1,\underline{\tilde{l}},m_l}+\mathrm{i}z^{(C,b)-+}_{\omega l\tilde{l}}\,\mu^{(C,b)}_{\omega-1,l+1,\underline{\tilde{l}},m_l}\\
z^{(C,b)--}_{\omega l\tilde{l}} &= -(\tilde{m}_++\omega+l-2)\,\chi^{(d-1)}_-(l,\tilde{l})\\
&= 2(\gamma^{C,b}-\beta^{C,b})\,\chi^{(d-1)}_-(l,\tilde{l})\\
z^{(C,b)-+}_{\omega l\tilde{l}} &= -(\tilde{m}_++\omega-l-d)\,\chi^{(d-1)}_+(l,\tilde{l})\\
&= 2\alpha^{C,b}\,\chi^{(d-1)}_+(l,\tilde{l}).
\end{aligned}\tag{C.309}$$

This coincides with (C.297). Thus all the boost actions on the C-modes that we calculated in the previous section are consistent with the boost actions on the S-modes.

C.4 Commutation of complex structures and isometries

In this section we are looking for complex structures J on spaces of Klein-Gordon solutions on AdS, whose actions on the solutions commute with the actions of the isometries as discussed in Section 2.4.5. That is, let K_{AB} the generator of any isometry on AdS, and let $\phi(t,\rho,\Omega)$ any Klein-Gordon solution on some region $\mathbb{M}$ of AdS. Then we require our complex structure to fulfill

$$\big(J\,(K_{AB}\triangleright\phi)\big)(t,\rho,\Omega) \stackrel{!}{=} \big(K_{AB}\triangleright(J\phi)\big)(t,\rho,\Omega) \qquad \forall\,\phi\in \mathrm{L}_{\mathbb{M}}\ . \tag{C.310}$$

Of the boosts, we shall only consider the infinitesimal d-boosts. This is sufficient, since the other boost's generators arise as Lie brackets of the d-boosts with rotation generators (and if two operators K and L each commute with J, then their Lie bracket $[K, L]$ again commutes with J).

C.4.1 AdS time-interval regions

Since for time-interval regions we only use the frequency representation, our complex structure is simply (the same as for Minkowski time-interval regions):

$$\big(J_{\Sigma_t}\,\phi\big)^{\pm}_{n\underline{l}\,m_l} \;=\; -\mathrm{i}\,\phi^{\pm}_{n\underline{l}\,m_l}\ , \tag{C.311}$$

which applied to the Jacobi expansion (2.201)

$$\phi(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l}\Big\{\phi^{+}_{n\underline{l}m_l}\,\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\phi^{-}_{n\underline{l}m_l}}\;\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\} \tag{C.312}$$

gives us the action

$$(J_{\Sigma_t}\phi)(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l}\Big\{-\mathrm{i}\phi^{+}_{n\underline{l}m_l}\,\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega) + \mathrm{i}\overline{\phi^{-}_{n\underline{l}m_l}}\;\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\}\ . \tag{C.313}$$

Next we check wether this choice makes J_{Σ_t} commute with the isometries, which for time translations and rotations turns out to be rather easy.

AdS time-interval region: time translations

For the (finite) time translation denoted by $k_{\Delta t}$ we obtain from (C.115)

$$\Big(J_{\Sigma_t}\big(k_{\Delta t}\triangleright\phi\big)\Big)(t,\rho,\Omega)=\sum_{n\underline{l}m_l}\Big\{-\mathrm{i}\Big(\phi^{+}_{n\underline{l}m_l}\mathrm{e}^{\mathrm{i}\omega^{\pm}_{nl}\Delta t}\Big)\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)+\mathrm{i}\Big(\overline{\phi^{-}_{n\underline{l}m_l}}\,\mathrm{e}^{-\mathrm{i}\omega^{\pm}_{nl}\Delta t}\Big)\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\}$$

$$\Big(k_{\Delta t}\triangleright(J_{\Sigma_t}\phi)\Big)(t,\rho,\Omega)=\sum_{n\underline{l}m_l}\Big\{\Big(-\mathrm{i}\phi^{+}_{n\underline{l}m_l}\Big)\mathrm{e}^{\mathrm{i}\omega^{\pm}_{nl}\Delta t}\,\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega) + \Big(\mathrm{i}\overline{\phi^{-}_{n\underline{l}m_l}}\Big)\,\mathrm{e}^{-\mathrm{i}\omega^{\pm}_{nl}\Delta t}\,\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\},$$

and thus

$$J_{\Sigma_t}\big(k_{\Delta t}\triangleright\phi\big) \;=\; k_{\Delta t}\triangleright\big(J_{\Sigma_t}\phi\big)\ . \tag{C.314}$$

AdS time-interval region: rotations

For the rotations, from (C.134) with (C.133) we have

$$\begin{aligned}(\hat{R}(\underline{\alpha})\triangleright\phi)(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l}\Bigg\{&\Bigg(\sum_{\underline{\tilde{l}}',\,m_l'}\phi^{+}_{nl\,\underline{\tilde{l}}'\,m_l'}\,\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\Big)_{m_l m_l'}\Bigg)\,\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)\\ &+\Bigg(\sum_{\underline{\tilde{l}}',\,m_l'}\overline{\phi^{-}_{nl\,\underline{\tilde{l}}'\,m_l'}}\;\overline{\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\Big)_{m_l m_l'}}\Bigg)\,\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Bigg\}\ .\end{aligned} \tag{C.315}$$

From this expansion we can derive

$$\Big(J_{\Sigma_t}\big(\hat{R}(\underline{\alpha})\triangleright\phi\big)\Big)(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l}\Bigg\{-\mathrm{i}\Bigg(\sum_{\underline{\tilde{l}}',\,m_l'}\phi^{+}_{nl\,\underline{\tilde{l}}'\,m_l'}\,\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\Big)_{m_l m_l'}\Bigg)\,\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)$$
$$+\mathrm{i}\Bigg(\sum_{\underline{\tilde{l}}',\,m_l'}\overline{\phi^{-}_{nl\,\underline{\tilde{l}}'\,m_l'}}\;\overline{\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\Big)_{m_l m_l'}}\Bigg)\,\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Bigg\}$$

$$\Big(\hat{R}(\underline{\alpha})\triangleright\big(J_{\Sigma_t}\phi\big)\Big)(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l}\Bigg\{\Bigg(\sum_{\underline{\tilde{l}}',\,m_l'}\Big(-\mathrm{i}\phi^{+}_{nl\,\underline{\tilde{l}}'\,m_l'}\Big)\,\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\Big)_{m_l m_l'}\Bigg)\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)$$
$$+\Bigg(\sum_{\underline{\tilde{l}}',\,m_l'}\Big(\mathrm{i}\overline{\phi^{-}_{nl\,\underline{\tilde{l}}'\,m_l'}}\Big)\,\overline{\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\Big)_{m_l m_l'}}\Bigg)\,\overline{\mu^{(\pm)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Bigg\}\,,$$

and thus

$$J_{\Sigma_t}\big(\hat{R}(\underline{\alpha})\triangleright\phi\big) \;=\; \hat{R}(\underline{\alpha})\triangleright\big(J_{\Sigma_t}\phi\big)\,. \tag{C.316}$$

AdS time-interval region: boosts

For the boosts the resulting expressions are a bit more lengthy, and therefore we will not consider the whole expansion here, but write down only the action on the momentum representation. From (C.180)

$$\big(K_{0d}\triangleright\phi\big)^{+}_{n\underline{l}m_l}=\tfrac{\mathrm{i}}{2}z^{(+)0-}_{n,l+1,\tilde{l}}\,\phi^{+}_{n,l+1,\underline{\tilde{l}},m_l}+\tfrac{\mathrm{i}}{2}z^{(+)-+}_{n+1,l-1,\tilde{l}}\,\phi^{+}_{n+1,l-1,\underline{\tilde{l}},m_l}+\tfrac{\mathrm{i}}{2}\tilde{z}^{(+)+-}_{n-1,l+1,\tilde{l}}\,\phi^{+}_{n-1,l+1,\underline{\tilde{l}},m_l}+\tfrac{\mathrm{i}}{2}\tilde{z}^{(+)0+}_{n,l-1,\tilde{l}}\,\phi^{+}_{n,l-1,\underline{\tilde{l}},m_l}$$

we can derive

$$\Big(J\big(K_{0d}\triangleright\phi\big)\Big)^{+}_{n\underline{l}m_l} \;=\; -\mathrm{i}\Big(\tfrac{\mathrm{i}}{2}z^{(+)0-}_{n,l+1,\tilde{l}}\,\phi^{+}_{n,l+1,\underline{\tilde{l}},m_l}+\tfrac{\mathrm{i}}{2}z^{(+)-+}_{n+1,l-1,\tilde{l}}\,\phi^{+}_{n+1,l-1,\underline{\tilde{l}},m_l}+\tfrac{\mathrm{i}}{2}\tilde{z}^{(+)+-}_{n-1,l+1,\tilde{l}}\,\phi^{+}_{n-1,l+1,\underline{\tilde{l}},m_l}+\tfrac{\mathrm{i}}{2}\tilde{z}^{(+)0+}_{n,l-1,\tilde{l}}\,\phi^{+}_{n,l-1,\underline{\tilde{l}},m_l}\Big)$$
$$\Big(K_{0d}\triangleright\big(J\,\phi\big)\Big)^{+}_{n\underline{l}m_l}=\tfrac{\mathrm{i}}{2}z^{(+)0-}_{n,l+1,\tilde{l}}\Big(-\mathrm{i}\phi^{+}_{n,l+1,\underline{\tilde{l}},m_l}\Big)+\tfrac{\mathrm{i}}{2}z^{(+)-+}_{n+1,l-1,\tilde{l}}\Big(-\mathrm{i}\phi^{+}_{n+1,l-1,\underline{\tilde{l}},m_l}\Big)+\tfrac{\mathrm{i}}{2}\tilde{z}^{(+)+-}_{n-1,l+1,\tilde{l}}\Big(-\mathrm{i}\phi^{+}_{n-1,l+1,\underline{\tilde{l}},m_l}\Big)+\tfrac{\mathrm{i}}{2}\tilde{z}^{(+)0+}_{n,l-1,\tilde{l}}\Big(-\mathrm{i}\phi^{+}_{n,l-1,\underline{\tilde{l}},m_l}\Big)$$

The corresponding similar relations for the action on $\overline{\phi^{-}_{n\underline{l}m_l}}$ respectively for the actions of $K_{d+1,d}$ can easily be checked in the same way. This means that we have

$$J_{\Sigma_t}(K_{0d}\triangleright\phi) \;=\; K_{0d}\triangleright(J_{\Sigma_t}\phi) \tag{C.317}$$
$$J_{\Sigma_t}(K_{d+1,d}\triangleright\phi) \;=\; K_{d+1,d}\triangleright(J_{\Sigma_t}\phi)\,. \tag{C.318}$$

We have thus shown that our choice for the complex structure commutes with the actions of all AdS isometries on the (real or complexified) space of bounded Klein-Gordon solutions on a time-interval region.

C.4.2 AdS rod regions

Here we start with the most general form of the complex structure, and then impose several conditions on it, which make its form more concrete. We shall work using the S-expansion (2.186):

$$\phi(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\phi^{S,a}_{\omega\underline{l}m_l}\,\mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega)+\phi^{S,b}_{\omega\underline{l}m_l}\,\mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega).\Big\} \tag{C.319}$$

Since all solutions on a tube region can be expanded in this way, and we want our complex structure J_ρ to map solutions to solutions, we can expand $J_\rho\phi$ in the same way:

$$\big(J_\rho\phi\big)(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l}\,\mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega)+\big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l}\,\mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega).\Big\} \tag{C.320}$$

Since J_ρ is linear, the most general form of J_ρ is

$$\begin{aligned}
\big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} &= \int\!\mathrm{d}\omega' \sum_{\underline{l}',m_l'} \Big\{ j^{S,aa}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \phi^{S,a}_{\omega'\underline{l}'m_l'} + j^{S,ab}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \phi^{S,b}_{\omega'\underline{l}'m_l'} \\
&\qquad + \tilde{j}^{S,aa}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \overline{\phi^{S,a}_{\omega'\underline{l}'m_l'}} + \tilde{j}^{S,ab}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \overline{\phi^{S,b}_{\omega'\underline{l}'m_l'}} \Big\} \\
\big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} &= \int\!\mathrm{d}\omega' \sum_{\underline{l}',m_l'} \Big\{ j^{S,ba}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \phi^{S,a}_{\omega'\underline{l}'m_l'} + j^{S,bb}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \phi^{S,b}_{\omega'\underline{l}'m_l'} \\
&\qquad + \tilde{j}^{S,ba}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \overline{\phi^{S,a}_{\omega'\underline{l}'m_l'}} + \tilde{j}^{S,bb}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \overline{\phi^{S,b}_{\omega'\underline{l}'m_l'}} \Big\}.
\end{aligned} \tag{C.321}$$

Here, the integral kernels $j^{S,aa}$ et cetera are complex functions on two sets of momenta, and completely determine the complex structure. At this point we require the action of J_ρ to be linear only with respect to real linear combinations of real solutions, for which (C.321) is the most general form. First, we can impose the usual essential properties of a complex structure:

0. J_ρ must map real solutions to real solutions.

1. $J_\rho^2 \stackrel{!}{=} -\mathbb{1}$ when acting on real solutions.

2. Compatibility with the symplectic structure: $\omega(J_\rho\eta,\, J_\rho\zeta) \stackrel{!}{=} \omega(\eta,\, \zeta)$ for all real solutions η, ζ.

These yield several integral conditions that the kernels $j^{S,aa}$ et cetera must fulfill. However, it is more effective to first require the complex structure J_ρ to commute with the actions of time translation and rotations, and to impose the essential properties only after doing this. As last step, we then require J_ρ to also commute with the boost's actions.

C.4.3 J_ρ: commutation with time-translations and spatial rotations

We begin by considering the action (C.120) of time translations:

$$\big(k_{\Delta t}\triangleright\phi\big)^{S,a}_{\omega\underline{l}m_l} = \mathrm{e}^{\mathrm{i}\omega\Delta t}\,\phi^{S,a}_{\omega\underline{l}m_l} \qquad \big(k_{\Delta t}\triangleright\phi\big)^{S,b}_{\omega\underline{l}m_l} = \mathrm{e}^{\mathrm{i}\omega\Delta t}\,\phi^{S,b}_{\omega\underline{l}m_l}. \tag{C.322}$$

This implies

$$\begin{aligned}
\big(J_\rho(k_{\Delta t}\triangleright\phi)\big)^{S,a}_{\omega\underline{l}m_l} &= \int\!\mathrm{d}\omega' \sum_{\underline{l}',m_l'} \Big\{ j^{S,aa}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \mathrm{e}^{\mathrm{i}\omega'\Delta t}\,\phi^{S,a}_{\omega'\underline{l}'m_l'} + j^{S,ab}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \mathrm{e}^{\mathrm{i}\omega'\Delta t}\,\phi^{S,b}_{\omega'\underline{l}'m_l'} \\
&\qquad + \tilde{j}^{S,aa}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega'\Delta t}\,\overline{\phi^{S,a}_{\omega'\underline{l}'m_l'}} + \tilde{j}^{S,ab}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega'\Delta t}\,\overline{\phi^{S,b}_{\omega'\underline{l}'m_l'}} \Big\} \\
\stackrel{!}{=} \big(k_{\Delta t}\triangleright(J_\rho\phi)\big)^{S,a}_{\omega\underline{l}m_l} &= \mathrm{e}^{\mathrm{i}\omega\Delta t} \int\!\mathrm{d}\omega' \sum_{\underline{l}',m_l'} \Big\{ j^{S,aa}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \phi^{S,a}_{\omega'\underline{l}'m_l'} + j^{S,ab}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \phi^{S,b}_{\omega'\underline{l}'m_l'} \\
&\qquad + \tilde{j}^{S,aa}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \overline{\phi^{S,a}_{\omega'\underline{l}'m_l'}} + \tilde{j}^{S,ab}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} \overline{\phi^{S,b}_{\omega'\underline{l}'m_l'}} \Big\}.
\end{aligned}$$

We want the upper and lower line to coincide for all solutions ϕ, which implies (with δ denoting Dirac deltas)

$$\begin{aligned}
j^{S,aa}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} &= \delta(\omega-\omega')\, j^{S,aa}_\omega\!\begin{pmatrix}\underline{l}\,m_l\\ \underline{l}'m_l'\end{pmatrix} & j^{S,ab}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} &= \delta(\omega-\omega')\, j^{S,ab}_\omega\!\begin{pmatrix}\underline{l}\,m_l\\ \underline{l}'m_l'\end{pmatrix} \\
\tilde{j}^{S,aa}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} &= \delta(\omega+\omega')\, \tilde{j}^{S,aa}_\omega\!\begin{pmatrix}\underline{l}\,m_l\\ \underline{l}'m_l'\end{pmatrix} & \tilde{j}^{S,ab}\!\begin{pmatrix}\omega\,\underline{l}\,m_l\\ \omega'\underline{l}'m_l'\end{pmatrix} &= \delta(\omega+\omega')\, \tilde{j}^{S,ab}_\omega\!\begin{pmatrix}\underline{l}\,m_l\\ \underline{l}'m_l'\end{pmatrix}.
\end{aligned} \tag{C.323}$$

In the same way

$$\big(J_\rho(k_{\Delta t}\triangleright\phi)\big)^{S,b}_{\omega\underline{l}m_l} \stackrel{!}{=} \big(k_{\Delta t}\triangleright(J_\rho\phi)\big)^{S,b}_{\omega\underline{l}m_l}$$

implies

$$\begin{aligned} j^{S,ba}\binom{\omega\,\underline{l}\,m_l}{\omega'\underline{l}\,'m_l'} &= \delta(\omega-\omega')\, j^{S,ba}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} \qquad & j^{S,bb}\binom{\omega\,\underline{l}\,m_l}{\omega'\underline{l}\,'m_l'} &= \delta(\omega-\omega')\, j^{S,bb}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} \\ \tilde{j}^{S,ba}\binom{\omega\,\underline{l}\,m_l}{\omega'\underline{l}\,'m_l'} &= \delta(\omega+\omega')\, \tilde{j}^{S,ba}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} \qquad & \tilde{j}^{S,bb}\binom{\omega\,\underline{l}\,m_l}{\omega'\underline{l}\,'m_l'} &= \delta(\omega+\omega')\, \tilde{j}^{S,bb}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}. \end{aligned} \tag{C.324}$$

Thus the actions (C.321) become a bit simpler:

$$\begin{aligned} \big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} &= \sum_{\underline{l}',m_l'}\Big\{ j^{S,aa}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\phi^{S,a}_{\omega\underline{l}'m_l'} + j^{S,ab}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\phi^{S,b}_{\omega\underline{l}'m_l'} \\ &\qquad + \tilde{j}^{S,aa}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\overline{\phi^{S,a}_{-\omega,\underline{l}'m_l'}} + \tilde{j}^{S,ab}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\overline{\phi^{S,b}_{-\omega,\underline{l}'m_l'}}\Big\} \\ \big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} &= \sum_{\underline{l}',m_l'}\Big\{ j^{S,ba}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\phi^{S,a}_{\omega\underline{l}'m_l'} + j^{S,bb}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\phi^{S,b}_{\omega\underline{l}'m_l'} \\ &\qquad + \tilde{j}^{S,ba}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\overline{\phi^{S,a}_{-\omega,\underline{l}'m_l'}} + \tilde{j}^{S,bb}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\overline{\phi^{S,b}_{-\omega,\underline{l}'m_l'}}\Big\}. \end{aligned} \tag{C.325}$$

Next we do the same for the action (C.151) of rotations:

$$\big(\hat{R}(\underline{\alpha})\triangleright\phi\big)^{S,a}_{\omega l\,\underline{\tilde{l}}\,m_l} = \sum_{\underline{\tilde{l}}',m_l'}\phi^{S,a}_{\omega l\,\underline{\tilde{l}}'\,m_l'}\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\Big)_{m_l m_l'} \qquad \big(\hat{R}(\underline{\alpha})\triangleright\phi\big)^{S,b}_{\omega l\,\underline{\tilde{l}}\,m_l} = \sum_{\underline{\tilde{l}}',m_l'}\phi^{S,b}_{\omega l\,\underline{\tilde{l}}'\,m_l'}\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}'}(\underline{\alpha})\Big)_{m_l m_l'}.$$

We thus find

$$\begin{aligned} \big(J_\rho(\hat{R}(\underline{\alpha})\triangleright\phi)\big)^{S,a}_{\omega\underline{l}m_l} &= \sum_{\underline{l}',m_l'}\sum_{\underline{\tilde{l}}'',m_l''}\Big\{ j^{S,aa}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\phi^{S,a}_{\omega l'\,\underline{\tilde{l}}''m_l''}\Big(D^{l'}_{\underline{\tilde{l}}',\underline{\tilde{l}}''}(\underline{\alpha})\Big)_{m_l'm_l''} + j^{S,ab}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\phi^{S,b}_{\omega l'\,\underline{\tilde{l}}''m_l''}\Big(D^{l'}_{\underline{\tilde{l}}',\underline{\tilde{l}}''}(\underline{\alpha})\Big)_{m_l'm_l''} \\ &\qquad + \tilde{j}^{S,aa}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\overline{\phi^{S,a}_{-\omega,l'\,\underline{\tilde{l}}''m_l''}}\,\overline{\Big(D^{l'}_{\underline{\tilde{l}}',\underline{\tilde{l}}''}(\underline{\alpha})\Big)_{m_l'm_l''}} + \tilde{j}^{S,ab}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'}\overline{\phi^{S,b}_{-\omega,l'\,\underline{\tilde{l}}''m_l''}}\,\overline{\Big(D^{l'}_{\underline{\tilde{l}}',\underline{\tilde{l}}''}(\underline{\alpha})\Big)_{m_l'm_l''}}\Big\} \\ \stackrel{!}{=} \big(\hat{R}(\underline{\alpha})\triangleright(J_\rho\phi)\big)^{S,a}_{\omega\underline{l}m_l} &= \sum_{\underline{l}',m_l'}\sum_{\underline{\tilde{l}}'',m_l''}\Big\{ j^{S,aa}_{\omega}\binom{l\,\underline{\tilde{l}}''m_l''}{l'\underline{\tilde{l}}'\,m_l'}\phi^{S,a}_{\omega l'\,\underline{\tilde{l}}'m_l'}\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}''}(\underline{\alpha})\Big)_{m_l m_l''} + j^{S,ab}_{\omega}\binom{l\,\underline{\tilde{l}}''m_l''}{l'\underline{\tilde{l}}'\,m_l'}\phi^{S,b}_{\omega l'\,\underline{\tilde{l}}'m_l'}\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}''}(\underline{\alpha})\Big)_{m_l m_l''} \\ &\qquad + \tilde{j}^{S,aa}_{\omega}\binom{l\,\underline{\tilde{l}}''m_l''}{l'\underline{\tilde{l}}'\,m_l'}\overline{\phi^{S,a}_{-\omega,l'\,\underline{\tilde{l}}'m_l'}}\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}''}(\underline{\alpha})\Big)_{m_l m_l''} + \tilde{j}^{S,ab}_{\omega}\binom{l\,\underline{\tilde{l}}''m_l''}{l'\underline{\tilde{l}}'\,m_l'}\overline{\phi^{S,b}_{-\omega,l'\,\underline{\tilde{l}}'m_l'}}\Big(D^{l}_{\underline{\tilde{l}},\underline{\tilde{l}}''}(\underline{\alpha})\Big)_{m_l m_l''}\Big\}. \end{aligned}$$

We want the upper and lower equality to coincide for all solutions ϕ, which implies (with δ denoting Kronecker deltas and using (A.37))

$$\begin{aligned} j^{S,aa}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} &= j^{S,aa}_{\omega l}\,\delta^{(d-2)}_{\underline{l},\underline{l}'}\,\delta_{m_l m_l'} \qquad & j^{S,ab}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} &= j^{S,ab}_{\omega l}\,\delta^{(d-2)}_{\underline{l},\underline{l}'}\,\delta_{m_l m_l'} \\ \tilde{j}^{S,aa}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} &= \tilde{j}^{S,aa}_{\omega l}\,\delta^{(d-2)}_{\underline{l},\underline{l}'}\,\delta_{m_l,-m_l'} \qquad & \tilde{j}^{S,ab}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} &= \tilde{j}^{S,ab}_{\omega l}\,\delta^{(d-2)}_{\underline{l},\underline{l}'}\,\delta_{m_l,-m_l'} \end{aligned} \tag{C.326}$$

In the same way we can show that also

$$\begin{aligned} j^{S,ba}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} &= j^{S,ba}_{\omega l}\,\delta^{(d-2)}_{\underline{l},\underline{l}'}\,\delta_{m_l m_l'} \qquad & j^{S,bb}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} &= j^{S,bb}_{\omega l}\,\delta^{(d-2)}_{\underline{l},\underline{l}'}\,\delta_{m_l m_l'} \\ \tilde{j}^{S,ba}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} &= \tilde{j}^{S,ba}_{\omega l}\,\delta^{(d-2)}_{\underline{l},\underline{l}'}\,\delta_{m_l,-m_l'} \qquad & \tilde{j}^{S,bb}_{\omega}\binom{\underline{l}\,m_l}{\underline{l}\,'m_l'} &= \tilde{j}^{S,bb}_{\omega l}\,\delta^{(d-2)}_{\underline{l},\underline{l}'}\,\delta_{m_l,-m_l'} \end{aligned} \tag{C.327}$$

Therefore we can now write the action (C.325) of our complex structure in a much simpler way:

$$\begin{aligned}\big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} &= j^{S,aa}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l} + \tilde{j}^{S,aa}_{\omega l}\,\overline{\phi^{S,a}_{-\omega,\underline{l},-m_l}} + j^{S,ab}_{\omega l}\,\phi^{S,b}_{\omega\underline{l}m_l} + \tilde{j}^{S,ab}_{\omega l}\,\overline{\phi^{S,b}_{-\omega,\underline{l},-m_l}} \\ \big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} &= j^{S,ba}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l} + \tilde{j}^{S,ba}_{\omega l}\,\overline{\phi^{S,a}_{-\omega,\underline{l},-m_l}} + j^{S,bb}_{\omega l}\,\phi^{S,b}_{\omega\underline{l}m_l} + \tilde{j}^{S,bb}_{\omega l}\,\overline{\phi^{S,b}_{-\omega,\underline{l},-m_l}}.\end{aligned} \tag{C.328}$$

For real solutions ϕ this can also be written as

$$\begin{aligned}\big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} &= \big(j^{S,aa}_{\omega l} + \tilde{j}^{S,aa}_{\omega l}\big)\,\phi^{S,a}_{\omega\underline{l}m_l} + \big(j^{S,ab}_{\omega l} + \tilde{j}^{S,ab}_{\omega l}\big)\,\phi^{S,b}_{\omega\underline{l}m_l} \\ \big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} &= \big(j^{S,ba}_{\omega l} + \tilde{j}^{S,ba}_{\omega l}\big)\,\phi^{S,a}_{\omega\underline{l}m_l} + \big(j^{S,bb}_{\omega l} + \tilde{j}^{S,bb}_{\omega l}\big)\,\phi^{S,b}_{\omega\underline{l}m_l}.\end{aligned} \tag{C.329}$$

We can now redefine $\big(j^{S,aa}_{\omega l} + \tilde{j}^{S,aa}_{\omega l}\big) \;\to\; j^{S,aa}_{\omega l}$ and ditto for the other j^S, and obtain

$$\begin{aligned}\big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} &= j^{S,aa}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l} + j^{S,ab}_{\omega l}\,\phi^{S,b}_{\omega\underline{l}m_l} \\ \big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} &= j^{S,ba}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l} + j^{S,bb}_{\omega l}\,\phi^{S,b}_{\omega\underline{l}m_l}.\end{aligned} \tag{C.330}$$

For real solutions the actions (C.330) and (C.328) are completely indentical while for complex solutions they are not. Because we are imposing all properties of the complex structure only for real solutions, we can choose between both actions at will. Since action (C.330) has the advantage of being linear also with respect to complex linear combinations of complex solutions, we shall choose this one. We could actually have imposed it right from the start, we can see now that our less restrictive ansatz (C.321) does not yield any advantage.

C.4.4 J_ρ: essential properties

We recall that real solutions ϕ are those with $\phi^{S,a}_{-\omega,\underline{l},-m_l} = \overline{\phi^{S,a}_{\omega\underline{l}\,m_l}}$ while $\phi^{S,b}_{-\omega,\underline{l},-m_l} = \overline{\phi^{S,b}_{\omega\underline{l}\,m_l}}$ in the S-expansion (2.186):

$$\phi(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega \sum_{\underline{l},m_l}\Big\{\phi^{S,a}_{\omega\underline{l}m_l}\,\mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^{S,b}_{\omega\underline{l}m_l}\,\mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega).\Big\} \tag{C.331}$$

Applying the action of J_ρ from (C.330) to this expansion, we can read off that the condition that J_ρ must turn real solutions into real solutions becomes

$$\begin{aligned} j^{S,aa}_{-\omega,l} &\overset{!}{=} \overline{j^{S,aa}_{\omega l}} \qquad & j^{S,ab}_{-\omega,l} &\overset{!}{=} \overline{j^{S,ab}_{\omega l}} \\ j^{S,ba}_{-\omega,l} &\overset{!}{=} \overline{j^{S,ba}_{\omega l}} \qquad & j^{S,bb}_{-\omega,l} &\overset{!}{=} \overline{j^{S,bb}_{\omega l}}.\end{aligned} \tag{C.332}$$

We note that this forces all j-factors to be real for $\omega = 0$. Applying the complex structure (C.330) twice to the expansion (C.331), it is straightforward to check that the property $J_\rho^2 = -\mathbb{1}$ amounts to the conditions

$$-1 \overset{!}{=} j^{S,aa}_{\omega l}\,j^{S,aa}_{\omega l} + j^{S,ab}_{\omega l}\,j^{S,ba}_{\omega l} \tag{C.333}$$

$$-1 \overset{!}{=} j^{S,bb}_{\omega l}\,j^{S,bb}_{\omega l} + j^{S,ab}_{\omega l}\,j^{S,ba}_{\omega l} \tag{C.334}$$

$$0 \overset{!}{=} j^{S,ab}_{\omega l}\left(j^{S,aa}_{\omega l} + j^{S,bb}_{\omega l}\right) \tag{C.335}$$

$$0 \overset{!}{=} j^{S,ba}_{\omega l}\left(j^{S,aa}_{\omega l} + j^{S,bb}_{\omega l}\right). \tag{C.336}$$

Requiring compatibility $\omega_\rho(J_\rho\eta, J_\rho\zeta) = \omega_\rho(\eta,\zeta)$ with the symplectic structure (2.195)

$$\omega_\rho(\eta,\zeta) \;=\; \pi R^{d-1}_{\text{AdS}}\int\!\mathrm{d}\omega\sum_{\underline{l},m_l}(2l+d-2)\left\{\eta^{S,a}_{\omega\underline{l}m_l}\,\zeta^{S,b}_{-\omega,\underline{l},-m_l} - \eta^{S,b}_{\omega\underline{l}m_l}\,\zeta^{S,a}_{-\omega,\underline{l},-m_l}\right\} \tag{C.337}$$

implies the conditions

$$+1 \overset{!}{=} j^{S,aa}_{\omega l}\,\overline{j^{S,bb}_{\omega l}} - j^{S,ba}_{\omega l}\,\overline{j^{S,ab}_{\omega l}}, \tag{C.338}$$

$$0 \overset{!}{=} j^{S,aa}_{\omega l}\,\overline{j^{S,ba}_{\omega l}} - \overline{j^{S,aa}_{\omega l}}\,j^{S,ba}_{\omega l}, \tag{C.339}$$

$$0 \overset{!}{=} j^{S,bb}_{\omega l}\,\overline{j^{S,ab}_{\omega l}} - \overline{j^{S,bb}_{\omega l}}\,j^{S,ab}_{\omega l}. \tag{C.340}$$

The last two equalities simply say that the products $j^{S,aa}_{\omega l}\,\overline{j^{S,ba}_{\omega l}}$ and $j^{S,bb}_{\omega l}\,\overline{j^{S,ab}_{\omega l}}$ must be real. Another property that we would like our complex structure to induce, is that the real g-product $g_\rho(\eta,\zeta) := \omega_\rho(\eta, J_\rho\zeta)$ becomes real for real solutions. Using the following property of real solutions η,ζ (resulting from reversing the integration and summation direction)

$$\begin{aligned}\int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \eta^{S,a}_{\omega\underline{l}m_l}\,\zeta^{S,b}_{-\omega,\underline{l},-m_l}\,j^{S,ba}_{-\omega,\underline{l},-m_l} &= \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \eta^{S,a}_{\omega\underline{l}m_l}\,\overline{\zeta^{S,b}_{\omega\underline{l}m_l}}\,\overline{j^{S,ba}_{\omega l}} \\ &= \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \Big(\tfrac{1}{2}\,\eta^{S,a}_{\omega\underline{l}m_l}\,\overline{\zeta^{S,b}_{\omega\underline{l}m_l}}\,\overline{j^{S,ba}_{\omega l}} + \tfrac{1}{2}\,\overline{\eta^{S,a}_{\omega\underline{l}m_l}}\,\zeta^{S,b}_{\omega\underline{l}m_l}\,j^{S,ba}_{\omega l}\Big)\end{aligned} \tag{C.341}$$

it can be quickly read off that the "real-to-real" property C.332 already assures that the real g-product $g(\eta,\zeta)$ is real for real solutions η,ζ, hence this condition induces no new equalities. This is expected, since the symplectic structure ω_ρ is real for real solutions, and hence $\omega_\rho(\eta, J_\rho\zeta)$ must be real because we already implemented that for real ζ we have $J_\rho\zeta$ real as well.

Thus we have the following conditions:

$$\text{(C.339)} \implies \quad j^{S,aa}_{\omega l}\,\overline{j^{S,ba}_{\omega l}} \overset{!}{=} \mathbb{R}\text{eal} \tag{C.342}$$

$$\text{(C.340)} \implies \quad j^{S,bb}_{\omega l}\,\overline{j^{S,ab}_{\omega l}} \overset{!}{=} \mathbb{R}\text{eal} \tag{C.343}$$

$$\text{(C.333)} \implies \quad \big(j^{S,aa}_{\omega l}\big)^2 + j^{S,ab}_{\omega l}\,j^{S,ba}_{\omega l} \overset{!}{=} -1 \tag{C.344}$$

$$\text{(C.334)} \implies \quad \big(j^{S,bb}_{\omega l}\big)^2 + j^{S,ab}_{\omega l}\,j^{S,ba}_{\omega l} \overset{!}{=} -1 \tag{C.345}$$

$$\text{(C.338)} \implies \quad j^{S,aa}_{\omega l}\,\overline{j^{S,bb}_{\omega l}} - j^{S,ba}_{\omega l}\,\overline{j^{S,ab}_{\omega l}} \overset{!}{=} +1 \tag{C.346}$$

$$\text{(C.335)} \implies \quad j^{S,ab}_{\omega l}\Big(j^{S,aa}_{\omega l} + j^{S,bb}_{\omega l}\Big) \overset{!}{=} 0 \tag{C.347}$$

$$\text{(C.336)} \implies \quad j^{S,ba}_{\omega l}\big(j^{S,aa}_{\omega l} + j^{S,bb}_{\omega l}\big) \overset{!}{=} 0. \tag{C.348}$$

One restrictive relation following from these is

$$\text{(C.344)} - \text{(C.345)} \implies \quad (j^{S,bb}_{\omega l})^2 \overset{!}{=} (j^{S,aa}_{\omega l})^2 \quad \implies \quad j^{S,bb}_{\omega l} \overset{!}{=} \pm j^{S,aa}_{\omega l}. \tag{C.349}$$

Inserting this into (C.346) implies that also

$$j^{S,ab}_{\omega l}\,\overline{j^{S,ba}_{\omega l}} \overset{!}{=} \mathbb{R}\text{eal}. \tag{C.350}$$

We collect the different solutions of these conditions through case distinctions. One way to do this, is to distinguish first between $\mathbb{R}\mathrm{e}\, j^{S,ab}_{\omega l} = 0$ and $\mathbb{R}\mathrm{e}\, j^{S,ab}_{\omega l} \neq 0$.

For the first case $\mathbb{R}\mathrm{e}\, j^{S,ab}_{\omega l} = 0$, that is, for $j^{S,ab}_{\omega l}$ purely imaginary, the conditions (C.350), (C.343) and (C.342) imply that the other three j-factors must be imaginary as well. To evaluate this further, we distinguish now also the two different cases of (C.349). The minus-case $j^{S,bb}_{\omega l} = -j^{S,aa}_{\omega l}$ together with all j-factors being imaginary leads to a contradiction between (C.344) and (C.346). Hence here we can only choose the plus-case $j^{S,bb}_{\omega l} = +j^{S,aa}_{\omega l}$. Then, with all j-factors being imaginary, taking the difference of (C.344) and (C.346) yields $(j^{S,aa}_{\omega l})^2 = -1$, and hence $j^{S,aa}_{\omega l} = \pm\mathrm{i}$. With (C.347) and (C.348) this implies that $j^{S,ab}_{\omega l} = j^{S,ba}_{\omega l} = 0$. Thus, our first case is:

$$\begin{aligned}
&\mathbb{R}\mathrm{e}\, j^{S,ab}_{\omega l} = 0 \qquad \Longleftrightarrow \qquad \mathbb{R}\mathrm{e}\, j^{S,ba}_{\omega l} = 0 \\
\Longrightarrow\quad & j^{S,aa}_{\omega l} = j^{S,bb}_{\omega l} = \pm\mathrm{i} \quad \text{with} \quad j^{S,aa}_{-\omega,l} = \overline{j^{S,aa}_{\omega,l}} = -j^{S,aa}_{\omega,l}, \\
\Longrightarrow\quad & j^{S,ab}_{\omega l} = j^{S,ba}_{\omega l} = 0.
\end{aligned} \tag{C.351}$$

We shall call it diagonal case because of the last line, and note that it is similar to the complex structure (C.311) on equal-time surfaces (but the symplectic structure on equal-time surfaces has a different form than the one we use on hypercylinders). We also note, that due to the second line this complex structure does not work at $\omega = 0$. Since so far we have only used the essentail properties of the complex structure plus commutation with time-translations and spatial rotations, this diagonal complex structure will occur not only for AdS, but rather for every spacetime that has time-translations and spatial rotations as isometries (for example Minkowski and Schwarzschild).

For the second case $\mathbb{R}\mathrm{e}\, j^{S,ab}_{\omega l} \neq 0$, we note that (C.347) and (C.348) immediately imply the minus-case $j^{S,bb}_{\omega l} = -j^{S,aa}_{\omega l}$ of (C.349), leaving us with only three undetermined j-factors. Next, we find that all j-factors must be real now. Suppressing for a moment the labels (ω, l), this can be shown by writing each j-factor as $j = j_{\mathbb{R}} + \mathrm{i} j_{\mathbb{I}}$ and going through the conditions. However, it is a bit faster to use polar coordinates: we write $j^{S,ab} = r_{ab}\,\mathrm{e}^{\mathrm{i}\varphi_{ab}}$, etc., with $\varphi_{ab} \in (-\pi, +\pi]$. Then, in order for getting real products in the conditions (C.342), (C.343) and (C.350), the differences of each two angles must be integer multiples of π, e.g.: $\varphi_{aa} - \varphi_{ba} = n\pi$ with $n \in \mathbb{Z}$. This means, that while all three remaining j-factors are complex numbers, they are real multiples of each other. That is, we can write $j^{S,aa}_{\omega l} = u\, j^{S,ab}_{\omega l}$ with $u \in \mathbb{R}$, and $j^{S,ba}_{\omega l} = v\, j^{S,ab}_{\omega l}$ with $v < 0$ due to (C.346). Then, (C.344) implies that $(j^{S,ab}_{\omega l})^2 \in \mathbb{R}$. Thus $j^{S,ab}_{\omega l}$ is either purely imaginary, or real. Since we consider the case $\mathbb{R}\mathrm{e}\, j^{S,ab}_{\omega l} \neq 0$, it cannot be purely imaginary, and therefore must be real, making the other j-factors real as well. Now conditions (C.344) and (C.346) coincide, and we cannot further simplify the j-factors for this case. Thus, our second case is:

$$\begin{aligned}
&\mathbb{R}\mathrm{e}\, j^{S,ab}_{\omega l} \neq 0 \qquad \Longleftrightarrow \qquad \mathbb{R}\mathrm{e}\, j^{S,ba}_{\omega l} \neq 0 \\
\Longrightarrow\quad & j^{S,bb}_{\omega l} = -j^{S,aa}_{\omega l} \quad \text{with all} \quad j^{S,\cdot\cdot}_{\omega l} \in \mathbb{R}, \\
\Longrightarrow\quad & (j^{S,aa}_{\omega l})^2 = -1 - j^{S,ab}_{\omega l} j^{S,ba}_{\omega l} \quad \text{with} \quad j^{S,ab}_{-\omega,l} = j^{S,ab}_{\omega,l}, \quad \text{and} \quad j^{S,ba}_{-\omega,l} = j^{S,ba}_{\omega,l}.
\end{aligned} \tag{C.352}$$

We shall call it nondiagonal case, which in the special case of $j^{S,aa}_{\omega l} = 0$ becomes antidiagonal. However, a priori nothing forces the diagonal element $j^{S,aa}$ to vanish. Further, we need $j^{S,ab}$ and $j^{S,ba}$ to have opposite sign: this is necessary to make $j^{S,aa}$ real. As opposed to the diagonal case, here no problem arises at $\omega = 0$.

It remains to study which further conditions are imposed on J_ρ by requiring a positive-definite g_ρ, and/or requiring it to commute with the actions of the boosts.

C.4.5 J_ρ: positive-definite induced g_ρ

For real solutions η, ζ we would like the real g-product $\mathrm{g}_\rho(\eta,\zeta)$ to be positive-definite. However, we note that in [?] it is shown that this is only a practical property, not a necessary one, because a consistent quantization can be set up for an indefinite g_ρ as well.

First, we consider the real g-product induced by the diagonal case (C.351). In particular, we choose $j^{S,aa}_{\omega,l} = +\mathrm{i}$ for $\omega > 0$ while $j^{S,aa}_{\omega,l} = -\mathrm{i}$ for $\omega < 0$. As remarked above, the point $\omega = 0$ cannot be included in this case. Using C.337 we obtain

$$\mathrm{g}_\rho(\eta,\zeta) \;=\; \mathrm{i}\pi R^{d-1}_{\mathrm{AdS}} \int_0^\infty \mathrm{d}\omega \sum_{\underline{l},m_l} (2l+d-2) \Big\{ -\eta^{S,a}_{\omega \underline{l} m_l}\, \zeta^{S,b}_{-\omega,\underline{l},-m_l} + \eta^{S,a}_{-\omega,\underline{l},-m_l}\, \zeta^{S,b}_{\omega \underline{l} m_l} + \eta^{S,b}_{\omega \underline{l} m_l}\, \zeta^{S,a}_{-\omega,\underline{l},-m_l} - \eta^{S,b}_{-\omega,\underline{l},-m_l}\, \zeta^{S,a}_{\omega \underline{l} m_l} \Big\}.$$

For real η, ζ this becomes

$$\begin{aligned}
\mathrm{g}_\rho(\eta,\zeta) &= \mathrm{i}\pi R_{\mathrm{AdS}}^{d-1}\int_0^\infty \mathrm{d}\omega \sum_{\underline{l},m_l}(2l+d-2)\Big\{-\eta^{S,a}_{\omega\underline{l}m_l}\overline{\zeta^{S,b}_{\omega\underline{l}m_l}}+\overline{\eta^{S,a}_{\omega\underline{l}m_l}}\zeta^{S,b}_{\omega\underline{l}m_l}+\eta^{S,b}_{\omega\underline{l}m_l}\overline{\zeta^{S,a}_{\omega\underline{l}m_l}}-\overline{\eta^{S,b}_{\omega\underline{l}m_l}}\zeta^{S,a}_{\omega\underline{l}m_l}\Big\}\\
&= 2\pi R_{\mathrm{AdS}}^{d-1}\int_0^\infty \mathrm{d}\omega \sum_{\underline{l},m_l}(2l+d-2)\ \mathbb{I}\mathrm{m}\Big(\eta^{S,a}_{\omega\underline{l}m_l}\overline{\zeta^{S,b}_{\omega\underline{l}m_l}}+\overline{\eta^{S,b}_{\omega\underline{l}m_l}}\zeta^{S,a}_{\omega\underline{l}m_l}\Big),
\end{aligned}$$

and thus for real ϕ

$$\mathrm{g}_\rho(\phi,\phi) = 2\pi R_{\mathrm{AdS}}^{d-1}\int_0^\infty \mathrm{d}\omega \sum_{\underline{l},m_l}(2l+d-2)\ \mathbb{I}\mathrm{m}\Big(2\mathbb{R}\mathrm{e}\,(\phi^{S,a}_{\omega\underline{l}m_l}\overline{\phi^{S,b}_{\omega\underline{l}m_l}})\Big) = 0.$$

Hence the diagonal case induces a real g-product which gives zero norm to all real solutions. (This is actually independent of the choice $j^{S,aa}_{\omega,l}=+\mathrm{i}$ for $\omega>0$ while $j^{S,aa}_{\omega,l}=-\mathrm{i}$ for $\omega<0$.) We want to avoid this, and therefore we shall prefer the nondiagonal case whenever possible.

Let us consider now the real g-product induced by the nondiagonal case. Using C.337 we obtain:

$$\begin{aligned}
\mathrm{g}_\rho(\eta,\zeta) = \pi R_{\mathrm{AdS}}^{d-1}\int \mathrm{d}\omega \sum_{\underline{l},m_l}(2l+d-2)\ &\Big\{\eta^{S,a}_{\omega\underline{l}m_l}\Big(j^{S,ba}_{\omega l}\,\zeta^{S,a}_{-\omega,\underline{l},-m_l}-j^{S,aa}_{\omega l}\zeta^{S,b}_{-\omega,\underline{l},-m_l}\Big)\\
&-\eta^{S,b}_{\omega\underline{l}m_l}\Big(j^{S,aa}_{\omega l}\zeta^{S,a}_{-\omega,\underline{l},-m_l}+j^{S,ab}_{\omega l}\zeta^{S,b}_{-\omega,\underline{l},-m_l}\Big)\Big\}.
\end{aligned}$$

For real η, ζ this becomes

$$\mathrm{g}_\rho(\eta,\zeta) = \pi R_{\mathrm{AdS}}^{d-1}\int \mathrm{d}\omega \sum_{\underline{l},m_l}(2l+d-2)\ \Big\{\eta^{S,a}_{\omega\underline{l}m_l}\Big(j^{S,ba}_{\omega l}\,\overline{\zeta^{S,a}_{\omega\underline{l}m_l}}-j^{S,aa}_{\omega l}\overline{\zeta^{S,b}_{\omega\underline{l}m_l}}\Big)-\eta^{S,b}_{\omega\underline{l}m_l}\Big(j^{S,aa}_{\omega l}\overline{\zeta^{S,a}_{\omega\underline{l}m_l}}+j^{S,ab}_{\omega l}\overline{\zeta^{S,b}_{\omega\underline{l}m_l}}\Big)\Big\},$$

and thus for real ϕ we get

$$\mathrm{g}_\rho(\phi,\phi)=\pi R_{\mathrm{AdS}}^{d-1}\int \mathrm{d}\omega \sum_{\underline{l},m_l}(2l+d-2)\ \Big\{j^{S,ba}_{\omega l}\big|\phi^{S,a}_{\omega\underline{l}m_l}\big|^2-j^{S,ab}_{\omega l}\big|\phi^{S,b}_{\omega\underline{l}m_l}\big|^2-2j^{S,aa}_{\omega l}\mathbb{R}\mathrm{e}\big(\phi^{S,a}_{\omega\underline{l}m_l}\overline{\phi^{S,b}_{\omega\underline{l}m_l}}\big)\Big\}. \tag{C.353}$$

By considering solutions ϕ with either $\phi^{S,a}$ or $\phi^{S,b}$ vanishing, this tells us that for a positive-definite real g-product we need

$$\mathbb{R}\mathrm{e}\,\big(j^{S,ba}_{\omega l}\big)>0 \qquad\qquad \mathbb{R}\mathrm{e}\,\big(j^{S,ab}_{\omega l}\big)<0. \tag{C.354}$$

In matrix notation (C.353) writes as a symmetric quadratic form:

$$\mathrm{g}_\rho(\phi,\phi)=\pi R_{\mathrm{AdS}}^{d-1}\int \mathrm{d}\omega \sum_{\underline{l},m_l}(2l+d-2)\ \Big(\overline{\phi^{S,a}_{\omega\underline{l}m_l}},\ \ \overline{\phi^{S,b}_{\omega\underline{l}m_l}}\Big)\begin{pmatrix}+j^{S,ba}_{\omega l} & -j^{S,aa}_{\omega l}\\ -j^{S,aa}_{\omega l} & -j^{S,ab}_{\omega l}\end{pmatrix}\begin{pmatrix}\phi^{S,a}_{\omega\underline{l}m_l}\\ \phi^{S,b}_{\omega\underline{l}m_l}\end{pmatrix}. \tag{C.355}$$

Sylvester's criterion tells us, that this form is positive-definite, if $j^{S,ba}_{\omega l}$ and the below determinant are both positive:

$$\begin{vmatrix}+j^{S,ba}_{\omega l} & -j^{S,aa}_{\omega l}\\ -j^{S,aa}_{\omega l} & -j^{S,ab}_{\omega l}\end{vmatrix} = -j^{S,ba}_{\omega l}j^{S,ab}_{\omega l}-(j^{S,aa}_{\omega l})^2>0. \tag{C.356}$$

The nondiagonal case (C.352) already includes that this determinant has value one. Hence for the nondiagonal case a positive-definite g_ρ is ensured by

$$j^{S,ab}_{\omega l} < 0 \qquad \text{with} \qquad j^{S,ba}_{\omega l} > 0. \tag{C.357}$$

For the nondiagonal case we thus have established the following complex structure, wherein the last line is only required for a positive-definite g_ρ:

$$\begin{aligned}
\big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} &= j^{S,aa}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l} + j^{S,ab}_{\omega l}\,\phi^{S,b}_{\omega\underline{l}m_l}\\
\big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} &= j^{S,ba}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l} - j^{S,aa}_{\omega l}\,\phi^{S,b}_{\omega\underline{l}m_l}\\
\big(j^{S,aa}_{\omega l}\big)^2 &= -j^{S,ab}_{\omega l}\,j^{S,ba}_{\omega l} - 1 \geq 0\\
j^{S,ab}_{-\omega,l} &= j^{S,ab}_{\omega l} \qquad\qquad j^{S,ba}_{-\omega,l} = j^{S,ba}_{\omega l}\\
j^{S,ab}_{\omega l} &< 0 \qquad\qquad\quad\; j^{S,ba}_{\omega l} > 0.
\end{aligned} \tag{C.358}$$

We remark that another reason for desiring a positive-definite g_ρ is that it prevents $j^{S,ab}_{\omega l}$ and $j^{S,ba}_{\omega l}$ from becoming zero, and thereby avoids the occurrence of the diagonal case for any frequency ω.

C.4.6 J_ρ: commutation with boosts

Next we check what restrictions the commutation condition induces for the j^S-factors. In Section C.3.3 we work out the action of the boost generators, resulting in:

$$\begin{aligned}
\big(K_{0d}\triangleright\phi\big)^{S,a}_{\omega\underline{l}m_l} &= \tfrac{\mathrm{i}}{2}\,\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2}\,\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2}\,z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2}\,z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l-1,\underline{\tilde{l}},m_l}\\
\big(K_{0d}\triangleright\phi\big)^{S,b}_{\omega\underline{l}m_l} &= \tfrac{\mathrm{i}}{2}\,\tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,b}_{\omega-1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2}\,\tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,b}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2}\,z^{(S,b)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,b}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \tfrac{\mathrm{i}}{2}\,z^{(S,b)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,b}_{\omega+1,l-1,\underline{\tilde{l}},m_l}
\end{aligned}$$

$$\begin{aligned}
\big(K_{d+1,d}\triangleright\phi\big)^{S,a}_{\omega\underline{l}m_l} &= -\tfrac{1}{2}\,\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l} - \tfrac{1}{2}\,\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \tfrac{1}{2}\,z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \tfrac{1}{2}\,z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l-1,\underline{\tilde{l}},m_l}\\
\big(K_{d+1,d}\triangleright\phi\big)^{S,b}_{\omega\underline{l}m_l} &= -\tfrac{1}{2}\,\tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,b}_{\omega-1,l+1,\underline{\tilde{l}},m_l} - \tfrac{1}{2}\,\tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,b}_{\omega-1,l-1,\underline{\tilde{l}},m_l} + \tfrac{1}{2}\,z^{(S,b)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,b}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \tfrac{1}{2}\,z^{(S,b)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,b}_{\omega+1,l-1,\underline{\tilde{l}},m_l}.
\end{aligned}$$

First, we consider the diagonal case (C.351) of the complx structure J_ρ:

$$\begin{aligned}
&j^{S,aa}_{\omega l} = j^{S,bb}_{\omega l} = \pm\mathrm{i} \quad \text{with} \quad j^{S,aa}_{-\omega,l} = -j^{S,aa}_{\omega,l},\\
&j^{S,ab}_{\omega l} = j^{S,ba}_{\omega l} = 0.
\end{aligned}$$

For simplicity we choose $j^{S,aa}_{\omega l} = \pm\mathrm{i}$ for all $\omega \gtrless 0$. We also recall that the diagonal case does not work at $\omega = 0$ due to frequency symmetry $j^{S,aa}_{-\omega,l} = -j^{S,aa}_{\omega,l}$. In order to verify whether the diagonal case commutes with the boosts' actions, let us compare the actions of $(J_\rho(K_{0d}\triangleright\phi))^{S,a}_{\omega\underline{l}m_l}$ and $(K_{0d}\triangleright(J_\rho\phi))^{S,a}_{\omega\underline{l}m_l}$. For clearer notation, we assume here that $\omega > 0$.

$$\begin{aligned}
\Big(J_\rho\big(K_{0d}\triangleright\phi\big)\Big)^{S,a}_{\omega\underline{l}m_l} = +\mathrm{i}\,\big(K_{0d}\triangleright\phi\big)^{S,a}_{\omega\underline{l}m_l} = +\mathrm{i}\,\Big\{&+\frac{\mathrm{i}}{2}\,\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l} + \frac{\mathrm{i}}{2}\,\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l}\\
&+\frac{\mathrm{i}}{2}\,z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \frac{\mathrm{i}}{2}\,z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l-1,\underline{\tilde{l}},m_l}\Big\}
\end{aligned} \tag{C.359}$$

$$\begin{aligned}
\Big(K_{0d}\triangleright\big(J_\rho\phi\big)\Big)^{S,a}_{\omega\underline{l}m_l} = &+\frac{\mathrm{i}}{2}\,j^{S,aa}_{\omega-1,l+1}\,\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l} + \frac{\mathrm{i}}{2}\,j^{S,aa}_{\omega-1,l-1}\,\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l}\\
&+\frac{\mathrm{i}}{2}\,j^{S,aa}_{\omega+1,l+1}\,z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l+1,\underline{\tilde{l}},m_l} + \frac{\mathrm{i}}{2}\,j^{S,aa}_{\omega+1,l-1}\,z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l-1,\underline{\tilde{l}},m_l}
\end{aligned} \tag{C.360}$$

We can read off, that for $\omega > 1$ in (C.360) all the factors $j^{S,aa}_{\omega\pm1,l\pm1}$ are simply $+\mathrm{i}$, making (C.359) agree with (C.360). However, for $\omega < 1$ the factors $j^{S,aa}_{\omega-1,l\mp1}$ become $-\mathrm{i}$, making (C.359) disagree with (C.360). For the infinitesimal boosts considered here, this disagreement only occurs for $|\omega| < 1$. However, if we proceed to consider finite boosts, then there appear not only frequency shifts of ± 1, but of ± 2, ± 3, and so on. Thus, for finite boosts the disagreement extends to the whole frequency range. (This does not depend on our simple choice of assigning $\pm\mathrm{i}$ to frequencies $\omega \gtrless 0$. We also note that by the same reasoning a particular nondiagonal complex structure in Section C.5.1 fails to commute with the boosts.) Hence, we find that the diagonal complex structure does not commute with the AdS boosts.

Second, we consider the nondiagonal case: from the J_ρ-action (C.358) we derive the combinations of complex structure and boost actions that we give below. We begin with the pair of actions

$$\begin{aligned}\Big(J_\rho\big(K_{0d}\triangleright\phi\big)\Big)^{S,a}_{\omega\underline{l}m_l} = {}& j^{S,aa}_{\omega l}\Big\{+\frac{\mathrm{i}}{2}\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l}+\frac{\mathrm{i}}{2}\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l}\\&+\frac{\mathrm{i}}{2}z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l+1,\underline{\tilde{l}},m_l}+\frac{\mathrm{i}}{2}z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l-1,\underline{\tilde{l}},m_l}\Big\}\\&+j^{S,ab}_{\omega l}\Big\{+\frac{\mathrm{i}}{2}\tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,b}_{\omega-1,l+1,\underline{\tilde{l}},m_l}+\frac{\mathrm{i}}{2}\tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,b}_{\omega-1,l-1,\underline{\tilde{l}},m_l}\\&+\frac{\mathrm{i}}{2}z^{(S,b)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,b}_{\omega+1,l+1,\underline{\tilde{l}},m_l}+\frac{\mathrm{i}}{2}z^{(S,b)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,b}_{\omega+1,l-1,\underline{\tilde{l}},m_l}\Big\}\end{aligned}\tag{C.361}$$

$$\begin{aligned}\Big(K_{0d}\triangleright\big(J_\rho\phi\big)\Big)^{S,a}_{\omega\underline{l}m_l} = {}& \frac{\mathrm{i}}{2}j^{S,aa}_{\omega-1,l+1}\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l}+\frac{\mathrm{i}}{2}j^{S,aa}_{\omega-1,l-1}\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l}\\&+\frac{\mathrm{i}}{2}j^{S,aa}_{\omega+1,l+1}z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l+1,\underline{\tilde{l}},m_l}+\frac{\mathrm{i}}{2}j^{S,aa}_{\omega+1,l-1}z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,a}_{\omega+1,l-1,\underline{\tilde{l}},m_l}\\&+\frac{\mathrm{i}}{2}j^{S,ab}_{\omega-1,l+1}\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\,\phi^{S,b}_{\omega-1,l+1,\underline{\tilde{l}},m_l}+\frac{\mathrm{i}}{2}j^{S,ab}_{\omega-1,l-1}\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\,\phi^{S,b}_{\omega-1,l-1,\underline{\tilde{l}},m_l}\\&+\frac{\mathrm{i}}{2}j^{S,ab}_{\omega+1,l+1}z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\,\phi^{S,b}_{\omega+1,l+1,\underline{\tilde{l}},m_l}+\frac{\mathrm{i}}{2}j^{S,ab}_{\omega+1,l-1}z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\,\phi^{S,b}_{\omega+1,l-1,\underline{\tilde{l}},m_l}\end{aligned}\tag{C.362}$$

We can read off that J_ρ and K_{0d} commute, that is: both equations coincide, if we can fix the j^S-factors such that the following equalities become fulfilled (by comparing the factors in front of $\phi^{S,a}_{\omega-1,l+1,\underline{\tilde{l}},m_l}$ in both equations, then those in front of $\phi^{S,a}_{\omega-1,l-1,\underline{\tilde{l}},m_l}$, and so on):

$$j^{S,aa}_{\omega l}\overset{!}{=}j^{S,aa}_{\omega-1,l+1}\overset{!}{=}j^{S,aa}_{\omega-1,l-1}\overset{!}{=}j^{S,aa}_{\omega+1,l+1}\overset{!}{=}j^{S,aa}_{\omega+1,l-1}\tag{C.363}$$

and

$$\begin{aligned}&j^{S,ab}_{\omega-1,l+1}\,\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}}\overset{!}{=}j^{S,ab}_{\omega l}\,\tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}} &&j^{S,ab}_{\omega-1,l-1}\,\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\overset{!}{=}j^{S,ab}_{\omega l}\,\tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\\&j^{S,ab}_{\omega+1,l+1}\,z^{(S,a)--}_{\omega+1,l+1,\tilde{l}}\overset{!}{=}j^{S,ab}_{\omega l}\,z^{(S,b)--}_{\omega+1,l+1,\tilde{l}} &&j^{S,ab}_{\omega+1,l-1}\,z^{(S,a)-+}_{\omega+1,l-1,\tilde{l}}\overset{!}{=}j^{S,ab}_{\omega l}\,z^{(S,b)+-}_{\omega+1,l-1,\tilde{l}}\end{aligned}\tag{C.364}$$

Further, we have the pairs of actions

$$\begin{aligned}\Big(J_\rho\big(K_{0d}\triangleright\phi\big)\Big)^{S,b}_{\omega\underline{l}m_l}&\overset{!}{=}\Big(K_{0d}\triangleright\big(J_\rho\phi\big)\Big)^{S,b}_{\omega\underline{l}m_l}\\\Big(J_\rho\big(K_{d+1,d}\triangleright\phi\big)\Big)^{S,a}_{\omega\underline{l}m_l}&\overset{!}{=}\Big(K_{d+1,d}\triangleright\big(J_\rho\phi\big)\Big)^{S,a}_{\omega\underline{l}m_l}\\\Big(J_\rho\big(K_{d+1,d}\triangleright\phi\big)\Big)^{S,b}_{\omega\underline{l}m_l}&\overset{!}{=}\Big(K_{d+1,d}\triangleright\big(J_\rho\phi\big)\Big)^{S,b}_{\omega\underline{l}m_l}.\end{aligned}$$

It is straightforward to check that these equalitites hold as well, provided that (C.363), (C.364) and the following equalities are met:

$$\begin{aligned}&j^{S,ba}_{\omega-1,l+1}\,\tilde{z}^{(S,b)+-}_{\omega-1,l+1,\tilde{l}}\overset{!}{=}j^{S,ba}_{\omega l}\,\tilde{z}^{(S,a)+-}_{\omega-1,l+1,\tilde{l}} &&j^{S,ba}_{\omega-1,l-1}\,\tilde{z}^{(S,b)++}_{\omega-1,l-1,\tilde{l}}\overset{!}{=}j^{S,ba}_{\omega l}\,\tilde{z}^{(S,a)++}_{\omega-1,l-1,\tilde{l}}\\&j^{S,ba}_{\omega+1,l+1}\,z^{(S,b)--}_{\omega+1,l+1,\tilde{l}}\overset{!}{=}j^{S,ba}_{\omega l}\,z^{(S,a)--}_{\omega+1,l+1,\tilde{l}} &&j^{S,ba}_{\omega+1,l-1}\,z^{(S,b)-+}_{\omega+1,l-1,\tilde{l}}\overset{!}{=}j^{S,ba}_{\omega l}\,z^{(S,a)+-}_{\omega+1,l-1,\tilde{l}}\end{aligned}\tag{C.365}$$

This means, that if we can find any solution $j^{S,ab}$ of (C.364), then setting $j^{S,ba} = -1/j^{S,ab}$ yields a solution of (C.365). Moreover, (C.364) and (C.365) give us the product

$$j^{S,ab}_{\omega\pm1,l\pm1}\, j^{S,ba}_{\omega\pm1,l\pm1} \;=\; j^{S,ab}_{\omega l}\, j^{S,ba}_{\omega l}.$$

This makes the boost condition (C.363) for $j^{S,aa}_{\omega l}$ compatible with those for $j^{S,ab}_{\omega l}$ and $j^{S,ba}_{\omega l}$:

$$(j^{S,aa}_{\omega\pm1,l\pm1})^2 \;=\; -1 - j^{S,ab}_{\omega\pm1,l\pm1}\, j^{S,ba}_{\omega\pm1,l\pm1} \;=\; -1 - j^{S,ab}_{\omega l}\, j^{S,ba}_{\omega l} \;=\; (j^{S,aa}_{\omega l})^2.$$

Therefore it is sufficient to study the solutions to the four conditions (C.364). Plugging the values of the z and $\tilde{z}$-factors (C.258), (C.259), (C.266) and (C.267) into (C.364), turns the four conditions into

$$j^{S,ab}_{\omega-1,l+1} \;\stackrel{!}{=}\; -j^{S,ab}_{\omega l}\; \frac{(\widetilde{m}_+ + \omega - l - d)\,(\widetilde{m}_+ - \omega + l)}{(2l+d)\,(2l+d-2)} \tag{C.366}$$

$$j^{S,ab}_{\omega l} \;\stackrel{!}{=}\; -j^{S,ab}_{\omega-1,l-1}\; \frac{(\widetilde{m}_+ - \omega - l - d+2)\,(\widetilde{m}_+ + \omega + l - 2)}{(2l+d-2)\,(2l+d-4)} \tag{C.367}$$

$$j^{S,ab}_{\omega+1,l+1} \;\stackrel{!}{=}\; -j^{S,ab}_{\omega l}\; \frac{(\widetilde{m}_+ - \omega - l - d)\,(\widetilde{m}_+ + \omega + l)}{(2l+d)\,(2l+d-2)} \tag{C.368}$$

$$j^{S,ab}_{\omega l} \;\stackrel{!}{=}\; -j^{S,ab}_{\omega+1,l-1}\; \frac{(\widetilde{m}_+ + \omega - l - d+2)\,(\widetilde{m}_+ - \omega + l - 2)}{(2l+d-2)\,(2l+d-4)}\,. \tag{C.369}$$

It is easy to read off that ${}^{\omega\,\to\,\omega+1}_{l\,\to\,l-1}$ turns (C.366) into (C.369), and ${}^{\omega\,\to\,\omega+1}_{l\,\to\,l+1}$ turns (C.367) into (C.368). Thus the only two conditions we need to consider are

$$j^{S,ab}_{\omega-1,l+1} \;\stackrel{!}{=}\; -j^{S,ab}_{\omega l}\; \frac{(\widetilde{m}_+ + \omega - l - d)\,(\widetilde{m}_+ - \omega + l)}{(2l+d)\,(2l+d-2)} \tag{C.370}$$

$$j^{S,ab}_{\omega+1,l+1} \;\stackrel{!}{=}\; -j^{S,ab}_{\omega l}\; \frac{(\widetilde{m}_+ - \omega - l - d)\,(\widetilde{m}_+ + \omega + l)}{(2l+d)\,(2l+d-2)}\,. \tag{C.371}$$

After some educated guessing, we find several solutions to these conditions, for example:

$$j^{S,ab}_{\omega l} \;=\; (-1)^l\, \frac{\Gamma(\alpha^{S,a})\,\Gamma(\beta^{S,a})}{\Gamma(\alpha^{S,b})\,\Gamma(\beta^{S,b})\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)} \tag{C.372}$$

$$j^{S,ab}_{\omega l} \;=\; (-1)^l\, \frac{\Gamma(1-\alpha^{S,b})\,\Gamma(1-\beta^{S,b})}{\Gamma(1-\alpha^{S,a})\,\Gamma(1-\beta^{S,a})\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)} \tag{C.373}$$

$$j^{S,ab}_{\omega l} \;=\; \frac{1}{\Gamma(\alpha^{S,b})\,\Gamma(\beta^{S,b})\,\Gamma(1-\alpha^{S,a})\,\Gamma(1-\beta^{S,a})\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)} \tag{C.374}$$

$$j^{S,ab}_{\omega l} \;=\; \frac{\Gamma(\alpha^{S,a})\,\Gamma(\beta^{S,a})\,\Gamma(1-\alpha^{S,b})\,\Gamma(1-\beta^{S,b})}{\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)} \tag{C.375}$$

We have

$$\begin{aligned}
\alpha^{S,a} &= +\tfrac{1}{2}\big(\tilde{m}_+ - \omega + l\big) & \beta^{S,a} &= +\tfrac{1}{2}\big(\tilde{m}_+ + \omega + l\big)\\
(1-\alpha^{S,a}) &= -\tfrac{1}{2}\big(\tilde{m}_+ - \omega + l - 2\big) & (1-\beta^{S,a}) &= -\tfrac{1}{2}\big(\tilde{m}_+ + \omega + l - 2\big)\\
\alpha^{S,b} &= +\tfrac{1}{2}\big(\tilde{m}_+ - \omega - l - d + 2\big) & \beta^{S,b} &= +\tfrac{1}{2}\big(\tilde{m}_+ + \omega - l - d + 2\big)\\
(1-\alpha^{S,b}) &= -\tfrac{1}{2}\big(\tilde{m}_+ - \omega - l - d\big) & (1-\beta^{S,b}) &= -\tfrac{1}{2}\big(\tilde{m}_+ + \omega - l - d\big).
\end{aligned} \tag{C.376}$$

C.5 Fixing J_ρ through amplitude equivalence

In order to further fix the complex structure J_ρ of the hypercylinder, we impose now the requirement that it be amplitude-equivalent to J_t as discussed in Section 3.1.10. (For Minkowski spacetime this is done in Section B.4.) That is, we want J_ρ to induce coincidence of the free amplitudes for time-interval and rod regions for coherent states whose characterizing functions are related through

$\xi_0 = \xi_{12}^{\rm R} - J_\rho \xi_{12}^{\rm I}$ as described below. Amplitude-equivalence is made precise in (3.71), saying that J_t and J_ρ must make the following amplitudes coincide for all global solutions η_1 and ζ_2:

$$\begin{aligned} \rho_{[t_1,t_2]}\Big(K^{\eta_1}_{\Sigma_1} \otimes \overline{K^{\zeta_2}_{\Sigma_2}}\Big) \;&=\; \exp\Big(-{\rm g}_t(\xi_{12}^{\rm I},\, \xi_{12}^{\rm I}) - {\rm i}{\rm g}_t(\xi_{12}^{\rm R},\, \xi_{12}^{\rm I})\Big) \\ \overset{!}{=}\; \rho_{\rho_0}\big(\overline{K^{\xi_0}_{\Sigma_0}}\big) \;&=\; \exp\Big(-\tfrac{1}{2}{\rm g}_\rho(\xi_{12}^{\rm I},\, \xi_{12}^{\rm I}) - \tfrac{\rm i}{2}{\rm g}_\rho(\xi_{12}^{\rm R},\, \xi_{12}^{\rm I})\Big), \end{aligned} \tag{C.377}$$

wherein (mind where we have J_t and where J_ρ!)

$$\xi_{12}^{\rm R} \;=\; \tfrac{1}{2}(\eta_1 + \zeta_2) \qquad\qquad \xi_{12}^{\rm I} \;=\; \tfrac{1}{2}(-J_t\eta_1 + J_t\zeta_2). \qquad\qquad \xi_0 \;=\; \xi_{12}^{\rm R} - J_\rho \xi_{12}^{\rm I}. \tag{C.378}$$

As discussed above (3.73), J_t and J_ρ are amplitude-equivalent for global solutions, precisely if for all global solutions η_1, ζ_2 they induce

$${\rm g}_t(\eta_1, \zeta_2) = \tfrac{1}{2}\, {\rm g}_\rho(\eta_1, \zeta_2). \tag{C.379}$$

We recall that the real g-product ${\rm g}_t$ for the equal-time hypersurface is given by (2.213), which using the normalization constant (C.67) becomes

$${\rm g}_t(\eta, \zeta) \;=\; R_{\rm AdS}^{d-1} \sum_{n\underline{l}m_l} \Big\{\overline{\eta^-_{n\underline{l}m_l}}\, \zeta^+_{n\underline{l}m_l} + \eta^+_{n\underline{l}m_l}\, \overline{\zeta^-_{n\underline{l}m_l}}\Big\}\, \frac{n!\,\Gamma(\gamma^{\rm S})^2\,\Gamma(n{+}\nu{+}1)}{\Gamma(n{+}\gamma^{\rm S})\,\Gamma(n{+}\nu{+}\gamma^{\rm S})} \tag{C.380}$$

$$=\; R_{\rm AdS}^{d-1} \sum_{n\underline{l}m_l} \Big\{\overline{\eta^-_{n\underline{l}m_l}}\, \zeta^+_{n\underline{l}m_l} + \eta^+_{n\underline{l}m_l}\, \overline{\zeta^-_{n\underline{l}m_l}}\Big\}\, \Gamma(\gamma^{\rm S})^2\, \frac{\Gamma(1{-}\alpha^{S,b})\,\Gamma(\beta^{S,b})}{\Gamma(1{-}\alpha^{S,a})\,\Gamma(\beta^{S,a})}\,. \tag{C.381}$$

In the last line, the parameters $\alpha^{S,\cdot}, \beta^{S,\cdot}$ and $\gamma^{\rm S}$ are understood as evaluated at the respective values of l and at positive magic frequencies $\omega = +\omega^+_{nl}$. The last line results from the first by plugging in relations (C.376). Our goal is thus to reproduce this result as $\frac{1}{2}\,{\rm g}_\rho(\eta_1, \zeta_2)$ thus starting from the real g-product ${\rm g}_\rho$ of the hypercylinder which is given by (2.64):

$${\rm g}_\rho(\eta, \zeta) \;:=\; 2\omega_\rho(\eta,\, J_\rho\zeta).$$

To this end we first compare expansions of global solutions near equal-time hypersurfaces and hypercylinders. We can expand any global solution ξ using the Jacobi expansion (2.201) of solutions near equal-time hypersurfaces:

$$\xi(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l} \Big\{\xi^+_{n\underline{l}m_l}\; \mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\xi^-_{n\underline{l}m_l}}\; \overline{\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\}\,. \tag{C.382}$$

The $\mu^{(+)}$-modes are defined in (2.173) as $\mu^{(a)}$-modes with magic frequencies ω^+_{nl}:

$$\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega) \;=\; \mu^{(S,a)}_{\omega^+_{nl}\underline{l}m_l}(t,\rho,\Omega) \;=\; {\rm e}^{-{\rm i}\omega^+_{nl}t}\; Y^{m_l}_{\underline{l}}(\Omega)\; J^{(+)}_{nl}(\rho) \qquad\qquad J^{(+)}_{nl}(\rho) \;=\; S^a_{\omega^+_{nl},l}(\rho)\,. \tag{C.383}$$

The $\mu^{(a)}$ and $\mu^{(b)}$-modes are defined in (2.167) as

$$\mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \;=\; {\rm e}^{-{\rm i}\omega t}\; Y^{m_l}_{\underline{l}}(\Omega)\; S^a_{\omega l}(\rho) \qquad\qquad \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \;=\; {\rm e}^{-{\rm i}\omega t}\; Y^{m_l}_{\underline{l}}(\Omega)\; S^b_{\omega l}(\rho). \tag{C.384}$$

These modes are used in the hypergeometric S-expansion (2.186) of general solutions ξ near hypercylinders:

$$\xi(t,r,\Omega) \;=\; \int\!{\rm d}\omega \sum_{\underline{l},m_l} \Big\{\xi^{S,a}_{\omega\underline{l}m_l}\; \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \xi^{S,b}_{\omega\underline{l}m_l}\; \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega)\Big\}. \tag{C.385}$$

Then, in order for ξ to be a global solution it must have $\xi^b_{\omega\underline{l}m_l} \equiv 0$. Further, for all frequencies ω that are not magic, ξ must also have $\xi^a_{\omega\underline{l}m_l} = 0$. In order to compare a global solution's expansions near an equal-time hypersurface and near a hypercylinder, we now convert the S-expansion (C.385) (which is continuous in ω) into a discrete sum over magic frequencies. Using Dirac delta functions, we write

$$\xi^{S,a}_{\omega\underline{l}m_l} = \sum_{n=0}^{\infty}\Big(\xi^a_{+n,\underline{l},m_l}\delta(\omega-\omega^+_{nl}) + \xi^a_{-n,\underline{l},m_l}\delta(\omega+\omega^+_{nl})\Big) \tag{C.386}$$

$$\xi^{S,b}_{\omega\underline{l}m_l} = \sum_{n=0}^{\infty}\Big(\xi^b_{+n,\underline{l},m_l}\delta(\omega-\omega^+_{nl}) + \xi^b_{-n,\underline{l},m_l}\delta(\omega+\omega^+_{nl})\Big). \tag{C.387}$$

With this the S-expansion (C.385) of a solution consisting of a and b-modes of magic frequencies becomes

$$\xi(t,r,\Omega) = \sum_{\underline{l},m_l}\sum_{n=0}^{\infty}\Big\{\xi^a_{+n,\underline{l},m_l}\;\mu^{(S,a)}_{\omega^+_{nl}\underline{l}m_l}(t,\rho,\Omega) + \xi^a_{-n,\underline{l},m_l}\;\mu^{(S,a)}_{-\omega^+_{nl},\underline{l},m_l}(t,\rho,\Omega) \tag{C.388}$$

$$+\,\xi^b_{+n,\underline{l},m_l}\;\mu^{(S,b)}_{\omega^+_{nl}\underline{l}m_l}(t,\rho,\Omega) + \xi^b_{-n,\underline{l},m_l}\;\mu^{(S,b)}_{-\omega^+_{nl},\underline{l},m_l}(t,\rho,\Omega)\Big\}. \tag{C.389}$$

Let us call this the discrete S-expansion. For global solutions we have $\xi^b_{\pm n,\underline{l},m_l} \equiv 0$, then the Jacobi and the discrete S-expansion are equivalent and we can transcribe them into each other:

$$\xi^a_{+n,\underline{l},m_l} = \xi^+_{n\underline{l}m_l} \qquad\qquad \xi^a_{-n,\underline{l},-m_l} = \overline{\xi^-_{n\underline{l}m_l}}. \tag{C.390}$$

It turns out useful to introduce yet another expansion: we now consider solutions consisting of a-modes with only magic frequencies and b-modes of all real frequencies. That is, we discretize only the a-part of the solution applying (C.386), while leaving its b-part continuous by not applying (C.387). The result of this is the hybrid S-expansion

$$\xi(t,r,\Omega) = \sum_{\underline{l},m_l}\sum_{n=0}^{\infty}\Big\{\xi^a_{+n,\underline{l},m_l}\;\mu^{(S,a)}_{\omega^+_{nl}\underline{l}m_l}(t,\rho,\Omega) + \xi^a_{-n,\underline{l},m_l}\;\mu^{(S,a)}_{-\omega^+_{nl},\underline{l},m_l}(t,\rho,\Omega)\Big\} + \int\mathrm{d}\omega\sum_{\underline{l},m_l}\xi^{S,b}_{\omega\underline{l}m_l}\;\mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega). \tag{C.391}$$

For global solutions the hybrid S-expansion as well has $\xi^{S,b}_{\omega\underline{l}m_l} \equiv 0$, and then becomes equivalent to the Jacobi expansion, relating to it through (C.390). Since the real g-product is defined via the symplectic structure

$$\mathrm{g}_\rho(\eta,\zeta) := 2\omega_\rho(\eta,\, J_\rho\zeta),$$

let us have a look at the latter now. The symplectic structure (2.195) for generic solutions η,ζ near hypercylinders is

$$\omega_\rho(\eta,\zeta) = R^{d-1}_{\mathrm{AdS}}\int\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\eta^{S,a}_{\omega\underline{l}m_l}\;\zeta^{S,b}_{-\omega,\underline{l},-m_l} - \eta^{S,b}_{\omega\underline{l}m_l}\;\zeta^{S,a}_{-\omega,\underline{l},-m_l}\Big\}\,\pi(2l+d-2). \tag{C.392}$$

(For global solutions η,ζ the symplectic structure ω_ρ vanishes, since these have $\eta^{S,b}_{\omega\underline{l}m_l} \equiv 0$.) For solutions consisting only of a and b-modes with magic frequencies, we can evaluate the sympletic structure by plugging the discrete S-representation (C.386)+ (C.387) into (C.392), resulting in

$$\omega_\rho(\eta,\zeta) = R^{d-1}_{\mathrm{AdS}}\sum_{\underline{l},m_l}\sum_{n=0}^{\infty}\Big\{\eta^a_{+n,\underline{l},m_l}\;\zeta^b_{-n,\underline{l},-m_l} + \eta^a_{-n,\underline{l},m_l}\;\zeta^b_{+n,\underline{l},-m_l} \tag{C.393}$$

$$-\,\eta^b_{+n,\underline{l},m_l}\;\zeta^a_{-n,\underline{l},-m_l} - \eta^b_{-n,\underline{l},m_l}\;\zeta^a_{+n,\underline{l},-m_l}\Big\}\,\tfrac{\pi}{2}\,(2l+d-2)\,\delta(0).$$

(The factor $\frac{1}{2}$ comes from the scaling property of the Dirac delta: $\delta(\omega^+_{nl} - \omega^+_{n',l}) = \delta(2n - 2n') = \frac{1}{2}\,\delta(n - n')$.) That is, if our solutions consist only of magic frequency modes, then the symplectic structure on the hypercylinder has a δ-divergence! The reason for this is the following: in the definition (2.194) of the symplectic structure

$$\omega_\rho(\eta,\zeta) \;=\; \tfrac{1}{2}\int \mathrm{d}t\,\mathrm{d}^{d-1}\Omega\; R^{d-1}_{\mathrm{AdS}}\tan^{d-1}\rho\;\left(\eta\,\partial_\rho\zeta - \zeta\,\partial_\rho\eta\right). \tag{C.394}$$

we integrate $\eta\,\partial_\rho\zeta - \zeta\,\partial_\rho\eta$ over the hypercylinder, which is infinite in t-direction. However, using only modes with the discrete magic frequencies we cannot form wave packets that are compactly supported (respectively decay sufficiently fast) on the hypercylinder. Hence the integral diverges. For the Minkowski hypercylinder the situation is different: there the global solutions cover a continuous range of frequencies, and therefore we obtain a finite symplectic structure for solutions with a and b-modes in this range.

This points to the following remedy: instead of the discrete S-expansion we use the hybrid one. Its continuous range of b-modes allows for wave packets of sufficient decay, while the discrete range of a-modes still lets us compare it to the global modes near an equal-time hypersurface. Plugging the hybrid S-representation (C.386) without (C.387) into (C.392) results in

$$\begin{aligned}\omega_\rho(\eta,\zeta) \;=\; R^{d-1}_{\mathrm{AdS}}\sum_{\underline{l},m_l}\sum_{n=0}^{\infty}\Big\{&\eta^a_{+n,\underline{l},m_l}\,\zeta^{S,b}_{-\omega^+_{nl},\underline{l},-m_l} + \eta^a_{-n,\underline{l},-m_l}\,\zeta^{S,b}_{\omega^+_{nl},\underline{l},m_l}\\ &- \eta^{S,b}_{\omega^+_{nl},\underline{l},m_l}\,\zeta^a_{-n,\underline{l},-m_l} - \eta^{S,b}_{-\omega^+_{nl},\underline{l},-m_l}\,\zeta^a_{+n,\underline{l},m_l}\Big\}\,\pi(2l+d-2).\end{aligned} \tag{C.395}$$

Hence using the hybrid S-expansion (C.391) keeps the symplectic structure on the hypercylinder finite. (The same would be achieved by keeping the a-part continuous and the b-part discrete, but then we could not compare the solution to global ones. Thus we are lead to making the a-part discrete and keeping the b-part continuous.)

With the symplectic structure (C.395) ready, we can come back to $\mathrm{g}_\rho(\eta,\zeta) := 2\omega_\rho(\eta,\,J_\rho\zeta)$. We let therein act on ζ a complex structure as in (C.358)

$$\begin{aligned}\left(J_\rho\phi\right)^{S,a}_{\omega\underline{l}m_l} &= j^{S,aa}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l} + j^{S,ab}_{\omega l}\,\phi^{S,b}_{\omega\underline{l}m_l} & j^{S,ab}_{-\omega,l} &= j^{S,ab}_{\omega l}\\ \left(J_\rho\phi\right)^{S,b}_{\omega\underline{l}m_l} &= j^{S,ba}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l} - j^{S,aa}_{\omega l}\,\phi^{S,b}_{\omega\underline{l}m_l} & j^{S,ba}_{-\omega,l} &= j^{S,ba}_{\omega l}.\\ & & \left(j^{S,aa}_{\omega l}\right)^2 &= -j^{S,ab}_{\omega l}\,j^{S,ba}_{\omega l} - 1 \geq 0.\end{aligned}$$

Using the hybrid representation, for global solutions η,ζ we then obtain (going from second to third line we use the frequency symmetry $j^{S,ba}_{\omega^+_{nl}l} = j^{S,ba}_{-\omega^+_{nl}l}$):

$$
\begin{aligned}
\tfrac{1}{2}\,\mathrm{g}_\rho(\eta,\zeta) &= \omega_\rho(\eta, J_\rho\zeta)\\
&= R^{d-1}_{\mathrm{AdS}} \sum_{n,\underline{l},m_l} \pi(2l+d-2)\,\Big\{\eta^a_{+n,\underline{l},m_l}\big(J_\rho\zeta\big)^{S,b}_{-\omega^+_{nl},\underline{l},-m_l} + \eta^a_{-n,\underline{l},-m_l}\big(J_\rho\zeta\big)^{S,b}_{\omega^+_{nl},\underline{l},m_l}\\
&\qquad - \underbrace{\eta^{S,b}_{\omega^+_{nl},\underline{l},m_l}}_{0}\big(J_\rho\zeta\big)^a_{-n,\underline{l},-m_l} - \underbrace{\eta^{S,b}_{-\omega^+_{nl},\underline{l},-m_l}}_{0}\big(J_\rho\zeta\big)^a_{+n,\underline{l},m_l}\Big\}\\
&= R^{d-1}_{\mathrm{AdS}} \sum_{n,\underline{l},m_l} \pi(2l+d-2)\,\Big\{\eta^a_{+n,\underline{l},m_l}\Big(j^{S,ba}_{\omega^+_{nl}l}\zeta^{S,a}_{-\omega^+_{nl},\underline{l},-m_l} - j^{S,aa}_{\omega^+_{nl}l}\underbrace{\zeta^{S,b}_{-\omega^+_{nl},\underline{l},-m_l}}_{0}\Big)\\
&\qquad + \eta^a_{-n,\underline{l},-m_l}\Big(j^{S,ba}_{\omega^+_{nl}l}\zeta^{S,a}_{\omega^+_{nl},\underline{l},m_l} - j^{S,aa}_{\omega^+_{nl}l}\underbrace{\zeta^{S,b}_{\omega^+_{nl},\underline{l},m_l}}_{0}\Big)\Big\}\\
&= R^{d-1}_{\mathrm{AdS}} \sum_{n,\underline{l},m_l} \pi(2l+d-2) j^{S,ba}_{\omega^+_{nl}l}\,\Big\{\eta^a_{+n,\underline{l},m_l}\zeta^{S,a}_{-\omega^+_{nl},\underline{l},-m_l} + \eta^a_{-n,\underline{l},-m_l}\zeta^{S,a}_{\omega^+_{nl},\underline{l},m_l}\Big\}\\
&= R^{d-1}_{\mathrm{AdS}} \sum_{n,\underline{l},m_l} \tfrac{\pi}{2}\,(2l+d-2)\,\delta(0)\,j^{S,ba}_{\omega^+_{nl}l}\,\Big\{\eta^a_{+n,\underline{l},m_l}\zeta^a_{-n,\underline{l},-m_l} + \eta^a_{-n,\underline{l},-m_l}\zeta^a_{+n,\underline{l},m_l}\Big\}.
\end{aligned}
$$

(Again, the factor $\frac{1}{2}$ comes from the scaling property of the Dirac delta: $\delta(\omega^+_{nl} - \omega^+_{n',l}) = \delta(2n - 2n') = \frac{1}{2}\,\delta(n-n')$.) Evaluating the real g-product g_ρ for global solutions, only the a-modes appear. Therefore, using the hybrid representation does not prevent it from diverging. As commented above, this divergence is due to only using discrete frequencies. The wave packets of these modes do not decay sufficiently fast and integrating them over the infinite hypercylinder results in divergence. This means, that in order to obtain a finite $g_\rho(\eta,\zeta)$, and thus finite rod amplitudes, it is necessary to use a continuous frequency range, that is: to use evanescent modes. Using only propagating modes is not sufficient on AdS.

Due to this circumstance, our plan to construct an amplitude equivalence between amplitudes for AdS time-interval and rod regions does not work out in the strict sense. We therefore have to accept the occurence of the $\delta(0)$-divergence, and construct only a weak amplitude equivalence that holds except for the $\delta(0)$-factor. On the other hand, we could have easily avoided this factor by not using the Dirac delta (distribution) but rather its square root (distribution) in (C.386). That is, the issue has to do with how exactly we bring into correspondence the modes on the hypercylinder with the modes on the equal-time hyperplane. The freedom is justified, since we are only after a *weak* amplitude equivalence and we shall thus remove the $\delta(0)$-factor from here onwards. Plugging translations (C.390), that is $\xi^a_{n\underline{l}m_l} = \xi^+_{n\underline{l}m_l}$ and $\xi^a_{-n,\underline{l},-m_l} = \overline{\xi^-_{n\underline{l}m_l}}$, into the last line, we obtain for global solutions η,ζ

$$
\begin{aligned}
\tfrac{1}{2}\,\mathrm{g}_\rho(\eta,\zeta) &= R^{d-1}_{\mathrm{AdS}} \sum_{n,\underline{l},m_l} \tfrac{\pi}{2}\,(2l+d-2)\,\delta(0)\,j^{S,ba}_{\omega^+_{nl}l}\,\Big\{\eta^+_{n\underline{l}m_l}\overline{\zeta^-_{n\underline{l}m_l}} + \overline{\eta^-_{n,\underline{l},-m_l}}\zeta^+_{n,\underline{l},-m_l}\Big\}\\
&= R^{d-1}_{\mathrm{AdS}} \sum_{n,\underline{l},m_l} \pi(\gamma^{\mathrm{S}}-1)\,\delta(0)\,j^{S,ba}_{\omega^+_{nl}l}\,\Big\{\eta^+_{n\underline{l}m_l}\overline{\zeta^-_{n\underline{l}m_l}} + \overline{\eta^-_{n\underline{l}m_l}}\zeta^+_{n\underline{l}m_l}\Big\}.
\end{aligned}
$$

We can read off that except for the $\delta(0)$-factor this agrees with

$$
\mathrm{g}_t(\eta,\zeta) = R^{d-1}_{\mathrm{AdS}} \sum_{n\underline{l}m_l} \Big\{\overline{\eta^-_{n\underline{l}m_l}}\,\zeta^+_{n\underline{l}m_l} + \eta^+_{n\underline{l}m_l}\,\overline{\zeta^-_{n\underline{l}m_l}}\Big\}\,\frac{n!\,\Gamma(\gamma^{\mathrm{S}})^2\,\Gamma(n+\nu+1)}{\Gamma(n+\gamma^{\mathrm{S}})\,\Gamma(n+\nu+\gamma^{\mathrm{S}})}\,. \qquad \text{(C.396)}
$$

precisely if

$$j^{S,ba}_{\omega^+_{nl}l} = j^{S,ba}_{-\omega^+_{nl}l} = \frac{1}{\pi(\gamma^{\mathrm{S}}-1)}\,\frac{n!\,\Gamma(\gamma^{\mathrm{S}})^2\,\Gamma(n+\nu+1)}{\Gamma(n+\gamma^{\mathrm{S}})\,\Gamma(n+\nu+\gamma^{\mathrm{S}})} \tag{C.397}$$

$$= \tfrac{1}{\pi}\,\Gamma(\gamma^{\mathrm{S}})\Gamma(\gamma^{\mathrm{S}}-1)\,\frac{\Gamma(1-\alpha^{S,b})\,\Gamma(\beta^{S,b})}{\Gamma(1-\alpha^{S,a})\,\Gamma(\beta^{S,a})}\,. \tag{C.398}$$

(Again, the parameters $\alpha^{S,\cdot}, \beta^{S,\cdot}$ and γ^{S} are understood as evaluated at the respective values of l and at positive magic frequencies $\omega = +\omega^+_{nl}$.) That is, as for Minkowski spacetime the requirement of amplitude equivalence fixes only the factor $j^{S,ba}$ of the complex structure while not fixing $j^{S,aa}$ and $j^{S,ab}$. Therefore, as for Minkowski spacetime we choose an anti-diagonal complex structure, that is: $j^{S,aa}_{\omega l} \equiv 0$. This induces $j^{S,ab} = -1/j^{S,ba}$ and also that J_ρ maps a-modes to b-modes and vice versa. This is a natural implementation of the property that J_ρ maps solutions well defined on the whole interior of the rod region to solutions that are well defined only near the boundary hypercylinder (and vice versa).

Further, we see that amplitude equivalence fixes the factor $j^{S,ba}$ only for the discrete set of magic frequencies $\pm\omega^+_{nl}$. In the following we shall make use of this remaining freedom to fix $j^{S,ba}$ also for nonmagic frequencies. First we recall all the properties that $j^{S,ba}$ is to fulfill. We use the anti-diagonal form of (C.358) that we discussed right above:

$$\big(J_\rho\phi\big)^{S,a}_{\omega\underline{l}m_l} = \frac{-1}{j^{S,ba}_{\omega l}}\,\phi^{S,b}_{\omega\underline{l}m_l} \qquad\qquad \big(J_\rho\phi\big)^{S,b}_{\omega\underline{l}m_l} = j^{S,ba}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l}.$$

Now the essential properties of compatibility with the symplectic structure, $J_\rho^2 = -\mathbb{1}$, and mapping real solutions to real solutions, together with commuting with time translations and spatial rotations are encoded already in the form (C.358). However,this holds only if J_ρ fulfills the first remaining requirement of frequency symmetry, that is

$$j^{S,ba}_{\omega,l} = j^{S,ba}_{-\omega,l}. \tag{C.399}$$

The second remaining requirement for J_ρ is to commute also with the boost generators, which amounts to (C.370)

$$\begin{aligned} j^{S,ba}_{\omega-1,l+1} &\stackrel{!}{=} -j^{S,ba}_{\omega l}\,\frac{(2l+d)\,(2l+d-2)}{(\widetilde{m}_+ +\omega-l-d)\,(\widetilde{m}_+ -\omega+l)} \\ j^{S,ba}_{\omega+1,l+1} &\stackrel{!}{=} -j^{S,ba}_{\omega l}\,\frac{(2l+d)\,(2l+d-2)}{(\widetilde{m}_+ -\omega-l-d)\,(\widetilde{m}_+ +\omega+l)}\,. \end{aligned} \tag{C.400}$$

We observe that this condition only relates factors $j^{S,ba}$ with a discrete difference in frequency ω and angular momentum l. (Compare this to Minkowski spacetime, where the corresponding condition related factors $j^{S,ba}$ with infinitesimally close frequencies.) This means that after fixing $j^{S,ba}$ for one frequency ω_0 and angular momentum l_0, this condition then does not fix $j^{S,ba}$ for all other ω and l, but only for those that are at discrete steps from ω_0 and l_0. The third and final remaining requirement is amplitude equivalence, represented by (C.397)

$$j^{S,ba}_{\omega^+_{nl}l} = j^{S,ba}_{-\omega^+_{nl}l} = \frac{1}{\pi(\gamma^{\mathrm{S}}-1)}\,\frac{n!\,\Gamma(\gamma^{\mathrm{S}})^2\,\Gamma(n+\nu+1)}{\Gamma(n+\gamma^{\mathrm{S}})\,\Gamma(n+\nu+\gamma^{\mathrm{S}})}\,. \tag{C.401}$$

For immediate use we now define two different candidate versions for the factor $j^{S,ba}_{\omega l}$, which we call the α-version respectively β-version:

$$j^{S,ba,\alpha}_{\omega l} = \tfrac{1}{\pi}\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)\,\frac{\Gamma(\alpha^{S,b})}{\Gamma(\alpha^{S,a})}\,\frac{\Gamma(1-\beta^{S,a})}{\Gamma(1-\beta^{S,b})} \tag{C.402}$$

$$j^{S,ba,\beta}_{\omega l} = \tfrac{1}{\pi}\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)\,\frac{\Gamma(\beta^{S,b})}{\Gamma(\beta^{S,a})}\,\frac{\Gamma(1-\alpha^{S,a})}{\Gamma(1-\alpha^{S,b})}\,. \tag{C.403}$$

We recall that in each line the parameters α, β and γ^{S} are calculated from ω and l. Since switching the sign of ω corresponds to interchanging α and β-parameters, we have

$$j^{S,ba,\alpha}_{\omega,l} = j^{S,ba,\beta}_{-\omega,l}.$$

Further, both (C.402) and (C.402) are boost-compatible in the sense of (C.365), that is: the actions of complex structure and boost generators commute. Trying out the factors of the $j^{S,ab}$-candidates (C.372), we find that the only possibility for $j^{S,ba}$ to fulfill (C.401) is setting

$$j^{S,ba}_{\omega^+_{nl},l} = j^{S,ba,\beta}_{\omega^+_{nl},l} = \tfrac{1}{\pi}\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)\,\frac{\Gamma(\beta^{S,b})}{\Gamma(\beta^{S,a})}\,\frac{\Gamma(1-\alpha^{S,a})}{\Gamma(1-\alpha^{S,b})} \tag{C.404}$$

$$j^{S,ba}_{-\omega^+_{nl},l} = j^{S,ba,\alpha}_{-\omega^+_{nl},l} = \tfrac{1}{\pi}\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)\,\frac{\Gamma(\alpha^{S,b})}{\Gamma(\alpha^{S,a})}\,\frac{\Gamma(1-\beta^{S,a})}{\Gamma(1-\beta^{S,b})}\,. \tag{C.405}$$

We recall that in (C.404) the parameters α, β are calculated from $+\omega^+_{nl}$, while in (C.405) they derive from $-\omega^+_{nl}$. Since switching the sign of ω corresponds to interchanging α and β-parameters, we can see directly that frequency symmetry becomes fulfilled by this choice:

$$\begin{aligned} j^{S,ba}_{\omega^+_{nl},l} &= j^{S,ba,\beta}_{\omega^+_{nl},l} = j^{S,ba,\alpha}_{-\omega^+_{nl},l} \\ &= j^{S,ba}_{-\omega^+_{nl},l}. \end{aligned}$$

Before extending this choice to nonmagic frequencies, let us verify that it indeed reproduces the amplitude equivalence condition (C.401). Plugging the definition (C.44) of the magic frequencies $\omega^+_{nl} = \tilde{m}_+ + 2n + l$ into the right hand side of (C.404) and using the parameter definitions (2.172) we obtain (C.401):

$$\begin{aligned} j^{S,ba}_{\omega^+_{nl},l} &= \tfrac{1}{\pi}\,\Gamma(\tfrac{1}{2}(2l+d))\,\Gamma(\tfrac{1}{2}(2l+d-2))\,\frac{\Gamma(\tilde{m}_+-\frac{d}{2}+n+1)}{\Gamma(\tilde{m}_++l+n)}\,\frac{\Gamma(n+1)}{\Gamma(n+l+\frac{d}{2})} \\ &= \tfrac{1}{\pi}\,\Gamma(\tfrac{1}{2}(2l+d))\,\frac{\Gamma(\frac{1}{2}(2l+d))}{\frac{1}{2}(2l+d-2)}\,\frac{\Gamma(n+\nu+1)}{\Gamma(n+\nu+\gamma^{\mathrm{S}})}\,\frac{n!}{\Gamma(n+\gamma^{\mathrm{S}})} \\ &= \tfrac{1}{\pi}\,\frac{\Gamma(\gamma^{\mathrm{S}})^2}{(2l+d-2)}\,\frac{\Gamma(n+\nu+1)}{\Gamma(n+\nu+\gamma^{\mathrm{S}})}\,\frac{2\,n!}{\Gamma(n+\gamma^{\mathrm{S}})}\,. \end{aligned}$$

Before extending this choice to the remaining frequencies and fixing also $j^{S,aa}_{\omega l}$ and $j^{S,ab}_{\omega l}$, we remark that already the existence of our choice is quite nontrivial, because the factors related therein have rather different origins. The factor appearing in amplitude equivalence condition (C.401) stems from integrating a global solution over an equal-time hyperplane Σ_t, while the factors in the boost conditions (C.400) stem from boost compatibility of the complex structure J_ρ for more general solutions near a hypercylinder Σ_ρ.

C.5.1 Two-branches choice $J^{\mathbf{two}}_\rho$

The problem at this point is that the α and β-versions of $j^{S,ba}$ are different functions of ω and l. For instance we can choose to define a first version $j^{S,ba,\mathrm{two}}$ by simply using choice (C.404) for all positive frequencies, while using (C.405) for all negative frequencies. That is, we use the β-version for the "right branch" consisting of *all* positive frequencies, and the α-version for the "left branch" of *all* negative frequencies. This results in an expression that contains ω only as $|\omega|$ and is thus manifestly frequency symmetric:

$$j^{S,ba,\text{two}}_{\omega l} = \begin{cases} j^{S,ba,\beta}_{\omega l} & \omega > 0 \\ j^{S,ba,\alpha}_{\omega l} & \omega < 0 \end{cases} \tag{C.406}$$

$$= \tfrac{1}{\pi}\,\Gamma(\tfrac{1}{2}(2l+d))\,\Gamma(\tfrac{1}{2}(2l+d-2))\,\frac{\Gamma(\frac{1}{2}(\tilde{m}_+ + |\omega| - l - d + 2))}{\Gamma(\frac{1}{2}(\tilde{m}_+ + |\omega| + l))}\,\frac{\Gamma(-\frac{1}{2}(\tilde{m}_+ - |\omega| + l - 2))}{\Gamma(-\frac{1}{2}(\tilde{m}_+ - |\omega| - l - d))}\,, \tag{C.407}$$

$$J^{\rho,\text{two}}_{\omega l} = \begin{pmatrix} 0 & -1/j^{S,ba,\text{two}}_{\omega l} \\ j^{S,ba,\text{two}}_{\omega l} & 0 \end{pmatrix}. \tag{C.408}$$

However, since the α and β-versions are different functions of ω, the gluing of the two versions at $\omega = 0$ is only continuous, but not smooth. While the non-smoothness of the choice $j^{S,ba,\text{two}}$ causes no problems, its definition through case distinction breaks the boosts-compatibility of (C.402) and (C.403). This breaking occurs for the boosts generators only for frequencies with $|\omega| < 1$, because for these frequencies the action of boosts generators creates frequencies $\omega \pm 1$ which cross the gluing point $\omega = 0$.

In order to see this, let us consider the example of a solution consisting of a single mode $\mu^{(S,a)}_{0.5,0}(t,\rho,\Omega)$ with frequency $\omega = 0.5$ and $l = 0$. We remark that when J_ρ directly acts on a mode (instead within the momentum representation) then we have to use the transposed matrix $(J^\rho_{\omega l})^\top$. This can be seen as follows. We first write the solution in the usual hypergeometric S-expansion, and switch to matrix notation:

$$\begin{aligned} \phi(t,r,\Omega) &= \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \phi^a_{\omega\underline{l}m_l}\ \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^b_{\omega\underline{l}m_l}\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\} \\ &= \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \begin{pmatrix} \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega), \\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \end{pmatrix}^{\!\top} \begin{pmatrix} \phi^a_{\omega\underline{l}m_l} \\ \phi^b_{\omega\underline{l}m_l} \end{pmatrix}. \end{aligned}$$

Now we let J_ρ act as usual on the momentum representation, and then move its action towards the modes:

$$\begin{aligned} \big(J_\rho\phi\big)(t,r,\Omega) &= \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \Big\{ \big(J_\rho\phi\big)^a_{\omega\underline{l}m_l}\ \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \big(J_\rho\phi\big)^b_{\omega\underline{l}m_l}\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \Big\} \\ &= \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \begin{pmatrix} \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega), \\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \end{pmatrix}^{\!\top} J^\rho_{\omega l} \begin{pmatrix} \phi^a_{\omega\underline{l}m_l} \\ \phi^b_{\omega\underline{l}m_l} \end{pmatrix} \\ &= \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \left[(J^\rho_{\omega l})^\top \begin{pmatrix} \mu^{(S,a)}_{\omega\underline{l}m_l}(t,\rho,\Omega), \\ \mu^{(S,b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \end{pmatrix} \right]^\top \begin{pmatrix} \phi^a_{\omega\underline{l}m_l} \\ \phi^b_{\omega\underline{l}m_l} \end{pmatrix}. \end{aligned} \tag{C.409}$$

The transposed matrix for general J_ρ is of course

$$(J^\rho_{\omega l})^\top = \begin{pmatrix} j^{S,aa}_{\omega l} & j^{S,ba}_{\omega l} \\ j^{S,ab}_{\omega l} & -j^{S,aa}_{\omega l} \end{pmatrix}, \tag{C.410}$$

and for our antidiagional complex structure becomes

$$(J^\rho_{\omega l})^\top = \begin{pmatrix} 0 & j^{S,ba}_{\omega l} \\ -1/j^{S,ba}_{\omega l} & 0 \end{pmatrix}. \tag{C.411}$$

Now let the complex structure (C.408) and the boost generator K_{0d} act on our mode. That is, we compare the actions of $(J^{\text{two}}_\rho\, K_{0d})$ and $(K_{0d}\, J^{\text{two}}_\rho)$ on $\mu^{(S,a)}_{0.5,0}(t,\rho,\Omega)$. We recall that $\chi^{(d-1)}_-(0,0) = 0$.

Using (C.407) and actions (C.270) together with (C.266) and (C.258), we then obtain

$$J^{\rm two}_\rho\left(K_{0d}\triangleright\mu^{(S,a)}_{0.5,\,0}\right) = \tfrac{\rm i}{2}\,\tilde z^{(S,a)++}_{0.5,\,0}\;j^{S,ba,\rm two}_{1.5,\,1}\;\mu^{(S,b)}_{1.5,\,1}+\tfrac{\rm i}{2}\,z^{(S,a)-+}_{0.5,\,0}\;j^{S,ba,\rm two}_{-0.5,\,1}\;\mu^{(S,b)}_{-0.5,\,1} \tag{C.412}$$

$$= -\tfrac{\rm i}{\pi}\,\chi^{(d-1)}_{+}(0,0)\,\Gamma^2(\tfrac{d}{2})\,\frac{\Gamma(\frac12(\tilde m_+-d+2.5))}{\Gamma(\frac12(\tilde m_++0.5))}\,\frac{\Gamma(-\frac12(\tilde m_+-2.5))}{\Gamma(-\frac12(\tilde m_+-d-0.5))}\;\mu^{(S,b)}_{1.5,\,1}$$

$$+\tfrac{\rm i}{\pi}\,\chi^{(d-1)}_{+}(0,0)\,\Gamma^2(\tfrac{d}{2})\,\frac{\Gamma(\frac12(\tilde m_+-d+1.5))}{\Gamma(\frac12(\tilde m_+-0.5))}\,\frac{\Gamma(-\frac12(\tilde m_+-1.5))}{\Gamma(-\frac12(\tilde m_+-d+0.5))}\;\mu^{(S,b)}_{-0.5,\,1},$$

while

$$K_{0d}\triangleright\left(J^{\rm two}_\rho\mu^{(S,a)}_{0.5,\,0}\right) = \tfrac{\rm i}{2}\,\tilde z^{(S,b)++}_{0.5,\,0}\;j^{S,ba,\rm two}_{0.5,\,0}\;\mu^{(S,b)}_{1.5,\,1}+\tfrac{\rm i}{2}\,z^{(S,b)-+}_{0.5,\,0}\;j^{S,ba,\rm two}_{0.5,\,0}\;\mu^{(S,b)}_{-0.5,\,1} \tag{C.413}$$

$$= -\tfrac{\rm i}{\pi}\,\chi^{(d-1)}_{+}(0,0)\,\Gamma^2(\tfrac{d}{2})\,\frac{\Gamma(\frac12(\tilde m_+-d+2.5))}{\Gamma(\frac12(\tilde m_++0.5))}\,\frac{\Gamma(-\frac12(\tilde m_+-2.5))}{\Gamma(-\frac12(\tilde m_+-d-0.5))}\;\mu^{(S,b)}_{1.5,\,1}$$

$$+\tfrac{\rm i}{\pi}\,\chi^{(d-1)}_{+}(0,0)\,\Gamma^2(\tfrac{d}{2})\,\frac{\Gamma(\frac12(\tilde m_+-d+2.5))}{\Gamma(\frac12(\tilde m_++0.5))}\,\frac{\Gamma(-\frac12(\tilde m_+-2.5))}{\Gamma(-\frac12(\tilde m_+-d-0.5))}\;\mu^{(S,b)}_{-0.5,\,1}.$$

We observe that the factors in front of $\mu^{(S,b)}_{1.5,\,1}$ coincide in (C.412) and (C.413). However, for $\mu^{(S,b)}_{-0.5,\,1}$ the frequency has crossed the gluing point of $\omega=0$, and the factors in front of it disagree in (C.412) and (C.413).

As we commented above, if we consider commutation of $J^{\rm two}_\rho$ only with the infinitesimal generators, then this breaking of the boost-compatibility of $J^{\rm two}_\rho$ occures only for frequencies $|\omega|<1$. However, if we proceed to consider finite boosts, then there appear not only frequency shifts of ±1, but of ±2, ±3, and so on. Thus, the compatibility becomes lost for the whole frequency range for finite boosts. In the next section we construct a complex structure J_ρ that is invariant under all AdS isometries, including the boosts.

Another point to remember is that the α and β-versions have zeros and singularities. At these points either $j^{S,ab}_{\omega l}$ or $j^{S,ba}_{\omega l}$ vanishes. Therefore, for all frequencies for which this occurs, the complex structure must be set to the diagonal form (C.351).

C.5.2 Isometry-invariant choice $J^{\rm iso}_\rho$

Here we actually make use of the fact that the conditions (C.400) encoding the boost compatibility of the complex structure

$$j^{S,ba}_{\omega-1,l+1} \overset{!}{=} -j^{S,ba}_{\omega l}\;\frac{(2l+d)\,(2l+d-2)}{(\widetilde m_++\omega-l-d)\,(\widetilde m_+-\omega+l)}$$

$$j^{S,ba}_{\omega+1,l+1} \overset{!}{=} -j^{S,ba}_{\omega l}\;\frac{(2l+d)\,(2l+d-2)}{(\widetilde m_+-\omega-l-d)\,(\widetilde m_++\omega+l)}$$

only relate factors $j^{S,ba}$ with integer frequency differences. In particular, if we consider only $j^{S,ba}_{\omega l}$ with l fixed, then these conditions relate only factors whose frequency difference is an *even* integer (since we have to apply them twice to get back the original l). Our starting point are again the conditions (C.404) and (C.405), which are induced by amplitude equivalence of time-interval and rod regions:

$$j^{S,ba}_{\omega^+_{nl},l} = j^{S,ba,\beta}_{\omega^+_{nl},l} \qquad\qquad j^{S,ba}_{-\omega^+_{nl},l} = j^{S,ba,\alpha}_{-\omega^+_{nl},l} = j^{S,ba,\beta}_{+\omega^+_{nl},l}.$$

That is, for the positive magic frequencies $+\omega^+_{nl}$ we need to choose the β-version, and for the negative magic frequencies $-\omega^+_{nl}$ we need to choose the α-version. This does not break boost compatibility as long as positive and negative magic frequencies are not separated by an even integer gap. Since $\omega^+_{nl}=\tilde m_++2n+l$, and if we fix l, then the difference between some positive magic frequency $\omega^+_{n_1l}$ and some negative magic frequency $-\omega^+_{n_2l}$ is $2\tilde m_++2(n_1+n_2)+2l$. This difference is thus an even integer only if $2\tilde m_+=d+2\nu$ is an even integer. That is, since d is odd, 2ν must be odd, too, which makes ν half-integer. Therefore, the complex structure we are to construct here will only be valid for values of ν that are not half-integer, that is, for values of $\tilde{m}_+$ that are non-integer. Since from the outset we only considered non-integer ν, this is an additional condition. Hence, we now only consider values of both $\tilde{m}_+$ and ν that are neither integer nor half-integer.

So far we thus have chosen the β-version for the positive magic frequencies $+\omega^+_{nl}$, and the α-version for the negative magic frequencies $-\omega^+_{nl}$:

$$j^{S,ba,\text{iso}}_{\omega l} = \begin{cases} j^{S,ba,\alpha}_{\omega l} & \omega = -\omega^+_{nl} \\ j^{S,ba,\beta}_{\omega l} & \omega = +\omega^+_{nl} \end{cases}. \tag{C.414}$$

Let us keep l fixed for the moment, for example at $l = 0$. The above boost compatibility conditions then induce the β-version for all frequencies at an even integer distance from the positive magic frequencies $+\omega^+_{n,0}$, and the α-version for all frequencies at an even integer distance from the negative magic frequencies $-\omega^+_{n,0}$. Apart from this discrete set of frequencies we are at liberty to choose our $j^{S,ba}_{\omega l}$ (as long as we respect frequency symmetry and boost compatibility). In any way, we obtain a pattern of interlaced frequency intervals, on some of which we choose $j^{S,b,\alpha}_{\omega l}$ while on others we choose $j^{S,ba,\beta}_{\omega l}$. We denote the result by $j^{S,ba,\text{iso}}_{\omega l}$, with the label (i) referring to the interlacing.

We shall now try to design our factor $j^{S,ba,\text{iso}}$ such that we avoid zeros and also singularities if possible. For some fixed l, the α-version vanishes if the frequency ω is either magic $\omega = +\omega^+_{nl}$, or if it is one of those which we call zero frequencies $\omega = +\omega^0_{nl}$:

$$j^{S,ba,\alpha}_{\omega l} = 0 \qquad \Longleftrightarrow \qquad \begin{array}{l} \omega \;=\; +\omega^+_{nl} \;:=\; +\tilde{m}_+ + 2n + l \;=\; +\nu + \frac{d}{2} + 2n + l \\ \omega \;=\; +\omega^0_{nl} \;:=\; -\tilde{m}_+ + d + 2n + l \;=\; -\nu + \frac{d}{2} + 2n + l \end{array}. \tag{C.415}$$

While the magic frequencies are always positive, the zero frequencies are positive for $2n+l > \nu - \frac{d}{2}$ and negative for $2n+l < \nu - \frac{d}{2}$. The β-version vanishes if the frequency ω is either negative-magic $\omega = -\omega^+_{nl}$, or "negative"-zero $\omega = -\omega^0_{nl}$:

$$j^{S,ba,\beta}_{\omega l} = 0 \qquad \Longleftrightarrow \qquad \begin{array}{l} \omega \;=\; -\omega^+_{nl} \;:=\; -\tilde{m}_+ - 2n - l \;=\; -\nu - \frac{d}{2} - 2n - l \\ \omega \;=\; -\omega^0_{nl} \;:=\; +\tilde{m}_+ - d - 2n - l \;=\; +\nu - \frac{d}{2} - 2n - l \end{array}. \tag{C.416}$$

If one version becomes zero for some frequency, then we have to choose the remaining version at this frequency. This leads to the following choice, giving us amplitude equivalence and avoiding zeros:

$$j^{S,ba,\text{iso}}_{\omega l} = \begin{cases} j^{S,ba,\alpha}_{\omega l} & \omega = -\omega^+_{nl} \\ j^{S,ba,\alpha}_{\omega l} & \omega = -\omega^0_{nl} \\ j^{S,ba,\beta}_{\omega l} & \omega = +\omega^+_{nl} \\ j^{S,ba,\beta}_{\omega l} & \omega = +\omega^0_{nl} \end{cases}. \tag{C.417}$$

However, we still have to verify that this choice is self-consistent, that is, that all frequencies associated to the β-version are not located at an even integer distance of any frequency associated to the α-version and vice versa. This amounts to: (1) the "negative" zero frequencies $-\omega^0_{nl}$ are not at an even integer distance from the positive magic frequencies $+\omega^+_{nl}$, and (2) the "positive" zero frequencies $+\omega^0_{nl}$ are not at an even integer distance from the negative magic frequencies $-\omega^+_{nl}$, and (3) the "negative" zero frequencies $-\omega^0_{nl}$ are not at an even integer distance from the "positive" zero frequencies $+\omega^0_{nl}$. Case (1) writes $+\omega^+_{n_1 l} - -\omega^0_{n_2 l} \neq 2z$ for any $z \in \mathbb{Z}$. Since $+\omega^+_{n_1 l} - -\omega^0_{n_2 l} = d + 2(n_1 + n_2 + l)$ and d is odd, case (1) never occurs. Case (2) never occurs for the same reason. Case (3) writes $+\omega^0_{n_1 l} - -\omega^0_{n_2 l} \neq 2z$ for any $z \in \mathbb{Z}$. Since $+\omega^0_{n_1 l} - -\omega^0_{n_2 l} = -2\tilde{m}_+ + 2(d + n_1 + n_2 + l) \neq 2z$, case (3) means just that $\tilde{m}_+$ must be non-integer, which we had already found above (C.414).

Next we try to achieve that $j_{\omega l}^{S,ba,\mathrm{iso}}$ remains free of singularities. For some fixed l, the α-version becomes singular if the frequency ω is either "sing" $\omega = +\omega_{nl}^{\mathrm{sing}}$ or "ular" $\omega = +\omega_{nl}^{\mathrm{ular}}$:

$$j_{\omega l}^{S,ba,\alpha} = \text{singular} \qquad \Longleftrightarrow \qquad \begin{array}{l} \omega \;=\; +\omega_{nl}^{\mathrm{sing}} \;:=\; +\tilde{m}_+ + 2n - l - d + 2 \;=\; +\nu - l - \frac{d}{2} + 2n + 2 \\ \omega \;=\; +\omega_{nl}^{\mathrm{ular}} \;:=\; -\tilde{m}_+ + 2n - l + 2 \;=\; -\nu - l - \frac{d}{2} + 2n + 2 \end{array} . \tag{C.418}$$

The "sing" frequencies are positive for $\nu + 2n + 2 > l + \frac{d}{2}$ and negative for $\nu + 2n + 2 < l + \frac{d}{2}$. The "ular" frequencies are positive for $2n + 2 > \nu + l + \frac{d}{2}$ and negative for $2n + 2 < \nu + l + \frac{d}{2}$. The β-version becomes singular if the frequency ω is either "negative-sing" $\omega = -\omega_{nl}^{\mathrm{sing}}$ or "negative-ular" $\omega = -\omega_{nl}^{\mathrm{ular}}$:

$$j_{\omega l}^{S,ba,\beta} = \text{singular} \qquad \Longleftrightarrow \qquad \begin{array}{l} \omega \;=\; -\omega_{nl}^{\mathrm{sing}} \;:=\; -\tilde{m}_+ - 2n + l + d - 2 \;=\; -\nu + l + \frac{d}{2} - 2n - 2 \\ \omega \;=\; -\omega_{nl}^{\mathrm{ular}} \;:=\; +\tilde{m}_+ - 2n + l - 2 \;=\; +\nu + l + \frac{d}{2} - 2n - 2 \end{array} . \tag{C.419}$$

If one version becomes singular for some frequency, then we have to choose the remaining version at this frequency. This leads to the following choice, hopefully uniting amplitude equivalence with avoiding zeros and singularities:

$$j_{\omega l}^{S,ba,\mathrm{iso}} \;=\; \begin{cases} j_{\omega l}^{S,ba,\alpha} & \omega = -\omega_{nl}^{+} \\ j_{\omega l}^{S,ba,\alpha} & \omega = -\omega_{nl}^{0} \\ j_{\omega l}^{S,ba,\alpha} & \omega = -\omega_{nl}^{\mathrm{sing}} \\ j_{\omega l}^{S,ba,\alpha} & \omega = -\omega_{nl}^{\mathrm{ular}} \\ j_{\omega l}^{S,ba,\beta} & \omega = +\omega_{nl}^{+} \\ j_{\omega l}^{S,ba,\beta} & \omega = +\omega_{nl}^{0} \\ j_{\omega l}^{S,ba,\beta} & \omega = +\omega_{nl}^{\mathrm{sing}} \\ j_{\omega l}^{S,ba,\beta} & \omega = +\omega_{nl}^{\mathrm{ular}} \end{cases} . \tag{C.420}$$

However, it remains to be checked whether this choice is really possible. To this end we have to verify again that all frequencies associated to the β-version are not located at an even integer distance of any frequency associated to the α-version. Unfortunately, we find that the "negative"-zero frequencies $-\omega_{n_2,l}^{0}$ are at an even integer distance from the "sing" frequencies $\omega_{n_1,l}^{\mathrm{sing}}$, and the negative magic frequencies are at even integer distance from the "ular" frequencies: $\omega_{n_1,l}^{\mathrm{sing}} - -\omega_{n_2,l}^{0} \;=\; 2(n_1 + n_2 + 1)$ and $\omega_{n_1,l}^{\mathrm{ular}} - -\omega_{n_2,l}^{+} \;=\; 2(n_1 + n_2 + 1)$. That is: the choice (C.420) is not self-consistent.

Since we cannot avoid the singularities of $j_{\omega l}^{S,ba}$, we cannot avoid the zeros of $j_{\omega l}^{S,ab,\beta} \;=\; (j_{\omega l}^{S,ba,\beta})^{-1}$. Hence there appears to be little benefit in avoiding the zeros of $j_{\omega l}^{S,ba,\beta}$. Therefore we stick to the simpler version (C.414):

$$j_{\omega l}^{S,ba,\mathrm{iso}} \;=\; \begin{cases} j_{\omega l}^{S,ba,\alpha} & \omega = -\omega_{nl}^{+} \\ j_{\omega l}^{S,ba,\beta} & \omega = +\omega_{nl}^{+} \end{cases} . \tag{C.421}$$

It remains to extend the choice (C.421) from its discrete set of frequencies $\omega \;=\; \pm\omega_{nl}^{+}$ to all $\omega \in \mathbb{R}$ for fixed l, for example $l = 0$. The values for the other l are then determined completely by the boost compatibility relations (C.400). The extension of the choice $j_{\omega l}^{S,ba,\mathrm{iso}}$ to all $\omega \in \mathbb{R}$ must fulfill three properties: first, include the magic frequencies as in (C.421). Second, be frequency-symmetric: $j_{\omega l}^{S,ba,\mathrm{iso}} \;=\; j_{-\omega,l}^{S,ba,\mathrm{iso}}$. Third, the pattern of interlaced intervals where we choose the α and β-versions must be translation-invariant for steps of 2 in ω-direction and l-direction. This is necessary in order to comply with the boost conditions (C.400), which relate $j_{\omega l}^{S,ba}$ to $j_{\omega\pm 2,l}^{S,ba}$ and $j_{\omega,l\pm 2}^{S,ba}$. Therefore, choosing the α-version for some frequency ω induces choosing the α-version for *all frequencies* $\omega \pm 2z$ with $z \in \mathbb{Z}$ (ditto for the β-version). The last two conditions imply that the interlaced intervals of the α and β-version alternate, and have the same length. Due to the step length of 2, this length can be set to values of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, ... For simplicity, we choose this length to be 1, see Figure C.422 (a). Therein, we have ω on the horizontal axis and l on the vertical. Intervals on which we choose the α-version appear in orange (light gray in monochrome), and intervals with the β-version are dark green (darker gray).

(a) Case α (b) Case β

Figure C.422: Interlaced complex structure J_ρ^{iso}: intervals in (ω, l)-space with α and β-version.

The first condition determines which of these intervals are associated to the α respectively β-version. This goes as follows. For $l = 0$, we associate the interval $(\lfloor \tilde{m}_+ \rfloor, \lceil \tilde{m}_+ \rceil]$ to the β-version. We use the standard notation of $\lfloor x \rfloor$ for the floor function (largest integer $\leq x$), and $\lceil x \rceil$ for ceiling (smallest integer $\geq x$). This choice already determines all other intervals: for $l = 0$ they alternate between α and β-version, and for higher l they are induced by the boost conditions. Thus the α and β-version alternate both horizontally (ω-direction) and vertically (l-direction), see Figure C.422 (a). In the (ω, l)-plane let us denote by I_β the set of intervals associated to the β-version as described above, and by I_α the set of intervals associated to the α-version. Then, our interlaced choice J_ρ^{iso} writes as

$$j_{\omega l}^{S,ba,\text{iso}} = \begin{cases} j_{\omega l}^{S,ba,\alpha} & (\omega, l) \in I_\alpha \\ j_{\omega l}^{S,ba,\beta} & (\omega, l) \in I_\beta \end{cases}, \tag{C.423}$$

wherein $j_{\omega l}^{S,ba,\alpha}$ and $j_{\omega l}^{S,ba,\beta}$ are those of (C.402). This implies that two different patterns are possible for our choice: for "Case α": $\tilde{m}_+ \in (d{+}2n,\ d{+}2n{+}1)$ with $n \in \mathbb{N}_0$ we have the unit interval $\omega \in (0,1)$ for $l = 0$ associated to the α-version, see Figure C.422 (a), while for "Case β": $\tilde{m}_+ \in (d{+}2n{+}1,\ d + 2n{+}2)$ we have it associated to the β-version. see Figure C.422 (b). The label of the case thus refers to which version occupies the unit interval $\omega \in (0,1)$ for $l = 0$. (We recall that d is odd, and that we only consider values of $\tilde{m}_+$ that are neither integer nor half-integer.) In both figures, the position of $\omega = +\tilde{m}_+$ is marked by a black disk, and that of $\omega = -\tilde{m}_+$ by a black circle. For $d = 3$ with $R_{\text{AdS}} = 1$, the example in Figure C.422 (a) arises from Klein-Gordon mass $m = 1$ giving $\tilde{m}_+ \approx 3.3$, the example (b) from $m = 2.5$ giving $\tilde{m}_+ \approx 4.4$. In any case, we choose the complex structure such that for $l = 0$ the black disk of $\omega = +\tilde{m}_+$ sits on a green (dark gray) interval of the β-version, and hence the circle of $\omega = -\tilde{m}_+$ on an orange (light gray) interval of the α-version.

We recall again that the α and β-versions have zeros and singularities, which happen to lie in some of the intervals we choose. At these points either $j_{\omega l}^{S,ab}$ or $j_{\omega l}^{S,ba}$ vanishes. Therefore, for all frequencies for which this occurs, the complex structure must be set to the diagonal form (C.351).

We thus have fixed completely the element $j_{\omega l}^{S,ba}$ of our complex structure J_ρ^{iso} through interlacing intervals on which we choose the α respectively β-version. While not very elegant, this is physically motivated: it makes our complex structure fulfill the essential properties, commute *with all* isometry actions, and induce amplitude equivalence between time-interval and rod amplitudes. We still have to fix $j_{\omega l}^{S,ab}$, which in turn fixes $j_{\omega l}^{S,aa}$ through $\left(j_{\omega l}^{S,aa}\right)^2 = -j_{\omega l}^{S,ab} j_{\omega l}^{S,ba} - 1$. As discussed in Appendix C.7, requiring the flat limit of the AdS real g-product g_ρ to reproduce the Minkowski real g-product g_r implies an anti-diagonal J_ρ, that is: $j_{\omega l}^{S,aa} \equiv 0$, which in turn implies $j_{\omega l}^{S,ab} = -1/j_{\omega l}^{S,ba}$. With this, fixing $j_{\omega l}^{S,ba}$ determines J_ρ completely.

C.6 Real g-products for AdS

Via (2.195), any anti-diagonal choice (that is: setting $j^{S,aa}_{\omega l} \equiv 0$) induces the real g-product

$$\begin{aligned} g_\rho(\eta,\zeta) &= 2\omega_\rho(\eta, J_\rho\zeta) \\ &= 2\pi R^{d-1}_{\text{AdS}} \int d\omega \sum_{\underline{l},m_l} \Big\{ \eta^a_{\omega\underline{l}m_l}\,(J_\rho\zeta)^b_{-\omega,\underline{l},-m_l} - \eta^b_{\omega\underline{l}m_l}\,(J_\rho\zeta)^a_{-\omega,\underline{l},-m_l} \Big\}\,(2l+d-2) \\ &= 2\pi R^{d-1}_{\text{AdS}} \int d\omega \sum_{\underline{l},m_l} \Big\{ \eta^a_{\omega\underline{l}m_l}\,\zeta^a_{-\omega,\underline{l},-m_l}\,j^{S,ba}_{\omega l} + \eta^b_{\omega\underline{l}m_l}\,\zeta^b_{-\omega,\underline{l},-m_l}/j^{S,ba}_{\omega l} \Big\}\,(2l+d-2). \end{aligned} \tag{C.424}$$

For real solutions ϕ we have $\phi^a_{-\omega,\underline{l},-m_l} = \overline{\phi^a_{\omega\underline{l}m_l}}$ and $\phi^b_{-\omega,\underline{l},-m_l} = \overline{\phi^b_{\omega\underline{l}m_l}}$ and thus obtain

$$g_\rho(\phi,\phi) = 2\pi R^{d-1}_{\text{AdS}} \int d\omega \sum_{\underline{l},m_l} \Big\{ \Big|\phi^a_{\omega\underline{l}m_l}\Big|^2 j^{S,ba}_{\omega l} + \Big|\phi^b_{\omega\underline{l}m_l}\Big|^2 / j^{S,ba}_{\omega l} \Big\}\,(2l+d-2). \tag{C.425}$$

We can read off that the real g-product is positive for modes with ω and such that $j^{S,ba}_{\omega l}$ is positive. Let us have a look where this is the case. Plotting the α and β-version reveals that the β-version is positive for all $\omega \geq (\tilde{m}_+ + l)$ while the α-version is positive for all $\omega \leq -(\tilde{m}_+ + l)$. For all other frequencies, both versions alternate between intervals with positive and negative sign. (See for example Figure C.430.)

Therefore, the interlaced version $j^{S,ba,\text{iso}}_{\omega l}$ alternates its sign quite frequently. However, it is positive for all magic frequencies $\pm\omega^+_{nl}$. For the two-branched version $j^{S,ba,\text{two}}_{\omega l}$ the situation is simpler: due to its definition, $j^{S,ba,\text{two}}_{\omega l}$ is positive for all frequencies with $|\omega| \geq (\tilde{m}_+ + l)$, while it alternates its sign for the remaining frequencies.

Let us compare this to the real g-product of a Minkowski hypercylinder. There we also have two complex structures: J^{pos}_r and J^{iso}_r. Both induce positive real g-products for the propagating modes (which there are those with $|\omega| > m$). Moreover, g^{pos}_r is positive for the evanescent modes as well, while g^{iso}_r alternates sign for evanescent modes ($|\omega| < m$). On AdS, both the interlaced and two-branched version have similar properties: for the propagating modes (which here are those with magic frequencies $\pm\omega^+_{nl}$) the real g-products become positive, while for the evanescent modes (all other frequencies) their sign alternates.

We now want to find explicit expressions that tell us where $j^{S,ba}_{\omega l}$ is positive, and where negative. Combining the four relations (C.366) respectively their versions for $j^{S,ba}_{\omega l}$, we find that

$$j^{S,ba}_{\omega-2,l} = j^{S,ba}_{\omega l}\,\kappa^-_{\omega l} \qquad \kappa^-_{\omega l} = \frac{(\tilde{m}_+ - \omega - l - d + 2)\,(\tilde{m}_+ + \omega + l - 2)}{(\tilde{m}_+ - \omega + l)\,(\tilde{m}_+ + \omega - l - d)} \tag{C.426}$$

$$j^{S,ba}_{\omega+2,l} = j^{S,ba}_{\omega l}\,\kappa^+_{\omega l} \qquad \kappa^+_{\omega l} = \frac{(\tilde{m}_+ + \omega - l - d + 2)\,(\tilde{m}_+ - \omega + l - 2)}{(\tilde{m}_+ + \omega + l)\,(\tilde{m}_+ - \omega - l - d)}\,. \tag{C.427}$$

Consistency is assured by $\kappa^+_{\omega l} = 1/\kappa^-_{\omega+2,l}$. These relations relate the signs of $j^{S,ba}_{\omega l}$ only for even integer frequency differences. In particular, they are fulfilled by both the α-version $j^{S,ba,\alpha}_{\omega l}$ and the β-version $j^{S,ba,\beta}_{\omega l}$. Since $j^{S,ba,\beta}_{\omega l} = j^{S,ba,\alpha}_{-\omega,l}$, it is sufficient to consider the signs of the α-version. We write

$$\kappa^+_{\omega l} = \frac{\kappa^1_{\omega l}\,\kappa^2_{\omega l}}{\kappa^3_{\omega l}\,\kappa^4_{\omega l}} \qquad \begin{aligned} \kappa^1_{\omega l} &= (\tilde{m}_+ + \omega - l - d + 2) & \kappa^2_{\omega l} &= (\tilde{m}_+ - \omega + l - 2) \\ \kappa^3_{\omega l} &= (\tilde{m}_+ + \omega + l) & \kappa^4_{\omega l} &= (\tilde{m}_+ - \omega - l - d), \end{aligned} \tag{C.428}$$

and denote by λ^a_l the unique frequency for which $\kappa^a_{\omega l}$ becomes zero (with l fixed):

$$\begin{aligned} \lambda^1_l &= -(\tilde{m}_+ - l - d + 2) & \lambda^2_l &= +(\tilde{m}_+ + l - 2) \\ \lambda^3_l &= -(\tilde{m}_+ + l) & \lambda^4_l &= +(\tilde{m}_+ - l - d). \end{aligned} \tag{C.429}$$

Then, $\kappa^1_{\omega l}$ is negative for $\omega < \lambda^1_l$ and $\kappa^3_{\omega l}$ is negative for $\omega < \lambda^3_l$, while $\kappa^2_{\omega l}$ is negative for $\omega > \lambda^2_l$ and $\kappa^4_{\omega l}$ is negative for $\omega > \lambda^4_l$. For all $l \geq 0$ we thus have $\lambda^3_l < \lambda^1_l < \lambda^2_l$ and $\lambda^3_l < \lambda^4_l < \lambda^2_l$, while $\lambda^1_l < \lambda^4_l$ for "small" $l < \nu - \frac{d-2}{2}$ but $\lambda^1_l > \lambda^4_l$ for "large" $l > \nu - \frac{d-2}{2}$. Hence for all $\omega < \lambda^3_l$ we have negative $\kappa^1_{\omega l}$ and $\kappa^3_{\omega l}$ but positive $\kappa^2_{\omega l}$ and $\kappa^4_{\omega l}$, making $\kappa^+_{\omega l}$ positive for all $\omega < \lambda^3_l$. Further, the α-version $j^{S,ba,\alpha}_{\omega l}$ is positive for all $\omega < \lambda^3_l + 2$ because of $j^{S,ba}_{\omega+2,l} = j^{S,ba}_{\omega l}\kappa^+_{\omega l}$.

We make this our starting point: the α-version $j^{S,ba,\alpha}_{\omega l}$ is positive for all $\omega < \lambda^3_l + 2 = -\tilde{m}_+ - l + 2$, and the sign for $\omega > \lambda^3_l + 2$ is determined by counting sign changes. In order to illustrate the following considerations, in Figure C.430 we plot the α-version and mark the values of the four λ's.

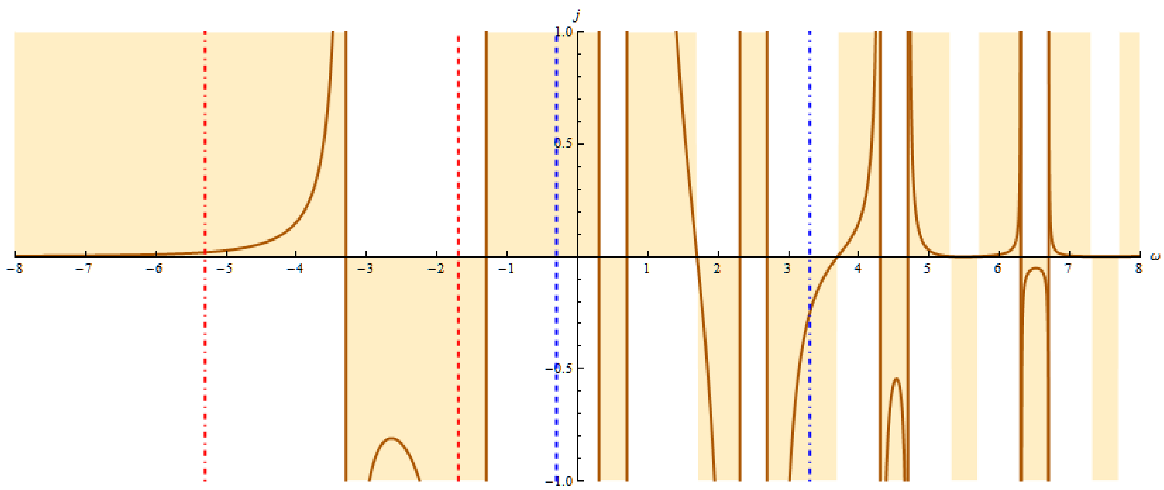


Figure C.430: Typical plot of the α-version $j^{S,ba,\alpha}_{\omega l}$ (orange continuous curves): here $d = 3$, $m = 1$, $R_{\text{AdS}} = 1$, $l = 2$, giving us $\tilde{m}_+ \approx 3.3$. Thus $\lambda^3_2 \approx -5.3$ (dot-dashed red line), $\lambda^4_2 \approx -1.7$ (dashed red line), $\lambda^1_2 \approx -0.3$ (dashed blue line) and $\lambda^2_2 \approx 3.3$ (dot-dashed blue line). The vertical orange lines at $\omega = \lambda^3_2 + 2n$ and $\omega = \lambda^4_2 + 2n$ with integer $n \geq 1$ are the poles of $j^{S,ba,\alpha}_{\omega,2}$. The background color indicates the sign: the α-version is positive where the orange background is above the ω-axis, and negative where the orange background lies below it.

We observe that the α-version changes sign at each singularity and at each zero. The singularities are caused by the factors $\kappa^3_{\omega l}$ and $\kappa^4_{\omega l}$ and appear to the right of λ^3_l and λ^4_l in steps of $\Delta\omega = 2$. We use the usual notation of $\lfloor x \rfloor$ for the floor function (largest integer $\leq x$), and $\theta(x)$ for the Heaviside step function which is 0 for all $x \leq 0$ and 1 for $x > 0$. The singularities induced by λ^3_l contribute $\sigma^3_{\omega l}$ sign changes between some frequency $\omega > \lambda^3_l$ and λ^3_l:

$$\sigma^3_{\omega l} \;=\; \theta(\omega - \lambda^3_l)\, \lfloor (\omega - \lambda^3_l)/2 \rfloor. \tag{C.431}$$

In the same way the singularities induced by λ^4_l contribute $\sigma^4_{\omega l}$ sign changes:

$$\sigma^4_{\omega l} \;=\; \theta(\omega - \lambda^4_l)\, \lfloor (\omega - \lambda^4_l)/2 \rfloor. \tag{C.432}$$

The zeros come from $\kappa^1_{\omega l}$ and $\kappa^2_{\omega l}$ and appear to the right of λ^1_l and λ^2_l, also in steps of $\Delta\omega = 2$. The zeros induced by λ^1_l and λ^2_l contribute $\sigma^1_{\omega l}$ and $\sigma^2_{\omega l}$ sign changes:

$$\sigma^1_{\omega l} \;=\; \theta(\omega - \lambda^1_l)\, \lfloor (\omega - \lambda^1_l)/2 \rfloor \tag{C.433}$$

$$\sigma^2_{\omega l} \;=\; \theta(\omega - \lambda^2_l)\, \lfloor (\omega - \lambda^2_l)/2 \rfloor. \tag{C.434}$$

The total of sign changes is the sum of the four σ's, resulting in the following formula for the sign of the α-version (the step functions ensure that this formula actually holds for all ω, not only for $\omega > \lambda^3_l + 2$):

$$\operatorname{sign} j^{S,ba,\alpha}_{\omega l} \;=\; (-1)^{\sigma^\alpha_{\omega l}} \qquad\qquad \sigma^\alpha_{\omega l} \;=\; \sigma^1_{\omega l} + \sigma^2_{\omega l} + \sigma^3_{\omega l} + \sigma^4_{\omega l}. \tag{C.435}$$

(This formula has been used to color the background in Figure C.430, we can see that the sign obtained by (C.435) agrees precisely with the sign of the plotted curve.) Hence the signs of choices $j^{S,ba,\mathrm{iso}}_{\omega l}$ and $j^{S,ba,\mathrm{two}}_{\omega l}$ result to be

$$\operatorname{sign} j^{S,ba,\mathrm{iso}}_{\omega l} \;=\; \begin{cases} (-1)^{\sigma^\alpha_{\omega l}} & (\omega,l)\in I_\alpha \\ (-1)^{\sigma^\alpha_{-\omega,l}} & (\omega,l)\in I_\beta \end{cases} \qquad \sigma^\alpha_{\omega l} \;=\; \sigma^1_{\omega l}+\sigma^2_{\omega l}+\sigma^3_{\omega l}+\sigma^4_{\omega l} \tag{C.436}$$

$$\operatorname{sign} j^{S,ba,\mathrm{two}}_{\omega l} \;=\; \begin{cases} (-1)^{\sigma^\alpha_{\omega l}} & \omega<0 \\ (-1)^{\sigma^\alpha_{-\omega,l}} & \omega>0 \end{cases}. \tag{C.437}$$

Formula (C.436) thus tells us whether the AdS hypercylinder's real g-product g_ρ in (C.425) is positive or negative for each mode $\mu^{(S,a)}_{\omega \underline{l} m_l}$ and $\mu^{(S,b)}_{\omega \underline{l} m_l}$.

C.7 Flat limits

C.7.1 Flat limits: time-intervals

Another property that indicates a good choice of complex structures for AdS is that the action of the complex structure on an AdS solution should "commute" with the process of taking the flat limit. That is, the two diagrams below should become commutative.

$$\begin{array}{ccc} \phi^{\mathrm{AdS}} & \xrightarrow{\;J_t^{\mathrm{AdS}}\;} & J_t^{\mathrm{AdS}}\phi^{\mathrm{AdS}} \\ \text{flat lim.}\big\downarrow & & \big\downarrow\text{flat lim.} \\ \phi^{\mathrm{Mink}} & \xrightarrow{\;J_t^{\mathrm{Mink}}\;} & J_t^{\mathrm{Mink}}\phi^{\mathrm{Mink}} \end{array} \qquad\qquad \begin{array}{ccc} \phi^{\mathrm{AdS}} & \xrightarrow{\;J_\rho\;} & J_\rho\phi^{\mathrm{AdS}} \\ \text{flat lim.}\big\downarrow & & \big\downarrow\text{flat lim.} \\ \phi^{\mathrm{Mink}} & \xrightarrow{\;J_r\;} & J_r\phi^{\mathrm{Mink}} \end{array}$$

Further, the induced real g-products should recover their Minkowski counterparts in the flat limit. That is, J_t of AdS induces the real g-product g_t, whose flat limit we wish to reproduce the real g-product g_t of the Minkowski equal-time plane. Ditto, J_ρ of AdS induces g_ρ, whose flat limit should reproduce g_r of the Minkowski hypercylinder. This section treats the equal-time hypersurface, and the next one the hypercylinder.

We proceed with the following three steps: first, to compute for a solution (near an equal-time plane Σ_t) the expansion that makes the solution's flat limit reproduce directly the expansion of a solution near a Minkowski equal-time plane. Second, check that the left diagram above is indeed commutative. Third, to check whether the flat limit of the real g-product $\mathrm{g}^{\mathrm{AdS}}_t(\eta,\zeta)$ (wherein we expand the solutions using the above expansion) reproduces directly the real g-product for solutions near a Minkowski equal-time plane. We will then perform the same two steps for solutions near an AdS hypercylinder Σ_ρ in the next section.

Step one is done already in Section 2.6.5. Let us recall the expansion (2.90) of a solution near a Minkowski equal-time plane:

$$\phi(t,r,\Omega) \;=\; \int\limits_0^\infty \mathrm{d}p \sum_{l,m_l} 2p\,(2\pi)^{-1/2}\, j_l(pr) \Big\{ \phi^+_{plm_l}\, \mathrm{e}^{-\mathrm{i}E_p t}\, Y^{m_l}_l(\Omega) + \overline{\phi^-_{plm_l}}\, \mathrm{e}^{\mathrm{i}E_p t}\, \overline{Y^{m_l}_l(\Omega)} \Big\}\,. \tag{C.438}$$

Our usual expansion (2.201) of a solution near an AdS equal-time plane is the (ordinary) Jacobi expansion:

$$\phi(t,\rho,\Omega) \;=\; \sum_{n\underline{l}m_l} \Big\{ \phi^+_{n\underline{l}m_l}\; \mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\phi^-_{n\underline{l}m_l}}\; \overline{\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)} \Big\}\,. \tag{C.439}$$

We can rescale the momentum representation as in (2.202)

$$\phi^{\pm}_{n\underline{l}m_l} = \phi^{\pm}_{\omega^+_{nl}\underline{l}m_l} = \phi^{\mathrm{F},\pm}_{n\underline{l}m_l}\,\frac{2}{R_{\mathrm{AdS}}}\,\frac{2\omega^+_{nl}}{\sqrt{2\pi}}\,\frac{(p^{\mathbb{R}}_{nl})^l}{(2l+d-2)!!} \tag{C.440}$$

$$\phi^{\mathrm{F},\pm}_{n\underline{l}m_l} = R^{-1}_{\mathrm{AdS}}\,\tilde{\phi}^{\mathrm{F},\pm}_{\tilde{p}\underline{l}m_l} \qquad \text{with here } \tilde{p} \text{ short for } \tilde{p}_{nl}, \tag{C.441}$$

obtaining what we shall call flat Jacobi expansion:

$$\phi(t,\rho,\Omega) = \sum_{n\underline{l}m_l}\Big\{\phi^{\mathrm{F},+}_{n\underline{l}m_l}\,\frac{2}{R_{\mathrm{AdS}}}\,\frac{2\omega^+_{nl}}{\sqrt{2\pi}}\,\frac{(p^{\mathbb{R}}_{nl})^l}{(2l+d-2)!!}\;\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega) + \overline{\phi^{\mathrm{F},-}_{n\underline{l}m_l}}\,\frac{2}{R_{\mathrm{AdS}}}\,\frac{2\omega^+_{nl}}{\sqrt{2\pi}}\,\frac{(p^{\mathbb{R}}_{nl})^l}{(2l+d-2)!!}\;\overline{\mu^{(+)}_{n\underline{l}m_l}(t,\rho,\Omega)}\Big\}\,. \tag{C.442}$$

Using (C.90) from Appendix C.2.8, we obtain that in the flat limit for $d=3$ the flat Jacobi expansion (C.442) becomes the Minkowski time-interval expansion (C.438):

$$\phi(t,r,\Omega) \xrightarrow[\text{lim.}]{\text{flat}} \int_0^\infty \mathrm{d}\tilde{p}\sum_{l,m_l} 2\tilde{p}\,(2\pi)^{-1/2}\,j_l(\tilde{p}r)\Big\{\tilde{\phi}^{\mathrm{F},+}_{\tilde{p}\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\tilde{\omega}_{\tilde{p}}\tau}\,Y^{m_l}_l(\Omega) + \overline{\tilde{\phi}^{\mathrm{F},-}_{\tilde{p}\underline{l}m_l}}\,\mathrm{e}^{\mathrm{i}\tilde{\omega}_{\tilde{p}}\tau}\,\overline{Y^{m_l}_l(\Omega)}\Big\}\,. \tag{C.443}$$

It is straightforward to verify that the flat limit commutes indeed with J_t, since for both Minkowski and AdS we have simply $(J_t\phi)^{\pm} = -\mathrm{i}\phi^{\pm}$. Thus we can proceed with the third step. We want to reproduce the result (2.98) of the real g-product for two solutions near an equal-time plane in Minkowski spacetime:

$$\mathrm{g}_t(\eta,\,\zeta) = \int_0^\infty \mathrm{d}p\sum_{l,m_l} 2E_p\left\{\overline{\eta^-_{plm_l}}\,\zeta^+_{plm_l} + \eta^+_{plm_l}\,\overline{\zeta^-_{plm_l}}\right\}\,. \tag{C.444}$$

To this end, we start with the real g-product (2.213) for two solutions near an AdS equal-time plane:

$$\mathrm{g}_t(\eta,\zeta) = \sum_{n\underline{l}m_l} 2\omega^+_{nl}R^{d-1}_{\mathrm{AdS}}\mathcal{N}^+_{nl}\left\{\overline{\eta^-_{n\underline{l}m_l}}\,\zeta^+_{n\underline{l}m_l} + \eta^+_{n\underline{l}m_l}\,\overline{\zeta^-_{n\underline{l}m_l}}\right\}\,. \tag{C.445}$$

The normalization factor stems from (C.67)

$$\mathcal{N}^+_{nl} = \frac{n!\,\Gamma(\gamma^{\mathrm{S}})^2\,\Gamma(n+\nu+1)}{2\omega^+_{nl}\,\Gamma(n+\gamma^{\mathrm{S}})\,\Gamma(n+\nu+\gamma^{\mathrm{S}})}\,. \tag{C.446}$$

Plugging the flat Jacobi expansion (C.459) into the AdS real g-product we get

$$\mathrm{g}_t(\eta,\zeta) = \sum_{n\underline{l}m_l}\left\{\overline{\eta^{\mathrm{F},-}_{\tilde{p}\underline{l}m_l}}\,\zeta^{\mathrm{F},+}_{\tilde{p}\underline{l}m_l} + \eta^{\mathrm{F},+}_{\tilde{p}\underline{l}m_l}\,\overline{\zeta^{\mathrm{F},-}_{\tilde{p}\underline{l}m_l}}\right\}R^{d-3}_{\mathrm{AdS}}\,\frac{n!\,\Gamma(\gamma^{\mathrm{S}})^2\,\Gamma(n+\nu+1)}{\Gamma(n+\gamma^{\mathrm{S}})\,\Gamma(n+\nu+\gamma^{\mathrm{S}})}\,\frac{(p^{\mathbb{R}}_{nl})^{2l}\,8(\omega^+_{nl})^2}{\pi\,((2l+d-2)!!)^2}\,.$$

Setting $d=3$, using $k! = \Gamma(k+1)$ and that for odd k we have [DLMF 5.4.2]

$$k!! = \Gamma(\tfrac{k}{2}+1)\,\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\,, \tag{C.447}$$

the real g-product becomes

$$\begin{aligned}\mathrm{g}_t(\eta,\zeta) &= \sum_{nlm_l}\left\{\overline{\eta^{\mathrm{F},-}_{\tilde{p}lm_l}}\,\zeta^{\mathrm{F},+}_{\tilde{p}lm_l} + \eta^{\mathrm{F},+}_{\tilde{p}lm_l}\,\overline{\zeta^{\mathrm{F},-}_{\tilde{p}lm_l}}\right\}\frac{\Gamma(n+1)\,\Gamma(l+\frac{3}{2})^2\,\Gamma(n+\nu+1)}{\Gamma(n+l+\frac{3}{2})\,\Gamma(n+\nu+l+\frac{3}{2})}\,\frac{(p^{\mathbb{R}}_{nl})^{2l}\,8(\omega^+_{nl})^2\,\pi}{\pi\,\Gamma(l+\frac{3}{2})^2\,2^{2l+2}}\\ &= \sum_{nlm_l}\left\{\overline{\eta^{\mathrm{F},-}_{\tilde{p}lm_l}}\,\zeta^{\mathrm{F},+}_{\tilde{p}lm_l} + \eta^{\mathrm{F},+}_{\tilde{p}lm_l}\,\overline{\zeta^{\mathrm{F},-}_{\tilde{p}lm_l}}\right\}\frac{\Gamma(n+1)\,\Gamma(n+\nu+1)}{\Gamma(n+l+\frac{3}{2})\,\Gamma(n+\nu+l+\frac{3}{2})}\,\frac{(p^{\mathbb{R}}_{nl})^{2l}\,8(\omega^+_{nl})^2}{2^{2l+2}}\,.\end{aligned}$$

In the next lines we perform the flat limit, which involves $\nu \;\to\; mR$ (again, R is short for R_{AdS}) and making the sum into an integral. To this end, from $\omega^+_{nl} \;:=\; \tilde m_+ + 2n + l$ we also substitute $n \;\to\; \frac{1}{2}(\omega^+_{nl} - \tilde m_+ - l)$.

$$
\begin{aligned}
\mathrm{g}_t(\eta,\zeta) \;=\; & \sum_{lm_l}\sum_{\omega=\omega^+_{0l}}^{\Delta\omega=2} \Big\{\overline{\tilde\eta^{\mathrm{F},-}_{\tilde p l m_l}}\,\tilde\zeta^{\mathrm{F},+}_{\tilde p l m_l} + \tilde\eta^{\mathrm{F},+}_{\tilde p l m_l}\,\overline{\tilde\zeta^{\mathrm{F},-}_{\tilde p l m_l}}\Big\}\,\frac{(p^{\mathbb{R}}_{nl})^{2l}\;8(\omega^+_{nl})^2}{R^2\,2^{2l+2}}\;\frac{\Gamma(\frac12(\omega^+_{nl}-\nu-l+\frac12))}{\Gamma(\frac12(\omega^+_{nl}-\nu+l+\frac32))}\,\frac{\Gamma(\frac12(\omega^+_{nl}+\nu-l+\frac12))}{\Gamma(\frac12(\omega^+_{nl}+\nu+l+\frac32))}\\
\xrightarrow[\text{lim.}]{\text{flat}}\; & \sum_{lm_l}\int_0^\infty \mathrm{d}\tilde p\;\frac{R}{2}\,\frac{\tilde p}{\tilde\omega_{\tilde p}}\Big\{\overline{\tilde\eta^{\mathrm{F},-}_{\tilde p l m_l}}\,\tilde\zeta^{\mathrm{F},+}_{\tilde p l m_l} + \tilde\eta^{\mathrm{F},+}_{\tilde p l m_l}\,\overline{\tilde\zeta^{\mathrm{F},-}_{\tilde p l m_l}}\Big\}\,\frac{p^{2l}\;8(\omega_p)^2}{R^2\,2^{2l+2}}\;\frac{\Gamma(\frac12(R(\tilde\omega-m)-l+\frac12))}{\Gamma(\frac12(R(\tilde\omega-m)+l+\frac32))}\,\frac{\Gamma(\frac12(R(\tilde\omega+m)-l+\frac12))}{\Gamma(\frac12(R(\tilde\omega+m)+l+\frac32))}\\
\approx & \sum_{lm_l}\int_0^\infty \mathrm{d}\tilde p\;\Big\{\overline{\tilde\eta^{\mathrm{F},-}_{\tilde p l m_l}}\,\tilde\zeta^{\mathrm{F},+}_{\tilde p l m_l} + \tilde\eta^{\mathrm{F},+}_{\tilde p l m_l}\,\overline{\tilde\zeta^{\mathrm{F},-}_{\tilde p l m_l}}\Big\}\,\frac{\tilde p}{\tilde\omega_{\tilde p}}\,\frac{p^{2l}\;4(\omega_p)^2}{R\,2^{2l+2}}\,\Big(\tfrac12 R(\tilde\omega-m)\Big)^{-l-\frac12}\Big(\tfrac12 R(\tilde\omega+m)\Big)^{-l-\frac12}
\end{aligned}
$$

Now we can simplify

$$
\begin{aligned}
\Big(\tfrac12 R(\tilde\omega-m)\Big)^{-l-\frac12}\Big(\tfrac12 R(\tilde\omega+m)\Big)^{-l-\frac12} \;&=\; \Big(\tfrac14 R^2(\tilde\omega^2-m^2)\Big)^{-l-\frac12} \;=\; \Big(\tfrac14 R^2\tilde p^2\Big)^{-l-\frac12} \;=\; \Big(\tfrac14 p^2\Big)^{-l-\frac12}\\
&=\; 2^{2l+1}\,p^{-2l-1}\;,
\end{aligned}
$$

giving us

$$
\begin{aligned}
\mathrm{g}_t(\eta,\zeta) \;\xrightarrow[\text{lim.}]{\text{flat}}\; & \sum_{lm_l}\int_0^\infty \mathrm{d}\tilde p\;\Big\{\overline{\tilde\eta^{\mathrm{F},-}_{\tilde p l m_l}}\,\tilde\zeta^{\mathrm{F},+}_{\tilde p l m_l} + \tilde\eta^{\mathrm{F},+}_{\tilde p l m_l}\,\overline{\tilde\zeta^{\mathrm{F},-}_{\tilde p l m_l}}\Big\}\,\frac{\tilde p}{\tilde\omega_{\tilde p}}\,\frac{p^{2l}\;4(\omega_p)^2}{R\,2^{2l+2}}\;2^{2l+1}\,p^{-2l-1}\\
=\; & \sum_{lm_l}\int_0^\infty \mathrm{d}\tilde p\;\Big\{\overline{\tilde\eta^{\mathrm{F},-}_{\tilde p l m_l}}\,\tilde\zeta^{\mathrm{F},+}_{\tilde p l m_l} + \tilde\eta^{\mathrm{F},+}_{\tilde p l m_l}\,\overline{\tilde\zeta^{\mathrm{F},-}_{\tilde p l m_l}}\Big\}\;2\tilde\omega_{\tilde p}\;.
\end{aligned}
$$

This is precisely the real g-product (C.444) for two solutions near a Minkowski equal-time plane. We have computed it here in the momentum representation: For $g_t(\eta,\zeta) = 2\omega_t(\eta, J_t\zeta)$ we have used the action of J_t in the momentum representation and plugged it into the momentum representation (2.209) of ω_t. Then we took the flat limit. In the diagram below this corresponds to starting top left, going down and then right.

$$
\begin{array}{ccc}
\mathrm{g}^{\mathrm{AdS}}_t(\eta, J_t\zeta) & \xrightarrow{\text{disc. flat lim.}} & \\
\Big\downarrow {\scriptstyle\int \mathrm{d}\rho} & & \Big\downarrow {\scriptstyle\int \mathrm{d}r}\\
 & \xrightarrow{\text{disc. flat lim.}} & \mathrm{g}^{\mathrm{Mink}}_\tau(\eta^{\mathrm{M}}, J_\tau\zeta^{\mathrm{M}})
\end{array}
$$

The other way of first going right and then down gives the same result, since it does so for the symplectic structure, see (2.197), and the complex structure J_t commutes with the flat limit (see the diagram at beginning of this subsection). Hence the above diagram commutes as well, indicating that our flat limit of g_t is self-consistent.

C.7.2 Flat limits: rod regions α and β-versions

Before also studying the flat limit of the real g-product g_ρ, we start by calculating the flat limit of $j^{S,ba,\mathrm{iso}}_{\omega l}$. That is, we calculate the flat limits of the α and β-versions $j^{S,ba,\alpha}_{\omega l}$ and $j^{S,ba,\beta}_{\omega l}$. We recall that the α-version writes

$$
j^{S,ba,\alpha}_{\omega l} \;=\; \tfrac{1}{\pi}\,\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)\,\frac{\Gamma(\alpha^{S,b})}{\Gamma(\alpha^{S,a})}\,\frac{\Gamma(1-\beta^{S,a})}{\Gamma(1-\beta^{S,b})}\;.
$$

The flat limit only affects the the last two quotients of Gamma functions. Since the Gammas' arguments are always real, the Gammas and their quotients must be real as well. Below we calculate the flat limit, replacing step by step some expressions by their values for large $R := R_{\text{AdS}}$, and setting $d = 3$. Note that we do not apply the flat limit directly to the arguments of the Gamma functions (but only after the factors emerge from the Gammas), in order to obtain the correct result for the overall sign.

$$\begin{aligned}
\frac{\Gamma(\alpha^{S,b})}{\Gamma(\alpha^{S,a})}\,\frac{\Gamma(1-\beta^{S,a})}{\Gamma(1-\beta^{S,b})} &= \frac{\Gamma(\frac{1}{2}(\tilde{m}_+-\omega-l-d+2))}{\Gamma(\frac{1}{2}(\tilde{m}_+-\omega+l))}\,\frac{\Gamma(-\frac{1}{2}(\tilde{m}_++\omega+l-2))}{\Gamma(-\frac{1}{2}(\tilde{m}_++\omega-l-d))} \\
&= \frac{\Gamma(-\frac{1}{2}(\omega-\tilde{m}_++l+1))}{\Gamma(-\frac{1}{2}(\omega-\tilde{m}_+-l))}\,\frac{\Gamma(-\frac{1}{2}(\omega+\tilde{m}_++l-2))}{\Gamma(-\frac{1}{2}(\omega+\tilde{m}_+-l-3))} \\
&= \frac{\Gamma(-\frac{1}{2}(\omega-\tilde{m}_+)-\frac{l}{2}-\frac{1}{2}))}{\Gamma(-\frac{1}{2}(\omega-\tilde{m}_+)+\frac{l}{2}))}\,\frac{\Gamma(-\frac{1}{2}(\omega+\tilde{m}_+)-\frac{l}{2}+1))}{\Gamma(-\frac{1}{2}(\omega+\tilde{m}_+)+\frac{l}{2}+\frac{3}{2}))} \\
&\approx \left(-\tfrac{1}{2}\left(\omega-\tilde{m}_+\right)\right)^{-l}\left(-\tfrac{1}{2}\left(\omega+\tilde{m}_+\right)\right)^{-l}\frac{\Gamma(-\frac{1}{2}(\omega-\tilde{m}_+)-\frac{l}{2}-\frac{1}{2}))}{\Gamma(-\frac{1}{2}(\omega-\tilde{m}_+)-\frac{l}{2}))}\,\frac{\Gamma(-\frac{1}{2}(\omega+\tilde{m}_+)-\frac{l}{2}+1))}{\Gamma(-\frac{1}{2}(\omega+\tilde{m}_+)-\frac{l}{2}+\frac{3}{2}))} \\
&= \left(\tfrac{R^2}{4}\left(\tilde{\omega}^2-m^2\right)\right)^{-l}\frac{\Gamma(X_-^\alpha)}{\Gamma(X_-^\alpha+\frac{1}{2})}\,\frac{\Gamma(X_+^\alpha)}{\Gamma(X_+^\alpha+\frac{1}{2})}\ ,
\end{aligned} \tag{C.448}$$

wherein

$$\begin{aligned}
X_-^\alpha &= -\tfrac{1}{2}(\omega-\tilde{m}_+)-\tfrac{l}{2}-\tfrac{1}{2} \\
X_+^\alpha &= -\tfrac{1}{2}(\omega+\tilde{m}_+)-\tfrac{l}{2}+1\ .
\end{aligned} \tag{C.449}$$

The calculation for the β-version is the same up to some signs, and yields

$$\frac{\Gamma(\beta^{S,b})}{\Gamma(\beta^{S,a})}\,\frac{\Gamma(1-\alpha^{S,a})}{\Gamma(1-\alpha^{S,b})} \approx \left(\tfrac{R^2}{4}\left(\tilde{\omega}^2-m^2\right)\right)^{-l}\frac{\Gamma(X_-^\beta)}{\Gamma(X_-^\beta+\frac{1}{2})}\,\frac{\Gamma(X_+^\beta)}{\Gamma(X_+^\beta+\frac{1}{2})}\ ,$$

wherein

$$\begin{aligned}
X_-^\beta &= +\tfrac{1}{2}(\omega-\tilde{m}_+)-\tfrac{l}{2}+1 \\
X_+^\beta &= +\tfrac{1}{2}(\omega+\tilde{m}_+)-\tfrac{l}{2}-\tfrac{1}{2}\ .
\end{aligned} \tag{C.450}$$

In the flat limit, for all $X_\pm^{\alpha,\beta}$ their absolute value $|X|$ becomes very large. Then, for positive X we can use

$$\frac{\Gamma(X)}{\Gamma(X+\frac{1}{2})} \approx X^{-1/2} = |X|^{-1/2}\ ,$$

while for negative $X_\pm$ we get

$$\frac{\Gamma(X)}{\Gamma(X+\frac{1}{2})} \approx |X|^{-1/2}\cdot\begin{cases}+1 & X\in(-n,-n+\frac{1}{2}) \\ -1 & X\in(-n+\frac{1}{2},-n+1)\end{cases} \qquad n\in\mathbb{N}^+$$

(because the Gamma function alternates its sign for negative arguments). By $\mathbb{N}^+$ we denote the positive natural numbers, thus excluding zero. Together this writes as follows for large $|X|$ with X positive or negative:

$$\frac{\Gamma(X)}{\Gamma(X+\frac{1}{2})} \approx |X|^{-1/2}\cdot\begin{cases}-1 & X\in(-n+\frac{1}{2},-n+1) \qquad n\in\mathbb{N}^+ \\ +1 & \text{else}\end{cases}$$

With this we can complete the process of taking the flat limit. For the α-version we obtain

$$j^{S,ba,\alpha}_{\omega l} \xrightarrow[\text{lim.}]{\text{flat}} \tfrac{1}{\pi}\,\Gamma(l+\tfrac{3}{2})\,\Gamma(l+\tfrac{1}{2})\left(\tfrac{R}{2}\tilde{p}_{\tilde{\omega}}\right)^{-2l}\left(\tfrac{R}{2}\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}\right)^{-1} q^{\alpha}_{-}\,q^{\alpha}_{+} \tag{C.451}$$

$$q^{\alpha}_{\mp} = \begin{cases} -1 & X^{\alpha}_{\mp} \in (-n+\tfrac{1}{2}, -n+1) \qquad n \in \mathbb{N}^{+} \\ +1 & \text{else} \end{cases} .$$

For the β-version we obtain the same result up to different sign factors q:

$$j^{S,ba,\beta}_{\omega l} \xrightarrow[\text{lim.}]{\text{flat}} \tfrac{1}{\pi}\,\Gamma(l+\tfrac{3}{2})\,\Gamma(l+\tfrac{1}{2})\left(\tfrac{R}{2}\tilde{p}_{\tilde{\omega}}\right)^{-2l}\left(\tfrac{R}{2}\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}\right)^{-1} q^{\beta}_{-}\,q^{\beta}_{+} \tag{C.452}$$

$$q^{\beta}_{\mp} = \begin{cases} -1 & X^{\beta}_{\mp} \in (-n+\tfrac{1}{2}, -n+1) \qquad n \in \mathbb{N}^{+} \\ +1 & \text{else} \end{cases} ,$$

In the following section we study what this implies for the real g-product.

C.7.3 Flat limits: rod regions $\mathrm{g}^{\mathrm{iso}}_{\rho}$

We recall that in Section C.7.1 the flat limit for the time-interval is considered only for the Jacobi modes (a discrete set with magic frequencies ω^{+}_{nl}), because only these are well-defined for all of space. In the flat limit the corresponding frequencies $\tilde{\omega} := \omega/R_{\text{AdS}}$ then become dense. For the hypercylinder we shall do something similar, because in the end it turns out that we can make use only of a discrete set of frequencies for which we have $q^{\alpha}_{-}\,q^{\alpha}_{+} = +1$ on intervals I_α respectively $q^{\beta}_{-}\,q^{\beta}_{+} = +1$ on intervals I_β. (See (C.423) for the intervals on which we choose α respectively β-version.)

That is, for $|\omega| \geq \tilde{m}_{+} + l$, we consider the flat limit only for hypergeometric S^a and S^b-modes with magic frequencies $\omega = \pm\omega^{+}_{nl}$. For the magic frequencies $-\omega^{+}_{nl}$ we always have $q^{\alpha}_{-}q^{\alpha}_{+} = +1$ on their intervals I_α, and for $+\omega^{+}_{nl}$ we have $q^{\beta}_{-}q^{\beta}_{+} = +1$ on I_β. In order to obtain $q^{\alpha,\beta}_{-}q^{\alpha,\beta}_{+} = +1$ also for frequencies with $|\omega| < \tilde{m}_{+} + l$, we also choose some discrete subset of frequencies, since $q^{\alpha,\beta}_{-}q^{\alpha,\beta}_{+}$ is not everywhere $+1$ for $|\omega| < \tilde{m}_{+} + l$. The magic frequencies ω^{+}_{nl} start at $\tilde{m}_{+} + l$, increasing by steps of $\Delta\omega = 2$. Hence, our first guess would be for $|\omega| < \tilde{m}_{+} + l$ to choose frequencies ω which start at $\tilde{m}_{+} + l$, but now decreasing by steps of 2. Unfortunately, this does not work, since precisely at these frequencies we encounter singularities of Gamma functions in our definitions of $j^{S,ba,\text{iso}}_{\omega l}$, making the overall sign ill-defined there.

Therefore we need to choose a different discrete subset of frequencies ω with $|\omega| < \tilde{m}_{+} + l$. The simplest choice we found for this are what we call δ-frequencies

$$\omega^{\delta}_{nl} := \tilde{m}_{\delta} + l - 2n \qquad\qquad n \in \{1, 2, \ldots, \lfloor \tilde{m}_{+} + l \rfloor\} \tag{C.453}$$

(such that always $\omega^{\delta}_{nl} > 0$), wherein

$$\tilde{m}_{\delta} := \lceil \tilde{m}_{+} \rceil - \tfrac{\delta}{2} \qquad \delta := |\tilde{m}_{+} - \text{int}(\tilde{m}_{+})| \qquad \text{int}(\tilde{m}_{+}) := \lfloor \tilde{m}_{+} + 0.5 \rfloor . \tag{C.454}$$

That is, δ is the distance of $\tilde{m}_{+}$ to its nearest integer $\text{int}(\tilde{m}_{+})$. For real numbers x, we use the usual notation of $\lfloor x \rfloor$ for the floor function (largest integer $\leq x$), and $\lceil x \rceil$ for ceiling (smallest integer $\geq x$). Then, the nearest integer of x is calculated through $\text{int}(x) := \lfloor x + 0.5 \rfloor$. In $\tilde{m}_{\delta} := \lceil \tilde{m}_{+} \rceil - \frac{1}{2}\,\delta$, instead of the factor $\frac{1}{2}$ we could choose any element of the open interval $(0, 1)$, our choice here is merely the simplest one. Like the magic ones, the δ-frequencies are a discrete set with $\Delta\omega = 2$.

It is straightforward to verify that with our choice the δ-frequencies ω^{δ}_{nl} always fall on frequency intervals for which we have chosen the β-version $j^{S,ba,\beta}_{\omega l}$, and that further we have both $q^{\beta}_{-} = +1$ and $q^{\beta}_{+} = +1$ for the δ-frequencies. (Hence the negative δ-frequencies $-\omega^{\delta}_{nl}$ always fall on intervals of the α-version, with $q^{\alpha}_{-} = +1$ and $q^{\alpha}_{+} = +1$ for the negative δ-frequencies.) Of course this choice of δ-frequencies is somewhat artificial, its only physical justification being that in the flat limit they give the signs that we need. We refer to magic and δ- frequencies together as flat frequencies. Thus, for flat frequencies we always have $q_-^{\alpha,\beta} q_+^{\alpha,\beta} = +1$.

Now we can repeat the procedure of Section C.7.1. To this end we use the following flat limit analoguous to (C.90), which says that summing the values of a function $f(\omega)$ over all flat frequencies turns in the flat limit into an integral over all positive frequencies:

$$\sum_n f(\omega_{nl}^{+,\delta}) \xrightarrow[\text{lim.}]{\text{flat}} \frac{R}{2} \int_0^\infty \mathrm{d}\tilde{\omega}\; f(R\tilde{\omega}). \tag{C.455}$$

Step one again consists in finding what we call the flat hypergeometric expansion, whose flat limit recovers the expansion (2.107) of a solution near a Minkowski hypercylinder:

$$\phi(t,r,\Omega) \;=\; \int \mathrm{d}E \sum_{l,m_l} \frac{p_E^{\mathbb{R}}}{4\pi} \left\{ \phi^a_{Elm_l} \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{\jmath}_{El}(r) + \phi^b_{Elm_l} \mathrm{e}^{-\mathrm{i}Et} Y_l^{m_l}(\Omega)\, \check{n}_{El}(r) \right\} . \tag{C.456}$$

Our usual expansion (2.186) of a solution near an AdS hypercylinder is the S-expansion:

$$\phi(t,r,\Omega) \;=\; \int \mathrm{d}\omega \sum_{\underline{l},m_l} \left\{ \phi^a_{\omega\underline{l}m_l}\; \mu^{(a)}_{\omega\underline{l}m_l}(t,\rho,\Omega) + \phi^b_{\omega\underline{l}m_l}\; \mu^{(b)}_{\omega\underline{l}m_l}(t,\rho,\Omega) \right\} . \tag{C.457}$$

Considering only flat frequencies, what remains is the discrete S-expansion

$$\begin{aligned} \phi(t,r,\Omega) \;=\; \sum_{n\underline{l}\,m_l} \Big\{ & \phi^a_{\omega_{nl}^{+,\delta}\underline{l}m_l}\; \mu^{(a)}_{\omega_{nl}^{+,\delta}\underline{l}m_l}(t,\rho,\Omega) + \phi^b_{\omega_{nl}^{+,\delta}\underline{l}m_l}\; \mu^{(b)}_{\omega_{nl}^{+,\delta}\underline{l}m_l}(t,\rho,\Omega) \\ & + \phi^a_{-\omega_{nl}^{+,\delta}\underline{l}m_l}\; \mu^{(a)}_{-\omega_{nl}^{+,\delta}\underline{l}m_l}(t,\rho,\Omega) + \phi^b_{-\omega_{nl}^{+,\delta}\underline{l}m_l}\; \mu^{(b)}_{-\omega_{nl}^{+,\delta}\underline{l}m_l}(t,\rho,\Omega) \Big\} . \end{aligned} \tag{C.458}$$

We can rescale the momentum representation as in (C.94)

$$\begin{aligned} \phi^a_{\omega\underline{l}\,m_l} \;&=\; \phi^{\mathrm{F},a}_{\omega\underline{l}m_l}\; \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\; \frac{(p^{\mathbb{R}}_{\omega})^l}{(2l{+}d{-}2)!!} \;=\; R^{-1}_{\mathrm{AdS}} \tilde{\phi}^{\mathrm{F},a}_{\tilde{\omega}\underline{l}m_l}\; \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\; \frac{(p^{\mathbb{R}}_{\omega})^l}{(2l{+}d{-}2)!!} \\ \phi^b_{\omega\underline{l}\,m_l} \;&=\; \phi^{\mathrm{F},b}_{\omega\underline{l}m_l}\; \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\; \frac{(2l{+}d{-}4)!!}{(p^{\mathbb{R}}_{\omega})^{l+1}} \;=\; R^{-1}_{\mathrm{AdS}} \tilde{\phi}^{\mathrm{F},b}_{\tilde{\omega}\underline{l}m_l}\; \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)}\; \frac{(2l{+}d{-}4)!!}{(p^{\mathbb{R}}_{\omega})^{l+1}} \;. \end{aligned} \tag{C.459}$$

obtaining what we shall call flat discrete S-expansion:

$$\begin{aligned} \phi(t,r,\Omega) = \sum_{n\underline{l}m_l} \Big\{ & \phi^{\mathrm{F},a}_{\omega_{nl}^{+,\delta}\underline{l}m_l} \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(4\pi)}\, \frac{(p^{\mathbb{R}}_{nl})^l}{(2l{+}d{-}2)!!}\, \mu^{(a)}_{\omega_{nl}^{+,\delta}\underline{l}m_l}(t,\rho,\Omega) + \phi^{\mathrm{F},b}_{\omega_{nl}^{+,\delta}\underline{l}m_l} \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(4\pi)}\, \frac{(2l{+}d{-}4)!!}{(p^{\mathbb{R}}_{nl})^{l+1}}\, \mu^{(b)}_{\omega_{nl}^{+,\delta}\underline{l}m_l}(t,\rho,\Omega) \\ & + \phi^{\mathrm{F},a}_{-\omega_{nl}^{+,\delta}\underline{l}m_l} \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(4\pi)}\, \frac{(p^{\mathbb{R}}_{nl})^l}{(2l{+}d{-}2)!!}\, \mu^{(a)}_{-\omega_{nl}^{+,\delta}\underline{l}m_l}(t,\rho,\Omega) + \phi^{\mathrm{F},b}_{-\omega_{nl}^{+,\delta}\underline{l}m_l} \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(4\pi)}\, \frac{(2l{+}d{-}4)!!}{(p^{\mathbb{R}}_{nl})^{l+1}}\, \mu^{(b)}_{-\omega_{nl}^{+,\delta}\underline{l}m_l}(t,\rho,\Omega) \Big\}. \end{aligned} \tag{C.460}$$

Therein, again p_{nl} is short for $p_{\omega_{nl}^{+,\delta}}$. Using (C.87) from Appendix C.2.7, together with (C.455) we obtain that in the flat limit for $d = 3$ the flat S-expansion (C.460) becomes the Minkowski hypercylinder expansion (C.456):

$$\phi(t,r,\Omega) \xrightarrow[\text{lim.}]{\text{flat}} \int \mathrm{d}\tilde{\omega} \sum_{l,m_l} \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{4\pi} \left\{ \tilde{\phi}^{\mathrm{F},a}_{\tilde{\omega}lm_l} \mathrm{e}^{-\mathrm{i}\tilde{\omega}\tau} Y_l^{m_l}(\Omega)\, \check{\jmath}_{\tilde{\omega}l}(r) + \tilde{\phi}^{\mathrm{F},b}_{\tilde{\omega}lm_l} \mathrm{e}^{-\mathrm{i}\tilde{\omega}\tau} Y_l^{m_l}(\Omega)\, \check{n}_{\tilde{\omega}l}(r) \right\} . \tag{C.461}$$

Next, let us verify that also for the hypercylinder the flat limit and complex structure "commute", that is, make the below diagram commutative:

$$\begin{array}{ccc} \phi^{\mathrm{AdS}} & \xrightarrow{J^{\mathrm{iso}}_{\rho}} & J_\rho \phi^{\mathrm{AdS}} \\ \Big\downarrow{\scriptstyle \text{flat lim.}} & & \Big\downarrow{\scriptstyle \text{flat lim.}} \\ \phi^{\mathrm{Mink}} & \xrightarrow{J^{\mathrm{iso}}_{r}} & J_r \phi^{\mathrm{Mink}} \end{array}$$

To this end, we need to show that the flat limit of $J^{\mathrm{iso}}_{\rho}\phi$ reproduces the action of J^{iso}_{r} on the flat limit of ϕ. We start from the discrete S-expansion:

$$\begin{aligned} \phi(t,r,\Omega) \;=\; \sum_{n\underline{l}\, m_l} \Big\{ & \phi^{S,a}_{\omega^{+,\delta}_{nl}\underline{l}m_l}\; \mu^{(S,a)}_{\omega^{+,\delta}_{nl}\underline{l}m_l}(t,\rho,\Omega) + \phi^{S,b}_{\omega^{+,\delta}_{nl}\underline{l}m_l}\; \mu^{(S,b)}_{\omega^{+,\delta}_{nl}\underline{l}m_l}(t,\rho,\Omega) \\ & + \phi^{S,a}_{-\omega^{+,\delta}_{nl}\underline{l}m_l}\; \mu^{(S,a)}_{-\omega^{+,\delta}_{nl}\underline{l}m_l}(t,\rho,\Omega) + \phi^{S,b}_{-\omega^{+,\delta}_{nl}\underline{l}m_l}\; \mu^{(S,b)}_{-\omega^{+,\delta}_{nl}\underline{l}m_l}(t,\rho,\Omega) \Big\}\,. \\ (J^{\mathrm{iso}}_{\rho}\phi)(t,r,\Omega) \;=\; \sum_{n\underline{l}\, m_l} \Big\{ & -(j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l})^{-1} \phi^{S,b}_{\omega^{+,\delta}_{nl}\underline{l}m_l}\; \mu^{(S,a)}_{\omega^{+,\delta}_{nl}\underline{l}m_l}(t,\rho,\Omega) + (j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l}) \phi^{S,a}_{\omega^{+,\delta}_{nl}\underline{l}m_l}\; \mu^{(S,b)}_{\omega^{+,\delta}_{nl}\underline{l}m_l}(t,\rho,\Omega) \\ & - (j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l})^{-1} \phi^{S,b}_{-\omega^{+,\delta}_{nl}\underline{l}m_l}\; \mu^{(S,a)}_{-\omega^{+,\delta}_{nl}\underline{l}m_l}(t,\rho,\Omega) + (j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l}) \phi^{S,a}_{-\omega^{+,\delta}_{nl}\underline{l}m_l}\; \mu^{(S,b)}_{-\omega^{+,\delta}_{nl}\underline{l}m_l}(t,\rho,\Omega) \Big\}\,, \end{aligned}$$

and switch to the flat discrete S-expansion with $d = 3$

$$\begin{aligned} (J^{\mathrm{iso}}_{\rho}\phi)(t,r,\Omega) \;=\; \sum_{nlm_l} \Big\{ & -(j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l})^{-1}\, \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(2\pi)}\, \frac{(2l-1)!!}{(p^{\mathbb{R}}_{\omega})^{l+1}}\, \phi^{F,b}_{\omega^{+,\delta}_{nl}lm_l}\; \mu^{(S,a)}_{\omega^{+,\delta}_{nl}lm_l}(t,\rho,\Omega) \\ & + (j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l})\, \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(2\pi)}\, \frac{(p^{\mathbb{R}}_{\omega})^{l}}{(2l+1)!!}\, \phi^{F,a}_{\omega^{+,\delta}_{nl}lm_l}\; \mu^{(S,b)}_{\omega^{+,\delta}_{nl}lm_l}(t,\rho,\Omega) \\ & - (j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l})^{-1}\, \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(2\pi)}\, \frac{(2l-1)!!}{(p^{\mathbb{R}}_{\omega})^{l+1}}\, \phi^{F,b}_{-\omega^{+,\delta}_{nl}lm_l}\; \mu^{(S,a)}_{-\omega^{+,\delta}_{nl}lm_l}(t,\rho,\Omega) \\ & + (j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l})\, \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(2\pi)}\, \frac{(p^{\mathbb{R}}_{\omega})^{l}}{(2l+1)!!}\, \phi^{F,a}_{-\omega^{+,\delta}_{nl}lm_l}\; \mu^{(S,b)}_{-\omega^{+,\delta}_{nl}lm_l}(t,\rho,\Omega) \Big\}\,. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} (J^{\mathrm{iso}}_{\rho}\phi)(t,r,\Omega) \;=\; \sum_{nlm_l} \frac{\tilde{p}^{\mathbb{R}}_{nl}}{(2\pi)} \Big\{ & -T^{-1} \phi^{F,b}_{\omega^{+,\delta}_{nl}lm_l}\; \frac{(p^{\mathbb{R}}_{nl})^{l}}{(2l+1)!!}\, \mu^{(S,a)}_{\omega^{+,\delta}_{nl}lm_l}(t,\rho,\Omega) \\ & + T\, \phi^{F,a}_{\omega^{+,\delta}_{nl}lm_l}\; \frac{(2l-1)!!}{(p^{\mathbb{R}}_{nl})^{l+1}}\, \mu^{(S,b)}_{\omega^{+,\delta}_{nl}lm_l}(t,\rho,\Omega) \\ & - T^{-1} \phi^{F,b}_{-\omega^{+,\delta}_{nl}lm_l}\; \frac{(p^{\mathbb{R}}_{nl})^{l}}{(2l+1)!!}\, \mu^{(S,a)}_{-\omega^{+,\delta}_{nl}lm_l}(t,\rho,\Omega) \\ & + T\, \phi^{F,a}_{-\omega^{+,\delta}_{nl}lm_l}\; \frac{(2l-1)!!}{(p^{\mathbb{R}}_{nl})^{l+1}}\, \mu^{(S,b)}_{-\omega^{+,\delta}_{nl}lm_l}(t,\rho,\Omega) \Big\}\,, \end{aligned}$$

wherein we have used the shorthand

$$T = \frac{j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l}\, (p^{\mathbb{R}}_{nl})^{2l+1}}{(2l+1)!!\,(2l-1)!!}\,.$$

Applying then the flat limits (C.455) and (C.87) we obtain

$$(J_\rho^{\mathrm{iso}}\phi)(t,r,\Omega) \xrightarrow[\text{lim.}]{\text{flat}} \int_0^\infty \mathrm{d}\tilde\omega \sum_{lm_l} \frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{(4\pi)} \Big\{ -T^{-1}\tilde\phi^{F,b}_{\tilde\omega l m_l}\, \mu^{(a)}_{\tilde\omega l m_l}(t,\rho,\Omega) + T\,\tilde\phi^{F,a}_{\tilde\omega l m_l}\, \mu^{(b)}_{\tilde\omega l m_l}(t,\rho,\Omega)$$
$$- T^{-1}\tilde\phi^{F,b}_{-\tilde\omega l m_l}\, \mu^{(a)}_{-\tilde\omega l m_l}(t,\rho,\Omega) + T\,\tilde\phi^{F,a}_{-\tilde\omega l m_l}\, \mu^{(b)}_{-\tilde\omega l m_l}(t,\rho,\Omega)\Big\}$$
$$= \int_{-\infty}^\infty \mathrm{d}\tilde\omega \sum_{lm_l} \frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{(4\pi)} \Big\{ -T^{-1}\tilde\phi^{F,b}_{\tilde\omega l m_l}\, \mu^{(a)}_{\tilde\omega l m_l}(t,\rho,\Omega) + T\,\tilde\phi^{F,a}_{\tilde\omega l m_l}\, \mu^{(b)}_{\tilde\omega l m_l}(t,\rho,\Omega)\Big\}$$

Therefore, it remains to compute the flat limit of the factor

$$T = \frac{(p^{\mathbb{R}}_{nl})^{2l+1}}{(2l-1)!!\,(2l+1)!!}\, j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l} \xrightarrow[\text{lim.}]{\text{flat}} \frac{(p^{\mathbb{R}}_{nl})^{2l+1}}{(2l-1)!!\,(2l+1)!!}\, \tfrac{1}{\pi}\,\Gamma(l+\tfrac{3}{2})\,\Gamma(l+\tfrac{1}{2}) \left(\tfrac{R}{2}\tilde p_{nl}\right)^{-2l} \left(\tfrac{R}{2}\tilde p^{\mathbb{R}}_{nl}\right)^{-1} q_-\, q_+ .$$

Therein, for $q_-\,q_+$ we have to use $q_-^\alpha\, q_+^\alpha$ for frequencies for which we choose $j^{S,ba,\alpha}_{\omega l}$, and $q_-^\beta\, q_+^\beta$ where we choose $j^{S,ba,\beta}_{\omega l}$. Since we have used only flat frequencies, we always have $q_-\,q_+ = 1$ and thus can omit this factor. (This is the reason for starting the whole process only with the flat frequencies.) Using again [DLMF 5.4.2] that for odd k we have $k!! = \Gamma(\frac{k}{2}+1)\,2^{\frac{k+1}{2}}/\sqrt{\pi}$, and thus $(2l-1)!!\,(2l+1)!! = \Gamma(l+\frac{3}{2})\Gamma(l+\frac{1}{2})\,2^{2l+1}/\pi$, the factor becomes

$$T = \frac{(p^{\mathbb{R}}_{nl})^{2l+1}}{(2l-1)!!\,(2l+1)!!}\, j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l} \xrightarrow[\text{lim.}]{\text{flat}} \frac{(\tilde p^{\mathbb{R}}_{nl})^{2l}}{(\tilde p_{nl})^{2l}} = \begin{cases} 1 & \tilde\omega^2 > m^2 \\ (-1)^l & \tilde\omega^2 < m^2 \end{cases} . \tag{C.462}$$

With this we find that the flat limit of $J_\rho^{\mathrm{iso}}\phi$ recovers the action of J_r^{iso} on the Minkowski solution which is the flat limit of ϕ:

$$(J_\rho^{\mathrm{iso}}\phi)(t,r,\Omega) \xrightarrow[\text{lim.}]{\text{flat}} \int_0^\infty \mathrm{d}\tilde\omega \sum_{lm_l} \frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{(4\pi)}\,(\pm 1)^l \Big\{ -\tilde\phi^{F,b}_{\tilde\omega l m_l}\, \mu^{(a)}_{\tilde\omega l m_l}(t,\rho,\Omega) + \tilde\phi^{F,a}_{\tilde\omega l m_l}\, \mu^{(b)}_{\tilde\omega l m_l}(t,\rho,\Omega)\Big\}.$$

As the second step, again we aim to reproduce the result (B.91) of the real g-product for two solutions near a hypercylinder in Minkowski spacetime:

$$\mathrm{g}_r^{\mathrm{iso}}(\eta,\zeta) = \int \mathrm{d}E \sum_{l,m_l} \frac{p^{\mathbb{R}}_E}{8\pi}\,(\pm 1)^l \left\{ \eta^a_{Elm_l}\zeta^a_{-E,l,-m_l} + \eta^b_{Elm_l}\zeta^b_{-E,l,-m_l}\right\} . \tag{C.463}$$

Our point of departure is the definition

$$\mathrm{g}_\rho^{\mathrm{iso}}(\eta,\zeta) = 2\omega_\rho(\eta, J_\rho^{\mathrm{iso}}\zeta). \tag{C.464}$$

We thus plug our complex structure J_ρ^{iso} (C.423) into the discrete symplectic structure (C.393) in which we omit the Dirac delta divergence, giving

$$\mathrm{g}_\rho^{\mathrm{iso}}(\eta,\zeta) = R^{d-1}_{\mathrm{AdS}} \sum_{n,\underline{l},m_l} \Big\{ \eta^{S,a}_{n\underline{l}m_l}\,(J_\rho^{\mathrm{iso}}\zeta)^{S,b}_{-n,\underline{l},-m_l} - \eta^{S,b}_{n,\underline{l},m_l}\,(J_\rho^{\mathrm{iso}}\zeta)^{S,a}_{-n,\underline{l},-m_l}$$
$$+ \eta^{S,a}_{-n,\underline{l},m_l}\,(J_\rho^{\mathrm{iso}}\zeta)^{S,b}_{n,\underline{l},-m_l} - \eta^{S,b}_{-n,\underline{l},m_l}\,(J_\rho^{\mathrm{iso}}\zeta)^{S,a}_{n,\underline{l},-m_l}\Big\}\,\pi(2l+d-2)$$
$$= R^{d-1}_{\mathrm{AdS}} \sum_{n,\underline{l},m_l} \Big\{ \eta^{S,a}_{n\underline{l}m_l}\,\zeta^{S,a}_{-n,\underline{l},-m_l} j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l} + \eta^{S,b}_{n,\underline{l},m_l}\,\zeta^{S,b}_{-n,\underline{l},-m_l}/j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l}$$
$$+ \eta^{S,a}_{-n,\underline{l},m_l}\,\zeta^{S,a}_{n,\underline{l},-m_l} j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l} + \eta^{S,b}_{-n,\underline{l},m_l}\,\zeta^{S,b}_{n,\underline{l},-m_l}/j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l}\Big\}\,\pi(2l+d-2).$$

Switching to the flat S-expansion gives for $d=3$

$$\begin{aligned}\mathrm{g}^{\mathrm{iso}}_\rho(\eta,\zeta) =& \sum_{n,l,m_l} R_{\mathrm{AdS}} \frac{\tilde{p}^{\mathbb{R}}_{nl}}{4\pi}\Big\{\eta^{F,a}_{nlm_l}\,\zeta^{F,a}_{-n,l,-m_l}\,T + \eta^{F,b}_{n,l,m_l}\,\zeta^{F,b}_{-n,l,-m_l}/T \\ &+ \eta^{F,a}_{-n,l,m_l}\,\zeta^{F,a}_{n,l,-m_l}\,T + \eta^{F,b}_{-n,l,m_l}\,\zeta^{F,b}_{n,l,-m_l}/T\Big\} \\ =& \sum_{n,l,m_l} R^{-1}_{\mathrm{AdS}} \frac{\tilde{p}^{\mathbb{R}}_{nl}}{4\pi}\Big\{\tilde{\eta}^{F,a}_{nlm_l}\,\tilde{\zeta}^{F,a}_{-n,l,-m_l}\,T + \tilde{\eta}^{F,b}_{n,l,m_l}\,\tilde{\zeta}^{F,b}_{-n,l,-m_l}/T \\ &+ \tilde{\eta}^{F,a}_{-n,l,m_l}\,\tilde{\zeta}^{F,a}_{n,l,-m_l}\,T + \tilde{\eta}^{F,b}_{-n,l,m_l}\,\tilde{\zeta}^{F,b}_{n,l,-m_l}/T\Big\}.\end{aligned}$$

Therein we again use the short

$$T = \frac{j^{S,ba,\mathrm{iso}}_{\omega^{+,\delta}_{nl},l}\,(p^{\mathbb{R}}_{nl})^{2l+1}}{(2l+1)!!\,(2l-1)!!}\,.$$

Applying now the flat limit (C.455) we get

$$\mathrm{g}^{\mathrm{iso}}_\rho(\eta,\zeta) = \int \mathrm{d}\tilde{\omega} \sum_{l,m_l} \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{8\pi}\Big\{\tilde{\eta}^{F,a}_{\tilde{\omega}lm_l}\,\tilde{\zeta}^{F,a}_{-\tilde{\omega},l,-m_l}\,T + \tilde{\eta}^{F,b}_{\tilde{\omega}lm_l}\,\tilde{\zeta}^{F,b}_{-\tilde{\omega},l,-m_l}/T\Big\}.$$

With the flat limit (C.462)

$$T \xrightarrow[\text{lim.}]{\text{flat}} \begin{cases} 1 & \tilde{\omega}^2 > m^2 \\ (-1)^l & \tilde{\omega}^2 < m^2 \end{cases}$$

the real g-product's flat limit reproduces the real g-product (C.463) of the Minkowski hypercylinder (again, $+1$ for $\tilde{\omega} > m$ and -1 for $\tilde{\omega} < m$):

$$\mathrm{g}^{\mathrm{iso}}_\rho(\eta,\zeta) \xrightarrow[\text{lim.}]{\text{flat}} \int \mathrm{d}\tilde{\omega} \sum_{lm_l} \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{8\pi}\,(\pm 1)^l \Big\{\tilde{\eta}^{\mathrm{F},a}_{\tilde{\omega}\underline{l}m_l}\,\tilde{\zeta}^{\mathrm{F},a}_{-\tilde{\omega},\underline{l},-m_l} + \tilde{\eta}^{\mathrm{F},b}_{\tilde{\omega}\underline{l}m_l}\,\tilde{\zeta}^{\mathrm{F},b}_{-\tilde{\omega},\underline{l},-m_l}\Big\}\,.$$

We need to compose our complex structure J^{iso}_ρ by interlacing two solutions of the conditions (C.370). A possible way to get rid of the need for interlacing might be of defining the radial functions S^a using a normalization constant, for instance the $N^{-1/2}$ as given in equation (2.5) of [43] by Limic, Niederle and Raczka. With our ν being their $-\mathrm{i}\Lambda$, in our notation their normalization writes

$$N^{-1/2} = \frac{1}{\sqrt{2\pi}}\left|\frac{\Gamma(\frac{1}{2}(l+\omega-\nu+\frac{d}{2}))\,\Gamma(\frac{1}{2}(l-\omega-\nu+\frac{d}{2}))}{\Gamma(l+\frac{d}{2})\,\Gamma(-\nu)}\right|.$$

Using absolute values did not work for us when solving conditions (C.370) which make the complex structure commute with the isometries. However, redefining the radial function S^a with $N^{-1/2}$ (and S^b with another suitable normalizer) induces different factors in the conditions (C.370). With some luck, there might exist a solution of these new conditions that is invariant under $\omega \to -\omega$, and that does not need to be interlaced with another solution in order to fulfill amplitude-equivalence (C.377), that is, making time-interval and rod amplitudes coincide.

Another possible strategy for finding a more elegant complex structure consists in giving up the simple anti-diagonal form (3.94), that is, allowing nonzero $j^{S,aa}_{\omega l}$. A particular way of realizing this is using a complex structure like that given by Ashtekar and Magnon on page 384 of [4]:

$$J_\Sigma = \frac{\mathcal{L}_n}{\sqrt{-\mathcal{L}_n^2}}. \tag{C.465}$$

Therein, Σ is a Cauchy hypersurface and n its normal vector field. Then the square of the Lie derivative $\mathcal{L}_n$ has negative eigenvalues when acting on the modes, making the root well-defined. Even though our hypercylinders are not Cauchy, one might try an Ashtekar-Magnon inspired complex structure like

$$J_\rho = \frac{\mathcal{L}_n}{|\mathcal{L}_n|} \tag{C.466}$$

wherein n is the normal vector field of the hypercylinder. The action of this complex structure then has to be calculated in the momentum representation, and commutation with isometries needs to be checked. Finally, amplitude-equivalence needs to be checked as well.

Another question is whether either of these two methods can give us a complex structure which makes our real g-product g_ρ positive-definite for modes of all frequencies, that is, for both propagating and evenescent modes.

C.7.4 Flat limits: rod regions $\mathrm{g}_\rho^{\mathbf{pos}}$

In the previous subsection we reproduce the real g-product g_r for a Minkowski hypercylinder as induced by the complex structure J_r^{iso}. For the AdS hypercylinder we therein use the interlaced complex structure J_ρ^{iso}. Since J_ρ^{iso} depends on the frequency ω, but J_r^{iso} does not, reproducing the Minkowski g_r only works for a discrete set of frequencies.

In this subsection we reproduce the real g-product g_r for a Minkowski hypercylinder as induced by the complex structure J_r^{pos}, which is independent of frequency ω and angular momentum l, and induces a positive-definite g_r. For the AdS hypercylinder we start with the two-branched complex structure J_ρ^{two}. We already found that this choice does not commute with AdS boosts. Hence we let go of this requirement (keeping thus only commutation with time translations and spatial rotations). Therefore we are no longer tied to using the α and β-versions for all frequencies, but only for the magic ones in order to keep our weak version of amplitude equivalence. We now make use of this new freedom, with the goal of modifying J_ρ^{two} such that it induces a positive-definite $\mathrm{g}_\rho^{\mathrm{pos}}$ for all frequencies.

As sketched in Section C.7.1, we start by constructing a complex structure J_ρ for AdS, whose action "commutes" with the process of taking the continuous flat limit:

$$\begin{array}{ccc} \phi^{\mathrm{AdS}} & \xrightarrow{\;J_\rho\;} & J_\rho\phi^{\mathrm{AdS}} \\ \Big\downarrow{\scriptstyle \text{cont. flat lim.}} & & \Big\downarrow{\scriptstyle \text{cont. flat lim.}} \\ \phi^{\mathrm{Mink}} & \xrightarrow{\;J_r^{\mathrm{pos}}\;} & J_r^{\mathrm{pos}}\phi^{\mathrm{Mink}} \end{array} \tag{C.467}$$

We recall the action (3.77) of the positive-definite complex structure of the Minkowski hypercylinder:

$$\big(J_r^{\mathrm{pos}}\phi\big)^a_{Elm_l} = -\,\phi^b_{Elm_l} \qquad \big(J_r^{\mathrm{pos}}\phi\big)^b_{Elm_l} = +\,\phi^a_{Elm_l} \tag{C.468}$$
$$\big(J_r^{\mathrm{pos}}\phi\big)(t,r,\Omega) = \int\!\mathrm{d}E\sum_{l,m_l}\frac{p_E^{\mathbb{R}}}{4\pi}\Big\{-\phi^b_{Elm_l}\mathrm{e}^{-\mathrm{i}Et}Y_l^{m_l}(\Omega)\,\check{\jmath}_{El}(r)+\phi^a_{Elm_l}\mathrm{e}^{-\mathrm{i}Et}Y_l^{m_l}(\Omega)\,\check{n}_{El}(r)\Big\}.$$

This is the action we aim to reproduce as the continuous flat limit of $J_\rho\phi$. An anti-diagonal complex structure acts in the S-expansion as

$$\begin{aligned}\big(J_\rho\phi\big)(t,r,\Omega) &= \int\!\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{(J_\rho\phi)^{S,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y_{\underline{l}}^{m_l}(\Omega)\,S^a_{\omega l}(\rho)+(J_\rho\phi)^{S,b}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y_{\underline{l}}^{m_l}(\Omega)\,S^b_{\omega l}(\rho)\Big\}\\ &= \int\!\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\frac{-1}{j^{S,ba}_{\omega l}}\phi^{S,b}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y_{\underline{l}}^{m_l}(\Omega)\,S^a_{\omega l}(\rho)+j^{S,ba}_{\omega l}\,\phi^{S,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y_{\underline{l}}^{m_l}(\Omega)\,S^b_{\omega l}(\rho)\Big\}.\end{aligned}$$

Switching to the flat S-representation (C.94), for $d=3$ this becomes

$$\big(J_\rho\phi\big)(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega \sum_{\underline{l},m_l} \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)} \Big\{ -T^{-1}\phi^{F,b}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)\,\frac{(p^{\mathbb{R}}_\omega)^l}{(2l+1)!!}S^a_{\omega l}(\rho)$$
$$+\,T\,\phi^{F,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\omega t}\,Y^{m_l}_{\underline{l}}(\Omega)\,\frac{(2l-1)!!}{(p^{\mathbb{R}}_\omega)^{l+1}}S^b_{\omega l}(\rho)\Big\}.$$

wherein again we use the shorthand

$$T = \frac{j^{S,ba}_{\omega l}\,(p^{\mathbb{R}}_\omega)^{2l+1}}{(2l+1)!!\,(2l-1)!!}\,.$$

Taking the flat limit we obtain

$$\big(J_\rho\phi\big)(t,r,\Omega) \xrightarrow[\text{lim.}]{\text{flat}} \int\!\mathrm{d}\tilde{\omega} \sum_{\underline{l},m_l} \frac{\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}}{(4\pi)} \Big\{ -T^{-1}\tilde{\phi}^{F,b}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\tilde{\omega}\tau}\,Y^{m_l}_{\underline{l}}(\Omega)\,\check{\jmath}_{El}(r) + T\tilde{\phi}^{F,a}_{\omega\underline{l}m_l}\,\mathrm{e}^{-\mathrm{i}\tilde{\omega}\tau}\,Y^{m_l}_{\underline{l}}(\Omega)\,\check{n}_{El}(r)\Big\}.$$

Thus in order to reproduce the action (C.468) of the Minkowski J^{pos}_r, we need to fix $j^{S,ba}_{\omega l}$ such that in the flat limit the factor T becomes unity. As found in Section C.6 above Figure C.430, the α-version is positive for $\omega \le -(\tilde{m}_+ + l - 2)$ and the β-version is positive for $\omega \ge (\tilde{m}_+ + l - 2)$. Hence from (C.451) +(C.452) we can read off that for these frequencies the flat limit of both versions is

$$j^{S,ba,\alpha/\beta}_{\omega l} \xrightarrow[\text{lim.}]{\text{flat}} \tfrac{1}{\pi}\,\Gamma(l+\tfrac{3}{2})\,\Gamma(l+\tfrac{1}{2})\,\Big(\tfrac{R}{2}\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}\Big)^{-2l-1} \tag{C.469}$$

$$=(2l-1)!!\,(2l+1)!!\,\Big(R\tilde{p}^{\mathbb{R}}_{\tilde{\omega}}\Big)^{-2l-1}. \tag{C.470}$$

Therein we use again [DLMF 5.4.2] that for odd k we have $k!! = \Gamma(\frac{k}{2}+1)\,2^{\frac{k+1}{2}}/\sqrt{\pi}$, and thus $\Gamma(l+\frac{3}{2})\Gamma(l+\frac{1}{2}) = (2l-1)!!\,(2l+1)!!2^{-2l-1}\pi$. Thus for "high" frequencies ω we can choose the α-version respectively β-version and obtain the correct flat limit: the flat limit of T becomes unity for these frequencies. For "low" frequencies we need to find an alternative expression for $j^{S,ba}_{\omega l}$ with the correct flat limit. We also need to fix a frequency $\omega^{\mathrm{split}}_l > (\tilde{m}_+ + l - 2)$ which distinguishes between high and low frequencies. Then we keep the α-version for $\omega \le -\omega^{\mathrm{split}}_l$ and the β-version for $\omega \ge +\omega^{\mathrm{split}}_l$. The shortest expression for $\omega^{\mathrm{split}}_l$ is setting it to $\omega^{\mathrm{split}}_l = \tilde{m}_+ + l$. For the remaining frequencies $|\omega| < \omega^{\mathrm{split}}_l$ the most obvious choice is

$$j^{S,ba,\mathrm{obv}}_{\omega l} = \tfrac{1}{\pi}\,\Gamma(\gamma^{\mathrm{S}})\,\Gamma(\gamma^{\mathrm{S}}-1)\,\Big(\tfrac{1}{2}p^{\mathbb{R}}_\omega\Big)^{-2l-1} \tag{C.471}$$

$$=(2l+d-2)!!\,(2l+d-4)!!\;2^{3-d}\Big(p^{\mathbb{R}}_\omega\Big)^{-2l-1}. \tag{C.472}$$

Then, for these frequencies with $d=3$ the factor T becomes unity trivially. Moreover, this choice has the advantage of being positive. It diverges at $|\omega| = \tilde{m}_+$ since there the radial momentum $p^{\mathbb{R}}_\omega := \sqrt{|\omega^2 - \tilde{m}_+^2|}$ becomes zero. This divergence is not a problem: actually it is even necessary to counter the corresponding zero in the factor T. We now define the following modification of the two-branched version and call it positive-definite version J^{pos}_ρ with $\omega^{\mathrm{split}}_l = \tilde{m}_+ + l$:

$$j^{S,ba,\mathrm{pos}}_{\omega l} = \begin{cases} j^{S,ba,\alpha}_{\omega l} & \omega \le -\omega^{\mathrm{split}}_l \\ j^{S,ba,\beta}_{\omega l} & \omega \ge +\omega^{\mathrm{split}}_l \\ j^{S,ba,\mathrm{obv}}_{\omega l} & \text{else} \end{cases} \tag{C.473}$$

As its name suggests, this version is positive for all frequencies ω. It is discontinuous at $\omega = \pm\omega^{\mathrm{split}}_l$. At $\omega = 0$ it is continuous (albeit not differentiable). At $\omega = \pm\tilde{m}_+$ it diverges as commented above.

(Thus for $l=0$ we have $\omega_l^{\rm split}=\tilde m_+$ and hence the divergence and discontinuity coincide.) This is shown (thick, continuous orange curve) in Figures 3.104 and 3.105, wherein we plot our different versions of $j^{S,ba}_{\omega l}$. For large $|\omega|$, we see that $j^{S,ba,\rm two}_{\omega l}$ and $j^{S,ba,\rm pos}_{\omega l}$ coincide and are positive, while for small $|\omega|$ only $j^{S,ba,\rm pos}_{\omega l}$ is positive and $j^{S,ba,\rm two}_{\omega l}$ alternates sign. Further, $j^{S,ba,\rm iso}_{\omega l}$ coincides with $j^{S,ba,\rm two}_{\omega l}$ only on half of the integer ω-intervals. This is so, because their respective associations of $j^{S,ba,\alpha}_{\omega l}$ and $j^{S,ba,\beta}_{\omega l}$ coincide for half of these intervals while they are opposite for the other half. We also see that all versions are frequency symmetric: $j^{S,ba}_{\omega l}=j^{S,ba}_{-\omega,l}$.

We remark that $j^{S,ba,\rm pos}_{\omega l}>0$ for all frequencies, thus inducing a positive-definite $g^{\rm pos}_\rho$. By construction our new J_ρ commutes with the continuous flat limit. We now study whether its induced real g-product $\mathrm{g}^{\rm pos}_\rho$ in the flat limit indeed reproduces the positive-definite Minkowski $g^{\rm pos}_r$. To this end, we start with the real g-product (C.424) for an anti-diagonal J_ρ with the solutions written in their S-expansions:

$$\begin{aligned}\mathrm{g}_\rho(\eta,\zeta) &= 2\omega_\rho(\eta,J_\rho\zeta)\\ &= 2\pi R_{\rm AdS}^{d-1}\int\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\eta^a_{\omega\underline{l}m_l}\,(J_\rho\zeta)^b_{-\omega,\underline{l},-m_l}-\eta^b_{\omega\underline{l}m_l}\,(J_\rho\zeta)^a_{-\omega,\underline{l},-m_l}\Big\}\,(2l+d-2)\\ &= 2\pi R_{\rm AdS}^{d-1}\int\mathrm{d}\omega\sum_{\underline{l},m_l}\Big\{\eta^a_{\omega\underline{l}m_l}\,\zeta^a_{-\omega,\underline{l},-m_l}\,j^{S,ba}_{\omega l}+\eta^b_{\omega\underline{l}m_l}\,\zeta^b_{-\omega,\underline{l},-m_l}/j^{S,ba}_{\omega l}\Big\}\,(2l+d-2).\end{aligned}\tag{C.474}$$

Substituting the flat S-expansion (C.94)

$$\begin{aligned}\phi^{S,a}_{\omega\underline{l}\,m_l} &= R^{-1}_{\rm AdS}\tilde\phi^{\mathrm{F},a}_{\tilde\omega\underline{l}m_l}\,\frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{(4\pi)}\,\frac{(p^{\mathbb{R}}_\omega)^l}{(2l+d-2)!!}\\ \phi^{S,b}_{\omega\underline{l}\,m_l} &= R^{-1}_{\rm AdS}\tilde\phi^{\mathrm{F},b}_{\tilde\omega\underline{l}m_l}\,\frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{(4\pi)}\,\frac{(2l+d-4)!!}{(p^{\mathbb{R}}_\omega)^{l+d-2}}\end{aligned}$$

yields for $d=3$

$$\mathrm{g}_\rho(\eta,\zeta)=\int\mathrm{d}\tilde\omega\sum_{\underline{l},m_l}\frac{\tilde p^{\mathbb{R}}_{\tilde\omega}}{8\pi}\Big\{\tilde\eta^{F,a}_{\omega\underline{l}m_l}\,\tilde\zeta^{F,a}_{-\omega,\underline{l},-m_l}\,T+\tilde\eta^{F,b}_{\omega\underline{l}m_l}\,\tilde\zeta^{F,b}_{-\omega,\underline{l},-m_l}T^{-1}\Big\},\tag{C.475}$$

wherein once more

$$T=\frac{j^{S,ba}_{\omega l}\,(p^{\mathbb{R}}_\omega)^{2l+1}}{(2l+1)!!\,(2l-1)!!}\,.$$

In order to reproduce the positive-definite Minkowski $g^{\rm pos}_r$ (B.91)

$$\mathrm{g}^{\rm pos}_r(\eta,\zeta)=\int\mathrm{d}E\sum_{l,m_l}\frac{p^{\mathbb{R}}_E}{8\pi}\Big\{\eta^a_{Elm_l}\zeta^a_{-E,l,-m_l}+\eta^b_{Elm_l}\zeta^b_{-E,l,-m_l}\Big\}\,,\tag{C.476}$$

we thus again need a $j^{S,ba}_{\omega l}$ such that in the flat limit the factor T becomes unity. Since our $j^{S,ba,\rm pos}_{\omega l}$ is constructed precisely to fulfill this condition, we have now verified that our complex structure $J^{\rm pos}_\rho$ in the flat limit indeed reproduces the positive-definite inner product $g^{\rm pos}_r$ of the Minkowski hypercylinder.

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