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# REAL-FORMAL ORBITAL RIGIDITY FOR GERMS OF REAL ANALYTIC VECTOR FIELDS ON $\mathbb{R}^2$.

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*A mi familia*

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# Contents

# Introducción

Sean $\nu, \omega : \left(\mathbb{R}^2, \widehat{0}\right) \longrightarrow \left(\mathbb{R}^2, \widehat{0}\right)$ dos gérmenes de campos analíticos reales. Los gérmenes de foliaciones singulares de dimension real 1 inducidas por $\nu$ y $\omega$ serán denotadas por $\mathscr{F}_\nu$ y $\mathscr{F}_\omega$, respectivamente.

*¿Cuándo $\nu$ y $\omega$ inducen foliaciones equivalentes?* Más específicamente, nos gustaría saber cuando existe un cambio de coordenadas analítico real que mande las hojas de la foliación $\mathscr{F}_\nu$ en las hojas de la foliación $\mathscr{F}_\omega$. Es decir, buscamos un germen de difeomorfismo analítico real $\boldsymbol{H} : \left(\mathbb{R}^2, \widehat{0}\right) \longrightarrow \left(\mathbb{R}^2, \widehat{0}\right)$ y un germen de función analítica real $\boldsymbol{K} : \left(\mathbb{R}^2, \widehat{0}\right) \longrightarrow \mathbb{R}$ distinto de cero en el origen, los cuales satisfacen

$$\omega = \boldsymbol{K}\, \boldsymbol{H}_*(\nu) := \boldsymbol{K}\left[(\mathrm{D}\, \boldsymbol{H} \cdot \nu) \circ \boldsymbol{H}^{-1}\right]. \tag{1}$$

En este caso, *$\nu$ y $\omega$ son real-analíticamente orbitalmente equivalentes.* Si el cambio de coordenadas analítico real preserva la parametrización de las hojas entonces $\boldsymbol{K} \equiv 1$. En este último caso diremos que *$\nu$ y $\omega$ son real-analíticamente equivalentes.* Una equivalencia (orbital) real-analítica es *estricta* si la parte lineal del difeomorfismo analítico real es la identidad (y la función escalar analítica real envía el origen a 1).

Al derivar ambos lados de la ecuación (1) obtenemos una *relación formal* entre las series de Taylor de $\nu$ and $\omega$ que es inducida por las series de Taylor de $\boldsymbol{H}$ y $\boldsymbol{K}$. En general, diremos que *$\nu$ y $\omega$ son real-formalmente orbitalmente equivalentes* si existen una transformación formal invertible $H \in \left(\mathbb{R}[[x, y]]\right)^2$ cuyas entradas coordenadas tienen término constante cero y una serie formal $K \in \mathbb{R}[[x, y]]$ con término constante distinto de cero, los cuales satisfacen

$$\widehat{\omega} = K\, H_*(\widehat{\nu}) := K\left[(\mathrm{D}\, H \cdot \widehat{\nu}) \circ H^{-1}\right],$$

donde $\widehat{\nu}$ y $\widehat{\omega}$ son series de Taylor de $\nu$ y $\omega$, respectivamente. Si $K \equiv 1$ decimos que los gérmenes $\nu$ y $\omega$ son *real-formalmente equivalentes.* De forma similar al caso analítico, una equivalencia (orbital) real-formal es *estricta* si la parte lineal de $H$ es la matriz identidad (y el término constante de $K$ es 1).

Como hemos visto, la existencia de una equivalencia (orbital) real-formal es una condición necesaria para la existencia de una equivalencia (orbital) real-analítica entre dos gérmenes. Ya que es más fácil probar que existe una equivalencia (orbital) real-formal, es importante saber cuando esta última es también una condición suficiente. Este fenómeno será llamado *rigidez (orbital) real-formal*.

Los bien conocidos resultados de Poincaré, Dulac, Siegel and Brjuno (véanse los teoremas 0.2.5 y 0.2.8) nos permiten concluir que la rigidez (orbital) real-formal ocurre en gérmenes de campos vectoriales analíticos reales cuya parte lineal es distinta de cero y satisface condiciones genéricas adicionales. Este fenómeno en general falla cuando los gérmenes tienen parte lineal distinta de cero pero ésta no cumple el resto de las hipótesis de dichos teoremas. Por ejemplo, el campo de Euler

$$x^2\frac{\partial}{\partial x}+(y-x)\frac{\partial}{\partial y}$$

tiene una única *separatriz compleja* (curva analítica irreducible que es invariante bajo el campo y pasa a través del origen), a saber la recta $\{x=0\}$, pero es real-formalmente (orbitalmente) equivalente a un campo vectorial analítico real (su forma normal (orbital) formal) con dos separatrices distintas (véase [IY08]). En la subsección 0.2.2 describiremos brevemente el caso general: *silla-nodos*.

El propósito de este trabajo es probar que para gérmenes analíticos reales sin parte lineal, el fenómeno de rigidez orbital real-formal ocurre bajo condiciones genéricas. Más específicamente, para $n \geqslant 2$ definimos $\mathcal{V}_n^{\mathbb{R}}$ como la clase de gérmenes de campos vectoriales analíticos reales en $\left(\mathbb{R}^2,\widehat{0}\right)$ con una singularidad aislada en el origen, con $(n-1)$-jet cero y $n$-jet distinto de cero. En esta tesis se prueba el siguiente resultado.

***Theorem* 0.0.1. *(Rigidez orbital real-formal)*.** *Bajo condiciones genéricas dos gérmenes analíticos reales $\nu,\omega \in \mathcal{V}_n^{\mathbb{R}}$ son (estrictamente) real-formalmente orbitalmente equivalentes si y sólo si son (estrictamente) real-analíticamente orbitalmente equivalentes.*

Dicho de otro modo, los gérmenes genéricos en $\mathcal{V}_n^{\mathbb{R}}$ que son real-formalmente orbitalmente equivalentes son real-analíticamente equivalentes. Es importante mencionar que el teorema 0.0.1 es la version analítica real del teorema de rigidez orbital formal de Voronin el cual aparece en esta introducción como el teorema 0.3.1 (más detalles acerca de estos teoremas son tratados en la sección 0.3).

Las condiciones genéricas del teorema 0.0.1 son específicadas en el capítulo 2 junto con un detallado esbozo de la prueba de este resultado.

En el resto de la introducción describiremos brevemente el contexto histórico de los problemas de rigidez (orbital) real-formal y rigidez (orbital) formal (esta última noción es definida en la sección 0.1 junto con otras que serán necesarias para discutir dichos fenómenos). Los teoremas de Poincaré y Brjuno serán discutidos en la sección 0.2 junto con algunos casos donde el fenómeno de rigidez formal no ocurre. En la sección 0.3 consideraremos gérmenes de campos vectoriales sin parte lineal; en particular describiremos algunos puntos clave usados en la prueba del teorema 0.0.1.

## 0.1 Rigidez real-formal y formal

Una forma natural de entender el fenómeno de rigidez (orbital) real-formal es mirando las *complejificaciones de* $\nu$ *y* $\omega$ pues de existir un difeomorfismo analítico real que conjugue las foliaciones generadas por $\nu$ y $\omega$ entonces dicha transformación debería preservar algunas propiedades que únicamente pueden ser observadas desde $\mathbb{C}^2$, como es el caso del *grupo de holonomía evanescente de las foliaciones inducidas por las complejificaciones de* $\nu$ *y* $\omega$ (esta noción es definida en el capítulo 1).

Por lo dicho previamente consideramos $v, w : (\mathbb{C}^2, \widehat{0}) \longrightarrow (\mathbb{C}^2, \widehat{0})$ dos gérmenes de campos vectoriales holomorfos. Ellos son *analíticamente orbitalmente equivalentes* si

$$w = \boldsymbol{\mathcal{K}}\, \boldsymbol{\mathcal{H}}_*(v) := \boldsymbol{\mathcal{K}}\big[(\mathrm{D}\, \boldsymbol{\mathcal{H}} \cdot v) \circ \boldsymbol{\mathcal{H}}^{-1}\big]\,, \tag{2}$$

donde $\boldsymbol{\mathcal{H}} : (\mathbb{C}^2, \widehat{0}) \longrightarrow (\mathbb{C}^2, \widehat{0})$ representa el germen de un biholomorfismo en el origen $\widehat{0}$ y $\boldsymbol{K} : (\mathbb{C}^2, \widehat{0}) \longrightarrow \mathbb{C}$ representa el germen de una función holomorfa que es distinta de cero en $\widehat{0}$. Siempre que $\boldsymbol{\mathcal{K}} \equiv 1$ diremos que $v$ y $w$ son *analíticamente equivalentes*. Si la ecuación (2) se satisface formalmente, diremos que *$v$ y $w$ son formalmente orbitalmente equivalentes*, o *formalmente equivalentes* cuando $\boldsymbol{\mathcal{K}} \equiv 1$. Si la equivalencia (orbital) formal implica la equivalencia (orbital) analítica diremos que ocurre el *fenómeno de rigidez (orbital) formal*.

Es necesario observar que la rigidez (orbital) real-formal no es una consecuencia inmediata de la rigidez (orbital) formal para gérmenes holomorfos: si $\nu, \omega : (\mathbb{R}^2, \widehat{0}) \longrightarrow (\mathbb{R}^2, \widehat{0})$ son real-formalmente (orbitalmente) equivalentes entonces sus complexificaciones son formalmente (orbitalmente) equivalentes. En el caso de concluir que las complejificaciones son analíticamente orbitalmente equivalentes, las transformaciones que realizan la equivalencia analítica podrían no ser las complejificaciones de mapeos analíticos reales, y en consecuencia, dichas transformaciones podrían no conjugar a las foliaciones originales $\mathscr{F}_v$ y $\mathscr{F}_w$ en el plano real.

A pesar de que la rigidez (orbital) real-formal no es un corolario de la rigidez (orbital) formal, esta última propiedad provee herramientas que serán útiles para el estudio del fenómeno de rigidez (orbital) real-formal.

## 0.2 Rigidez real-formal y formal en singularidades esenciales

En esta sección discutiremos los teoremas de Poincaré-Dulac y Brjuno (junto con sus versiones analíticas reales). Estos teoremas enuncian que el fenómeno de rigidez (real-)formal ocurre para gérmenes genéricos de campos vectoriales con parte lineal distinta de cero. A pesar de que estos resultados son válidos para $\mathbb{C}^m$ y $\mathbb{R}^m$, únicamente nos enfocaremos en $\mathbb{C}^2$ y $\mathbb{R}^2$. Al final de esta sección consideraremos *casos especiales* donde el fenómeno de rigidez no ocurre.

### 0.2.1 Teoremas de Poincaré-Dulac y Brjuno

En lo siguiente $v, w : \left(\mathbb{C}^2, \widehat{0}\right) \longrightarrow \left(\mathbb{C}^2, \widehat{0}\right)$ serán gérmenes de campos vectoriales holomorfos y $A$ será la parte lineal de $v$.

***Definition* 0.2.1** (Pares (no) resonantes y campos vectoriales resonantes)**.** La pareja ordenada $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ es llamada *resonante*, o más precisamente, *resonante aditiva* si existe una pareja $(m_1, m_2) \in \mathbb{N}^2$ tal que $m_1 + m_2 \geqslant 2$ y

$$\lambda_i = m_1\lambda_1 + m_2\lambda_2 \,, \quad \text{para algún} \quad i \in \{1, 2\} \,. \tag{3}$$

La igualdad previa será llamada *resonancia* y el campo vectorial $x_1^{m_1} x_2^{m_2} \frac{\partial}{\partial x_i}$ será *el campo vectorial resonante correspondiente a la resonancia* (3). Si no existe una pareja de números naturales que satisfaga la propiedad (3) diremos que $(\lambda_1, \lambda_2)$ *es no resonante.*

Cuando $v$ tiene parte lineal distinta de cero, diremos que $v$ *es resonante* si el espectro de su parte lineal es resonante. Si el espectro es no resonante $v$ será *no resonante.*

Si $v$ tiene parte lineal distinta de cero es posible demostrar que existe un cambio formal de coordenadas $\boldsymbol{\mathcal{H}}$ tal que $\boldsymbol{\mathcal{H}}_*(v) = A + \widetilde{v}$, donde $\widetilde{v}$ es un campo vectorial formal de orden mayor que 1 cuyo desarrollo de Taylor consta únicamente de campos vectoriales resonantes que corresponden a las resonancias del espectro de $A$: los campos vectoriales no resonantes pueden ser eliminados recursivamente por medio de una transformación polinomial adecuada. Más específicamente, el campo vectorial monomial no resonante $x_1^{n_1} x_2^{n_2} \frac{\partial}{\partial x_i}$ es eliminado por una transformación polinomial que tiene un coeficiente con factor $(\lambda_i - n_1\lambda_1 - n_2\lambda_2)^{-1}$, donde $\lambda_1, \lambda_2$ son los valores propios de $A$; el número $\lambda_i - n_1\lambda_1 - n_2\lambda_2$ será llamado *pequeño denominador.*

Este campo vectorial formal es llamado *la forma normal formal de $v$* y es el único respecto a las propiedades descritas en el párrafo anterior. En particular $\widetilde{v} \equiv 0$ siempre que el espectro de $A$ es no resonante, es decir, si $v$ es no resonante entonces es formalmente linealizable. Se pueden encontrar pruebas completas de las afirmaciones previas en [Arn83] y [IY08].

Las siguientes propiedades serán útiles para precisar cuando o no ocurre el fenómeno de rigidez (orbital) formal.

***Definition* 0.2.2** (Dominios de Poincaré y Siegel)**.** *El dominio de Poincaré* es la colección de parejas ordenadas $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ tales que $0 \in \mathbb{C}$ no pertenece al casco convexo de $\lambda_1, \lambda_2$ en $\mathbb{C}$. El *dominio de Siegel* es el complemento del dominio de Poincaré en $\mathbb{C}^2$.

**Remark 0.2.3.** De la definición se tiene que $(\lambda_1, \lambda_2)$ pertenece al dominio de Poincaré si y sólo si $\lambda_1, \lambda_2 \neq 0$ y su cociente es un número real no negativo. Por lo tanto pertenecer al dominio de Poincaré es una propiedad genérica.

Se sigue directamente que $(\lambda_1, \lambda_2)$ tiene un número finito de resonancias cuando dicha pareja pertenece al dominio de Poincaré. Como consecuencia, $v$ tiene una forma normal formal que es polinomial siempre que el espectro de su parte lineal $A$ pertenece al dominio de Poincaré.

Si $(\lambda_1, \lambda_2)$ pertenece al dominio de Siegel y $\lambda_1, \lambda_2 \neq 0$ tiene cociente irracional negativo, entonces la pareja es no resonante. De otra forma, $(\lambda_1, \lambda_2)$ tiene un número infinito de resonancias.

Los teoremas 0.2.4 y 0.2.7 señalan que el fenómeno de rigidez (orbital) formal ocurre bajo condiciones genéricas sobre la parte lineal de gérmenes de campos vectoriales holomorfos no degenerados. Las versiones analíticas reales de dichos resultados (teoremas 0.2.5 y 0.2.8) no son corolarios inmediatos, como se verá en el esbozo de sus pruebas que aparecen al final de esta subsección.

Recordemos que $v, w : \left(\mathbb{C}^2, \widehat{0}\right) \longrightarrow \left(\mathbb{C}^2, \widehat{0}\right)$ son gérmenes de campos vectoriales holomorfos y $A$ es la parte lineal de $v$.

***Theorem* 0.2.4** (Poincaré-Dulac)**.** *Si el espectro de $A$ pertenece al dominio de Poincaré entonces $v$ y $w$ son formalmente (orbitalmente) equivalentes si y sólo si son analíticamente (orbitalmente) equivalentes.*

***Theorem* 0.2.5** (Poincaré-Dulac, versión analítica real)**.** *Si $v$ y $w$ son las complejificaciones de gérmenes de campos analíticos reales los cuales son formalmente (orbitalmente) equivalentes, entonces bajo las condiciones del teorema de Poincaré-Dulac ellos (las foliaciones que ellos inducen) son conjugados(as) por el germen de un biholomorfismo que resulta ser el germen de la complejificación de un difeomorfismo analítico real.*

La siguiente propiedad nos permitirá formular el teorema de Brjuno.

***Definition* 0.2.6.** [Condición de Brjuno] Una pareja no resonante $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ satisface *la condición de Brjuno* si existen $C, \epsilon > 0$ tales que

$$\left|\lambda_i - (m_1\lambda_1 + m_2\lambda_2)\right|^{-1} \leqslant C \exp(|m|^{1-\epsilon}) \,,$$

para $m := (m_1, m_2) \in \mathbb{N}^2$, con $|m| := m_1 + m_2$ suficientemente grande.

***Theorem* 0.2.7** (Brjuno). *Supongamos que el espectro de $A$ pertenece al dominio de Siegel, es no resonante y satisface la condición de Brjuno. Entonces $v$ es analíticamente linealizable. Como consecuencia, si $v$ y $w$ son formalmente (orbitalmente) equivalentes entonces son analíticamente (orbitalmente) equivalentes.*

Un resultado previo al teorema de Brjuno es el teorema de Siegel. Este último tiene las mismas conclusiones que el teorema 0.2.7 bajo las hipótesis de que el espectro de $A$ es *diofantino*. A pesar de que esta condición es más restrictiva que la condición de Brjuno, las parejas en $\mathbb{C}^2$ que no son diofantinas tienen medida de Lebesgue cero (véase [Arn83]).

***Theorem* 0.2.8** (Brjuno, versión analítica real). *Si $v$ y $w$ son las complejificaciones de gérmenes de campos analíticos reales los cuales son formalmente (orbitalmente) equivalentes, entonces bajo las condiciones del teorema de Brjuno ellos (las foliaciones que ellos inducen) son conjugados(as) por el germen de un biholomorfismo que resulta ser el germen de la complejificación de un difeomorfismo analítico real.*

Las pruebas completas de los resultados mencionados arriba pueden ser encontradas en [Po1878], [Du1904], [Brj71], [Arn83] y [IY08]. En el resto de la subsección esbozaremos brevemente las ideas principales de dichas pruebas.

Bajo las condiciones de los teoremas 0.2.4 y 0.2.7 se puede concluir que la forma normal formal de $v$ es un campo vectorial polinomial. Más aún, la transformación formal $\boldsymbol{\mathcal{H}}$ que manda a $v$ en su forma normal formal es convergente, dado que en estos casos los pequeños denominadores decrecen menos rápido que $\frac{1}{C}\exp(|m|^{\epsilon-1})$ y en consecuencia, el crecimiento de los coeficientes de $\boldsymbol{\mathcal{H}}$ es controlado.

Si $v$ y $w$ son formalmente (orbitalmente) equivalentes, entonces podemos suponer que ellos tiene el mismo $k$-jet, siendo $k$ tan grande como deseemos. Lo anterior se obtiene después de un cambio de coordenadas polinomial (multiplicado por un polinomio con término constante distinto de cero en caso de que la equivalencia sea orbital), el cual es inducido por un jet finito adecuado de las transformaciones formales que realizan la equivalencia. Como consecuencia, ellos tienen la misma forma normal formal (por unicidad de la forma normal formal). Por lo tanto $w$ será analíticamente equivalente a esta forma normal formal por la misma razón que lo es $v$, y de esta manera uno puede concluir los teoremas 0.2.4 y 0.2.7.

Las versiones analíticas reales de los teoremas de Poincaré-Dulac y Brjuno (teoremas 0.2.5 y 0.2.8) son probados con los argumentos del caso general, esto después de demostrar que la forma normal formal de la complejificación de un campo vectorial analítico real tiene coeficientes reales al igual que el cambio de coordenadas (orbital) formal que los conjuga.

### 0.2.2 Casos en los que la rigidez formal no ocurre

*¿Qué ocurre cuando las condiciones de los teoremas Poincaré-Dulac y Brjuno no se satisfacen?* En esta subsección mencionaremos brevemente algunos casos en los que estas condiciones no se satisfacen y en los que el fenómeno de rigidez formal no ocurre: *sillas de Cremer, sillas complejas resonantes y silla-nodos complejas.*

Un germen de campo vectorial holomorfo en $\widehat{0} \in \mathbb{C}^2$ será llamado *silla compleja* si su parte lineal tiene dos valores propios distintos de cero los cuales pertenecen al dominio de Siegel. Por el teorema de Hadamard-Perron este germen tiene dos separatrices. Después de un cambio de coordenadas analítico es posible asumir que dichas separatrices son los ejes, y tras multiplicar el campo vectorial resultante por una constante distinta de cero podemos suponer que su parte lineal es $\frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}$, donde $\lambda$ es el respectivo cociente de los valores propios. En lo siguiente nos referiremos al mapeo de holonomía sobre un pequeño lazo sobre el eje $x$ que gira en sentido positivo como *el mapeo de holonomía de la silla compleja.* Note que el coeficiente lineal de este mapeo de holonomía es $\exp(2\pi i\lambda) \in \mathbb{S}^1$.

Existen sillas complejas, llamadas *sillas de Cremer*, las cuales son formalmente linealizables pero no analíticamente linealizables. La clasificación analítica de dichas sillas complejas es desconocida aún. En la siguiente parte indicaremos brevemente los argumentos que garantizan la existencia de este tipo de sillas complejas.

El único invariante analítico de las sillas complejas con la misma parte lineal es su mapeo de holomía ([EI84], [IY08]). Por otro lado tenemos el siguiente teorema de realización cuya prueba puede ser consultada en [PMY94] o en [EISV93].

***Theorem* 0.2.9.** *Para cualquier germen conforme $f(z) = \exp(2\pi i\phi)z + O(z^2)$, $\phi \in \mathbb{R}$, y cualquier $\lambda < 0$ tal que $\lambda \equiv \phi \bmod \mathbb{Z}$, existe un campo vectorial con parte lineal $\frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}$ cuyo mapeo de holonomía coincide con $f$.*

De los resultados anteriores y del siguiente teorema de Yoccoz ([Yoc88], [Yoc95]), se sigue la existencia de las sillas complejas que son formalmente linealizables pero no son analíticamente linealizables.

***Theorem* 0.2.10** (Yoccoz)**.** *Si el número complejo $\mu = \exp(2\pi il)$, con $l \in \mathbb{R}$, no satisface la condición de Brjuno*[1]*, entonces existe un germen holomorfo en $0 \in \mathbb{C}$, $z \mapsto \mu z + O(|z|^2)$, que no es analíticamente linealizable.*

Ahora mencionaremos el caso de *las sillas complejas resonantes* (los valores propios tienen cociente racional negativo). La clasificación (formal) orbital de estas sillas depende de parámetros escalares.

[1] El número complejo $\mu = \exp(2\pi il)$, con $l \in \mathbb{R}$, satisface la *condición de Brjuno* si no es raíz de la unidad y existen $\epsilon, C > 0$ tales que para todo $k \in \mathbb{N}$, $| \mu^k - 1 |^{-1} < C \exp(k^{1-\epsilon})$.

Una clase de equivalencia formal de sillas complejas resonantes tiene moduli funcional de clasificación analítica que es conocido como *moduli de Ecalle-Voronin* (estos invariantes de clasificación fueron descubiertos independientemente por Ecalle y Voronin, [Eca75], [Vor81], [MR83], [Ily93], [VG96]). La existencia del moduli funcional implica que el fenómeno de rigidez formal no ocurre.

Este moduli funcional es obtenido a partir de la clasificación analítica de los mapeos de holonomía de las sillas complejas resonantes (que a su vez se obtiene a través de la clasificación analítica de *gérmenes parabólicos*[2]). Una exposición completas de estos resultados se puede encontrar en [IY08].

Finalmente discutiremos brevemente el caso de los gérmenes de campos vectoriales holomorfos en $\widehat{0} \in \mathbb{C}^2$ cuya parte lineal tiene un valor propio cero y otro distinto de cero. Éste será llamado *silla-nodo*.

Una silla nodo es formalmente orbitalmente equivalente a

$$\frac{z^{r+1}}{1+ax^r}\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

para alguna $a \in \mathbb{C}$ (los ejes coordenados son separatrices de este campo vectorial). En general una silla-nodo no tiene separatriz holomorfa relacionada al valor propio cero. Más aún, J. Martinet and J.-P. Ramis obtuvieron un moduli de clasificación analítica orbital para toda clase de equivalencia orbital formal de silla-nodos. Este es un moduli funcional (equivalente al moduli de Ecalle-Voronin) y es conocido como *moduli de Martinet-Ramis* ([MR83], [Eca81], [Ily93], [VM02]). Así la clasificación orbital formal tiene moduli de dimensión 1 ($a \in \mathbb{C}$), y la clasificación orbital analítica tiene moduli funcional. Por lo tanto el fenómeno de rigidez orbital formal no ocurre en este caso.

Excepcionalmente, el fenómeno de rigidez orbital formal ocurre para los gérmenes de campos vectoriales que tienen parte lineal nilpotente, es decir, la clasificación orbital formal y analítica coinciden ([Tak73], [CM88], [Mat91], [EISV93], [Lor99], [SZ02]).

## 0.3 Rigidez orbital en singularidades sin parte lineal

Dado un número natural $n \geqslant 2$ denotaremos como $\mathcal{V}_n$ a la clase de gérmenes de campos vectoriales holomorfos en $\widehat{0} \in \mathbb{C}^2$ con una singularidad aislada en el origen, $(n-1)$-jet cero y $n$-jet distinto de cero. De manera análoga, $\mathcal{V}_n^{\mathbb{R}}$ será la clase de gérmenes de campos vectoriales analíticos reales en $\widehat{0} \in \mathbb{R}^2$ con una singularidad aislada en el origen, $(n-1)$-jet cero y $n$-jet distinto de cero.

En [Vo97] Voronin probó que el fenómeno de rigidez orbital formal ocurre bajo condiciones genéricas en gérmenes de campos vectoriales no dicríticos que

[2] Un germen conforme $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ es *parabólico* si $f'(0) = 1$.

pertenecen a la clase $\mathcal{V}_n$ (Teorema 0.3.1). Más tarde Ortiz, Rosales y Voronin probaron que el fenómeno de rigidez formal también ocurre bajo condiciones genéricas en el caso de campos vectoriales en $\mathcal{V}_n$ ([ORV01]). Después de esto, ellos probaron que la rigidez formal y la rigidez orbital formal ocurre bajo condiciones genéricas en gérmenes de campos vectoriales dicríticos que pertenecen a la clase $\mathcal{V}_n$, obteniendo adicionalmente los invariantes mínimos para la clasificación orbital analítica *estricta*[3] de dichos gérmenes ([ORV05]). Recientemente ellos obtuvieron, bajo condiciones genéricas, los invariantes mínimos para la clasificación orbital analítica estricta de gérmenes de campos vectoriales no dicríticos ([ORV12]). Es importante mencionar que con la finalidad de obtener los invariantes mínimos, ellos construyeron formas normales orbitales formales tanto para gérmenes dicríticos como para no dicríticos, los cuales son de hecho formas normales orbitales analíticas para casos genéricos, es decir, dichas formas normales orbitales formales de hecho convergen en casos genéricos ([ORV08], [ORV14]).

El siguiente resultado es el teorema de rigidez orbital formal de Voronin:

***Theorem* 0.3.1** (Voronin)**.** *Bajo condiciones genéricas*[4]*, $v, w \in \mathcal{V}_n$ son formalmente orbitalmente equivalentes si y sólo si ellos son analíticamente orbitalmente equivalentes.*

Como hemos dicho previamente, el objetivo de esta tesis es probar el teorema 0.0.1 que es la versión analítica real del teorema 0.3.1. La prueba está totalmente inspirada en las construcciones que hizo Voronin para mostrar el teorema 0.3.1.

Para finalizar la sección discutiremos brevemente los puntos clave de la prueba del teorema 0.0.1. Para este propósito consideraremos $\nu$ y $\omega$ como en el enunciado de dicho teorema.

Las desingularizaciones de las complejificaciones de los gérmenes $\nu$ y $\omega$ inducen foliaciones en un vecindad del *divisor excepcional complejo* $\mathbb{D}_{\mathbb{C}}$ en la *banda de Möbius compleja* (véanse la definiciones de estas nociones en el capítulo 1).

Obtendremos un biholomorfismo local que conjugue a dichas desingularizaciones dejando invariante *la banda de Möbius real* $\mathbb{M}_{\mathbb{R}}$, es decir, cuyo *blow-down* preserve el plano real $\mathbb{R}^2$. Este biholomorfismo es construido gracias a una *conjugación transversal analítica* que es la complejificación de un difeomorfismo local analítico real. Esta conjugación transversal es lograda por la equivalencia orbital real-formal entre $\nu$ y $\omega$ y la condición genérica de no solubilidad de los grupos de holonomía (teoremas 2.2.1 y 2.2.2). Con el próposito de extender esta conjugación transversal analítica a una vecindad del divisor excepcional complejo $\mathbb{D}_{\mathbb{C}}$, se construye una *foliación auxiliar* adecuada. Esta foliación es la desingularización de la complejificación de un campo vectorial analítico real que es dicrítico (Lemma 5.1.1). Entonces la extensión se puede realizar mediante

[3] Aquí *estricta* significa que en la equivalencia orbital expresada en (2), la parte lineal de $\boldsymbol{\mathcal{H}}$ es la matriz identidad y el término constante de $\boldsymbol{\mathcal{K}}$ es 1.

[4] Voronin consideró gérmenes no dicríticos en $\mathcal{V}_n$ que satisfacen que sus blow-ups tienen $n+1$ puntos singulares diferentes dos a dos en $\mathbb{D}_{\mathbb{C}}$ (con número característico fuera de $\mathbb{Q}^+ \cup \{0\}$), y grupos de holonomía no solubles.

continuaciones analíticas, excepto en los puntos singulares de las desingularizaciones de las complejificaciones de $\nu$ y $\omega$, y en un número finito de puntos donde la foliación auxiliar no es transversal al divisor excepcional complejo $\mathbb{D}_{\mathbb{C}}$ (capítulo 6). Después de un meticuloso trabajo, el mapeo que se obtuvo mediante las continuaciones analíticas puede ser extendido alrededor de dichos *puntos especiales*, dejando invariante la banda de Möbius real $\mathbb{M}_{\mathbb{R}}$ (capítulos 7 y 8).

Como hemos mencionado antes, en el capítulo 2 se dará un esbozo detallado de la prueba del teorema 0.0.1.

# Introduction

Let $\nu, \omega : (\mathbb{R}^2, \widehat{0}) \longrightarrow (\mathbb{R}^2, \widehat{0})$ be two germs of real analytic vector fields, and $\mathscr{F}_\nu$, $\mathscr{F}_\omega$ the germs of singular foliations of real dimension 1 induced by $\nu$ and $\omega$, respectively.

When do $\nu$ and $\omega$ *lead to the same or equivalent foliations*? More specifically, we would like to know when does there exist a real analytic coordinate change mapping the leaves of $\mathscr{F}_\nu$ into the leaves of $\mathscr{F}_\omega$. That is, we look for the germ of a real analytic diffeomorphism $\boldsymbol{H} : (\mathbb{R}^2, \widehat{0}) \longrightarrow (\mathbb{R}^2, \widehat{0})$, and the germ of a real analytic function $\boldsymbol{K} : (\mathbb{R}^2, \widehat{0}) \longrightarrow \mathbb{R}$ which is nonzero at the origin, satisfying

$$\omega = \boldsymbol{K}\, \boldsymbol{H}_*(\nu) := \boldsymbol{K}\left[(\mathrm{D}\, \boldsymbol{H} \cdot \nu) \circ \boldsymbol{H}^{-1}\right]. \tag{4}$$

In this case, $\nu$ *and* $\omega$ *are real-analytically orbitally equivalent.* If the real analytic change of coordinates preserves the parametrization of the leaves then $\boldsymbol{K} \equiv 1$. In such a case we shall say that $\nu$ *and* $\omega$ *are real-analytically equivalent.* A real-analytic (orbital) equivalence is *strict* if the linear part of the real analytic diffeomorphism is the identity matrix (and the real analytic scalar function sends the origin to 1).

Differentiating both sides of the equality (4) we obtain a *formal relation* between the Taylor series of $\nu$ and $\omega$ induced by the Taylor series of $\boldsymbol{H}$ and $\boldsymbol{K}$. In general, we say that $\nu$ *and* $\omega$ *are real-formally orbitally equivalent* when there exist an invertible formal transformation $H \in \left(\mathbb{R}[[x,y]]\right)^2$ with zero constant terms in each coordinate and a formal series $K \in \mathbb{R}[[x,y]]$ with nonzero constant term satisfying

$$\widehat{\omega} = K\, H_*(\widehat{\nu}) := K\left[(\mathrm{D}\, H \cdot \widehat{\nu}) \circ H^{-1}\right],$$

where $\widehat{\nu}$ and $\widehat{\omega}$ are the Taylor series of $\nu$ and $\omega$, respectively. If $K \equiv 1$ we say that the germs $\nu$ and $\omega$ are *real-formally equivalent.* Similar to the previous definition, a real-formal (orbital) equivalence is *strict* if the linear part of $H$ is the identity matrix (and the constant term of $K$ is 1).

As we have seen, the existence of a real-formal (orbital) equivalence is a necessary condition for the existence of a real-analytic (orbital) equivalence between two germs. Since it is easier to prove that there exists a real-formal (orbital) equivalence, it is important to know whether it is also a sufficient condition. This phenomenon will be called *real-formal (orbital) rigidity*.

The well-known results due to Poincaré, Dulac, Siegel and Brjuno (see Theorems 0.5.5 and 0.5.8) allow us to conclude that the real-formal (orbital) rigidity takes place for germs of real analytic vector fields with generic nonzero linear part. But this phenomenon in general fails for germs with nonzero linear part which do not satisfy the genericity assumptions of these rigidity theorems. For example, the Euler vector field

$$x^2\frac{\partial}{\partial x}+(y-x)\frac{\partial}{\partial y}$$

has a unique *complex separatrix* (invariant irreducible analytic curve passing through the origin), namely the line $\{x=0\}$, but it is real-formally (orbitally) equivalent to a real analytic vector field (its *formal (orbital) normal form*) with two different separatrices (see [IY08]). In Subsection 0.5.2 we shall return to saddle-nodes.

The purpose of this work is to prove that for real analytic germs with zero linear part, the real-formal orbital rigidity phenomenon takes place under generic conditions. More specifically, for $n\geqslant 2$ we consider $\mathcal{V}_n^{\mathbb{R}}$ the class of germs of real analytic vector fields on $\left(\mathbb{R}^2,\widehat{0}\right)$ with an isolated singularity at the origin, having zero $(n-1)$-jet and nonzero $n$-jet. In this dissertation, we shall prove the following result

***Theorem* 0.3.2. *(Real-Formal Orbital Rigidity)*.** *Two generic real analytic germs $\nu,\omega\in\mathcal{V}_n^{\mathbb{R}}$ are (strictly) real-formally orbitally equivalent if and only if they are (strictly) real-analytically orbitally equivalent.*

Unless otherwise stated, the foliations induced by real-formally orbitally equivalent generic germs in $\mathcal{V}_n^{\mathbb{R}}$ are real-analytically orbitally equivalent. It is important to mention that Theorem 0.3.2 is the real analytic version of Voronin's formal orbital rigidity theorem which appears as Theorem 0.6.1 below (more details about these theorems are treated in Section 0.6).

The genericity conditions of Theorem 0.3.2 are specified in Chapter 2 together with a detailed outline of the proof of this result.

In the rest of this introduction we shall briefly describe the historical context of the problem of the formal and real-formal (orbital) rigidity phenomena (the notion of formal (orbital) rigidity is defined in Section 0.4 together with other necessary notions to discuss these phenomena). Poincaré-Dulac's and Brjuno's theorems are discussed in Section 0.5 together with some cases where the formal rigidity phenomenon fails. Further, in Section 0.6, we shall consider germs of vector fields with zero linear part; in particular we shall describe some key points used in the proof of Theorem 0.3.2.

## 0.4 Formal and Real-Formal Rigidity

A natural way to understand the real-formal (orbital) rigidity phenomenon is to look to the *complexifications of* $\nu$ *and* $\omega$: if there is a real analytic diffeomorphism conjugating the foliations generated by $\nu$ and $\omega$ then this transformation should preserve some properties which can only be observed from $\mathbb{C}^2$; this is the case of *the vanishing holonomy group of foliations induced by the complexifications of* $\nu$ *and* $\omega$ (this notion is defined in Chapter 1).

In this way, we consider $v, w : (\mathbb{C}^2, \widehat{0}) \longrightarrow (\mathbb{C}^2, \widehat{0})$ germs of holomorphic vector fields. They are *analytically orbitally equivalent* if

$$w = \boldsymbol{\mathcal{K}}\,\boldsymbol{\mathcal{H}}_*(v) := \boldsymbol{\mathcal{K}}\big[(\mathrm{D}\,\boldsymbol{\mathcal{H}} \cdot v) \circ \boldsymbol{\mathcal{H}}^{-1}\big]\,, \tag{5}$$

where $\boldsymbol{\mathcal{H}} : (\mathbb{C}^2, \widehat{0}) \longrightarrow (\mathbb{C}^2, \widehat{0})$ represents the germ of a biholomorphism at $\widehat{0}$ and $\boldsymbol{K} : (\mathbb{C}^2, \widehat{0}) \longrightarrow \mathbb{C}$ represents the germ of a holomorphic function which is nonzero at $\widehat{0}$. Whenever $\boldsymbol{\mathcal{K}} \equiv 1$, we shall say that $v$ and $w$ are *analytically equivalent*. If the equation (5) is formally satisfied, $v$ and $w$ will be called *formally orbitally equivalent*, or *formally equivalent* whenever $\boldsymbol{\mathcal{K}} \equiv 1$. If the formal (orbital) equivalence implies the analytic (orbital) equivalence then we shall say that there is a *formal (orbital) rigidity phenomenon.*

It is necessary to observe that real-formal (orbital) rigidity for real analytic germs is not an immediate consequence of formal (orbital) rigidity for holomorphic germs: if $\nu, \omega : (\mathbb{R}^2, \widehat{0}) \longrightarrow (\mathbb{R}^2, \widehat{0})$ are real-formally (orbitally) equivalent then their complexifications are formally (orbitally) equivalent. But if we conclude that the complexifications are analytically (orbitally) equivalent, *the transformations realizing the analytic equivalence are not necessarily the complexifications of real analytic maps*, and as a consequence, they could not conjugate the original foliations $\mathscr{F}_v$ and $\mathscr{F}_w$ in the real plane.

Even though the real-formal (orbital) rigidity is not a corollary of the formal (orbital) rigidity, the latter provides notions and tools which are useful for the study of real-formal (orbital) rigidity.

## 0.5 Formal and Real-Formal Rigidity for Essential Singularities

In this section we shall discuss Poincaré-Dulac's and Brjuno's theorems (together with their real analytic versions). These theorems state that the (real-)formal rigidity phenomenon takes place for generic germs of vector fields with nonzero linear part (even though these results are valid for $\mathbb{C}^n$ and $\mathbb{R}^n$, we are only focused on $\mathbb{C}^2$ and $\mathbb{R}^2$). At the end of this section we shall consider *special cases* where the formal rigidity phenomenon does not take place.

### 0.5.1 Poincaré-Dulac's and Brjuno's Theorems

In what follows $v, w : (\mathbb{C}^2, \widehat{0}) \longrightarrow (\mathbb{C}^2, \widehat{0})$ will be germs of holomorphic vector fields and $A$ will be the linear part of $v$.

***Definition* 0.5.1** ((Non)resonant pairs and resonant vector monomials)**.** The pair $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ is called *resonant*, or more precisely, *additive resonant* if there exists $(m_1, m_2) \in \mathbb{N}^2$ such that $m_1 + m_2 \geqslant 2$, satisfying

$$\lambda_i = m_1\lambda_1 + m_2\lambda_2\,, \quad \text{for some} \quad i \in \{1, 2\}\,. \tag{6}$$

The above equality will be called *resonance* and the monomial vector field $x_1^{m_1} x_2^{m_2} \frac{\partial}{\partial x_i}$ will be *the resonant vector monomial corresponding to the resonance* (6). If there does not exist a pair of natural numbers satisfying the property (6), we shall say that $(\lambda_1, \lambda_2)$ *is nonresonant*.

When $v$ has nonzero linear part, we shall say that $v$ *is resonant* if the spectrum of its linear part is resonant. Otherwise, if the spectrum is nonresonant $v$ will be *nonresonant*.

If $v$ has nonzero linear part, it is possible to show that there exists a formal change of coordinates $\boldsymbol{\mathcal{H}}$ such that $\boldsymbol{\mathcal{H}}_*(v) = A + \widetilde{v}$, where $\widetilde{v}$ is a formal vector field with order greater than 1 having only resonant vector monomials corresponding to the resonances of the spectrum of $A$: the nonresonant vector monomials can be eliminated recursively by means of suitable polynomial transformations. More specifically, the resonant vector monomial $x_1^{n_1} x_2^{n_2} \frac{\partial}{\partial x_i}$ is eliminated by a polynomial transformation having a coefficient with factor $(\lambda_i - n_1\lambda_1 - n_2\lambda_2)^{-1}$, where $\lambda_1, \lambda_2$ are the eigenvalues of $A$; the number $\lambda_i - n_1\lambda_1 - n_2\lambda_2$ will be called *small denominator*.

This formal vector field is called *the formal normal form of* $v$ and it is unique with respect to such properties. In particular $\widetilde{v} \equiv 0$ whenever the spectrum of $A$ is nonresonant, that is, if $v$ is nonresonant then it is formally linearizable. Accurate proofs of these assertions can be found in [Arn83] and [IY08].

The following properties will be useful to describe whenever or not the formal (orbital) rigidity phenomenon takes place.

***Definition* 0.5.2** (Poincaré and Siegel Domains)**.** *The Poincaré domain* is the collection of ordered pairs $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ such that $0 \in \mathbb{C}$ does not belong to the convex hull of $\lambda_1, \lambda_2$ in $\mathbb{C}$. The *Siegel domain* is the complement of the Poincaré domain in $\mathbb{C}^2$.

**Remark 0.5.3.** By the definition, $(\lambda_1, \lambda_2)$ belongs to the Poincaré domain if and only if $\lambda_1, \lambda_2 \neq 0$ and its quotient is not a negative real number. Therefore this is a generic property.

It follows directly that $(\lambda_1, \lambda_2)$ has a finite number of resonances when it belongs to the Poincaré domain. As a consequence $v$ has a polynomial formal normal form whenever the spectrum of its linear part $A$ belongs to the Poincaré domain.

If $(\lambda_1, \lambda_2)$ belongs to the Siegel domain and $\lambda_1, \lambda_2 \neq 0$ has a negative irrational quotient, then the pair is nonresonant. Otherwise $(\lambda_1, \lambda_2)$ has an infinite number of resonances.

Theorems 0.5.4 and 0.5.7 state that the formal (orbital) rigidity phenomenon takes place under generic conditions over the linear part of germs of nondegenerate holomorphic vector fields. The real analytic versions of such results (Theorems 0.5.5 and 0.5.8) are not their immediate corollaries, as will be seen from the outlines of their proof appearing at the end of this subsection.

We recall that $v, w : (\mathbb{C}^2, \widehat{0}) \longrightarrow (\mathbb{C}^2, \widehat{0})$ are germs of holomorphic vector fields and $A$ is the linear part of $v$.

***Theorem* 0.5.4** (Poincaré-Dulac)**.** *If the spectrum of $A$ belongs to the Poincaré domain then $v$ and $w$ are formally (orbitally) equivalent if and only if they are analytically (orbitally) equivalent.*

***Theorem* 0.5.5** (Poincaré-Dulac, real analytic version)**.** *If $v$ and $w$ are the complexifications of germs of real analytic vector fields, being formally (orbitally) equivalent, under the condition of Poincaré-Dulac's theorem they (their induced foliations) are conjugated by the germ of a biholomorphism which is the complexification of the germ of real analytic diffeomorphism.*

The following property allows us to formulate Brjuno's theorem.

***Definition* 0.5.6.** [Brjuno's condition] A nonresonant pair $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ satisfies *the Brjuno's condition* if there exist $C, \epsilon > 0$ such that

$$\left|\lambda_i - (m_1\lambda_1 + m_2\lambda_2)\right|^{-1} \leqslant C \exp(|m|^{1-\epsilon})$$

for $m := (m_1, m_2) \in \mathbb{N}^2$, with $|m| := m_1 + m_2$ large enough.

***Theorem* 0.5.7** (Brjuno)**.** *If the spectrum of $A$, being nonresonant, belongs to the Siegel domain and satisfies the Brjuno's condition, then $v$ is analytically linearizable. As a consequence, if $v$ and $w$ are formally (orbitally) equivalent then they are analytically (orbitally) equivalent.*

An earlier result is the Siegel's theorem which has the same conclusion as Brjuno's theorem assuming that the spectrum of $A$ is *Diophantine*. Even though this condition is more restrictive than the Brjuno's condition, the pairs in $\mathbb{C}^2$ which are not Diophantine have Lebesgue measure zero (see [Arn83]).

***Theorem* 0.5.8** (Brjuno, real analytic version)**.** *If $v$ and $w$ are the complexifications of germs of real analytic vector fields which are formally (orbitally) equivalent, under the conditions of Brjuno's theorem, they (their induced foliations) are conjugated by the germ of a biholomorphism which is the complexification of a germ of real analytic diffeomorphism.*

In what follows we shall briefly outline the main ideas of the proofs of the above results. Accurate proofs of these assertions can be found in [Po1878], [Du1904], [Brj71], [Arn83] and [IY08]. Under the conditions of Theorems 0.5.4 and 0.5.7 we can conclude that the formal normal form of $v$ is a polynomial vector field. Moreover, the formal transformation $\boldsymbol{\mathcal{H}}$ mapping $v$ into its formal normal form is convergent, since in these cases the small denominators decay no faster than $\frac{1}{C}\exp(|m|^{\epsilon-1})$ and as a consequence, the growth of the coefficients of $\boldsymbol{\mathcal{H}}$ is controlled.

If $v$ and $w$ are formally (orbitally) equivalent, then we can suppose that they have the same $k$-jet for an arbitrary finite order $k$ after a polynomial change of coordinates (multiplied by a polynomial with nonzero constant term if the equivalence is orbital), which is induced by a suitable finite jet of the formal transformations realizing the equivalence. As a consequence, they have the same formal normal form (by the uniqueness of the formal normal form). Therefore $w$ will be analytically equivalent to this formal normal form, and in this way one can conclude Theorems 0.5.4 and 0.5.7.

The real analytic versions of Poincaré-Dulac's and Brjuno's theorems (Theorems 0.5.5 and 0.5.8) are proved by the previous arguments, after proving that the formal normal form of the complexification of a real analytic vector field has real coefficients, in the same way as the formal (orbital) change of coordinates which conjugates them.

### 0.5.2 Cases where Formal Rigidity Fails

*What happens when the conditions of Poincaré-Dulac's and Brjuno's theorems are not satisfied?* We shall briefly mention some cases where this conditions are violated and as a consequence, the formal rigidity phenomenon does not take place: *Cremer saddles, resonant complex saddles and complex saddle-nodes.*

We consider the germ of a holomorphic vector field at $\widehat{0} \in \mathbb{C}^2$. It will be called *complex saddle* if its linear part has nonzero eigenvalues belonging to the Siegel domain. By Hadamard-Perron's theorem this germ has two separatrices. After an analytic change of coordinates we may assume that these separatrices are the axes, and multiplying this latter equation by a nonzero constant we can assume that its linear part is $\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y}$, where $\lambda$ is the respective quotient of the eigenvalues. In what follows the holonomy map over a positive small loop on the $x$-axis will be referred as *the holonomy map of the complex saddle.* Note that the linear coefficient of this holonomy map is $\exp(2\pi i\lambda) \in \mathbb{S}^1$.

There exist complex saddles, called *Cremer saddles*, which are formally linearizable but are not analytically linearizable. The analytic classification of such complex saddles is unknown. Below we shall briefly indicate the arguments to guarantee the existence of this type of complex saddles.

The only analytic invariant of the complex saddles with the same linear part is their holonomy map ([EI84], [IY08]). On the other hand we have the following realization theorem whose proof can be found in [PMY94] or [EISV93].

***Theorem* 0.5.9.** *For any conformal germ* $f(z) = \exp(2\pi i\phi)z + O(z^2)$, $\phi \in \mathbb{R}$ *and any* $\lambda < 0$ *such that* $\lambda \equiv \phi \bmod \mathbb{Z}$, *there exists a vector field with linear part* $\frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}$ *whose holonomy map coincides with* $f$.

From the above results and the following theorem due to Yoccoz ([Yoc88], [Yoc95]), the existence of the complex saddles which are formally linearizable but are not analytically linearizable follows.

***Theorem* 0.5.10** (Yoccoz)**.** *If the complex number* $\mu = \exp(2\pi i l)$, *with* $l \in \mathbb{R}$, *violates the Brjuno's condition*[5], *then there exists a holomorphic germ at* $0 \in \mathbb{C}$, $z \mapsto \mu z + O(|z|^2)$, *which is not analytically linearizable.*

Now we shall mention the case of *resonant complex saddles* (eigenvalues have rational negative ratio). The formal (orbital) classification of these saddles depends on scalar parameters (see [Ily85], [Ily93]).

Given a formal equivalence class of resonant complex saddles, there is a functional modulus of analytic classification which is known as *Ecalle-Voronin's modulus* (this modulus was discovered independently by Ecalle and Voronin, [Eca75], [Vor81], [MR83], [Ily93], [VG96]). The existence of such functional moduli implies that the formal rigidity phenomenon cannot take place.

This modulus is obtained from the analytic classification of the holonomy maps of the resonant complex saddles (through the analytic classification of *parabolic germs*[6]). A complete exposition of these results can be found in [IY08].

Finally we shall briefly discuss the germs of holomorphic vector fields at $\widehat{0} \in \mathbb{C}^2$ whose linear part has one zero and one nonzero eigenvalue. It will be called *saddle-node*.

A saddle-node is formally orbitally equivalent to

$$\frac{z^{r+1}}{1+ax^r}\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

for some $a \in \mathbb{C}$ (the coordinate axes are separatrices of these vector fields). In general a saddle-node does not have a holomorphic separatrix related to the zero eigenvalue. Moreover, J. Martinet and J.-P. Ramis obtained the moduli of analytic classification for every formal equivalence class of saddle-nodes. Such moduli are functional moduli (equivalent to Ecalle-Voronin's moduli) and it is known as *Martinet-Ramis's moduli* ([MR83], [Eca81], [Ily93], [VM02]). Hence the formal orbital classification has one-dimensional moduli ($a \in \mathbb{C}$), and the analytic orbital classification has functional moduli. Thus, the formal rigidity phenomenon cannot take place.

Exceptionally, formal orbital rigidity phenomenon takes place for germs of vector fields having nilpotent linear part, that is, the formal and analytic orbital classification coincide ([Tak73], [CM88], [Mat91], [EISV93], [Lor99], [SZ02]).

[5] We shall say that the complex number $\mu = \exp(2\pi i l)$, with $l \in \mathbb{R}$, satisfies the *Brjuno's condition* if it is not a root of unity and there exists $\epsilon, C > 0$ such that for all $k \in \mathbb{N}$, $|\mu^k - 1|^{-1} < C\exp(k^{1-\epsilon})$.

[6] Given a conformal germ $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ we shall say that it is *parabolic* if $f'(0) = 1$.

## 0.6 Orbital Rigidity for Singularities with Zero Linear Part

We shall denote by $\mathcal{V}_n$ the class of germs of holomorphic vector fields at $\widehat{0} \in \mathbb{C}^2$ with an isolated singularity at the origin, zero $(n-1)$-jet and nonzero $n$-jet, for $n \geqslant 2$. Similarly, $\mathcal{V}_n^{\mathbb{R}}$ will be the class of germs of real analytic vector fields at $\widehat{0} \in \mathbb{R}^2$ with an isolated singularity at the origin, zero $(n-1)$-jet and nonzero $n$-jet.

In [Vo97] Voronin proved that the formal orbital rigidity phenomenon takes place for generic nondicritical germs of vector fields in the class $\mathcal{V}_n$ (Theorem 0.6.1). Later Ortiz, Rosales and Voronin proved that the rigidity phenomenon also takes place for vector fields, that is, there is formal rigidity for generic germs of vector fields in $\mathcal{V}_n$ ([ORV01]). After that, they proved that formal rigidity and formal orbital rigidity takes place for generic dicritical germs of vector fields in $\mathcal{V}_n$, obtaining in addition the minimal invariants for the *strict* analytic orbital classification [7] of such germs ([ORV05]). Recently they gave the minimal invariants for strict analytic orbital classification of generic nondicritical germs ([ORV12]). It is important to mention that in order to obtain the minimal invariants, they constructed formal orbital normal forms for both dicritical and nondicritical germs, which are in fact analytic normal forms for generic cases ([ORV08], [ORV14]).

The the following result is Voronin's formal orbital rigidity theorem:

***Theorem* 0.6.1** (Voronin)**.** *Under generic conditions*[8]*, $v, w \in \mathcal{V}_n$ are formally orbitally equivalent if and only if they are analytically orbitally equivalent.*

As we have said previously, the main goal of this work is to prove the real analytic version of Theorem 0.6.1 which appears as Theorem 0.3.2. The proof of the latter is highly inspired in the proof of Voronin's result.

We briefly discuss the key points of the proof of Theorem 0.3.2. For that purpose we consider $\nu$ and $\omega$ as in the statement of the Theorem.

The desingularizations of the complexifications of the germs $\nu$ and $\omega$ induce foliations on a neighborhood of *the complex exceptional divisor* $\mathbb{D}_{\mathbb{C}}$ on *the complex Möbius band* $\mathbb{M}_{\mathbb{C}}$ (see the definitions of these notions in Chapter 1).

We shall obtain a local biholomorphism which conjugates those desingularizations leaving invariant the real Möbius band $\mathbb{M}_{\mathbb{R}}$, that is, whose blow-down preserves the real plane $\mathbb{R}^2$. This biholomorphism is constructed by an *analytic transversal conjugation* which is the complexification of a real analytic local diffeomorphism. This transversal conjugation is achieved by the real-formal orbital equivalence between $\nu$ and $\omega$, and the genericity assumption of nonsolvability

[7] Here *strict* means that the analytic orbital equivalence (5) satisfies the linear part of $\boldsymbol{\mathcal{H}}$ is the identity matrix and the constant term of $\boldsymbol{\mathcal{K}}$ is 1.

[8] Voronin considered nondicritical germs in $\mathcal{V}_n$ satisfying their blow-ups have $n+1$ pairwise different singular points in $\mathbb{D}_{\mathbb{C}}$ (with characteristic number outside $\mathbb{Q}^+ \cup \{0\}$), and nonsolvable holonomy groups.

of the holonomy groups (Theorems 2.2.1 and 2.2.2). In order to extend this analytic transversal conjugation on a neighborhood of the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$, a suitable *auxiliary foliation* is constructed. This foliation will be the desingularization of the complexification of a real analytic dicritical vector field (Lemma 5.1.1). Then the extension will be possible by analytic continuation, except for the singular points of the desingularizations of the complexifications of $\nu$ and $\omega$, and a finite number of points where the auxiliary foliation is not transversal to the exceptional divisor $\mathbb{D}_{\mathbb{C}}$ (Chapter 6). After a meticulous work, this map can also be extended around these *special points* leaving invariant the real Möbius band (Chapters 7 and 8).

As we have mentioned, a detailed outline of the proof of Theorem 0.3.2 is given in Chapter 2.

# Chapter 1

# Basic notions and definitions

In this chapter we shall give notions and definitions which will be useful to prove Theorem 0.3.2. The genericity assumptions and a detailed outline of the proof of the theorem is given in Chapter 2, while its complete proof is achieved in the rest of the chapters.

The notions of *real and complex Möbius bands* will be specified in Section 1.1. The concepts of *vanishing holonomy group of a foliation* and the *strong conjugation of vanishing holonomy groups* will be defined in Sections 1.2 and 1.3, respectively.

Let $f : \big(\mathbb{R}^2, \widehat{0}\big) \longrightarrow \big(\mathbb{R}, c\big)$ be the germ of a real analytic function at $\widehat{0} \in \mathbb{R}^2$ and $v : \big(\mathbb{R}^2, \widehat{0}\big) \longrightarrow \big(\mathbb{R}^2, \widehat{0}\big)$ the germ of a real analytic vector field at $\widehat{0} \in \mathbb{R}^2$. We will refer to their respective complexifications as $f^{\mathbb{C}} : \big(\mathbb{C}^2, \widehat{0}\big) \longrightarrow \big(\mathbb{C}, c\big)$ and $v^{\mathbb{C}} : \big(\mathbb{C}^2, \widehat{0}\big) \longrightarrow \big(\mathbb{C}^2, \widehat{0}\big)$.

In what follows we shall denote a germ (of a vector field or a function) and its representatives in the same way.

## 1.1 Real and Complex Möbius Bands

*The complex Möbius band* $\mathbb{M}_{\mathbb{C}}$ is the closure in $\mathbb{C}^2 \times \mathbb{CP}^1$ of the set

$$\Big\{ \big((x, y), (x; y)\big) \in \mathbb{C}^2 \times \mathbb{CP}^1 \,\Big|\, (x, y) \neq \widehat{0} \Big\}.$$

The boundary of this set in $\mathbb{C}^2 \times \mathbb{CP}^1$ is $\mathbb{D}_{\mathbb{C}} := \{\widehat{0}\} \times \mathbb{CP}^1$ and it will be called *the complex exceptional divisor*. The following embeddings induce an analytic manifold structure on $\mathbb{M}_{\mathbb{C}}$

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\ \Phi\ } & \mathbb{C}^2 \times \mathbb{CP}^1 \\ (x,u) & \longmapsto & ((x,xu),(1;u)) \end{array} \quad , \quad \begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\ \Psi\ } & \mathbb{C}^2 \times \mathbb{CP}^1 \\ (v,y) & \longmapsto & ((vy,y),(v;1)) \end{array}$$

The projection $\pi^{\mathbb{C}} : \mathbb{M}_{\mathbb{C}} \longrightarrow \mathbb{C}^2$ is called the *complex blow-down* (or *the standard complex monoidal map*). Its inverse map $\left(\pi^{\mathbb{C}}\right)^{-1}\colon \mathbb{C}^2 \smallsetminus \{\widehat{0}\} \longrightarrow \mathbb{M}_{\mathbb{C}} \smallsetminus \mathbb{D}_{\mathbb{C}}$ is defined as $(x,y) \mapsto \big((x,y),(x;y)\big)$. This map is known as *the complex blow-up*.

The set $\mathbb{M}_{\mathbb{C}} \cap \left(\mathbb{R}^2 \times \mathbb{RP}^1\right)$ will be called *the real Möbius band* and it will be denoted by $\mathbb{M}_{\mathbb{R}}$. Its boundary in $\mathbb{R}^2 \times \mathbb{RP}^1$ is $\{\widehat{0}\} \times \mathbb{RP}^1 = \mathbb{D}_{\mathbb{C}} \cap \left(\mathbb{R}^2 \times \mathbb{RP}^1\right)$. It will be called *the real exceptional divisor* and it will be denoted by $\mathbb{D}_{\mathbb{R}}$. The restrictions of the embeddings $\Phi$ and $\Psi$ on $\mathbb{R}^2$ give a real analytic atlas on $\mathbb{M}_{\mathbb{R}}$, that is, $\mathbb{M}_{\mathbb{R}}$ is a real analytic manifold.

The canonical projection $\pi : \mathbb{M}_{\mathbb{R}} \longrightarrow \mathbb{R}^2$ is called *the real blow-down* (or *the standard real monoidal map*). Its inverse map $\pi^{-1} : \mathbb{R}^2 \smallsetminus \{\widehat{0}\} \longrightarrow \mathbb{M}_{\mathbb{R}} \smallsetminus \mathbb{D}_{\mathbb{R}}$ is defined as $(x,y) \mapsto \big((x,y),(x;y)\big)$ and it is known as *the real blow-up*.

In what follows, the map $\Phi$ ($\Psi$) together with its domain will be called *the coordinate chart* $(x, u = y/x)$ (*the coordinate chart* $(v = x/y, y)$).

It is important to underline that *the complex Möbius band* $\mathbb{M}_{\mathbb{C}}$ *is the complexification of the real Möbius band* $\mathbb{M}_{\mathbb{R}}$ in the following sense:

If $M$ is a complex analytic manifold of complex dimension $n$ and $N \subseteq M$ is a real analytic manifold of real dimension $n$, then we say that $M$ is *the complexification of* $N$ if there exist $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ an atlas on $N$ and $\{(\mathscr{U}_\alpha, \Phi_\alpha)\}_{\alpha \in A}$ an atlas on $M$ such that $\mathscr{U}_\alpha \cap N = U_\alpha$ for all $\alpha \in A$ and whenever $U_\alpha \cap U_\beta \neq \emptyset$, $\Phi_\beta \circ \Phi_\alpha^{-1}$ is the complexification of $\phi_\beta \circ \phi_\alpha^{-1}$. The easiest example arises when we consider $M = \mathbb{C}^n$ and $N = \mathbb{R}^n$. Using the standard atlas we can conclude that $\mathbb{CP}^n$ is the complexification of $\mathbb{RP}^n$.

## 1.2 Vanishing Holonomy Group

Recall that, given a natural number $n \geq 2$, $\mathcal{V}_n^{\mathbb{R}}$ will be the class of germs of real analytic vector fields at $\widehat{0} \in \mathbb{R}^2$ with an isolated singularity at the origin, having zero $(n-1)$-jet and nonzero $n$-jet.

If we consider $\upsilon \in \mathcal{V}_n^{\mathbb{R}}$, its integral curves generate a partition of a neighborhood of the origin on $\mathbb{R}^2$ into disjoint connected subsets of real dimension 1 except for the singular solution $\widehat{0}$. This partition will be called *the (singular) foliation induced by* $\upsilon$ and it will be denoted by $\mathscr{F}_\upsilon$. In the same way, the integral curves of the complexification of $\upsilon$ generate a partition of a neighborhood of the origin on $\mathbb{C}^2$ into disjoint connected subsets of complex dimension 1 except for the singular solution $\widehat{0}$. This partition will be called *the (singular) foliation induced by* $\upsilon^{\mathbb{C}}$ and it will be denoted by $\mathscr{F}_\upsilon^{\mathbb{C}}$.

The pullback of the foliation $\mathscr{F}_\upsilon$ with respect to the real blow-down generates a partition of a neighborhood of the real exceptional divisor $\mathbb{D}_{\mathbb{R}}$ on the real

Möbius band $\mathbb{M}_{\mathbb{R}}$ into disjoint connected subsets of real dimension 1, except for a finite number of isolated singularities on $\mathbb{D}_{\mathbb{R}}$. This partition is a singular foliation called *the blow-up of the foliation* $\mathscr{F}_\upsilon$ and will be denoted by $\widetilde{\mathscr{F}_\upsilon}$. In a similar way, the pullback of the foliation $\mathscr{F}_\upsilon^{\mathbb{C}}$ with respect to the complex blow-down generates a partition of a neighborhood of the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$ on the complex Möbius band $\mathbb{M}_{\mathbb{C}}$ into disjoint connected subsets of complex dimension 1, except for a finite number of isolated singularities on $\mathbb{D}_{\mathbb{C}}$. This partition is a singular foliation called *the blow-up of the foliation* $\mathscr{F}_\upsilon^{\mathbb{C}}$ and will be denoted by $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$.

If the foliation $\widetilde{\mathscr{F}_\upsilon}$ ($\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$) has only elementary singularities on real exceptional divisor $\mathbb{D}_{\mathbb{R}}$ (on complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$), this foliation will be called *the desingularization of* $\mathscr{F}_\upsilon$ (*the desingularization of* $\mathscr{F}_\upsilon^{\mathbb{C}}$).

***Definition* 1.2.1.** The $n$-jet of $\upsilon \in \mathcal{V}_n^{\mathbb{R}}$, say $(P_n, Q_n)$, will be called *the principal part of* $\upsilon$. If the homogeneous polynomial $xQ_n - yP_n$ is nonzero we shall say that $\upsilon$ is *nondicritical*. Otherwise $\upsilon$ will be called *dicritical*.

Let $\upsilon \in \mathcal{V}_n^{\mathbb{R}}$ be a nondicritical germ. In this case the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$ is an invariant set of the foliation $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$. Furthermore, this foliation has only a finite number of singular points $\mathfrak{d}_1, \dots, \mathfrak{d}_m$ in $\mathbb{D}_{\mathbb{C}}$.

We consider a path $\gamma : [0,1] \longrightarrow \mathbb{D}_{\mathbb{C}} \setminus \{\mathfrak{d}_1, \dots, \mathfrak{d}_m\}$ beginning at $\mathfrak{d}_0$ and ending at $\mathfrak{d}_0'$, and two *cross-sections*[1] $T_{\mathfrak{d}_0}$ and $T'_{\mathfrak{d}_0'}$ to $\mathbb{D}_{\mathbb{C}}$ at $\mathfrak{d}_0$ and $\mathfrak{d}_0'$, respectively.

***Definition* 1.2.2. (Correspondence map over a path)** The map obtained by the lift of the path $\gamma$ on each leaf sufficiently close to the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$ will be called *the correspondence map for the foliation* $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$ *over the path* $\gamma$ *from* $T_{\mathfrak{d}_0}$ *to* $T'_{\mathfrak{d}_0'}$ and it will be denoted by

$$\Delta_\gamma^\upsilon : \big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big) \longrightarrow \big(T'_{\mathfrak{d}_0'}, \mathfrak{d}_0'\big)$$

.

A careful exposition of the previous concept and their resulting properties can be found in [G-MO89] and [IY08].

The correspondence map $\Delta_\gamma^\upsilon$ is the germ of a biholomorphism whose inverse map is $\Delta_{\gamma^{-1}}^\upsilon : \big(T'_{\mathfrak{d}_0'}, \mathfrak{d}_0'\big) \longrightarrow \big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big)$, where $\gamma^{-1}$ is defined as $t \mapsto \gamma(1-t)$.

For $\alpha \in \Pi_1\big(\mathbb{D}_{\mathbb{C}} \setminus \{\mathfrak{d}_1, \dots, \mathfrak{d}_{n+1}\}, \mathfrak{d}_0\big)$ and $\sigma, \widetilde{\sigma} \in \alpha$, the correspondence maps over $\sigma$ and $\widetilde{\sigma}$ coincide, that is, $\Delta_\sigma^\upsilon = \Delta_{\widetilde{\sigma}}^\upsilon : \big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big) \longrightarrow \big(T'_{\mathfrak{d}_0}, \mathfrak{d}_0\big)$. As a consequence we shall denote by $\Delta_\alpha^\upsilon : \big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big) \longrightarrow \big(T'_{\mathfrak{d}_0'}, \mathfrak{d}_0\big)$ the correspondence map over any representative of $\alpha$. In particular, if $T_{\mathfrak{d}_0} = T'_{\mathfrak{d}_0}$, then the map $\Delta_\alpha^\upsilon$ will be called *the holonomy map*[2] *for the foliation* $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$ *over* $\alpha$.

[1]A *cross-section to* $\mathbb{D}_{\mathbb{C}}$ *at a point* $\mathfrak{d} \in \mathbb{D}_{\mathbb{C}}$ is the germ of a complex curve which is contained in $\mathbb{M}_{\mathbb{C}}$ and intersects $\mathbb{D}_{\mathbb{C}}$ transversally at $\mathfrak{d}$.

[2]This map is defined modulo conjugacy, the latter given by the changes of cross-sections.

***Definition* 1.2.3. (Vanishing holonomy group)** The *vanishing holonomy group of* $\widetilde{\mathscr{F}}_{\upsilon}^{\mathbb{C}}$ *on* $\big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big)$ is defined as the group

$$\mathcal{G}_{\upsilon} = \Big\{ \Delta_{\alpha}^{\upsilon} : \big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big) \longrightarrow \big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big) \,\big|\, \alpha \in \Pi_1\big(\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{d}_1, \ldots, \mathfrak{d}_m\}, \mathfrak{d}_0\big) \Big\} .$$

This group modulo a simultaneous conjugacy of all holonomy maps will be referred to as *holonomy group of* $\widetilde{\mathscr{F}}_{\upsilon}^{\mathbb{C}}$ (that is, it is independent of a cross-section or even a base point).

## 1.3 Strong Conjugation of Vanishing Holonomy Groups

We consider two germs of nondicritical vector fields $\upsilon, \zeta \in \mathcal{V}_n^{\mathbb{R}}$ such that their principal part coincides. This implies that the foliations $\widetilde{\mathscr{F}}_{\upsilon}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_{\zeta}^{\mathbb{C}}$ have the same singular points $\mathfrak{d}_1, \ldots, \mathfrak{d}_m$ on the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$. We fix $\mathfrak{d}_0 \in \mathbb{D}_{\mathbb{C}} \smallsetminus \big\{\mathfrak{d}_1, \ldots, \mathfrak{d}_m\big\}$ and a cross-section $T_{\mathfrak{d}_0}$ to $\mathbb{D}_{\mathbb{C}}$ at $\mathfrak{d}_0$. As before, the vanishing holonomy groups of $\widetilde{\mathscr{F}}_{\upsilon}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_{\zeta}^{\mathbb{C}}$ on $\big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big)$ will be denoted by $\mathcal{G}_{\upsilon}$ and $\mathcal{G}_{\zeta}$, respectively.

***Definition* 1.3.1.** (**Strong conjugation**) The holonomy groups $\mathcal{G}_{\upsilon}$ and $\mathcal{G}_{\zeta}$ are *analytically strongly conjugated* if there exists $h : \big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big) \longrightarrow \big(T_{\mathfrak{d}_0}, \mathfrak{d}_0\big)$ the germ of a biholomorphism such that for any $\alpha \in \Pi_1\big(\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{d}_1, \ldots, \mathfrak{d}_m\}, \mathfrak{d}_0\big)$

$$h \circ \Delta_{\alpha}^{\upsilon} \circ h^{-1} = \Delta_{\alpha}^{\zeta} . \tag{1.1}$$

In the same way, $\mathcal{G}_{\upsilon}$ and $\mathcal{G}_{\zeta}$ are *formally strongly conjugated* if there is a local parametrization of the complex curve $T_{\mathfrak{d}_0}$ by a parameter $z$ and an invertible formal transformation $h \in \mathbb{C}[[z]]$ with zero constant term, such that for any $\alpha \in \Pi_1\big(\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{d}_1, \ldots, \mathfrak{d}_m\}, \mathfrak{d}_0\big)$ the equality (1.1) is satisfied.

In Chapter 4 we shall prove that the strict formal orbital equivalence between two germs of holomorphic vector fields at $\widehat{0} \in \mathbb{C}^2$ implies the strong formal conjugacy of their holonomy groups (Lemma 4.4.1). This property will be very relevant to the proof of Theorem 0.3.2.

# Chapter 2

# Generic Conditions and Outline of the Main Theorem

Recall that given $n \geq 2$, $\mathcal{V}_n^{\mathbb{R}}$ is the class of germs of real analytic vector fields at $\widehat{0} \in \mathbb{R}^2$ with an isolated singularity at the origin, having zero $(n-1)$-jet and nonzero $n$-jet. The foliation induced by $\upsilon \in \mathcal{V}_n^{\mathbb{R}}$ on $(\mathbb{R}^2, \widehat{0})$ is denoted by $\mathscr{F}_\upsilon$ and its blow-up is denoted by $\widetilde{\mathscr{F}_\upsilon}$. In the same way $\mathscr{F}_\upsilon^{\mathbb{C}}$ is the foliation induced by $\upsilon^{\mathbb{C}}$ on $(\mathbb{C}^2, \widehat{0})$ and $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$ is the blow-up of this foliation on $(\mathbb{M}_{\mathbb{C}}, \mathbb{D}_{\mathbb{C}})$.

## 2.1 The Class $\Sigma_n^{\mathbb{R}}$

A germ $\upsilon \in \mathcal{V}_n^{\mathbb{R}}$ belongs to $\Sigma_n^{\mathbb{R}}$ if it satisfies the following conditions:

i. The germ $\upsilon$ is nondicritic. Furthermore, if $(P_n, Q_n)$ is the principal part of $\upsilon$, the homogeneous polynomial $xQ_n - yP_n$ has $n+1$ simple linear factors (not necessarily real).

As a consequence the foliation $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$ has $n+1$ pairwise different singular points $\mathfrak{d}_1, \cdots, \mathfrak{d}_{n+1}$ on the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$. Given $1 \leq j \leq n+1$ we consider $\lambda_j^1$, $\lambda_j^2$ the eigenvalues of the linear part of the foliation $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$ at the singular point $\mathfrak{d}_j$, being $\lambda_j^2$ the eigenvalue related to the complex exceptional divisor. It can verified that $\lambda_j^2$ is nonzero. The ratio $\lambda_j := \lambda_j^1/\lambda_j^2$ is *the characteristic number (or Camacho-Sad index) of the foliation* $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$ *at the singular point* $\mathfrak{d}_j$.

ii. For all $1 \leq j \leq n+1$, $\quad \lambda_j \in \mathbb{C} \smallsetminus \big(\mathbb{Q}_+ \cup \{\widehat{0}\}\big)$.

iii. The vanishing holonomy group of the foliation $\widetilde{\mathscr{F}_\upsilon^{\mathbb{C}}}$ is nonsolvable.

The conditions i and iii are generic on the class $\mathcal{V}_n^{\mathbb{R}}$ in the algebraic sense: the $n$-jet of a germ in $\mathcal{V}_n^{\mathbb{R}}$ violating the condition i satisfies a finite number of real polynomial identities, and if a germ in $\mathcal{V}_n^{\mathbb{R}}$ does not meet the condition iii, its $(n+2)$-jet satisfies a finite number of real polynomial identities (see [Sh84]). In the sense of Lebesgue measure, the condition ii is satisfied by most germs in $\mathcal{V}_n^{\mathbb{R}}$ whose blow-ups have singularities with positive characteristic numbers.

Before starting the outline of the proof of Theorem 0.3.2, we shall conclude an immediate corollary. For that purpose we shall refer to two germs in $\Sigma_n^{\mathbb{R}}$ as *$C^\infty$-smoothly orbitally equivalent* if an infinitely smooth change of coordinates brings one of them into the other being multiplied by the invertible germ of a infinitely smooth scalar function. This equivalence will be called *strict* if the linear part of the infinitely smooth change of coordinates is the identity matrix and the constant term of the infinitely smooth scalar function is 1.

***Corollary* 2.1.1. *($C^\infty$-smooth Orbital Rigidity)*.** *Two germs $\nu, \omega \in \Sigma_n^{\mathbb{R}}$ are (strictly) $C^\infty$-smoothly orbitally equivalent if and only if they are (strictly) real-analytically orbitally equivalent.*

## 2.2 Outline of the Proof of the Main Theorem

The proof of Theorem 0.3.2 is achieved in the following chapters. For that purpose we shall consider $\nu, \omega \in \Sigma_n^{\mathbb{R}}$ as in the statement of aforementioned theorem.

By the conditions i and ii the foliations $\widetilde{\mathscr{F}}_\nu^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_\omega^{\mathbb{C}}$ have elementary singularities on the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$, that is, they are the *desingularizations of the foliations $\mathscr{F}_\nu^{\mathbb{C}}$ and $\mathscr{F}_\omega^{\mathbb{C}}$*. Moreover, from Poincaré's linearization theorem and Hadamard-Perron's theorem (see [IY08]), $\mathscr{F}_\nu^{\mathbb{C}}$ and $\mathscr{F}_\omega^{\mathbb{C}}$ have $n+1$ pairwise transversal *separatrices*[1]. Except for linear changes of coordinates with real coefficients, we may assume without loss of generality that the separatrices of both germs are not tangent to the line $\{x=0\}$ at the origin.

We consider $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1} \in \mathbb{D}_{\mathbb{C}}$, the singular points of the foliation $\widetilde{\mathscr{F}}_\nu^{\mathbb{C}}$, and $\lambda_1, \ldots, \lambda_{n+1}$ their respective characteristic numbers. Since the separatrices are not tangent to the line $\{x=0\}$, the separatrix of $\nu^{\mathbb{C}}$ whose desingularization passes through $\mathfrak{p}_i$, say $S_i$, can be described as $\{y = \phi_i(x)\}$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ belongs to the domain of the coordinate chart $(x, u = y/x)$.

It is important to notice that when $\mathfrak{p}_i \in \mathbb{D}_{\mathbb{R}}$, $S_i$ is the complexification of a real analytic curve, that is, the Taylor series of $\phi_i$ has real coefficients. On the other hand, if $\mathfrak{p}_i \notin \mathbb{D}_{\mathbb{R}}$ then there exists another singular point $\mathfrak{p}_j \notin \mathbb{D}_{\mathbb{R}}$ which is *the conjugate point of* $\mathfrak{p}_i$ (this means that $\mathfrak{p}_i = (0, p_i)$ and $\mathfrak{p}_j = (0, \bar{p}_i)$ with respect to the chart $(x, u)$). Furthermore, $S_j$ is *the conjugate curve of* $S_i$, that is, if $\phi_i(x) = \sum_{k \geqslant 1} a_k x^k$ then $\phi_j(x) = \sum_{k \geqslant 1} \overline{a}_k x^k$.

[1] Recall that a separatrix is an invariant irreducible analytic curve passing through the origin.

In Chapter 3 we shall show that without loss of generality, we can assume that $\nu$ and $\omega$ are strictly real-formally orbitally equivalent and moreover, they have the same separatrices.

In Chapter 4 we will prove that the vanishing holonomy groups of $\widetilde{\mathscr{F}_\nu^{\mathbb{C}}}$ and $\widetilde{\mathscr{F}_\omega^{\mathbb{C}}}$ are formally strongly conjugated. More accurately, we will obtain the following result as a consequence of the real analyticity of germs $\nu$ and $\omega$.

***Theorem* 2.2.1.** *(**Strong Real-Formal Conjugation**.) Let $\mathfrak{p}_0 \in \mathbb{D}_{\mathbb{R}}$ be a nonsingular point belonging to the domain of the chart $(x,u)$. Consider the line $\Gamma_{\mathfrak{p}_0} = \big\{u = p_0\big\}$, where $p_0 = u(\mathfrak{p}_0)$. Then there exists an invertible formal transformation $h_{\mathfrak{p}_0} \in \mathbb{R}[[z]]$ with zero constant term, which strongly conjugates the holonomy groups of $\widetilde{\mathscr{F}_\nu^{\mathbb{C}}}$ and $\widetilde{\mathscr{F}_\omega^{\mathbb{C}}}$ on $(\Gamma_{\mathfrak{p}_0}, \mathfrak{p}_0)$.*

The following rigidity theorem for holomorphic self-maps on $(\mathbb{C},0)$ allows us to conclude that the formal transformation of Theorem 2.2.1 is convergent:

***Theorem* 2.2.2.** *([CM88],[R89]) Let $G$ and $\widetilde{G}$ be nonsolvable finitely generated groups of germs of holomorphic self-maps on $(\mathbb{C},0)$. Suppose that there exists an invertible formal transformation with zero constant term $h \in \mathbb{C}[[z]]$ which conjugates them, that is,*

$$h\,G\,h^{-1} := \Big\{h \circ f \circ h^{-1} \,\big|\, f \in G\Big\} = \widetilde{G}\,.$$

*Then $h$ is a convergent series.*

This real analytic conjugation between the vanishing holonomy groups of $\widetilde{\mathscr{F}_\nu^{\mathbb{C}}}$ and $\widetilde{\mathscr{F}_\omega^{\mathbb{C}}}$ can be extended along the leaves of the foliations $\widetilde{\mathscr{F}_\nu^{\mathbb{C}}}$ and $\widetilde{\mathscr{F}_\omega^{\mathbb{C}}}$ by analytic continuation. For this purpose, in Chapter 5 we will construct a dicritical holomorphic foliation on $\big(\mathbb{C}^2, \widehat{0}\big)$ (which will be the complexification of the real analytic germ of a vector field), whose blow-up on $\big(\mathbb{M}_{\mathbb{C}}, \mathbb{D}_{\mathbb{C}}\big)$ will allow us to provide the analytic continuation leaving invariant the real Möbius band. The latter will be called the *auxiliary foliation*. At the end of Chapter 6 we define the biholomorphism which carries the foliation $\widetilde{\mathscr{F}_\nu^{\mathbb{C}}}$ to the foliation $\widetilde{\mathscr{F}_\omega^{\mathbb{C}}}$, but far from finitely many *special points*. Chapters 7 and 8 are focused on the extension of this biholomorphism to such special points leaving invariant the real Möbius band.

# Chapter 3

# Equivalence of Separatrices

In this chapter we prove that without loss of generality we can suppose that $\nu$ and $\omega$ have the same separatrices and their $N$-jets coincide, for $N$ a sufficiently large natural number.

***Lemma* 3.0.3.** *Given a natural number $N \geqslant n$ there exists $\omega_N \in \Sigma_n^{\mathbb{R}}$ with the same separatrices and the same $N$-jet as the germ $\nu$, and which is real-analytically orbitally equivalent to $\omega$, and as a consequence, is real-formally orbitally equivalent to $\nu$.*

*Moreover, the invertible formal transformation and the scalar formal function realizing the equivalence between $\omega_N$ and $\nu$ have their $N$-jets equal to the identity and the constant function $1$, respectively.*

**Proof**. We consider $\boldsymbol{H} \in \big(\mathbb{R}[[x,y]]\big)^2$ the invertible formal transformation whose components do not have constant term, and $\boldsymbol{K} \in \mathbb{R}[[x,y]]$ with nonzero constant term such that

$$\omega = \tfrac{1}{\boldsymbol{K}}\Big[\big(\mathrm{D}\,\boldsymbol{H}\cdot \nu\big)\circ \boldsymbol{H}^{-1}\Big]. \tag{3.1}$$

We take $k := n + N + 1$ and denote by $\mathcal{H}_k$ the $k$-jet of $\boldsymbol{H}^{-1}$ and by $\mathcal{K}_{k-1}$ the $(k-1)$-jet of $\frac{1}{\boldsymbol{K}\circ \mathcal{H}_k^{-1}}$.

The germ of a real analytic vector field

$$\omega_0 := \tfrac{1}{\mathcal{K}_{k-1}}\Big[\big(\mathrm{D}\,\mathcal{H}_k\cdot \omega\big)\circ \mathcal{H}_k^{-1}\Big]$$

belongs to the class $\Sigma_n^{\mathbb{R}}$, since it is real-analytically orbitally equivalent to $\omega$. Moreover, it has the same $(2n+N)$-jet as $\nu$. In particular, $\widetilde{\mathscr{F}}_{\nu}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_{\omega_0}^{\mathbb{C}}$ have the same singular points $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ in the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$.

Now we will modify the germ $\omega_0$ by a real analytic coordinate change which will map the separatrices of $\omega_0$ to the separatrices of $\nu$ without changing the $N$-jet of $\omega_0$.

We consider a separatrix of the complexification of a germ in $\Sigma_n^{\mathbb{R}}$. Then the $m$-jet of this separatrix (that is, the $m$-jet of the irreducible holomorphic scalar function which defines this curve) is determined by the $(m+n)$-jet of the germ of a vector field. In consequence, if $S_i = \{y = \phi_i(x)\}$ and $\{y = \psi_i(x)\}$ are the separatrices of $\nu^{\mathbb{C}}$ and $\omega_0^{\mathbb{C}}$ whose desingularizations intersect the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$ at $\mathfrak{p}_i$, since $\nu$ and $\omega$ have the same $(2n+N)$-jet, then $\text{ord}_{\widehat{0}}\left(\phi_i - \psi_i\right) \geqslant n + N + 1$.

Thus the germ of a function at $\widehat{0} \in \mathbb{C}^2$ defined as: $\mathcal{G}(0,y) = (0,y)$ and

$$\mathcal{G}(x,y) := \left( x, y + \sum_{i=1}^{n+1} \frac{\left(\phi_i(x)-\psi_i(x)\right)\prod_{\substack{j=1\\ j\neq i}}^{n+1}\left(y-\psi_j(x)\right)}{\prod_{\substack{j=1\\ j\neq i}}^{n+1}\left(\psi_i(x)-\psi_j(x)\right)} \right) \quad \text{when } x \neq 0\,,$$

is an invertible holomorphic function with $N$-jet equal to the identity. Moreover, it has real coefficients since for each summand there is another term *conjugated* to it[1]. In this way, the invertible formal transformation $\mathcal{G} \circ \mathcal{H}_k \circ \boldsymbol{H}$ has real coefficients and its $N$-jet is equal to the identity.

The germ of a real analytic vector field defined as

$$\begin{aligned} \omega_N &:= \left(\mathrm{D}\,\mathcal{G}\cdot \omega_0\right)\circ \mathcal{G}^{-1} \\ &= \frac{1}{\left(\mathcal{K}_{k-1}\circ\mathcal{G}^{-1}\right)\left(\boldsymbol{K}\circ\mathcal{H}_k^{-1}\circ\mathcal{G}^{-1}\right)}\left[\left(\mathrm{D}\left(\mathcal{G}\circ\mathcal{H}_k\circ\boldsymbol{H}\right)\cdot\nu\right)\circ\left(\mathcal{G}\circ\mathcal{H}_k\circ\boldsymbol{H}\right)^{-1}\right] \end{aligned}$$

belongs to the class $\Sigma_n^{\mathbb{R}}$ since it is strictly real-analytically equivalent to $\omega_0$, and satisfies $\text{ord}_{\widehat{0}}(\nu - \omega_N) \geqslant n + \text{ord}_{\widehat{0}}\Big(\big(\mathcal{G}\circ\mathcal{H}_k\circ\boldsymbol{H}\big) - \text{id}\Big) - 1 \geqslant n + N$. Moreover, it has the same separatrices as $\nu$ since $\mathcal{G}(x, \psi_i(x)) = (x, \phi_i(x))$ for all $i \in \{1, \ldots, n\}$ and it is strictly real-formally orbitally equivalent to $\nu$.

■

In what follows we shall suppose that $\nu$ and $\omega$ have the same separatrices and are strictly real-formally orbitally equivalent. Moreover, we shall assume that the formal transformations $\boldsymbol{H}$ and $\boldsymbol{K}$ realizing the equivalence, have $N$-jet equal to the identity and the constant function 1, respectively, where $N$ is a natural number independent of $\omega$ (it will be specified in Chapter 6).

[1]Let $f$ be the germ of a holomorphic function at $\widehat{0} \in \mathbb{C}^n$. *The conjugated function of* $f$, denoted by $\overline{f(\bar{z})}$, is the germ of a holomorphic function at $\widehat{0} \in \mathbb{C}^n$ satisfying the following condition: if $f(z) = \sum_{\mathbf{k}\in\mathbb{N}^n} a_{\mathbf{k}} z^{\mathbf{k}}$ then $\overline{f(\bar{z})} = \sum_{\mathbf{k}\in\mathbb{N}^n} \overline{a}_{\mathbf{k}} z^{\mathbf{k}}$. In this context, the function $f$ is the complexification of a real analytic germ if and only if it is the conjugated function of itself.

# Chapter 4

# Formal Conjugation of the Holonomy Group

In this chapter we achieve the proof of Theorem 2.2.1. More specifically, we shall prove that the strict formal orbital equivalence between two germs of holomorphic vector fields at $\widehat{0} \in \mathbb{C}^2$ implies the strong formal conjugacy of their holonomy groups (Lemma 4.4.1), and moreover, if they are the complexifications of real analytic germs and the equivalence is real-formal then the conjugacy has real coefficients (Remark 4.4.3).

Along the proof will be necessary to work with holonomy maps on formal cross-sections. For this purpose we shall develop preliminary notions and properties in Sections 4.1, 4.2 and 4.3.

## 4.1 Existence and Uniqueness of Solutions of Formal Equations

In this section we shall establish the notions of *solution of a formal differential equation* and *initial condition of a formal solution*. Besides, we shall conclude the existence and uniqueness of a formal equation in Lemma 4.1.1.

We consider $P(u, x, z) \in \mathbb{C}[[u, x, z]]$. This formal series induces a *nonautonomous formal differential equation with time $u$ and parameter $z$* which will be represented as

$$\frac{\mathrm{d}\, x}{\mathrm{d}\, u} = P(u, x, z)\,. \tag{4.1}$$

More specifically, a formal series $\mathbf{x}(u, x, z) \in \mathbb{C}[[u, x, z]]$ is called a *solution of the formal equation* (4.1) if the following formal equality is satisfied

$$\frac{\mathrm{d}\, \mathbf{x}}{\mathrm{d}\, u} = P\big(u, \mathbf{x}(u, x, z), z\big)\,.$$

This last equality has to be understood as a termwise equality. We consider a family of formal curves $\widehat{\ } = \big\{(u,x) = \ \Psi(x,z), \Phi(x,z))\big\}$, where $\Psi(x,z) = \sum_{k\geqslant 1} \psi_k(z)\, x^k$ and $\Phi(x,z) = \sum_{k\geqslant 1} \phi_k(z)\, x^k$ are formal series with coefficients in $\mathbb{C}[[z]]$. We shall say that the solution of the formal equation (4.1) intersects the family $\widehat{\ }$ if

$$\Phi(x,z) = \mathbf{x}(\Psi(x,z), x, z)\,. \tag{4.2}$$

If the series $P(u,x,z)$, $\Psi(x,z)$ and $\Phi(x,z)$ are convergent, then $\mathbf{x}(u,x,z)$ is the flow map of the holomorphic differential equation $\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = P(u,x,z)$ with initial conditions in the holomorphic curve $\widehat{\ }$ (see Lemma 4.2.1). Figure 4.1 is inspired by this special case.

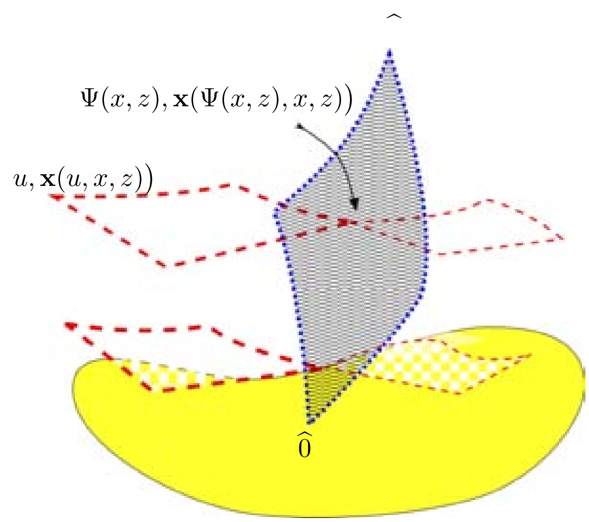


Figure 4.1: Schematic Figure in the coordinates $(x,u)$ of the formal solution of (4.1) intersecting $\widehat{\ }$.

It is important to stress that all the figures appearing in this chapter were done in order to help the reader. In particular, all dotted curves are formal (not necessarily convergent).

***Lemma* 4.1.1.** *There exists a unique solution of the formal differential equation (4.1) intersecting* $\widehat{\ }$.

**Proof**. First of all, we will prove *the uniqueness*. Let $\mathbf{x}(u,x,z)$ be a solution of the formal differential equation (4.1) satisfying the equality (4.2). We can express its formal Taylor series in the following way:

$$\mathbf{x}(u,x,z) = \Phi(x,z) + \sum_{m\geqslant 1} \tfrac{1}{m!}\tfrac{\partial^m \mathbf{x}}{\partial\, u^m}\Big|_{(\Psi(x,z),x,z)}(u \quad \Psi(x,z))^m\,.$$

We will stress that $\frac{\partial^m \mathbf{x}}{\partial u^m}\Big|_{(\Psi(x,z),x,z)}$ is independent of $\mathbf{x}$ (see the equality (4.3)). For this purpose it will be necessary to introduce some definitions.

We take $m \geqslant 0$ and the set of variables $\mathcal{C}_m = \{z_{(k,l)} \,:\, 0 \leqslant k, l,\, k+l \leqslant m\}$ ordered as follows: $z_{(k,l)} < z_{(\tilde{k},\tilde{l})}$ if and only if

- $k + l < \widetilde{k} + \widetilde{l}$, or
- $k + l = \widetilde{k} + \widetilde{l}$, and in such a case $k < \widetilde{k}$.

In what follows $q(z_{(k,l)})_{k+l\leqslant m}$ represents a polynomial with coefficients in $\mathbb{C}$ and variables in the ordered set $\mathcal{C}_m$. We recursively define a family of polynomials: $q_1(z_{(0,0)}) := z_{(0,0)}$; for $m \geqslant 1$,

$$q_{m+1}(z_{(k,l)})_{k+l\leqslant m} := \sum_{k+l\leqslant m-1} \left[ \frac{\partial\, q_m}{\partial\, z_{(k,l)}} \big( z_{(k+1,l)} + z_{(k,l+1)} z_{(0,0)} \big) \right].$$

These polynomials have real coefficients. It is directly verified by induction in $m \geqslant 1$ that

$$\frac{\partial^m \mathbf{x}}{\partial\, u^m}\Big|_{(\Psi(x,z),x,z)} = q_m \big( P_{u^k\, x^l} (\Psi(x,z), \Phi(x,z), z) \big)_{l+k\leqslant m-1}\,, \tag{4.3}$$

where $P_{u^0\, x^0} := P$, $P_{u^k\, x^l} := \frac{\partial^{l+k} P}{\partial\, u^k \partial\, x^l}$. Then we can conclude the uniqueness.

Inspired by the above arguments we define the formal series $\widetilde{\mathbf{x}}(u,x,z)$ as

$$\Phi(x,z) + \sum_{m\geqslant 1} \tfrac{1}{m!}\; q_m \big( P_{u^k\, x^l} (\Psi(x,z), \Phi(x,z), z) \big)_{k+l\leqslant m-1} (u - \Psi(x,z))^m\,. \tag{4.4}$$

By comparing the formal series $\frac{\partial\, \widetilde{\mathbf{x}}}{\partial\, u}$ with $P(u, \widetilde{\mathbf{x}}(u,x,z), z)$, it is possible to prove that $\widetilde{\mathbf{x}}$ is a solution of the formal differential equation (4.1) intersecting $\widehat{\Gamma}$.

■

**Remark 4.1.2.** Later we shall use formal differential equations induced by elements from $\mathbb{C}[[u-u_0, x-x_0, z-z_0]]$, for $(u_0, x_0, z_0) \in \mathbb{C}^3$. Below we shall define such formal equations and their solutions. The existence and uniqueness of their solutions will be obtained by Lemma 4.1.1, except for a *translation.*

We consider a formal series $\widetilde{P}(u,x,z) \in \mathbb{C}[[u-u_0, x-x_0, z-z_0]]$. This formal series induces a *nonautonomous formal differential equation with time $u$, parameter $z$ and base point* $(u_0, x_0, z_0)$ which will be represented as

$$\frac{\mathrm{d}\, x}{\mathrm{d}\, u} = \widetilde{P}(u,x,z)\,. \tag{4.5}$$

More specifically, a formal series $\widetilde{\mathbf{x}}(u,x,z) \in \mathbb{C}[[u-u_0, x-x_0, z-z_0]]$ will be a solution of the formal differential equation (4.5) if the following formal equality is satisfied

$$\frac{\mathrm{d}\,\widetilde{\mathbf{x}}}{\mathrm{d}\,u} = \widetilde{P}(u, \widetilde{\mathbf{x}}(u, x, z), z)\,.$$

Given a family of formal curves $\widetilde{\Gamma} = \big\{(u, x) = \big(u_0 + \Psi(x, z), x_0 + \Phi(x, z)\big)\big\}$, where $\Psi(x, z)$ and $\Phi(x, z)$ are formal series in $\mathbb{C}[[x - x_0, z - z_0]]$ with zero constant term, we shall say that the solution $\widetilde{\mathbf{x}}(u, x, z)$ of the formal differential equation (4.5) intersects $\widetilde{\Gamma}$ if

$$x_0 + \Phi(x, z) = \widetilde{\mathbf{x}}(u_0 + \Psi(x, z), x, z)\,.$$

We have existence and uniqueness of solutions of formal differential equations with base point $(0, 0, 0)$ by Lemma 4.1.1. This assertion can be extended to the general case, since a formal series $\widetilde{\mathbf{x}}(u, x, z)$ is a solution of the formal differential equation (4.5) intersecting $\widetilde{\Gamma}$ if and only if the formal series $\widetilde{\mathbf{x}}(u + u_0, x + x_0, z + z_0) - x_0 \in \mathbb{C}[[u, x, z]]$ is a solution of the formal differential equation with base point $(0, 0, 0)$

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = \widetilde{P}(u + u_0, x + x_0, z + z_0)\,,$$

intersecting the family of formal curves

$$\big\{(u, x) = \big(\Psi(x + x_0, z + z_0), \Phi(x + x_0, z + z_0)\big)\big\}\,,$$

where the series $\Psi(x + x_0, z + z_0), \Phi(x + x_0, z + z_0)$ are elements from $\mathbb{C}[[x, z]]$ with zero constant term.

## 4.2 Correspondence Maps on Formal Curves

We consider a nonautonomous differential equation

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = Q(u, x)\,, \tag{4.6}$$

where $Q(u, x)$ is a holomorphic function defined in an open neighborhood $\widetilde{U} \subseteq \mathbb{C}^2$ of $(u_0, x_0)$. Let $\widehat{\Gamma} = \big\{(u, x) = \big(u_0 + \Psi(x), x_0 + \Phi(x)\big)\big\}$ be a formal curve, where $\Psi(x) = \sum_{k \geqslant 1} \psi_k\,(x - x_0)^k$, $\Phi(x) = \sum_{k \geqslant 1} \phi_k\,(x - x_0)^k \in \mathbb{C}[[x - x_0]]$.

We will say that a formal series $\mathbf{x}(u, x) \in \mathbb{C}[[u - u_0, x - x_0]]$ is a *solution of the differential equation (4.6) intersecting* $\widehat{\Gamma}$ if it satisfies

$$\frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,u} = Q(u, \mathbf{x}(u, x)) \qquad \text{and} \qquad x_0 + \Phi(x) = \mathbf{x}(u_0 + \Psi(x), x)\,.$$

***Lemma* 4.2.1.** *There exists a unique solution* $\mathbf{x}(u,x) = \sum_{k\geqslant 0} c_k(u)(x \quad x_0)^k$ *of the holomorphic differential equation (4.6) intersecting* $\widehat{\ }$. *Moreover, there exists* $U \subseteq \mathbb{C}$ *an open neighborhood of* $u_0$ *where* $c_k$ *is convergent for all* $k \geqslant 1$.

**Proof**. On the one hand, the existence and the uniqueness are obtained by Remark 4.1.2. On the other hand, the additional requirement will be a consequence of the following geometric construction of $\mathbf{x}$.

Since the differential equation (4.6) is holomorphic, there exist $U, V \subseteq \mathbb{C}$ open discs centered at $u_0$ and $x_0$, respectively, such that

$$\begin{array}{ccc} U^2 \times V & \xrightarrow{\ \mathtt{X}\ } & \mathbb{C} \\ (u, \widetilde{u}, x) & \longmapsto & \mathtt{X}(u, \widetilde{u}; x) \end{array}$$

is a well defined holomorphic function, where $u \mapsto \mathtt{X}(u, \widetilde{u}; x)$ is a solution of the equation (4.6) passing through $x$ for the time $\widetilde{u}$.

Observe that $\mathtt{X}(u, u_0 + \Psi(x); x_0 + \Phi(x))$ is a formal series in the variable $x \quad x_0$ whose coefficientes are holomorphic functions defined in $U$. Moreover, $\mathtt{X}$ is a solution of the differential equation (4.6) intersecting $\widehat{\ }$. As a consequence of the uniqueness it must be $\mathbf{x}(u,x)$. ∎

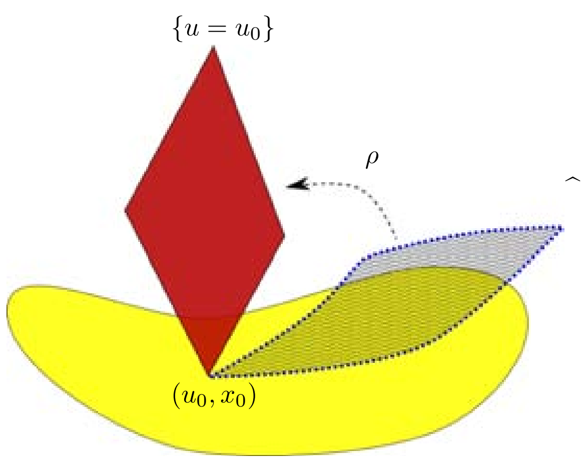


Figure 4.2: The shift from $\widehat{\ }$ to $\{u = u_0\}$.

From now on we will consider $Q(u, x_0) \equiv 0$; as a consequence $\{x = x_0\}$ will be an invariant curve of the equation (4.6). Moreover, we shall suppose that $\phi_1$, the coefficient of the linear part of the series $\Phi(x)$, is nonzero; we shall say that $\widehat{\ }$ *is a formal cross-section to* $\{x = x_0\}$ *at the point* $(u_0, x_0)$.

Under these hypothesis, if $\mathbf{x}(u,x) = \sum_{k\geqslant 0} c_k(u)(x \quad x_0)^k$ is the solution of the holomorphic differential equation (4.6) intersecting $\widehat{\ }$, we can conclude that $c_0(u) \equiv x_0$ and $c_1(u_0)$ is nonzero. As a consequence $\rho(x) := \mathbf{x}(u_0, x)$ is an invertible formal transformation. In what follows $\rho$ will be called *the shift from the formal cross-section* $\widehat{\quad}$ *to the line* $\{u = u_0\}$ *with respect to the differential equation* (4.6). To emphasize this shift we shall write $\rho : \widehat{\quad} \to \{u = u_0\}$ as is usually done for functions (see Figure 4.2).

***Definition* 4.2.2** (Correspondence maps on formal curves)**.** We consider numbers $w_1, w_2 \in \mathbb{C}$ such that $(w_1, x_0), (w_2, x_0)$ belong to the domain $\widetilde{U}$ of $Q$. For $i = 1, 2$ we choose a formal cross-section with base point $(w_i, x_0)$

$$\widehat{\quad}_i = \left\{ \ w_i + \Psi_i(x), x_0 + \Phi_i(x)\big) \right\},$$

where $\Psi_i(x) = \sum_{k\geqslant 1} \psi_{i,k} (x \quad x_0)^k$, $\Phi_i(x) = \sum_{k\geqslant 1} \phi_{i,k} (x \quad x_0)^k$, with $\phi_{i,1} \neq 0$, are formal series in $\mathbb{C}[[x \quad x_0]]$. We shall denote by $\rho_i : \widehat{\quad}_i \to \quad_i$ the shift from the formal cross-section $\widehat{\quad}_i$ to the line $\quad_i = \{u = w_i\}$.

Given a path $\tau$ connecting $(w_1, x_0)$ with $(w_2, x_0)$ and whose image is contained in $\{x = x_0\} \cap \widetilde{U}$, we shall denote by $\Delta_\tau : \quad_1 \to \quad_2$ the correspondence map for the equation (4.6) over the path $\tau$.

We define *the correspondence map over the path* $\tau$ *from the formal cross-section* $\widehat{\quad}_1$ *to the formal cross-section* $\widehat{\quad}_2$ as the composition $\rho_2^{\ 1} \circ \Delta_\tau \circ \rho_1 =: \widehat{\Delta}_\tau$. In what follows it will be denoted by $\widehat{\Delta}_\tau : \widehat{\quad}_1 \to \widehat{\quad}_2$ (see Figure 4.3).

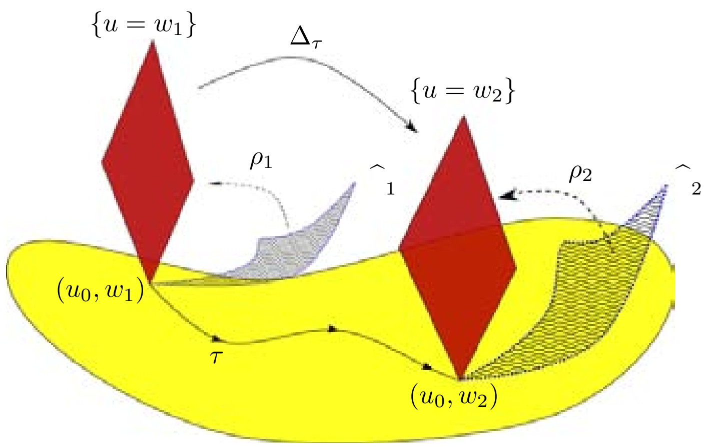


Figure 4.3: The Correspondence Map.

**Remark 4.2.3.** We consider $\widehat{\mathbf{x}}(u, x) = x_0 + \sum_{k\geqslant 1} \widehat{c}_k\,(u)(x \quad x_0)^k$ the solution of the differential equation (4.6) intersecting $\widehat{\quad}_1$, and an open disc $\widehat{U}$ centered at $w_1$ such that $\widehat{c}_k$ is convergent in $\widehat{U}$ for every $k \geqslant 1$. From Lemma 4.2.1 and its proof we can choose $\widehat{U}$ such that there exists an open disc $\widehat{V}$ centered at $x_0$ satisfying that $\widetilde{\mathtt{X}}$

$$\begin{array}{rcl} \widehat{U}^2 \times \widehat{V} & \xrightarrow{\widetilde{\mathtt{X}}} & \mathbb{C} \\ (u, \widetilde{u}, x) & \longmapsto & \widetilde{\mathtt{X}}(u, \widetilde{u}; x) \end{array}$$

is a well defined holomorphic function, where $u \mapsto \widetilde{\mathtt{X}}(u, \widetilde{u}; x)$ is the solution of the equation (4.6) passing through $x$ at the time $\widetilde{u}$. By uniqueness $\widehat{\mathbf{x}}(u, x) = \widetilde{\mathtt{X}}(u, w_1 + \Psi_1(x); x_0 + \Phi_1(x))$ (Lemma 4.2.1).

Now we shall consider $w_2 \in \widehat{U}$ and $\Psi_2(x) = \sum_{k \geqslant 1} \psi_{i,k}(x - x_0)^k \in \mathbb{C}[[x - x_0]]$ as in Definition 4.2.2. Then we obtain a new formal cross-section with base point $(w_2, x_0)$,

$$\widehat{\widehat{\Gamma}}_2 = \left\{ \big(w_2 + \Psi_2(x), \widehat{\mathbf{x}}(w_2 + \Psi_2(x), x)\big) \right\}.$$

We will prove that in this case, the correspondence map over $\tau$, beginning at $\widehat{\Gamma}_1$ and ending at $\widehat{\widehat{\Gamma}}_2$, is the identity map. This is equivalent to $\Delta_\tau \circ \rho_1 = \widehat{\rho}_2$, where $\widehat{\rho}_2 : \widehat{\widehat{\Gamma}}_2 \longrightarrow \Gamma_2$ is the shift from $\widehat{\widehat{\Gamma}}_2$ to the line $\Gamma_2$.

We have by definition

$$\begin{aligned} \rho_1(x) &= \widehat{\mathbf{x}}(w_1, x) = \widetilde{\mathtt{X}}\big(w_1, w_1 + \Psi_1(x); x_0 + \Phi_1(x)\big), \\ \widehat{\rho}_2(x) &= \widetilde{\mathtt{X}}\big(w_2, w_2 + \Psi_2(x); \widehat{\mathbf{x}}(w_2 + \Psi_2(x), x)\big), \\ \Delta_\tau(x) &= \widetilde{\mathtt{X}}(w_2, w_1; x) \end{aligned}$$

By uniqueness we obtain the following equalities:

$$\begin{aligned} &\widetilde{\mathtt{X}}\big(u, w_2 + \Psi_2(x); \widehat{\mathbf{x}}(w_2 + \Psi_2(x), x)\big) \\ =&\widetilde{\mathtt{X}}\Big(u, w_2 + \Psi_2(x); \widetilde{\mathtt{X}}\big(w_2 + \Psi_2(x), w_1 + \Psi_1(x); x_0 + \Phi_1(x)\big)\Big) \\ =&\widetilde{\mathtt{X}}\big(u, w_1 + \Psi_1(x); x_0 + \Phi_1(x)\big), \end{aligned}$$

and moreover,

$$\begin{aligned} \widetilde{\mathtt{X}}\big(u, w_1; \widehat{\mathbf{x}}(w_1, x)\big) =& \widetilde{\mathtt{X}}\Big(u, w_1; \widetilde{\mathtt{X}}\big(w_1, w_1 + \Psi_1(x); x_0 + \Phi_1(x)\big)\Big) \\ =&\widetilde{\mathtt{X}}\big(u, w_1 + \Psi_1(x); x_0 + \Phi_1(x)\big), \end{aligned}$$

Therefore, the assertion is concluded considering $u = w_2$.

## 4.3 Holonomy Maps on Formal Curves

From Remark 4.2.3 we will now obtain an expression of a correspondence map over a loop (that is, of a *holonomy map*) on formal curves.

In what follows we shall use the notation introduced in above. We consider $\gamma : [0,1] \to \widetilde{U}$ a loop with base point $(u_0, x_0)$ whose image is contained in the line $\{x = x_0\}$. Given a partition of $[0,1]$, $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$, if we consider $\gamma(t_j) = (u_j, x_0)$ for $0 \leqslant j \leqslant k$, there exist open discs $U_j, V_j \subseteq \mathbb{C}$ centered at $u_j$ and $x_0$ respectively, which satisfies

$$\begin{array}{ccc} U_j^2 \times V_j & \xrightarrow{\mathtt{X}_j} & \mathbb{C} \\ (u, \widetilde{u}, x) & \longmapsto & \mathtt{X}_j(u, \widetilde{u}; x) \end{array}$$

is a well defined holomorphic function, where $u \mapsto \mathtt{X}_j(u, \widetilde{u}; x)$ is the solution of the differential equation (4.6) passing through $x$ at the time $\widetilde{u}$. We choose this partition in such a way that $\gamma(t_{j+1}) \in U_j \times V_j$ for $0 \leqslant j < k$.

Given $0 \leqslant j < k$ and $\gamma_j = \gamma|_{[t_j, t_{j+1}]}$, we shall denote by $\Delta_{\gamma_j} : \ _j \to \ _{j+1}$ the correspondence map of the equation (4.6) over $\gamma_j$, where $\ _l := \{u = u_l\}$ for $0 \leqslant l \leqslant k$ (see Figure 4.4).

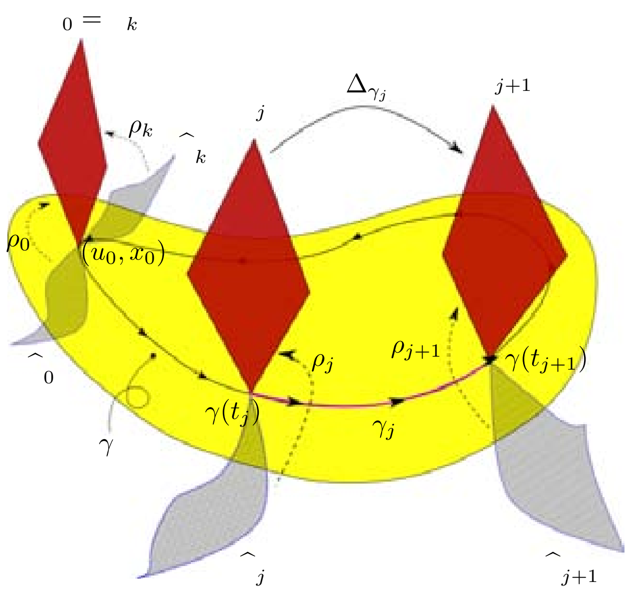


Figure 4.4: The Holonomy Map.

For each $0 \leqslant j \leqslant k$ we consider a formal series $\Psi_j(x) \in \mathbb{C}[[x - x_0]]$ with zero constant term. We shall recursively define a family of invertible formal series with zero constant term. We begin with an arbitrary series satisfying these properties $\Phi_0(x) \in \mathbb{C}[[x-x_0]]$. For $0 \leqslant j < k$ we define

$$\Phi_{j+1}(x) = \mathtt{X}_j(u_{j+1} + \Psi_{j+1}(x), u_j + \Psi_j(x); x_0 + \Phi_j(x)) - x_0 \in \mathbb{C}[[x-x_0]]\,,$$

an invertible formal series with zero constant term.

For $0 \leqslant j \leqslant k$ we define a formal cross-section with base point $\gamma(t_j)$

$$\widehat{\Gamma}_j = \big\{\big(u_j + \Psi_j(x), x_0 + \Phi_j(x)\big)\big\}\,.$$

The formal cross-section $\widehat{\Gamma}_0$ was arbitrarily chosen and the others were constructed from this. We shall denote by $\rho_j : \widehat{\Gamma}_j \longrightarrow \Gamma_j$ the shift from the formal cross-section $\widehat{\Gamma}_j$ to the line $\Gamma_j$ (see Figure 4.4). By the Remark 4.2.3, for every $0 \leqslant j < k$ we obtain:

$$\Delta_{\gamma_j} \circ \rho_j = \rho_{j+1}\,.$$

By induction we conclude that for every $0 \leqslant j < k$

$$\Delta_{\gamma_j} \circ \cdots \circ \Delta_{\gamma_0} \circ \rho_0 = \rho_{j+1}\,,$$

As a consequence, the holomorphic function $\Delta_\gamma = \Delta_{\gamma_j} \circ \cdots \circ \Delta_{\gamma_0}$, being the holonomy map over $\gamma$ on the line $\Gamma_0$, satisfies $\Delta_\gamma \circ \rho_0 = \rho_k$, this is,

$$\Delta_\gamma\big(\mathtt{X}_0\big(u_0, u_0 + \Psi_0(x); x_0 + \Phi_0(x)\big)\big) = \mathtt{X}_0\big(u_0, u_0 + \Psi_k(x); x_0 + \Phi_k(x)\big)\,.$$

Therefore the holonomy map over $\gamma$ on the formal cross-section $\widehat{\Gamma}_0$ is

$$\rho_0^{-1} \circ \Delta_\gamma \circ \rho_0 = \rho_0^{-1}\rho_k\,.$$

## 4.4 Formal Conjugation of the Vanishing Holonomy Group of a Holomorphic Foliation

In this section we shall prove Lemma 4.4.1 which states that the holonomy groups of two germs of nondicritic holomorphic foliations on $(\mathbb{C}^2, \widehat{0})$ being strictly formally equivalent, are *formally strongly conjugate*. Moreover, we shall conclude that the conjugacy has real coefficients whenever the germs of holomorphic foliations are the complexifications of real analytic germs and the strict formal equivalence between them has real coefficients (Remark 4.4.3).

We consider $\nu_1, \nu_2$ germs of holomorphic vector fields at the origin of $\mathbb{C}^2$ with an isolated singularity at $\widehat{0}$. Given $i = 1, 2$, the Taylor series of $\nu_i$ around the origin is

$$\big(P_n^{(i)} + P_{n+1}^{(i)} + \cdots,\, Q_n^{(i)} + Q_{n+1}^{(i)} + \cdots\big)\,,$$

where $P_j^{(i)}, Q_j^{(i)}$ are homogeneous polynomials of degree $j \geqslant n$. We suppose that $\nu_1, \nu_2$ are *strictly formally orbitally equivalent*, that is, there exist $K \in \mathbb{C}[[x, y]]$ with constant term 1 and $H \in (\mathbb{C}[[x, y]])^2$ such that $H(\widehat{0}) = \widehat{0}$ and whose linear part is the identity matrix, satisfying

$$\big(\mathrm{D}\, H \cdot \nu_1\big)\circ H^{-1}(x, y) := \mathrm{D}\, H_{(H^{-1}(x,y))}\, \nu_1(H^{-1}(x, y)) = K(x, y)\, \nu_2(x, y)\,.$$

We will write $H = (H_1, H_2) = \big(x + H_{1,2} + H_{1,3} + \cdots,\, y + H_{2,2} + H_{2,3} + \cdots\big)$. For $i = 1, 2$, we shall denote by $\mathscr{F}_{\nu_i}$ the foliation induced by $\nu_i$ in a neighborhood of $\widehat{0}$ and by $\widetilde{\mathscr{F}_{\nu_i}}$ the desingularization of $\mathscr{F}_{\nu_i}$ on $(\mathbb{M}_{\mathbb{C}}, \mathbb{D}_{\mathbb{C}})$.

We shall suppose that $\nu_1$ is nondicritical, that is, $xQ_n^{(1)}(x, y) - yP_n^{(1)}(x, y)$ is nonzero. By the strict formal equivalence it follows that $P_n^{(1)} = P_n^{(2)}$ and $Q_n^{(1)} = Q_n^{(2)}$, and as a consequence $\nu_2$ is also nondicritical. Besides, the singular points of the foliations $\widetilde{\mathscr{F}_{\nu_1}}$ and $\widetilde{\mathscr{F}_{\nu_2}}$ on $\mathbb{D}_{\mathbb{C}}$ are the same; they will be denoted by $\mathfrak{d}_1, \ldots, \mathfrak{d}_m$. We choose a point $\mathfrak{u}_0 \in \mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{d}_1, \ldots, \mathfrak{d}_m\}$ belonging to the coordinate chart $(x, u = y/x)$, where its expression is $(0, u_0)$. The line with base point $\mathfrak{u}_0$ is written $\Gamma_0 = \{u = u_0\}$ in this coordinates and is transversal to the divisor $\mathbb{D}_{\mathbb{C}}$. Let $\gamma$ be a loop with base point $\mathfrak{u}_0$ and whose image is contained in $\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{d}_1, \ldots, \mathfrak{d}_m\}$, we shall denote by $\Delta_\gamma^{(i)} : \big(\Gamma_0, \mathfrak{u}_0\big) \longrightarrow \big(\Gamma_0, \mathfrak{u}_0\big)$ the holonomy map for the foliation $\widetilde{\mathscr{F}_{\nu_i}}$ over $\gamma$, for $i = 1, 2$.

In what follows we will prove that there exists $h(x) \in \mathbb{C}[[x]]$ an invertible formal transformation satisfying $h \circ \Delta_\gamma^{(1)} \circ h^{-1} = \Delta_\gamma^{(2)}$ for every loop $\gamma$ with base point $\mathfrak{u}_0$ and whose image is contained in $\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{d}_1, \ldots, \mathfrak{d}_m\}$ (see the equality (4.7)).

In the coordinates $(x, u)$ the foliation $\widetilde{\mathscr{F}_{\nu_i}}$ is generated by the holomorphic vector field $\widetilde{\nu}_i(x, u) = \big(\widetilde{\nu}_{i,1}(x, u), \widetilde{\nu}_{i,2}(x, u)\big)$ defined as

$$\Big(x(P_n^{(i)}(1, u) + xP_{n+1}^{(i)}(1, u) + \cdots),\, R_{n+1}^{(i)}(1, u) + xR_{n+2}^{(i)}(1, u) + \cdots\Big)\,,$$

where $R_{n+j+1}^{(i)}(x, y) = xQ_{n+j}^{(i)}(x, y) - yP_{n+j}^{(i)}(x, y)$ for $j \geqslant 0$.

We shall denote the desingularization of $H$ by $\widetilde{H} = \big(\widetilde{H}_1, \widetilde{H}_2\big)$. In the coordinates $(x, u)$ its expression is

$$\begin{aligned}\big(\widetilde{H}_1(x,u),\widetilde{H}_2(x,u)\big) &= \left(H_1(x,xu),\frac{H_2(x,xu)}{H_1(x,xu)}\right)\\ &= \left(x+x^2H_{1,2}(1,u)+\cdots\,,\ \frac{u+xH_{2,2}(1,u)+\cdots}{1+xH_{1,2}(1,u)+\cdots}\right),\end{aligned}$$

as a consequence $\widetilde{H}_1(x,u)$, $\widetilde{H}_2(x,u)$ are formal in $x$ with polynomial coefficients in $u$. Since $H$ conjugates $\nu_1$ and $\nu_2$, it follows that $\big(\mathrm{D}\,\widetilde{H}\cdot\widetilde{\nu}_1\big)\circ\widetilde{H}^{-1}(x,u) = \widetilde{K}(x,u)\,\widetilde{\nu}_2(x,u)$, where $\widetilde{K}(x,u) := K(x,xu)$.

Now we will study the holonomy groups of $\widetilde{\mathscr{F}}_{\nu_1}$ and $\widetilde{\mathscr{F}}_{\nu_2}$. For $i=1,2$ we shall define the nonautonomous equation

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = \frac{\widetilde{\nu}_{i,1}(x,u)}{\widetilde{\nu}_{i,2}(x,u)} = \sum_{k\geqslant 1} S_{i,k}(u)x^k = P_i(x,u)\,, \qquad (*_i)$$

where $S_{i,k}$ is a rational function in $u$ with poles at the roots of the polynomial $R_{n+1}(1,u)$. We consider the formal cross-section with base point $\mathfrak{u}_0$, $\widehat{\Gamma}_0 := \widetilde{H}(\Gamma_0) = \big\{\big(\widetilde{H}_1(x,u_0),\widetilde{H}_2(x,u_0)\big)\big\}$.

***Lemma 4.4.1.*** *The holonomy groups of $\widetilde{\mathscr{F}}_{\nu_1}$ and $\widetilde{\mathscr{F}}_{\nu_2}$ on $\big(\Gamma_0,\mathfrak{u}_0\big)$ are formally strongly conjugated by $\rho_0:\widehat{\Gamma}_0\longrightarrow\Gamma_0$, the shift from $\widehat{\Gamma}_0$ to $\Gamma_0$ with respect to the differential equation $(*_2)$. That is,*

$$\rho_0\circ\Delta_\gamma^{(1)}\circ\rho_0^{-1} = \Delta_\gamma^{(2)}\,, \qquad (4.7)$$

*for any loop $\gamma$ contained in $\mathbb{D}_{\mathbb{C}}\smallsetminus\{\mathfrak{d}_1,\ldots,\mathfrak{d}_m\}$ with base point $\mathfrak{u}_0$ .*

To prove the equality (4.7) it will be necessary to give an explicit description of the relation between the solutions of the equations $(*_1)$ and $(*_2)$ (Lemma 4.4.2).

To give such a description we first choose a nonsingular point $\widetilde{\mathfrak{u}}_0\in\mathbb{D}_{\mathbb{C}}$ which belongs to the domain of the coordinate chart $(x,u)$ and whose expression in this coordinates is $(0,\widetilde{u}_0)$. Without loss of generality we can choose open discs $U,V\subseteq\mathbb{C}$ centered at $\widetilde{u}_0$ and $0$ respectively, satisfying the product $V\times U$ is contained in the domains of $P_1$ and $P_2$, and

$$\begin{array}{rcl} U^2\times V & \xrightarrow{\ \mathtt{X}^{(i)}\ } & \mathbb{C}\\ (u,\widetilde{u},x) & \longmapsto & \mathtt{X}^{(i)}(u,\widetilde{u};x)\end{array}$$

is a well defined holomorphic function, where $u\mapsto\mathtt{X}^{(i)}(u,\widetilde{u};x)$ is the solution of the differential equation $(*_i)$ passing through $x$ at the time $\widetilde{u}$.

We consider $\Gamma = \big\{(x,u) = \big(\Phi(x), \widetilde{u}_0 + \Psi(x)\big)\big\}$ a cross-section to $\{x = 0\}$ at the base point $(0, \widetilde{u}_0)$ (it can be either holomorphic or formal). We consider $\mathbf{x}(u,x) := \mathtt{X}^{(1)}\big(u, \widetilde{u}_0 + \Psi(x); \Phi(x)\big)$ the solution of the equation $(*_1)$ intersecting $\Gamma$.

On the other hand, we take the formal cross-section $\widehat{\Gamma} = \widetilde{H}(\Gamma)$ with base point $u_0$. Then

$$\widetilde{\mathbf{x}}(u,x) := \mathtt{X}^{(2)}\big(u,\ \widetilde{H}_2\big(\Phi(x), \widetilde{u}_0 + \Psi(x)\big);\ \widetilde{H}_1\big(\Phi(x), \widetilde{u}_0 + \Psi(x)\big)\big)$$

is the solution of the equation $(*_2)$ intersecting $\widehat{\Gamma}$.

***Lemma* 4.4.2.**

$$\widetilde{H}_1(\mathbf{x}(u,x),u) = \widetilde{\mathbf{x}}\big(\widetilde{H}_2(\mathbf{x}(u,x),u),x\big)\,.$$

**Proof**. To conclude this assertion we will prove that both sides of the equality are solutions of a nonautonomous formal differential equation depending on the parameter $x$ and intersecting the same formal cross-section.

We have $\big(\mathrm{D}\,\widetilde{H}\cdot\widetilde{\nu}_1\big)\circ\widetilde{H}^{-1}(x,u) = \widetilde{K}(x,u)\,\widetilde{\nu}_2(x,u)$. This is equivalent to the following equality

$$\begin{pmatrix}\widetilde{H}_{1,x} & \widetilde{H}_{1,u}\\ \widetilde{H}_{2,x} & \widetilde{H}_{2,u}\end{pmatrix}\begin{pmatrix}P_1\\ 1\end{pmatrix} = \frac{\widetilde{K}\circ\widetilde{H}}{\widetilde{\nu}_{1,2}}\ \widetilde{\nu}_{2,2}\circ\widetilde{H}\ \begin{pmatrix}P_2\circ\widetilde{H}\\ 1\end{pmatrix}.$$

By the above expression and the chain rule we obtain the following equalities

$$\frac{\partial\,\widetilde{H}_1(\mathbf{x}(u,x),u)}{\partial\,u}\bigg|_{(x,u)} = \big(\widetilde{H}_{1,x},\widetilde{H}_{1,u}\big)\Big|_{(\mathbf{x}(u,x),u)}\big(P_1(\mathbf{x}(u,x),u),\ 1\big)$$
$$= \left(\frac{\widetilde{K}\circ\widetilde{H}}{\widetilde{\nu}_{1,2}}\ \widetilde{\nu}_{2,2}\circ\widetilde{H}\right)\bigg|_{(\mathbf{x}(u,x),u)} P_2\big(\widetilde{H}_1(\mathbf{x}(u,x),u),\widetilde{H}_2(\mathbf{x}(u,x),u)\big)\,.$$

$$\frac{\partial\,\widetilde{\mathbf{x}}\big(\widetilde{H}_2(\mathbf{x}(u,x),u),x\big)}{\partial\,u}\bigg|_{(x,u)}$$
$$= \frac{\partial\,\widetilde{\mathbf{x}}}{\partial\,u}\bigg|_{\left(\widetilde{H}_2(\mathbf{x}(u,x),u),x\right)}\big(\widetilde{H}_{2,x},\ \widetilde{H}_{2,u}\big)\Big|_{(\mathbf{x}(u,x),u)}\big(P_1(\mathbf{x}(u,x),u),\ 1\big)$$
$$= \left(\frac{\widetilde{K}\circ\widetilde{H}}{\widetilde{\nu}_{1,2}}\ \widetilde{\nu}_{2,2}\circ\widetilde{H}\right)\bigg|_{(\mathbf{x}(u,x),u)} P_2\big(\widetilde{\mathbf{x}}\big(\widetilde{H}_2(\mathbf{x}(u,x),u),x\big),\ \widetilde{H}_2(\mathbf{x}(u,x),u)\big)\,.$$

Therefore $\widetilde{H}_1(\mathbf{x}(u,x),u)$ and $\widetilde{\mathbf{x}}(\widetilde{H}_2(\mathbf{x}(u,x),u),x)$ are solutions of the nonautonomous formal differential equation with parameter $x$

$$\frac{\mathrm{d}\,w}{\mathrm{d}\,u} = \left(\frac{\widetilde{K}\circ\widetilde{H}}{\widetilde{\nu}_{1,2}}\;\widetilde{\nu}_{2,2}\circ\widetilde{H}\right)\Bigg|_{(\mathbf{x}(u,x),u)} P_2\big(w,\ \widetilde{H}_2(\mathbf{x}(u,x),u)\big)\,.$$

Besides, by definition both series intersect the formal cross-section $\big\{(u,w) = \big(u_0+\Psi(x),\ \widetilde{H}_1(\Phi(x),u_0+\Psi(x))\big)\big\}$. By Lemma 4.1.1 they must coincide.

■

**Proof of Lemma 4.4.1.** Given the invariance of the holonomy maps over homotopic loops, we stress that to prove the equality (4.7) it is sufficient to take loops whose images are contained in the domain of the coordinate chart $(x,u)$ free from singular points. In this way we consider $\gamma$ a loop with base point $\mathfrak{u}_0$ whose image is contained in the domain of the coordinate chart $(x,u)$ free from singular points.

Recall that the holonomy map of the foliation $\widetilde{\mathscr{F}_{\nu_i}}$ over $\gamma$ defined on $\big(\Gamma_0,\mathfrak{u}_0\big)$, denoted by $\Delta_\gamma^{(i)}$, is the same that the holonomy map of the equation $(*_i)$ over $\gamma$ defined on $\big(\Gamma_0,\mathfrak{u}_0\big)$.

Without loss of generality we shall suppose that the domain of $\gamma$ is the interval $[0,1]$. Given a partition $0=t_0<t_1<\cdots<t_{k-1}<t_k=1$, if we consider $\gamma(t_j)=(0,u_j)$ seen from the coordinates $(x,u)$, for $0\leqslant j\leqslant k$ there exist open discs $U_j, V_j\subseteq\mathbb{C}$ centered at $u_j$ and 0 respectively, such that $V_j\times U_j\subseteq \operatorname{dom}P_1$, $\operatorname{dom}P_2$ and, for $i=1,2$,

$$\begin{array}{rcl} U_j^2\times V_j & \xrightarrow{\ \mathbf{x}_j^{(i)}\ } & \mathbb{C} \\ (u,\widetilde{u},x) & \longmapsto & \mathbf{x}_j^{(i)}(u,\widetilde{u};x) \end{array}\qquad,$$

is a well defined holomorphic function, where $u\mapsto \mathbf{x}_j^{(i)}(u,\widetilde{u};x)$ is the solution of the differential equation $(*_i)$ passing through $x$ at the time $\widetilde{u}$. We choose this partition in such a way that $\gamma(t_{j+1})\in V_j\times U_j$ for $0\leqslant j<k$.

Given $0\leqslant j<k$ and $\gamma_j=\gamma|_{[t_j,t_{j+1}]}$, we shall denote by

$$\Delta_{\gamma_j}^{(i)} : \big(\Gamma_j,\gamma(t_j)\big)\longrightarrow\big(\Gamma_{j+1},\gamma(t_{j+1})\big)$$

the holonomy map of the equation $(*_i)$ over the loop $\gamma_j$ , where $\Gamma_l := \{u=u_l\}$ for $0\leqslant l\leqslant k$ (see Figure 4.5).

We choose a *special* parametrization of $\Gamma_j$, for $0\leqslant j<k$. We recursively define: $\Phi_0^{(1)}(x)=x$; for $0\leqslant j<k$

$$\Phi_{j+1}^{(1)}(x) = \mathbf{x}_j^{(1)}\big(u_{j+1},u_j;\Phi_j^{(1)}(x)\big)\,,$$

Thus we consider $\big\{\big(\Phi_j^{(1)}(x),u_j\big)\big\}$ the parametrization of $\Gamma_j$ for $0\leqslant j<k$. By

induction it follows that $\Phi^{(1)}_{j+1}(x) = \Delta^{(1)}_{\gamma_j} \circ \cdots \Delta^{(1)}_{\gamma_0}(x)$ for all $0 \leqslant j < k$. Then $\Phi^{(1)}_k(x) = \Delta^{(1)}_{\gamma}(x)$.

For $0 \leqslant j < k$ we consider

$$\widehat{\ }_j = \ \Phi^{(2)}_j(x), u_j + \Psi^{(2)}_j(x)) := \widetilde{H}(\ _j) = \{\ \widetilde{H}_1\ \Phi^{(1)}_j(x), u_j), \widetilde{H}_2\ \Phi^{(1)}_j(x), u_j))\},$$

a formal cross-section with base point $(0, u_j)$ and

$$\widehat{\ }_k = \{\ \widetilde{H}_1\ \Phi^{(1)}_k(x), u_0), \widetilde{H}_2\ \Phi^{(1)}_k(x), u_0))\}$$

another formal cross-section with base point $(0, u_0)$. For $0 \leqslant j \leqslant k$ we shall write $\rho_j : \widehat{\ }_j \ \rightarrow \ _j$, the shift from $\widehat{\ }_j$ to $\ _j$ with respect to the differential equation $(*_2)$. We denote by $\widehat{\Delta}^{(2)}_{\gamma}$ the holonomy map of the equation $(*_2)$ on the formal curve $\widehat{\ }_0$ (see Figure 4.5).

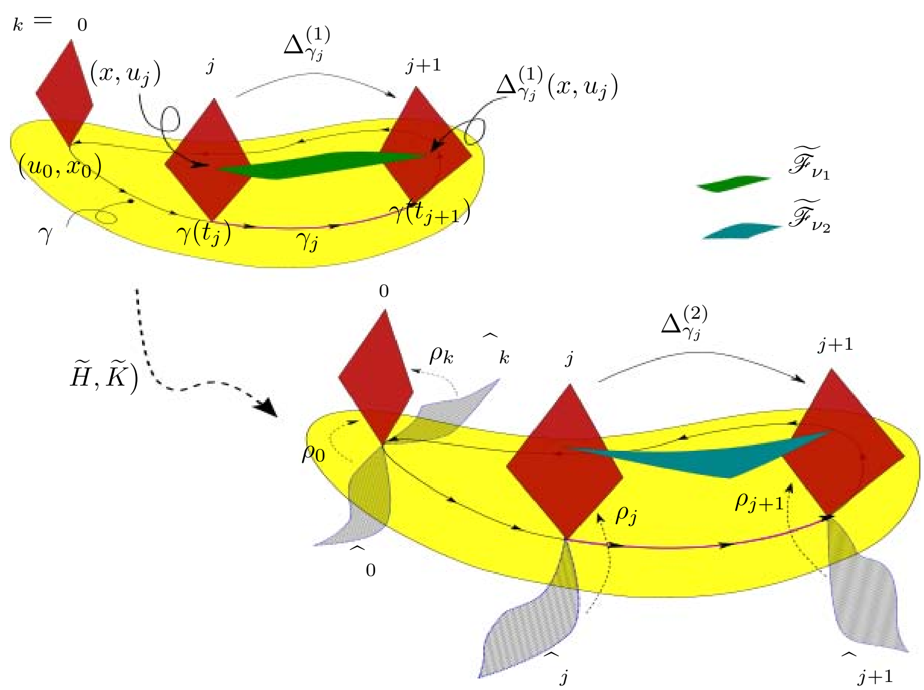


Figure 4.5: The Formal Conjugation of the Holonomy Groups.

It is important to note that, for $0 \leqslant j < k$,

$$\Phi^{(2)}_{j+1}(x) = \mathbf{x}^{(2)}_j\ \ u_{j+1} + \Psi^{(2)}_{j+1}(x), u_j + \Psi^{(2)}_j(x); \Phi^{(2)}_j(x)\big), \tag{4.8}$$

which is obtained from

$$\begin{aligned}&\widetilde{H}_1\big(\mathbf{x}_j^{(1)}\big(u,u_j;\Phi_j^{(1)}(x)\big),u\big)\\ &=\mathbf{x}_j^{(2)}\Big(\widetilde{H}_2\big(\mathbf{x}_j^{(1)}\big(u,u_j;\Phi_j^{(1)}(x)\big),u\big),\widetilde{H}_2\big(\Phi_j^{(1)}(x),u_j\big);\widetilde{H}_1\big(\Phi_j^{(1)}(x),u_j\big)\Big)\,,\end{aligned}$$

taking $u=u_{j+1}$, which in turn is a consequence of Lemma 4.4.2.

As it was shown in 4.3, for all $0\leqslant j<k$, $\Delta_{\gamma_j}^{(2)}\circ\rho_j=\rho_{j+1}$. Hence $\Delta_{\gamma}^{(2)}\circ\rho_0=\rho_k$. Therefore

$$\widehat{\Delta}_{\gamma}^{(2)}=\rho_0^{-1}\circ\Delta_{\gamma}^{(2)}\circ\rho_0=\rho_0^{-1}\circ\rho_k\,. \tag{4.9}$$

In what follows we will prove that $\Delta_{\gamma}^{(1)}=\rho_0^{-1}\circ\rho_k$ (this together with the equivalence (4.9) give rise to the equality (4.7)).

By definition

$$\begin{aligned}\rho_0(x)&=\mathbf{x}_0^{(2)}\big(u_0,\widetilde{H}_2(x,u_0);\widetilde{H}_1(x,u_0)\big)\,,\\ \rho_k(x)&=\mathbf{x}_0^{(2)}\big(u_0,\widetilde{H}_2(\Phi_k^{(1)}(x),u_0);\widetilde{H}_1(\Phi_k^{(1)}(x),u_0)\big)\,;\end{aligned}$$

since $\Phi_k^{(1)}(x)=\Delta_{\gamma}^{(1)}(x)$, it follows immediately that $\rho_0\circ\Delta_{\gamma}^{(1)}=\rho_k$.

■

**Remark 4.4.3.** Let $\nu_1$, $\nu_2$, $H$ and $K$ be as above, under the additional assumptions that $\nu_1$ and $\nu_2$ are the complexifications of real analytic vector fields defined on a neighborhood of $\widehat{0}\in\mathbb{R}^2$ and the transformations $H$ and $K$ realizing the strict formal orbital equivalence have real coefficients (that is, $H\in(\mathbb{R}[[x,y]])^2$ and $K\in\mathbb{R}[[x,y]]$). If $u_0\in\mathbb{R}$ then $\widehat{\Gamma}_0=\big\{\big(\widetilde{H}_1(x,u_0),\widetilde{H}_2(x,u_0)\big)\big\}$ is a formal cross-section with real coefficients and $\mathbf{x}_0^{(2)}$ is the complexification of a real analytic series. As a consequence the invertible formal series $\rho_0(x)=\mathbf{x}_0^{(2)}\big(u_0,\widetilde{H}_2(x,u_0);\widetilde{H}_1(x,u_0)\big)$ has real coefficients.

# Chapter 5

# The Auxiliary Foliation and the Polar Curves

In order to motivate the rest of the proof of Theorem 0.3.2, we shall make a brief digression. Let $\mathfrak{p}_0 \in \mathbb{D}_{\mathbb{R}}$ be a nonsingular point belonging to the chart $(x,u)$ and $\Gamma_{\mathfrak{p}_0}$ be the line passing through $\mathfrak{p}_0$, that is, with respect to this chart $\Gamma_{\mathfrak{p}_0} = \{u = p_0\}$, where $p_0 = u(\mathfrak{p}_0)$. By Theorems 2.2.1 and 2.2.2 there exists the germ of a real analytic diffeomorphism $h : (\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ whose complexification strongly conjugates the vanishing holonomy groups of $\widetilde{\mathscr{F}}_\nu^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_\omega^{\mathbb{C}}$ on $(\Gamma_{\mathfrak{p}_0}, \mathfrak{p}_0)$, that is, for all $\alpha \in \Pi_1(\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}, \mathfrak{p}_0)$

$$h^{\mathbb{C}} \circ \Delta_\alpha^\nu \circ \left(h^{\mathbb{C}}\right)^{-1} = \Delta_\alpha^\omega\,, \tag{5.1}$$

where $\Delta_\alpha^\nu, \Delta_\alpha^\omega : (\Gamma_{\mathfrak{p}_0}, \mathfrak{p}_0) \longrightarrow (\Gamma_{\mathfrak{p}_0}, \mathfrak{p}_0)$ are the holonomy maps for the foliations $\widetilde{\mathscr{F}}_\nu^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_\omega^{\mathbb{C}}$ over $\alpha$, respectively. If $\widetilde{\mathfrak{p}}_0 \in \mathbb{D}_{\mathbb{C}}$ is another nonsingular point and $\Gamma_{\widetilde{\mathfrak{p}}_0}$ is a cross-section to $\mathbb{D}_{\mathbb{C}}$ at $\widetilde{\mathfrak{p}}_0$, we denote in a similar way the correspondence maps over the paths on $\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}$ connecting $\mathfrak{p}_0$ and $\widetilde{\mathfrak{p}}_0$ which are defined between $\Gamma_{\mathfrak{p}_0}$ and $\Gamma_{\widetilde{\mathfrak{p}}_0}$.

By the equality 5.1, for any two paths $\tau, \delta$ on $\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}$, beginning at $\mathfrak{p}_0$ and ending at $\widetilde{\mathfrak{p}}_0$,

$$\Delta_\tau^\omega \circ h^{\mathbb{C}} \circ \Delta_{\tau^{-1}}^\nu = \Delta_\delta^\omega \circ h^{\mathbb{C}} \circ \Delta_{\delta^{-1}}^\nu\,. \tag{5.2}$$

If we obtained a dicritical holomorphic foliation in $(\mathbb{M}_{\mathbb{C}}, \mathbb{D}_{\mathbb{C}})$ such that the desingularization of the separatrices $S_i = \{y = \phi_i(x)\}$ are invariant subsets of this foliation and whose leaves cross the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$ transversally, then we would construct a biholomorphism far from the singular points defined by means of analytic continuation of the leaves of $\widetilde{\mathscr{F}}_\nu^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_\omega^{\mathbb{C}}$ along any path free from singularities, as it is expressed in the equation 5.2. Then we would need to holomorphically extend this biholomorphism to a neighborhood of the desingularization of each separatrix $S_i$ and to prove that the biholomorphism leaves invariant the real Möbius band. In this way we would conclude that the restriction of the blow-down of this transformation onto $\mathbb{R}^2$ conjugates the foliations $\mathscr{F}_\nu$ and $\mathscr{F}_\omega$. Unfortunately, the existence of such a dicritical holomorphic foliation in $(\mathbb{M}_\mathbb{C}, \mathbb{D}_\mathbb{C})$ is equivalent to the existence of a holomorphic change of coordinates which simultaneously rectifies the separatrices (transforms all separatrices into straight lines), but in general this is impossible (see [Gra79]).

This chapter is devoted to prove the existence of an *auxiliary foliation* (Lemma 5.1.1) which provides a way to extend the transversal conjugation obtained in Theorems 2.2.1 and 2.2.2. By the above arguments the transversality of the auxiliary foliation with the divisor $\mathbb{D}_\mathbb{C}$ cannot be achieved at all points. It happens that there will be a finite number of unavoidable points of tangency of the divisor with the auxiliary foliation. Even more, there is a locus of tangency (*polar curves*) between the original foliation and the auxiliary one, as is stated in Lemma 5.1.2. This locus of tangency is unavoidable as well.

## 5.1 Existence of the Auxiliary Foliation and the Polar Curves

Recall that $S_i = \{y = \phi_i(x)\}$ is the separatrix of $\nu^\mathbb{C}$ whose desingularization passes through $\mathfrak{p}_i$, for $1 \le j \le n+1$

***Lemma* 5.1.1.** *Let $\mathfrak{q}_1, \dots, \mathfrak{q}_{n-2} \in \mathbb{D}_\mathbb{R}$ be $n-2$ distinct nonsingular points belonging to the chart $(x,u)$. Then there exists the germ of a real analytic vector field $\nu_A = P_A \frac{\partial}{\partial x} + Q_A \frac{\partial}{\partial y}$ such that the blow-up of the foliation induced by $\nu_A^\mathbb{C}$ is a dicritical foliation on $(\mathbb{M}_\mathbb{C}, \mathbb{D}_\mathbb{C})$, which will be denoted by $\widetilde{\mathscr{F}}^\mathbb{C}$, satisfying the following properties:*

*(a) $\widetilde{\mathscr{F}}^\mathbb{C}$ does not have any singular point on $(\mathbb{M}_\mathbb{C}, \mathbb{D}_\mathbb{C})$.*

*(b) The leaves of $\widetilde{\mathscr{F}}^\mathbb{C}$ intersect the exceptional divisor $\mathbb{D}_\mathbb{C}$ transversally, except at the $n-2$ points $\mathfrak{q}_1, \dots, \mathfrak{q}_{n-2}$.*

*(c) For all $1 \le i \le n+1$ the desingularization of $S_i = \{y = \phi_i(x)\}$ is an invariant subset of $\widetilde{\mathscr{F}}^\mathbb{C}$.*

*(d) In the chart $(x,u)$, the foliation $\widetilde{\mathscr{F}}^\mathbb{C}$ around the point $\mathfrak{q}_j$ is described by the differential equation*

$$\frac{dx}{du} = f_j^\mathbb{C}(u) + g_j^\mathbb{C}(x,u)\,, \tag{5.3}$$

*where $f_j, g_j$ are real analytic functions defined on a neighborhood of $u(\mathfrak{q}_j) = q_j$ and $(0, q_j)$, respectively. Moreover, $g_j(x,u) = O(x)$ and $ord_{q_j}(f_j(u)) = 1$; that is, the leaf of the foliation $\widetilde{\mathscr{F}}^\mathbb{C}$ passing through $\mathfrak{q}_j$ has a simple tangency with $\mathbb{D}_\mathbb{C}$.*

**Proof**. We consider the real analytic function

$$R(x,y)=\prod_{i=1}^{n+1}(y-\phi_i(x))=R_{n+1}(x,y)+R_{n+2}(x,y)+\cdots\,,$$

where $R_i \in \mathbb{R}[x,y]$ is an homogeneous polynomial of degree $i$, for $i \geqslant n+1$.

The union of the curves $S_1,\ldots,S_{n+1}$ are invariant by a real analytic vector field $\widetilde{P}\frac{\partial}{\partial x}+\widetilde{Q}\frac{\partial}{\partial y}$ if and only if there exists a real analytic scalar function $C$ such that

$$\widetilde{P}R_x+\widetilde{Q}R_y=RC\,. \tag{5.4}$$

We shall prove now that there exist a dicritical real analytic vector field satisfying (5.4), together with the conditions a and b.

Namely, let $h(x,y)=\prod_{j=1}^{n-2}(y-q_jx)$, $C=(n+1)h$, $\widetilde{P}(x,y)=xh(x,y)+\widehat{P}(x,y)$ and $\widetilde{Q}(x,y)=yh(x,y)+\widehat{Q}(x,y)$. By substituing these expressions into the equation (5.4), we obtain

$$\widehat{P}R_x+\widehat{Q}R_y=h\big((n+1)R-xR_x-yR_y\big)\,. \tag{5.5}$$

The right hand side of the equation (5.5) has order greater or equal to $2n$. Then it belongs to $\mathcal{I}_{\nabla R}:=\big\{PR_x+QR_y\,:\,P,Q\in\mathbb{R}\langle x,y\rangle\big\}$ the ideal generated by the partial derivatives $R_x$ and $R_y$ in the local ring of germs of real analytic scalar functions $\mathbb{R}\langle x,y\rangle$ (see Section 5.2 at the end of this chapter for the proof of this assertion). As a consequence there exist $\widehat{P},\widehat{Q}$ satisfying the equality (5.5).

Since $R_{n+1,x}$ and $R_{n+1,y}$ have no common factors in the local ring $\mathbb{R}\langle x,y\rangle$, we can conclude that $\mathrm{ord}_{\hat{0}}\,\widehat{P}$ and $\mathrm{ord}_{\hat{0}}\,\widehat{Q}$ is greater or equal to $n$.

Considering $\widetilde{P},\widetilde{Q},C$ as before and $c\in\mathbb{R}$, the real analytic functions

$$\widetilde{P}_c=\widetilde{P}-cR_y\,,\quad \widetilde{Q}_c=\widetilde{Q}+cR_x$$

satisfy the equation (5.4), that is, $\widetilde{P}_cR_x+\widetilde{Q}_cR_y=RC$. Now we obtain $c$ such that the complexification of the real analytic vector field $\nu_c=\widetilde{P}_c\frac{\partial}{\partial x}+\widetilde{Q}_c\frac{\partial}{\partial y}$ satisfies all conditions described in the statement. For this purpose we shall analyze the desingularization of the foliation induced by the complexification of $\nu_c$. In the coordinate chart $(x,u=y/x)$ such a desingularization is generated by the vector field

$$\begin{aligned}\dot{x}&=h^{\mathbb{C}}(1,u)+\sum_{k\geq 0}\Big(P^{\mathbb{C}}_{n+k}(1,u)-cR^{\mathbb{C}}_{n+k+1,y}(1,u)\Big)x^{k+1}\\ \dot{u}&=\sum_{k\geq 0}\Big(Q^{\mathbb{C}}_{n+k}(1,u)-uP^{\mathbb{C}}_{n+k}(1,u)+c(n+k+1)R^{\mathbb{C}}_{n+k+1}(1,u)\Big)x^{k}\,,\end{aligned} \tag{5.6}$$

where $\widehat{P} = P_n + P_{n+1} + P_{n+2} + \cdots$, $\widehat{Q} = Q_n + Q_{n+1} + Q_{n+2} + \cdots$.

We choose $c_0 \in \mathbb{R}$ satisfying

$$Q_n(1, q_j) - q_j P_n(1, q_j) + c_0(n+1)R_{n+1}(1, q_j) \neq 0\,,$$

for all $1 \leq j \leq n-2$. Thus the foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$ obtained by the desingularization of $\nu^{\mathbb{C}}_{c_0}$, satisfies (c) by construction. Moreover, this foliation satisfies (a) and (b) since in the coordinate chart $(x, u = y/x)$ the foliation is transversal to $\mathbb{D}_{\mathbb{C}}$ except at the points $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}$, as is expressed in (5.6), and it does not have a tangency point at the origin of the chart $(v = x/y, y)$ since $h(x, y)$ is not divided by $x$. Below we verify that the foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$ also satisfies the property (d).

Near any point $\mathfrak{q}_j$ the foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$ is described as the extended phase space of the nonautonomous real analytic ordinary differential equation

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = \frac{h^{\mathbb{C}}(1,u)+\sum_{k\geq 0}\Big(P^{\mathbb{C}}_{n+k}(1,u)-c_0 R^{\mathbb{C}}_{n+k+1,y}(1,u)\Big)x^{k+1}}{\sum_{k\geq 0}\Big(Q^{\mathbb{C}}_{n+k}(1,u)-uP^{\mathbb{C}}_{n+k}(1,u)+c_0(n+k+1)R^{\mathbb{C}}_{n+k+1}(1,u)\Big)x^{k}} \tag{5.7}$$

Therefore we can choose a neighborhood of $\mathfrak{q}_j$ where the foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$ is determined by the complexification of a real analytic differential equation $\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = f^{\mathbb{C}}_j(u) + g^{\mathbb{C}}_j(x, u)$, where $g^{\mathbb{C}}_j(x, u) = \mathrm{O}(x)$ and

$$f^{\mathbb{C}}_j(u) = \frac{h^{\mathbb{C}}(1, u)}{Q^{\mathbb{C}}_n(1, u) - uP^{\mathbb{C}}_n(1, u) + c_0(n+1)R^{\mathbb{C}}_{n+1}(1, u)}\,.$$

Then $f^{\mathbb{C}}_j(q_j) = 0$ and

$$f^{\mathbb{C}}_{j,u}(q_j) = \frac{\prod_{i=1,\, i\neq j}^{n-2}(q_j - q_i)}{Q_n(1, q_j) - uP_n(1, q_j) + c_0(n+1)R_{n+1}(1, q_j)} \neq 0\,.$$

■

In what follows the foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$ described in Lemma 5.1.1 will be called the *auxiliary foliation.*

The next result states the existence of the locus $\widetilde{T}$ of points on $(\mathbb{M}_{\mathbb{C}}, \mathbb{D}_{\mathbb{C}})$ where the leaves of the foliations $\widetilde{\mathscr{F}}^{\mathbb{C}}_{\nu}$ and $\widetilde{\mathscr{F}}^{\mathbb{C}}$ are tangent. It is important to identify this set since we shall have to pay special attention to extend the biholomorphism over all these points.

***Lemma* 5.1.2.** *The set* $\widetilde{T}$ *consists of the desingularization of the separatrices of* $\nu^{\mathbb{C}}$ *and the complexification of* $n-2$ *real analytic curves* $T_1, \ldots, T_{n-2}$ *which cross the exceptional divisor* $\mathbb{D}_{\mathbb{C}}$ *transversally at the points* $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}$*, respectively.*

**Proof**. It is enough to analyze only what happens around each point in $\widetilde{T} \cap \mathbb{D}_{\mathbb{C}} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}, \mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}\}$.

We shall denote by $\widetilde{\nu}$ and $\widetilde{\nu}_A$ the germs of vector fields with isolated singularities generating, in the coordinate chart $(x, u = y/x)$, the foliations $\widetilde{\mathscr{F}_\nu}$ and $\widetilde{\mathscr{F}}$, being $\widetilde{\mathscr{F}}$ the desingularization of the foliation generated by $\nu_A$.

We consider the germ of the function described in the coordinate chart $(x, u)$ as

$$\begin{array}{ccc} (\mathbb{C}^2, \{x = 0\}) \times \mathbb{C} & \xrightarrow{\Delta^{\mathbb{C}}} & \mathbb{C}^2 \\ ((x, u), \lambda) & \longmapsto & \widetilde{\nu}^{\mathbb{C}}(x, u) - \lambda \widetilde{\nu}_A^{\mathbb{C}}(x, u) \end{array}, \tag{5.8}$$

which is the complexification of the real analytic function $\Delta = (\Delta_1, \Delta_2) := \widetilde{\nu} - \lambda \widetilde{\nu}_A = (\widetilde{\nu}_1 - \lambda \widetilde{v}_1, \widetilde{\nu}_2 - \lambda \widetilde{v}_2)$.

In what follows we consider a representative of the germ $\Delta^{\mathbb{C}}$, which will be denoted in the same way. Thus $\alpha \in \widetilde{T}$ if and only if $\alpha$ belongs to the coordinate chart $(x, u)$ and $\Delta^{\mathbb{C}}(\alpha, \lambda(\alpha)) = \widehat{0}$ for some $\lambda(\alpha) \in \mathbb{C}$.

The derivative of $\Delta^{\mathbb{C}}$ at the point $(x_0, u_0, \lambda_0)$ is, by definition,

$$\mathrm{D}\, \Delta^{\mathbb{C}}(x_0, u_0, \lambda_0) = \begin{bmatrix} \widetilde{\nu}_{1,x}^{\mathbb{C}} - \lambda \widetilde{v}_{1,x}^{\mathbb{C}} & \widetilde{\nu}_{1,u}^{\mathbb{C}} - \lambda \widetilde{v}_{1,u}^{\mathbb{C}} & -\widetilde{v}_1^{\mathbb{C}} \\ \widetilde{\nu}_{2,x}^{\mathbb{C}} - \lambda \widetilde{v}_{2,x}^{\mathbb{C}} & \widetilde{\nu}_{2,u}^{\mathbb{C}} - \lambda \widetilde{v}_{2,u}^{\mathbb{C}} & -\widetilde{v}_2^{\mathbb{C}} \end{bmatrix} \Bigg|_{(x_0, u_0, \lambda_0)}.$$

Let us denote by $u(\mathfrak{p}_i) = p_i$ the $u$-coordinate of the point $\mathfrak{p}_i$ for $1 \leq i \leq n+1$. Then $\Delta^{\mathbb{C}}(0, p_i, 0) = (0, 0)$ and

$$\mathrm{D}\, \Delta^{\mathbb{C}}(0, p_i, 0) = \begin{bmatrix} \widetilde{\nu}_{1,x}^{\mathbb{C}}(0, p_i) & \widetilde{\nu}_{1,u}^{\mathbb{C}}(0, p_i) & -\widetilde{v}_1^{\mathbb{C}}(0, p_i) \\ \widetilde{\nu}_{2,x}^{\mathbb{C}}(0, p_i) & \widetilde{\nu}_{2,u}^{\mathbb{C}}(0, p_i) & -\widetilde{v}_2^{\mathbb{C}}(0, p_i) \end{bmatrix}.$$

Since $\mathrm{D}\, \widetilde{\nu}^{\mathbb{C}}(0, p_i)$ is an invertible matrix, by the implicit function theorem there exist $U_i \subseteq \mathbb{C}$ and $V_i \subseteq \mathbb{C}^2$ open neighborhoods of $0$ and $(0, p_i)$, respectively, and a holomorphic function $h_i : U_i \longrightarrow V_i$ such that for all $(x, u, \lambda) \in V_i \times U_i$, $\Delta^{\mathbb{C}}(x, u, \lambda) = 0$ if and only if $h_i(\lambda) = (x, u)$. Hence, in a neighborhood of $\mathfrak{p}_i$ on $\mathbb{M}$ the complexification of $S_i$ is the unique invariant curve for both foliations $\widetilde{\mathscr{F}_\nu^{\mathbb{C}}}$ and $\widetilde{\mathscr{F}^{\mathbb{C}}}$.

On the other hand, we consider $u(\mathfrak{q}_j) = q_j$ for $1 \leq j \leq n-2$. By Lemma 5.1.1 we know that there exists $\lambda_j \in \mathbb{R}$ such that $\Delta^{\mathbb{C}}(0, q_j, \lambda_j) = 0$. Thus

$$\begin{aligned} \mathrm{D}\, \Delta^{\mathbb{C}}(0, q_j, \lambda_j) = & \begin{bmatrix} \widetilde{\nu}_{1,x}^{\mathbb{C}}(0, q_j) & \widetilde{\nu}_{1,u}^{\mathbb{C}}(0, q_j) & 0 \\ \widetilde{\nu}_{2,x}^{\mathbb{C}}(0, q_j) & \widetilde{\nu}_{2,u}^{\mathbb{C}}(0, q_j) & 0 \end{bmatrix} \\ & - \begin{bmatrix} \lambda_j \widetilde{v}_{1,x}^{\mathbb{C}}(0, q_j) & \lambda_j \prod_{l=1\, l \neq j}^{n-2} (q_j - q_l) & 0 \\ \lambda_j \widetilde{v}_{2,x}^{\mathbb{C}}(0, q_j) & \lambda_j \widetilde{v}_{2,u}^{\mathbb{C}}(0, q_j) & \widetilde{v}_2^{\mathbb{C}}(0, q_j) \end{bmatrix}, \end{aligned}$$

where $\widetilde{\nu}_{1,u}^{\mathbb{C}}(0, q_j) = 0$ and $\widetilde{v}_2^{\mathbb{C}}(0, q_j) \neq 0$ (since $\widetilde{\nu}_A^{\mathbb{C}}$ does not have any singular point in $\mathbb{D}_{\mathbb{C}}$). Then, eliminating the first column of $\mathrm{D}\,\Delta^{\mathbb{C}}(0, q_j, \lambda_j)$ we obtain an invertible matrix. By the implicit function theorem (real analytic version) there exist $W_j \subseteq \mathbb{C}$ and $Z_j \subseteq \mathbb{C}^2$, open neighborhoods of 0 and $(q_j, \lambda_j)$, respectively, and a real analytic function $\widehat{g}_j = (g_{j,1}, g_{j,2})$ such that $\widehat{g}_j^{\mathbb{C}} : W_j \longrightarrow Z_j$ and for all $(x, u, \lambda) \in W_j \times Z_j$, $\Delta^{\mathbb{C}}(x, u, \lambda) = 0$ if and only if $\widehat{g}_j^{\mathbb{C}}(x) = (u, \lambda)$. Therefore, in a neighborhood of $\mathfrak{q}_i$ on $\mathbb{M}$ the curve $T_j^{\mathbb{C}} := \{u = g_{j,1}^{\mathbb{C}}(x)\}$ is the unique invariant curve for both foliations $\widetilde{\mathscr{F}_\nu}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}^{\mathbb{C}}$.

■

The complexification of the real analytic curve $T_j$ described in Lemma 5.1.2 will be called *polar curve*.

## 5.2 Ideals in the local ring $\mathbb{R}\langle x, y\rangle$

In order to conclude the proof of Lemma 5.1.1, our main goal is to prove Proposition 5.2.1. For this purpose we shall use Lemmas 5.2.2 and 5.2.3.

We consider $L \in \mathbb{R}\langle x, y\rangle$ such that $\mathrm{ord}_{\widehat{0}}(L) = n + 1 \geq 2$. We assume that $L$ has $n + 1$ pairwise transversal smooth analytic branches, but not necessarily real analytic. We shall denote by $\mathfrak{M}_{\mathbb{R}\langle x,y\rangle}$ the maximal ideal of $\mathbb{R}\langle x, y\rangle$ (the convergent series with zero constant term) and by $\mathcal{I}_{\nabla L}$ the ideal of $L$ generated by the partial derivatives $L_x$ and $L_y$ in the local ring $\mathbb{R}\langle x, y\rangle$ (that is, $\mathcal{I}_{\nabla L} = \{PL_x + QL_y \;:\; P,\, Q \in \mathbb{R}\langle x, y\rangle\,\}$).

***Proposition* 5.2.1.**

$$\mathfrak{M}_{\mathbb{R}\langle x,y\rangle}{}^{2n-1} \subseteq \mathcal{I}_{\nabla L}\,.$$

Now we shall prove Lemmas 5.2.2 and 5.2.3 to prove Proposition 5.2.1.

***Lemma* 5.2.2.** *The real vector space $\mathbb{R}\langle x, y\rangle/\mathcal{I}_{\nabla L}$ is finite-dimensional, say $s \in \mathbb{N}$. Moreover,*

$$\mathfrak{M}_{\mathbb{R}\langle x,y\rangle}{}^{s} \subseteq \mathcal{I}_{\nabla L}\,.$$

**Proof**. We suppose that $\dim_{\mathbb{R}} \mathbb{R}\langle x, y\rangle/\mathcal{I}_{\nabla L} = s < \infty$. We take $\phi_1, \ldots, \phi_s$ elements from $\mathfrak{M}_{\mathbb{R}\langle x,y\rangle}$, and we consider the set $\{1, \phi_1, \phi_1\phi_2, \ldots, \phi_1 \cdots \phi_s\}$. Then there exist $l_0, l_1, \ldots, l_s \in \mathbb{R}$, at least one nonzero, such that

$$l_0 + l_1\phi_1 + \cdots + l_s\phi_1 \cdots \phi_s \in \mathcal{I}_{\nabla L}\,.$$

Note that $l_0$ is zero, since

$$l_0 = -(l_1\phi_1 + \cdots + l_s\phi_1 \cdots \phi_s) \in \mathfrak{M}_{\mathbb{R}\langle x,y\rangle}\,.$$

Let $1 \leq r \leq s$ be the smallest number such that $l_r \neq 0$, then

$$\phi_1\phi_2\cdots\phi_r(l_r + l_{r+1}\phi_{r+1} + \cdots + l_s\phi_{r+1}\cdots\phi_s) \in \mathcal{I}_{\nabla L}\,.$$

Since $l_r + l_{r+1}\phi_{r+1} + \cdots + l_s\phi_{r+1}\cdots\phi_s$ is an invertible element from $\mathbb{R}\langle x, y\rangle$, then $\phi_1\phi_2\cdots\phi_r \in \mathcal{I}_{\nabla L}$, and as a consequence $\phi_1\phi_2\cdots\phi_s \in \mathcal{I}_{\nabla L}$.

Now we shall prove that $\mathbb{R}\langle x, y\rangle/\mathcal{I}_{\nabla L}$ is a finite-dimensional real vector space. Since $L$ has $n+1$ pairwise transversal smooth analytic branches, then $L_x$ and $L_y$ have no common roots in $\mathbb{C}\langle x, y\rangle$. As a consequence the origin is an isolated element from $\{L_x = 0\} \cap \{L_y = 0\}$. Thus $\dim_{\mathbb{C}} \mathbb{C}\langle x, y\rangle/\langle L_x, L_y\rangle_{\mathbb{C}\langle x,y\rangle}$ is finite, where $\langle L_x, L_y\rangle_{\mathbb{C}\langle x,y\rangle}$ is the ideal in $\mathbb{C}\langle x, y\rangle$ generate by $L_x$ and $L_y$ (see [AG-ZV85]). Because of

$$\dim_{\mathbb{R}} \mathbb{R}\langle x, y\rangle/\mathcal{I}_{\nabla L} = \dim_{\mathbb{C}} \mathbb{C}\langle x, y\rangle/\langle L_x, L_y\rangle_{\mathbb{C}\langle x,y\rangle}\,,$$

the proof is complete.

■

***Lemma* 5.2.3.** *Let $T \in \mathbb{R}[x, y]$ be a homogeneous polynomial of degree $n+1$ which is split into $n+1$ pairwise different lines, but not necessarily real. Then for every $i, j \geq 0$ such that $i + j = 2n - 1$, there exist $P_{i,j}, Q_{i,j} \in \mathbb{R}[x, y]$ unique homogeneous polynomials of degree $n-1$ satisfying*

$$x^i y^j = P_{i,j}T_x + Q_{i,j}T_y\,.$$

**Proof**. Since $T_x, T_y$ are homogeneous polynomials of degree $n$ and have no common roots, we obtain the *uniqueness*: if $P_{i,j}, P'_{i,j}$ and $Q_{i,j}, Q'_{i,j}$ are homogeneous polynomials of degree $n-1$ belonging to $\mathbb{R}[x, y]$ and satisfying

$$P'_{i,j}T_x + Q'_{i,j}T_y = x^i y^j = P_{i,j}T_x + Q_{i,j}T_y\,,$$

that is, $(P_{i,j} - P'_{i,j})T_x = -(Q_{i,j} - Q'_{i,j})T_y$, then $P_{i,j} - P'_{i,j} \equiv 0$ and $Q_{i,j} - Q'_{i,j} \equiv 0$.

In order to prove *the existence* we shall consider two cases.

**Case 1:** the monomial $x$ is not a factor of $T$. In this case $x$ is not a factor of $T_y$, since $(n+1)T = xT_x + yT_y$.

We consider $A(u) := T_x(1, u)$, $B(u) := T_y(1, u)$; thus $\deg A \leq n$ and $\deg B = n$. These polynomials have not common roots because of $x^n A(y/x) = T_x(x, y)$, $x^n B(y/x) = T_y(x, y)$. Since $\mathbb{R}[u]$ is an Euclidean domain with respect to the function *degree*, there exist $P, Q \in \mathbb{R}[u]$ such that

$$1 = P(u)A(u) + Q(u)B(u)\,. \tag{5.9}$$

By division algorithm there exist $\widetilde{P}, P_0 \in \mathbb{R}[u]$ unique polynomials satisfying

$P(u) = \widetilde{P}(u)B(u) + P_0(u)$, where $\deg P_0 < \deg B \leq n$. Denoting by $Q_0 = A\widetilde{P} + Q$, from the equation (5.9) we obtain

$$\begin{aligned} 1 &= (\widetilde{P}B + P_0)A + QB = P_0A + (\widetilde{P}A + Q)B \\ &= P_0A + Q_0B\,. \end{aligned}$$

Since $\deg B + \deg Q_0 = \deg A + \deg P_0 < \deg A + \deg B$, we have $\deg Q_0 < \deg A \leq n$. Thus,

$$\begin{aligned} x^{2n-1} &= (x^{n-1}P_0(y/x))(x^nA(y/x)) + (x^{n-1}Q_0(y/x))(x^nB(y/x)) \\ &= (x^{n-1}P_0(y/x))T_x(x,y) + (x^{n-1}Q_0(y/x))T_y(x,y)\,. \end{aligned}$$

Now we consider $1 \leq k \leq 2n-1$. From the equation (5.9) it follows

$$u^k = u^kP(u)A(u) + u^kQ(u)B(u)\,. \tag{5.10}$$

There exist $\widetilde{P}_k, P_k$ unique polynomials satisfying $u^kP(u) = \widetilde{P}_k(u)B(u) + P_k(u)$, being $\deg P_k < \deg B \leq n$. Denoting by $Q_k(u) = \widetilde{P}_k(u)A(u) + u^kQ(u)$, from the equation (5.10) we have

$$\begin{aligned} u^k &= (\widetilde{P}_k(u)B(u) + P_k(u))A(u) + u^kQ(u)B(u) \\ &= P_k(u)A(u) + (\widetilde{P}_k(u)A(u) + u^kQ(u))B(u) \\ &= P_k(u)A(u) + Q_k(u)B(u)\,. \end{aligned}$$

The degree of $Q_k$ is less than $n$, otherwise $\deg(Q_kB) = \deg Q_k + \deg B \geq 2n > \deg A + \deg P_k = \deg(P_kA)$, obtaining $\deg(Q_kB + P_kA) \geq 2n$, leading to a contradiction.

Taking $u = y/x$ and multiplying by $x^{2n-1}$ to both sides of the above equality, we obtain

$$\begin{aligned} x^{2n-1-k}y^k &= (x^{n-1}P_k(y/x))(x^nA(y/x)) + (x^{n-1}Q_k(y/x))(x^nB(y/x)) \\ &= (x^{n-1}P_k(y/x))T_x(x,y) + (x^{n-1}Q_k(y/x))T_y(x,y)\,. \end{aligned}$$

Choosing $P_{i,j}(x,y) = x^{n-1}P_j(y/x)$, $Q_{i,j}(x,y) = x^{n-1}Q_j(y/x)$, in this case we conclude the existence.

**Case 2:** the monomial $x$ is a factor of $T$. In such a case $x$ is no a factor of $T_x$ because of $x$ a simple factor of $T$. We consider $A(u) := T_x(1,u)$, $B(u) := T_y(1,u)$; thus $\deg A = n$ and $\deg B \leq n$. The assertion follows in the same way as before, using $A(u)$ instead of $B(u)$ in the previous arguments.

■

**Remark 5.2.4.** By Lemma 5.2.3 it follows that for $S \in \mathbb{R}[x, y]$ a homogeneous polynomial of degree $m \geq 2n - 1$ there exist $P, Q \in \mathbb{R}[x, y]$ homogeneous polynomials of degree $m - n$ such that $PT_x + QT_y = S$.

Now we consider $\widetilde{T} = T_{n+1} + T_{n+2} + \cdots \in \mathbb{R}[[x, y]]$ such that $\mathrm{ord}_{\widehat{0}}(\widetilde{T}) = n+1$ at the origin and whose $(n + 1)$-jet satisfies the conditions of Lemma 5.2.3. Applying this last result, it follows by induction that for all $i_0, j_0 \geq 0$ such that $i_0 + j_0 = 2n - 1$ there exist $P_r, Q_r \in \mathbb{R}[x, y]$ homogeneous polynomials of degree $r \geq n - 1$ such that for all $k \geq 0$

$$x^{i_0} y^{j_0} + \mathrm{O}(\|(x, y)\|^{2n+k+1}) = \\ (P_{n-1} + P_n + \cdots + P_{n+k})\widetilde{T}_x + (Q_{n-1} + Q_n + \cdots + Q_{n+k})\widetilde{T}_y \,.$$

As a consequence, we have obtained the formal version of Proposition 5.2.1.

**Proof of Proposition 5.2.1.** By our assumptions on $L = L_{n+1} + \cdots$, its $(n + 1)$-jet satisfies the hypothesis of Lemma 5.2.3. As a consequence for all $i_0, j_0 \geq 0$ such that $i_0 + j_0 = 2n - 1$, we can obtain $\widetilde{P}_r, \widetilde{Q}_r \in \mathbb{R}[x, y]$ homogeneous polynomials of degree $r \geq n - 1$ with the following properties

$$x^{i_0} y^{j_0} + \mathrm{O}(\|(x, y)\|^{2n+k+1}) = \\ (\widetilde{P}_{n-1} + \widetilde{P}_n + \cdots + \widetilde{P}_{n+k})L_x + (\widetilde{Q}_{n-1} + \widetilde{Q}_n + \cdots + \widetilde{Q}_{n+k})L_y \,,$$

for all $k \geq 0$. Choosing $k_0 \geq 1$ such that $2n + k_0 + 1$ is greater or equal to the dimension of the real vector space $\mathbb{R}\langle x, y\rangle/\mathcal{I}_{\nabla L}$, there exist $\widetilde{P}, \widetilde{Q} \in \mathbb{R}\langle x, y\rangle$ satisfying $\widetilde{P}L_x + \widetilde{Q}L_y$ is equal to

$$x^{i_0} y^{j_0} - (\widetilde{P}_{n-1} + \widetilde{P}_n + \cdots + \widetilde{P}_{n+k_0})L_x - (\widetilde{Q}_{n-1} + \widetilde{Q}_n + \cdots + \widetilde{Q}_{n+k_0})L_y \,.$$

By Lemma 5.2.2, $x^{i_0} y^{j_0} \in \mathcal{I}_{\nabla L}$.

■

# Chapter 6

# Standardization of the Auxiliary Foliation

The main result of this chapter is given in Proposition 6.0.5 which states that, except for a real analytic change of coordinates, we can suppose that $\nu$ and $\omega$ are strictly real-formally equivalent and their complexifications, $\nu^{\mathbb{C}}$ and $\omega^{\mathbb{C}}$, have coinciding separatrices and their corresponding holonomies coincide as well. At the end of this chapter a biholomorphism between $\widetilde{\mathscr{F}}_{\nu}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_{\omega}^{\mathbb{C}}$ far from the singular points $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ and the tangency points $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}$, is given.

Without loss of generality we suppose that the tangency points $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}$ between the leaves of the desingularization of the auxiliary foliation and the exceptional divisor satisfy $u(\mathfrak{q}_1) < \cdots < u(\mathfrak{q}_{n-2})$.

From now on we may assume that $u(\mathfrak{q}_{n-2})$ is less than $u(\mathfrak{p}_i)$, for every singular point $\mathfrak{p}_i$ belonging to the real exceptional divisor[1] $\mathbb{D}_{\mathbb{R}}$.

We consider $\mathfrak{q}_0 \in \mathbb{D}_{\mathbb{R}} \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}$ in the domain of the coordinate chart $(x, u)$ such that $u(\mathfrak{q}_0) = q_0 \neq 0$ is less than $u(\mathfrak{q}_1)$. The leaf of the auxiliary foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$ passing through $\mathfrak{q}_0$ is the complexification of a real analytic curve $T_0$ which is transversal to the complex exceptional divisor. By Theorems 2.2.1 and 2.2.2 there exists a real analytic diffeomorphism $\mathfrak{h}$ on $\big(T_0, \mathfrak{q}_0\big)$ whose complexification, $\mathfrak{h}^{\mathbb{C}}$, conjugates the holonomy groups of $\widetilde{\mathscr{F}}_{\nu}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_{\omega}^{\mathbb{C}}$ strongly. That is, for every $\alpha \in \Pi_1(\mathbb{D}_{\mathbb{C}} \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}, \mathfrak{q}_0)$,

$$\mathfrak{h}^{\mathbb{C}} \circ \Delta_{\alpha}^{\nu} \circ (\mathfrak{h}^{\mathbb{C}})^{-1} = \Delta_{\alpha}^{\omega}\,,$$

where $\Delta_{\alpha}^{\nu}, \Delta_{\alpha}^{\omega} : \big(T_0^{\mathbb{C}}, \mathfrak{q}_0\big) \longrightarrow \big(T_0^{\mathbb{C}}, \mathfrak{q}_0\big)$ are the holonomy maps over $\alpha$ for the foliations $\widetilde{\mathscr{F}}_{\nu}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_{\omega}^{\mathbb{C}}$, respectively.

[1] We can prove the real-formal orbital rigidity theorem without this condition, but in this case the proofs of the following properties would be a little more complicated.

Given $j = 1, \ldots, n-2$ we choose the path $\gamma_j$ beginning at $\mathfrak{q}_j$ and ending at $\mathfrak{q}_0$ such that in the coordinate chart $(x, u)$ is the path $[0,1] \ni t \mapsto (0, tq_0 + (1-t)q_j)$, where $q_j = u(\mathfrak{q}_j)$. The germs $\Delta^{\nu}_{\gamma_j}, \Delta^{\omega}_{\gamma_j} : (T^{\mathbb{C}}_j, \mathfrak{q}_j) \longrightarrow (T^{\mathbb{C}}_0, \mathfrak{q}_0)$ are the correspondence maps over the path $\gamma_j$ for the foliations $\widetilde{\mathscr{F}}^{\mathbb{C}}_{\nu}$ and $\widetilde{\mathscr{F}}^{\mathbb{C}}_{\omega}$, respectively.

***Proposition* 6.0.5.** *Except for a real analytic change of coordinates defined on $(\mathbb{R}^2, \widehat{0})$ whose linear part is the identity matrix, we can suppose that $\omega$ has the following properties:*

*(a) $\nu$ and $\omega$ are strictly real-formally orbitally equivalent.*

*(b) $\nu^{\mathbb{C}}$ and $\omega^{\mathbb{C}}$ have the same separatrices $S_1, \ldots, S_{n+1}$.*

*(c) $\widetilde{T}$ is the set of points on $(\mathbb{M}_{\mathbb{C}}, \mathbb{D}_{\mathbb{C}})$ where the leaves of the foliations $\widetilde{\mathscr{F}}^{\mathbb{C}}_{\omega}$ and $\widetilde{\mathscr{F}}^{\mathbb{C}}$ are tangent.*

*(d) For any $\alpha \in \Pi_1(\mathbb{D}_{\mathbb{C}} \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}, \mathfrak{q}_0)$, the holonomy maps $\Delta^{\nu}_{\alpha}, \Delta^{\omega}_{\alpha}$ coincide.*

*(e) For $j = 1, \ldots, n-2$, the correspondence maps over the path $\gamma_j$, $\Delta^{\nu}_{\gamma_j}, \Delta^{\omega}_{\gamma_j}$ coincide.*

The properties (a) and (b) have been proved in Chapter 3. The rest of the proof relies on Lemma 6.0.6 and below. Now we shall introduce some necessary notions.

Consider $\mathcal{L}_1, \ldots, \mathcal{L}_m$ smooth pairwise transversal germs of complex analytic curves at the origin of $\mathbb{C}^2$ which are not tangent to the axis $y$. Therefore, given $i = 1, \ldots, m$, there exists $\psi_i$ the germ of a holomorphic function at $0 \in \mathbb{C}$ such that $\mathcal{L}_i = \{y = \psi_i(x)\}$ satisfying $\psi_i'(0) \neq \psi_j'(0)$ if $i \neq j$.

These curves have some additional properties:

- There exists $1 \leqslant k \leqslant m$ such that $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are the complexifications of real analytic curves, that is, $\psi_i(x)$ has real coefficients for $1 \leqslant i \leqslant k$.
- The product $\Psi(x, y) := \prod_{i=1}^{m}(y - \psi_i(x))$ is the complexification of a real analytic function at the origin of $\mathbb{C}^2$. As a consequence, given $k+1 \leqslant i \leqslant m$ there exists $k+1 \leqslant j \leqslant m$ such that $\mathcal{L}_j$ is the *conjugate* of $\mathcal{L}_i$, that is, if $\psi_i(x) = \sum_{r \geqslant 1} a_r x^r$ then $\psi_j(x) = \sum_{r \geqslant 1} \bar{a}_r x^r$, where $\bar{a}$ is the complex conjugate of $a \in \mathbb{C}$.

For each $i = 1, \ldots, m$ we consider $F_i : (\mathcal{L}_i, \widehat{0}) \longrightarrow (\mathcal{L}_i, \widehat{0})$ the germ of a biholomorphism with the following property. If $\mathcal{L}_j$ is the conjugate of $\mathcal{L}_i$ then $F_j$ is the conjugate of $F_i$. That is, since $F_i(x) = (f_i(x), \psi_i \circ f_i(x))$ where $f_i$ is the germ of a biholomorphism at $0 \in \mathbb{C}$, the coefficients of the Taylor series of $f_j$ are the complex conjugates of the respective coefficients of the Taylor series of $f_i$. As a consequence, $F_l$ is the complexification of the germ of a real analytic diffeomorphism whenever $1 \leqslant l \leqslant k$.

We consider $\vartheta \in \Sigma_n^{\mathbb{R}}$ the germ of a real analytic vector field which is not tangent to any real analytic curve $\mathcal{L}_1 \cap \mathbb{R}^2, \ldots, \mathcal{L}_k \cap \mathbb{R}^2$. Therefore, denoting by $\mathrm{L}_\vartheta \Psi$ the Lie derivative of the germ $\Psi$ along the vector field $\vartheta$, for $1 \leqslant j \leqslant k$ we have the following real analytic germ which does not vanish identically

$$\mathrm{L}_\vartheta \Psi\,(x, \psi_j(x)) = \prod_{\substack{i=1,\\ i\neq j}}^{m} (\psi_j(x) - \psi_i(x)) \Big[(-\psi_j(x), 1)\cdot \vartheta(x, \psi_j(x))\Big].$$

As a consequence, $M := \max_{1\leqslant j\leqslant k}\{\mathrm{ord}_0\, \mathrm{L}_\vartheta \Psi(x, \psi_j(x))\}$ is a natural number.

***Lemma* 6.0.6.** *Suppose that* $\mathrm{ord}_0\,(f_i(x) - x)$ *is greater or equal to the maximum between* $m+1$ *and* $M+k+m-n$ *for all* $i = 1, \ldots, m$.

*If* $\upsilon \in \Sigma_n^{\mathbb{R}}$ *is a germ satisfying* $\mathrm{ord}_{\widehat{0}}(\vartheta - \upsilon)$ *is greater or equal to* $M+k$*, there exists a real analytic diffeomorphism* $\mathscr{H} : (\mathbb{R}^2, \widehat{0}) \longrightarrow (\mathbb{R}^2, \widehat{0})$ *whose complexification brings* $\upsilon^{\mathbb{C}}$ *to a vector field coinciding with* $\vartheta^{\mathbb{C}}$ *on the subset* $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_k$*. Besides, its linear part is the identity and* $\mathscr{H}^{\mathbb{C}}\big|_{\mathcal{L}_i} = F_i$ *for all* $i = 1, \ldots, m$.

**Proof**. Firstly we shall consider $\psi_i'(0)$ being nonzero for $i = 1, \ldots, m$. We shall deal with the other case at the end of the proof.

We shall bring $\upsilon^{\mathbb{C}}$ to a vector field $\widehat{\upsilon}^{\mathbb{C}}$ by means of the complexification of a real analytic diffeomorphism $\widehat{\mathscr{H}} : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ whose restriction on $\mathcal{L}_i$ will be $F_i$ for $i = 1, \ldots, m$. Then we shall transform $\widehat{\upsilon}^{\mathbb{C}}$ to another vector field which coincides with $\vartheta^{\mathbb{C}}$ on $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_k$ via the complexification of another real analytic diffeomorphism whose restriction on $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_m$ will be the identity map.

We consider the germs of functions at $\widehat{0} \in \mathbb{C}^2$

$$\widehat{\mathscr{H}_1}(x, y) := \sum_{i=1}^{m} \frac{(f_i(x) - x) \prod_{\substack{j=1,\\ j\neq i}}^{m} \big(y - \psi_j(x)\big)}{\prod_{\substack{j=1,\\ j\neq i}}^{m} \big(\psi_i(x) - \psi_j(x)\big)}$$

$$\widehat{\mathscr{H}_2}(x, y) := \sum_{i=1}^{m} \frac{(\psi_i \circ f_i \circ \psi_i^{-1}(y) - y) \prod_{\substack{j=1,\\ j\neq i}}^{m} \big(y - \psi_j(x)\big)}{\prod_{\substack{j=1,\\ j\neq i}}^{m} \big(y - \psi_j \circ \psi_i^{-1}(y)\big)},$$

which are germs of holomorphic functions whose summands have order greater or equal to $\mathrm{ord}_0(f_i(x) - x) - m + 1$ (where $\mathrm{ord}_0(f_i(x) - x) = \mathrm{ord}_0\big(\psi_i \circ f_i \circ \psi_i^{-1}(y) - y\big)$ is greater or equal to the maximum between $m+1$ and $M+k+m-n$, by hypothesis). Furthermore, since for all $1 \leqslant i \leqslant m$ there is $1 \leqslant j \leqslant m$ such that the germs of functions $\psi_j$ and $f_j$ are the conjugated functions of $\psi_i$ and

$f_i$, respectively, thus for each summand of $\widehat{\mathscr{H}_1}$ there is another term conjugated to it, hence $\widehat{\mathscr{H}_1}$ has real coefficients. The same arguments prove that $\widehat{\mathscr{H}_2}$ has real coefficients. Therefore $\widehat{\mathscr{H}_1}$ and $\widehat{\mathscr{H}_2}$ are complexifications of germs of real analytic functions at $\widehat{0} \in \mathbb{R}^2$ and its orders at the origin are greater or equal to the maximum between 2 and $M + k - n + 1$.

Because of the previous properties, we define $\widehat{\mathscr{H}}^{\mathbb{C}}$ as a representative of the germ $(x + \widehat{\mathscr{H}_1}, y + \widehat{\mathscr{H}_2})$ .

We define $\widehat{\upsilon} := \big(\mathrm{D}\,\widehat{\mathscr{H}} \cdot \upsilon\big) \circ \widehat{\mathscr{H}}^{-1}$. A direct calculation shows that $\mathrm{ord}_{\widehat{0}}\big(\upsilon - \widehat{\upsilon}\big)$ is greater or equal to $n + \mathrm{ord}_{\widehat{0}}\big(\widehat{\mathscr{H}} - \mathrm{id}\big) - 1 \geqslant M + k$. Thus $\mathrm{ord}_{\widehat{0}}\big(\vartheta - \widehat{\upsilon}\big) \geqslant M + k$.

Now we look for a real analytic diffeomorphism $\widehat{\mathscr{G}}(x, y)$ whose complexification is the identity on $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_m$ and such that for all $1 \leqslant j \leqslant k$

$$\Big(\mathrm{D}\,\widehat{\mathscr{G}}^{\mathbb{C}} \cdot \vartheta^{\mathbb{C}}\Big)(x, \psi_j(x)) = \widehat{\upsilon}^{\mathbb{C}}\Big(\widehat{\mathscr{G}}^{\mathbb{C}}(x, \psi_j(x))\Big) . \tag{6.1}$$

For this purpose let us define $\widehat{\mathscr{G}}(x, y) := (x, y) + \Psi(x, y)\mathscr{G}(x, y)$, where $\mathscr{G}$ is a real analytic vector function defined on $\big(\mathbb{R}^2, \widehat{0}\big)$ whose order at the origin is greater or equal to 1. Since $\Psi(x, y) = \prod_{i=1}^{m}(y - \psi_i(x))$, the complexification of $\widehat{\mathscr{G}}(x, y)$ is the identity on $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_m$. Furthermore, the equations (6.1) are equivalent to the following equalities

$$\mathscr{G}^{\mathbb{C}}(x, \psi_j(x)) = \frac{\big(\widehat{\upsilon}^{\mathbb{C}} - \vartheta^{\mathbb{C}}\big)(x, \psi_j(x))}{\big(\mathrm{L}_{\vartheta}\,\Psi\big)^{\mathbb{C}}(x, \psi_j(x))} =: \big(\mathscr{G}^{\mathbb{C}}_{1,j}(x), \mathscr{G}^{\mathbb{C}}_{2,j}(x)\big) , \tag{6.2}$$

where $\mathscr{G}_{1,j}$ and $\mathscr{G}_{2,j}$ are real analytic scalar functions defined on $(\mathbb{R}, 0)$ and its order at the origin is greater or equal to $k$ since the order at the origin of $\widehat{\upsilon} - \vartheta$ is greater or equal to $M + k$.

To provide a proper definition of $\mathscr{G}$, let $\mathscr{G}_1$ and $\mathscr{G}_2$ two germs of holomorphic functions at the origin of $\mathbb{C}^2$

$$\mathscr{G}_1(x, y) := \sum_{j=1}^{k} \frac{\mathscr{G}^{\mathbb{C}}_{1,j}(x) \prod_{\substack{l=1,\\ l \neq j}}^{k} \big(y - \psi_l(x)\big)}{\prod_{\substack{l=1,\\ l \neq j}}^{k} \big(\psi_j(x) - \psi_l(x)\big)} ,$$

$$\mathscr{G}_2(x, y) := \sum_{j=1}^{k} \frac{\mathscr{G}^{\mathbb{C}}_{2,j}(x) \prod_{\substack{l=1,\\ l \neq j}}^{k} \big(y - \psi_l(x)\big)}{\prod_{\substack{l=1,\\ l \neq j}}^{k} \big(\psi_j(x) - \psi_l(x)\big)} .$$

Both are the complexification of germs of real analytic functions whose order at the origin is greater or equal to 1. Thus, if we can choose $\mathscr{G}^{\mathbb{C}}$ as a representative of $\big(\mathscr{G}_1, \mathscr{G}_2\big)$ and $\widehat{\mathscr{G}}$ as before, the function $\mathscr{H} := \widehat{\mathscr{G}}^{-1} \circ \widehat{\mathscr{H}}$ is a real analytic diffeomorphism satisfying the required conditions.

To complete the proof we consider the case where there is $i \in \{1, \dots, m\}$ such that $\psi_i'(0) = 0$. We choose $b \in \mathbb{R}$ such that $b \neq \psi_j'(0)$ for $1 \leqslant j \leqslant m$.

Then the complexification of the real linear transformation $A(x,y)=(x,bx+y)$ brings each curve $\mathcal{L}_j$ to the curve

$$\widetilde{\mathcal{L}_j} := A^{\mathbb{C}}(\mathcal{L}_j) = \big\{ y = \big(b - \psi_j'(0)\big)x + \big(\psi_j(x) - \psi_j'(0)x\big) =: \widetilde{\psi}_j(x) \big\}$$

which has nonzero linear term. For all $\widetilde{\mathcal{L}}_j$ we consider the germ of the biholomorphism $\widetilde{F}_j : (\widetilde{\mathcal{L}}_j, 0) \longrightarrow (\widetilde{\mathcal{L}}_j, 0)$ defined as $\widetilde{F}_j(x) = \big(f_j(x), \widetilde{\psi}_j \circ f_j(x)\big)$. Applying the previous construction to the germs of vector fields $\widetilde{\vartheta} := A \circ \vartheta \circ A^{-1}$ and $\widetilde{\upsilon} := A \circ \upsilon \circ A^{-1}$, we get the germ of a real analytic diffeomorphism $\widetilde{\mathscr{H}}$ whose complexification satisfies $\widetilde{\mathscr{H}}^{\mathbb{C}}|_{\widetilde{\mathcal{L}}_i} = \widetilde{F}_i$ for all $j = 1, \ldots, m$, and brings $\widetilde{\upsilon}^{\mathbb{C}}$ to the germ of a vector field which coincides with $\widetilde{\vartheta}^{\mathbb{C}}$ on $\widetilde{\mathcal{L}}_1 \cup \cdots \cup \widetilde{\mathcal{L}}_k$. Therefore $A^{-1} \circ \widetilde{\mathscr{H}} \circ A$ is the germ of a real analytic diffeomorphism satisfying the properties described in the statement.

■

**Proof of Proposition 6.0.5.** We now apply Lemma 6.0.6 to the *blow-down* of the polar curves $\mathcal{T}_0^{\mathbb{C}}, \ldots, T_{n-2}^{\mathbb{C}}$ and the separatrices $S_1, \ldots, S_{n+1}$. For that purpose we consider the *blow-down* of the germs

$$\begin{aligned} &(\mathfrak{h}^{\mathbb{C}})^{-1} : \big(T_0^{\mathbb{C}}, \mathfrak{q}_0\big) \longrightarrow \big(T_0^{\mathbb{C}}, \mathfrak{q}_0\big)\,, \\ &\big(\Delta^{\omega}_{\gamma_j^{-1}} \circ \mathfrak{h}^{\mathbb{C}} \circ \Delta^{\nu}_{\gamma_j}\big)^{-1} : \big(T_j^{\mathbb{C}}, \mathfrak{q}_j\big) \longrightarrow \big(T_j^{\mathbb{C}}, \mathfrak{q}_j\big)\,, \quad \text{for} \quad j = 1, \ldots, n-2\,, \end{aligned} \tag{6.3}$$

and the identity map between each separatrix.

In this case $\Psi$ will be the product of the holomorphic functions generating the separatrices and the blow-down of $T_0^{\mathbb{C}}, \ldots, T_{n-2}^{\mathbb{C}}$, and $M$ the maximum of the orders of the Lie derivative $\mathrm{L}_\nu \Psi$ evaluated at the curves $T_0^{\mathbb{C}}, \ldots, T_{n-2}^{\mathbb{C}}$.

Let $\widetilde{M}$ be a natural number. If the formal transformations $\mathcal{H}$ and the formal series $\mathcal{K}$ conjugating $\nu$ and $\omega$ satisfy

- the invertible formal transformation $\mathcal{H}$ has a $(2\widetilde{M} + n)$-jet equal to the identity, and
- the formal series $\mathcal{K}$ has a $(2\widetilde{M} + n)$-jet equal to the constant function 1,

then, by the equalities (4.4) and (4.7), the blow-down of the defined transformations (6.3) have $\widetilde{M}$-jet equal to the identity. By Lemma 3.0.3 we can assume that $\mathcal{H}$ and $\mathcal{K}$ have $(2\widetilde{M}+n)$-jet equal to the identity. So we can apply Lemma 6.0.6 considering $\widetilde{M} = M + 2n + 1$.

Therefore, there exists $\mathscr{H} : (\mathbb{R}^2, \widehat{0}) \longrightarrow (\mathbb{R}^2, \widehat{0})$ a real analytic diffeomorphism satisfying

- its linear part at the origin is the identity matrix,
- its complexification $\mathscr{H}^{\mathbb{C}}$ coincides with the identity map on $S_1 \cup \cdots \cup S_{n+1}$, and
- its complexification $\mathscr{H}^{\mathbb{C}}$ coincides on the curve $T_j^{\mathbb{C}}$ with the blow-down of the germ described in the equation (6.3), for all $j = 0, \ldots, n-2$.

Thus the germ $\widetilde{w} := \big(\mathrm{D}\,\mathscr{H} \cdot \omega\big) \circ \mathscr{H}^{-1}$ belongs to the class $\Sigma_n^{\mathbb{R}}$ and is strictly real-formally orbitally equivalent to $\nu$. Moreover, the restriction of its complexification on $T_0^{\mathbb{C}} \cup \cdots \cup T_{n-2}^{\mathbb{C}}$ is equal to the germ $\nu^{\mathbb{C}}$ (as a consequence $\widetilde{T}$ is the set of tangency points between $\widetilde{\mathscr{F}}_{\widetilde{\omega}}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}^{\mathbb{C}}$).

Finally, we shall analyze the holonomy maps for the foliation $\widetilde{\mathscr{F}}_{\widetilde{\omega}}^{\mathbb{C}}$. We shall denote the desingularization of $\mathscr{H}^{\mathbb{C}}$ by $\widetilde{\mathscr{H}}^{\mathbb{C}}$. Given a homotopy class $\alpha \in \Pi_1(\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}, \mathfrak{q}_0)$, we shall denote by $\Delta_\alpha^{\widetilde{\omega}} : \big(T_0^{\mathbb{C}}, \mathfrak{q}_0\big) \longrightarrow \big(T_0^{\mathbb{C}}, \mathfrak{q}_0\big)$ the holonomy map over $\alpha$ for the foliation $\widetilde{\mathscr{F}}_{\widetilde{\omega}}^{\mathbb{C}}$, then

$$\Delta_\alpha^{\widetilde{\omega}} = \widetilde{\mathscr{H}}^{\mathbb{C}}\big|_{T_0^{\mathbb{C}}} \circ \Delta_\alpha^{\omega} \circ \Big(\widetilde{\mathscr{H}}^{\mathbb{C}}\big|_{T_0^{\mathbb{C}}}\Big)^{-1} = (\mathfrak{h}^{\mathbb{C}})^{-1} \circ \Delta_\alpha^{\omega} \circ \mathfrak{h}^{\mathbb{C}} = \Delta_\alpha^{\nu} \,.$$

Moreover, considering $\Delta_{\gamma_j}^{\widetilde{\omega}} : \big(T_j^{\mathbb{C}}, \mathfrak{q}_j\big) \longrightarrow \big(T_0^{\mathbb{C}}, \mathfrak{q}_0\big)$ the holonomy map over the path $\gamma_j$ for the foliation $\widetilde{\mathscr{F}}_{\widetilde{\omega}}^{\mathbb{C}}$, we have

$$\begin{aligned} \Delta_{\gamma_j}^{\widetilde{\omega}} &= \widetilde{\mathscr{H}}^{\mathbb{C}}\big|_{T_0^{\mathbb{C}}} \circ \Delta_{\gamma_j}^{\omega} \circ \Big(\widetilde{\mathscr{H}}^{\mathbb{C}}\big|_{T_j^{\mathbb{C}}}\Big)^{-1} \\ &= (\mathfrak{h}^{\mathbb{C}})^{-1} \circ \Delta_{\gamma_j}^{\omega} \circ \Big(\Delta_{\gamma_j^{-1}}^{\omega} \circ \mathfrak{h}^{\mathbb{C}} \circ \Delta_{\gamma_j}^{\nu}\Big) = \Delta_{\gamma_j}^{\nu} \,. \end{aligned}$$

■

Now we are ready to construct the biholomorphism between $\widetilde{\mathscr{F}}_{\nu}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_{\omega}^{\mathbb{C}}$ on a neighborhood of $\mathbb{D}_{\mathbb{C}}$ but far from the singular points $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ and the tangency points $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}$.

Let $\mathfrak{q} \in \mathbb{D}_{\mathbb{C}}$ be a nonsingular point different from $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}$. We denote by $\mathscr{L}_{\mathfrak{q}}$ the leaf of the ary foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$ passing through $q$. We consider a path $\gamma$ which is contained in $\mathbb{D}_{\mathbb{C}} \smallsetminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}$, beginning at $\mathfrak{q}_0$ and ending at $\mathfrak{q}$, then $\Delta_\gamma^{\nu}, \Delta_\gamma^{\omega} : \big(T_0^{\mathbb{C}}, \mathfrak{q}_0\big) \longrightarrow \big(\mathscr{L}_{\mathfrak{q}}, \mathfrak{q}\big)$ will be the holonomy maps over $\gamma$ for the foliations $\widetilde{\mathscr{F}}_{\nu}^{\mathbb{C}}$ and $\widetilde{\mathscr{F}}_{\omega}^{\mathbb{C}}$, respectively.

We define a biholomorphism on $(\mathscr{L}_{\mathfrak{q}}, \mathfrak{q})$

$$\boldsymbol{\mathcal{H}}\big|_{\mathscr{L}_{\mathfrak{q}}} := \Delta_{\gamma_{\mathfrak{q}}}^{\omega} \circ \Delta_{\gamma_{\mathfrak{q}}^{-1}}^{\nu} \,. \tag{6.4}$$

From these biholomorphisms we obtain a biholomorphism $\boldsymbol{\mathcal{H}}$ defined on a neighborhood of $\mathbb{D}_{\mathbb{C}}$ except for open sets around the singular points $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ and the tangency points $\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}$, where the leaves of $\widetilde{\mathscr{F}^{\mathbb{C}}}$ are tangent to the exceptional divisor (such open sets are assume to be as small as needed).

In the following chapters we shall prove that this biholomorphism is extended around the singular points and the tangency points, and moreover, the real Möbius band is invariant by this biholomorphism.

# Chapter 7

# Extension around the Polar Curves

In what follows we consider two germs of vector fields $\vartheta, \upsilon \in \Sigma_n^{\mathbb{R}}$ such that their respective desingularizations $\widetilde{\mathscr{F}_\vartheta}$ and $\widetilde{\mathscr{F}_\upsilon}$ do not have a singular point at $\mathfrak{q}_j$ and whose set of tangency points with the auxiliar foliation $\widetilde{\mathscr{F}}$ is the polar curve $T_j$. The goal of this section is to prove the existence of a unique real analytic diffeomorphism defined in a neighbordhood in $\mathbb{M}_{\mathbb{R}}$ of a point of tangency $\mathfrak{q}_j$ sending (locally) the pair of foliations $\left(\widetilde{\mathscr{F}_\vartheta}, \widetilde{\mathscr{F}}\right)$ to the pair $\left(\widetilde{\mathscr{F}_\upsilon}, \widetilde{\mathscr{F}}\right)$ and being the identity map on $T_j \cup \mathbb{D}_{\mathbb{R}}$ (Lemma 7.0.11 and Corollary 7.0.12). The proof will require Lemmas 7.0.7 and 7.0.8.

***Lemma* 7.0.7.** *Around every point $\mathfrak{q}_j$ the desingularization of the foliation induced by $\nu_A^{\mathbb{C}}$ (which is described in Lemma 5.1.1) has a first integral $\mathcal{J}_j^{\mathbb{C}}$ which is the complexification of a real analytic germ. In the coordinate chart $(x,u)$ $\mathcal{J}_j^{\mathbb{C}}$ satisfies*

$$\mathcal{J}_j^{\mathbb{C}}(0,q_j) = \mathcal{J}_{j,u}^{\mathbb{C}}(0,q_j) = 0\,, \quad \mathcal{J}_{j,x}^{\mathbb{C}}(0,q_j)\mathcal{J}_{j,u^2}^{\mathbb{C}}(0,q_j) \neq 0\,. \tag{7.1}$$

*where $u(\mathfrak{q}_j) = q_j$.*

**Proof**. We consider a fixed natural number $1 \leq j \leq n-2$. By the proof of Lemma 5.1.1 we have a local description of $\widetilde{\mathscr{F}}^{\mathbb{C}}$ around $\mathfrak{q}_j$, namely, the complexification of the following real analytic equation

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = f_j(u) + g_j(x,u)\,, \tag{7.2}$$

where $f_j, g_j$ are real analytic functions defined around neighborhoods of $u(\mathfrak{q}_j) = q_j$ and $(0,q_j)$, respectively, satisfying $g_j(x,u) = \mathrm{O}(x)$ and $\mathrm{ord}_{q_j}(f_j(u)) = 1$.

Let $\phi_j(u,x)$ be the flow of the nonautonomous equation (7.2) satisfying $\phi_j(q_j,x) = x$, then the real analytic function

$$(x,u) \overset{H_j}{\longmapsto} (\phi_j(u,x),u) \ ,$$

is invertible in a neighborhood of $(0,q_j)$, since $\mathrm{D}\, H_j(0,q_j)$ is the identity matrix $\left(\frac{\partial \phi_j}{\partial x}\big|_{(q_j,0)} = 1 \text{ and } \frac{\partial \phi_j}{\partial u}\big|_{(q_j,0)} = 0\right)$.

The complexification of $H_j$ maps the trivial foliation $\{x = c\}_{c\in(\mathbb{C},0)}$ to the foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$. Thus the complexification of the real analytic function $\mathcal{J}_j = \pi_x \circ H_j^{-1}$ is a first integral of $\widetilde{\mathscr{F}}^{\mathbb{C}}$ around $(0,q_j)$ (here $\pi_x$ is the projection $(x,u) \mapsto x$).

From the properties of $f_j$, $g_j$ and $\phi_j$ we obtain $\mathcal{J}_j(0,q_j) = \mathcal{J}_{j,u}(0,q_j) = 0$, $\mathcal{J}_{j,x}(0,q_j) = 1$ and

$$\mathcal{J}_{j,u^2}(0,q_j) = -f_{j,u}(q_j) = -\prod_{\substack{l=1,\\ l\neq j}}^{n-2}(q_j - q_l)\,.$$

Since for every $k,l \geq 0$ $\frac{\partial^{k+l}\mathcal{J}_j^{\mathbb{C}}}{\partial x^k \partial u^l}$ is the complexification of $\frac{\partial^{k+l}\mathcal{J}_j}{\partial x^k \partial u^l}$, we can conclude the result from the previous equalities.

■

From now on the desingularization of the foliation induced by $\nu_A$ will be denoted by $\widetilde{\mathscr{F}}$. We consider the germ $\vartheta \in \Sigma^n_{\mathbb{R}}$ described at the beginning of this chapter.

***Lemma* 7.0.8.** *Let $H_0$ be a real analytic diffeomorphism defined on a neighborhood of the point $\mathfrak{q}_j$ on $\mathbb{M}_{\mathbb{R}}$ such that $H_0\big|_{\mathbb{D}_{\mathbb{R}}} = id_{\mathbb{D}_{\mathbb{R}}}$ and sends $T_j$ to the line $T'_j$ described as $\{u = q_j\}$ in the coordinates $(x,u)$.*

*Then there exists a unique real analytic diffeomorphism $\widetilde{H}_j$ defined on a neighborhood of the point $\mathfrak{q}_j$ on $\mathbb{M}_{\mathbb{R}}$ satisfying*

*i.* $\widetilde{H}_j\big|_{T_j} = H_0\big|_{T_j}$, $\widetilde{H}_j\big|_{\mathbb{D}_{\mathbb{R}}} = id_{\mathbb{D}_{\mathbb{R}}}$.

*ii. It carries the foliation $\widetilde{\mathscr{F}_\vartheta}$ to the trivial foliation $\{x = c\}_{c\in(\mathbb{R},0)}$, and the foliation $\widetilde{\mathscr{F}}$ to $\{x + t_j(u) = c\}_{c\in(\mathbb{R},0)}$, where $t_j : (\mathbb{R},q_j) \longrightarrow (\mathbb{R},0)$ is a nonconstant real analytic function satisfying $t_j(q_j) = t_{j,u}(q_j) = 0$.*

Considering the complexification of the real analytic diffeomorphism $\widetilde{H}_j$, we immediately obtain the following corollary.

***Corollary* 7.0.9.** *Let $H_0$ be a real analytic diffeomorphism defined on a neighborhood of the point $\mathfrak{q}_j$ on $\mathbb{M}_{\mathbb{R}}$ such that $H_0\big|_{\mathbb{D}_{\mathbb{R}}} = id_{\mathbb{D}_{\mathbb{R}}}$ and sends $T_j$ to the line $T'_j$ described as $\{u = q_j\}$ in the coordinates $(x,u)$.*

*Then there exists a unique real analytic diffeomorphism $\widetilde{H}_j$ defined on a neighborhood of the point $\mathfrak{q}_j$ on $\mathbb{M}_{\mathbb{R}}$ and whose complexification satisfies* *i.* $\widetilde{H}_j^{\mathbb{C}}\big|_{T_j^{\mathbb{C}}} = H_0^{\mathbb{C}}\big|_{T_j^{\mathbb{C}}}$, $\widetilde{H}_j^{\mathbb{C}}\big|_{\mathbb{D}_{\mathbb{C}}} = id_{\mathbb{D}_{\mathbb{C}}}$.

*ii. The complexification* $\widetilde{H}_j^{\mathbb{C}}$ *carries the foliation* $\widetilde{\mathscr{F}}_{\vartheta}^{\mathbb{C}}$ *to the trivial foliation* $\{x = c\}_{c\in(\mathbb{C},0)}$, *and the foliation* $\widetilde{\mathscr{F}}^{\mathbb{C}}$ *to the foliation* $\{x + t_j^{\mathbb{C}}(u) = c\}_{c\in(\mathbb{C},0)}$, *where* $t_j : (\mathbb{R}, q_j) \longrightarrow (\mathbb{R}, 0)$ *is a nonconstant real analytic function satisfying* $t_j(q_j) = t_{j,u}(q_j) = 0$.

**Proof of Lemma 7.0.8.** We begin proving the existence of $\widetilde{H}_j$. We will work with the foliations $H_0(\widetilde{\mathscr{F}}_{\vartheta})$ and $H_0(\widetilde{\mathscr{F}})$.

We consider the nonautonomous real analytic equation $\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = \vartheta_j(x, u)$ whose extended phase portrait coincides with the foliation $H_0(\widetilde{\mathscr{F}}_{\vartheta})$. Since the real exceptional divisor is a leaf of this foliation then $\vartheta_j(x, u) = \mathrm{O}(\|x\|)$. We shall denote by $\phi_{\vartheta_j}(u, x)$ the flow of the nonautonomous equation satisfying $\phi_{\vartheta_j}(q_j, x) = x$.

The function $(x, u) \mapsto (\phi_{\vartheta_j}(u, x), u)$ is a real analytic diffeomorphism defined on a neighborhood of $(0, q_j)$. By definition this diffeomorphism maps the trivial foliation $\{x = c\}_{c\in(\mathbb{R},0)}$ to the foliation $H_0(\widetilde{\mathscr{F}}_{\vartheta})$ and its restrictions on the lines $\{x = 0\}$ and $\{u = q_j\}$ are the identity. We shall denote its inverse by $\widetilde{H}_0$ (see Figure 7.1).

Now we shall analyze the foliation $\widetilde{H}_0 \circ H_0(\widetilde{\mathscr{F}})$. Considering $\mathcal{J}_j$ as in Lemma 7.0.7, the scalar function $\mathcal{J}_j \circ H_0^{-1} \circ \widetilde{H}_0^{-1}$ is a first integral for this foliation. Below this scalar function will be changed for another first integral (equation (7.3) which will be similar to that given in the statement ii. Finally, after applying another coordinate change, we will obtain the required properties.

We have the real analytic function $\phi(x) := \mathcal{J}_j \circ H_0^{-1}(x, q_j)$ is invertible in a neighborhood of 0 by the chain rule and the inverse function theorem. As a consequence $\widetilde{\mathcal{J}}_j := \phi^{-1} \circ \mathcal{J}_j \circ H_0^{-1} \circ \widetilde{H}_0^{-1}$ is a first integral of $\widetilde{H}_0 \circ H_0(\widetilde{\mathscr{F}})$ satisfying $\widetilde{\mathcal{J}}_j(x, q_j) = x$. Moreover, we have $\widetilde{\mathcal{J}}_{j,u}(x, q_j) = 0$, since the level curves of $\widetilde{\mathcal{J}}_j$ are tangent to the trivial foliation $\{x = c\}_{c\in(\mathbb{R},0)}$ at every point in the line $\{u = q_j\}$. Then

$$\widetilde{\mathcal{J}}_j(x, u) = x + (u - q_j)^2 \widehat{\mathcal{J}}_j(x, u)\,, \tag{7.3}$$

where $\widehat{\mathcal{J}}_j$ is a real analytic function defined on a neighborhood of $(0, q_j)$ (see Figure 7.1).

In what follows we will express the function $(u - q_j)^2 \widehat{\mathcal{J}}_j(x, u)$ in terms of one variable by means of a change of coordinates. For this purpose we will prove that $\widehat{\mathcal{J}}_j(0, q_j) \neq 0$.

By the expression (7.3) we have

$$\begin{aligned}\widetilde{\mathcal{J}}_{j,u^2}(0, q_j) = \big(&2\widehat{\mathcal{J}}_j(x, u) + 4(u - q_j)\widehat{\mathcal{J}}_{j,u}(x, u)\\ &+ (u - q_j)^2 \widehat{\mathcal{J}}_{j,u^2}(x, u)\big)\big|_{(0,q_j)} = 2\widehat{\mathcal{J}}_j(0, q_j)\,,\end{aligned}$$

that is, $\widehat{\mathcal{J}}_j(0,q_j)\neq 0$ if and only if $\widetilde{\mathcal{J}}_{j,u^2}(0,q_j)\neq 0$.

On the other hand, since the restriction of the composition $H_0^{-1}\circ \widetilde{H}_0^{-1}$ on the line $\{x=0\}$ is the identity, then by the definition of $\widetilde{\mathcal{J}}_j$ it follows that

$$\begin{aligned}\widetilde{\mathcal{J}}_{j,u}(0,\widetilde{u}) &= \lim_{u\to\widetilde{u}}\frac{\phi^{-1}\circ\mathcal{J}_j(0,u)-\phi^{-1}\circ\mathcal{J}_j(0,\widetilde{u})}{u-\widetilde{u}} = \left.\frac{\partial\left(\phi^{-1}\circ\mathcal{J}_j\right)}{\partial u}\right|_{(0,\widetilde{u})}\\ &= (\phi^{-1})'(\mathcal{J}_j(0,\widetilde{u}))\mathcal{J}_{j,u}(0,\widetilde{u})\,.\end{aligned}$$

As a consequence, $\widetilde{\mathcal{J}}_{j,u^2}(0,q_j)=(\phi^{-1})'(0)\mathcal{J}_{j,u^2}(0,q_j)$. Since $\phi$ is invertible in a neighborhood of 0 and $\mathcal{J}_{j,u^2}(0,q_j)\neq 0$ (Lemma 7.0.7), this is nonzero. Therefore $\widehat{\mathcal{J}}_j(0,q_j)\neq 0$.

Now we will construct the change of coordinates. For this purpose we will consider two cases.

**Case 1:** the number $\widehat{\mathcal{J}}_j(0,q_j)$ is positive. In this case there exists a neighborhood of $(0,q_j)$ where $\widehat{\mathcal{J}}_j$ takes only positive values. For this reason there is a real analytic function $G_j$ defined on a neighborhood of $(0,q_j)$ satisfying

$$(G_j(x,u))^2=(u-q_j)^2\widehat{\mathcal{J}}_j(x,u)\,. \tag{7.4}$$

Because of $\widehat{\mathcal{J}}_j(0,q_j)$ is nonzero, we can conclude $G_{j,u}(0,q_j)$ is nonzero. Then the function $\mathcal{G}_j(u):=G_j(0,u)$ is invertible in a neighborhood of $q_j$.

Since the derivative in $u$ of $\mathcal{G}_j^{-1}\circ G_j(x,u)$ at the point $(0,q_j)$ is 1, the function $\widehat{H}_0$ defined by $(x,u)\mapsto(x,\mathcal{G}_j^{-1}\circ G_j(x,u))$ is invertible in a neighborhood of $(0,q_j)$. By definition $\widehat{H}_0$ preserves the trivial foliation $\{x=c\}_{c\in(\mathbb{R},0)}$ and $\widehat{H}_0\big|_{\{x=0\}}=\mathrm{id}_{\{x=0\}}$, $\widehat{H}_0\big|_{\{u=q_j\}}=\mathrm{id}_{\{u=q_j\}}$.

Therefore, the composition $\widehat{H}_0\circ\widetilde{H}_0\circ H_0$ is a real analytic diffeomorphism defined on a neighborhood of $(0,q_j)$ which coincides with $H_0$ on the curve $T_j$ and whose restriction on $\{x=0\}$ is the identity. Moreover, this diffeomorphism maps the foliation $\widetilde{\mathscr{F}_\vartheta}$ to the trivial foliation $\{x=c\}_{c\in(\mathbb{R},0)}$, and sends leaves of the foliation $\widetilde{\mathscr{F}}$ into level curves of $\widetilde{\mathcal{J}}_j\circ\widehat{H}_0^{-1}$ (see Figure 7.1). Finally, we shall analyze this last scalar function.

Denoting $\widehat{H}_0^{-1}(x,u)=(x,h(x,u))$, we have $\mathcal{G}_j^{-1}\circ G_j(x,h(x,u))=u$. This is equivalent to $G_j(x,h(x,u))=\mathcal{G}_j(u)$. Then

$$\widetilde{\mathcal{J}}_j\circ\widehat{H}_0^{-1}(x,u)=\widetilde{\mathcal{J}}_j(x,h(x,u))=x+\big(G_j(x,h(x,u))\big)^2=x+\big(\mathcal{G}_j(u)\big)^2\,.$$

**Case 2:** the number $\widehat{\mathcal{J}}_j(0,q_j)$ is negative. In this case there exists a neighborhood of $(0,q_j)$ where the function $\widehat{\mathcal{J}}_j$ takes only negative values. We consider the real analytic function $G_j$ satisfying

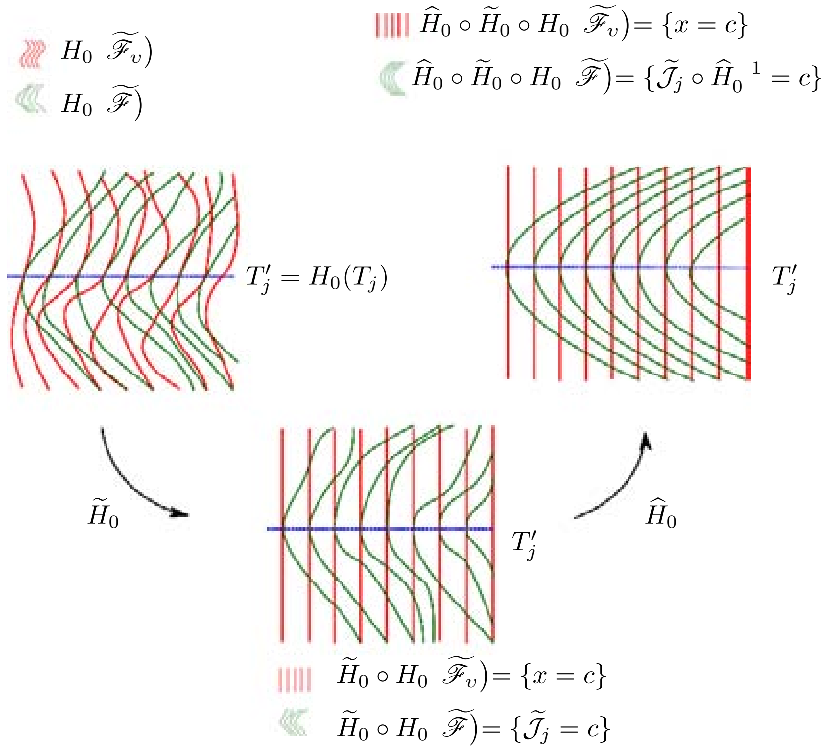


Figure 7.1: The Formal Conjugation of the Holonomy Groups.

$$(G_j(x,u))^2 = \quad (u \quad q_j)^2 \widehat{\mathcal{J}}_j(x,u)\,. \tag{7.5}$$

Defining $\mathcal{G}_j$ and $\widehat{H}_0$ as in the previous case we obtain real analytic functions with the same properties as before, except for the description of the first integral of $\widehat{H}_0 \circ \widetilde{H}_0 \circ H_0(\widetilde{\mathscr{F}})$. In this case this scalar function is

$$\widetilde{\mathcal{J}}_j \circ \widehat{H}_0 \ ^1(x,u) = \widetilde{\mathcal{J}}_j(x, h(x,u)) = x \qquad G_j(x, h(x,u)))^2 = x \qquad \mathcal{G}_j(u))^2\,.$$

We have concluded the *existence*. In what follows we will prove *the uniqueness*. We consider two real analytic diffeomorphisms $\widetilde{H}_1, \widetilde{H}_2$ satisfying the statements i and ii. In particular $\widetilde{H}_i$ maps the foliation $\widetilde{\mathscr{F}}$ to the foliation $\{x + t_i(u) = c\}_{c \in (\mathbb{R},0)}$, where $t_i$ is a nonconstant real analytic function defined on a neighborhood of $q_j$ which satisfies $t_i(q_j) = t'_i(q_j) = 0$.

As a consequence $\widetilde{H} := \widetilde{H}_2 \circ \widetilde{H}_1 \ ^1$ is a real analytic diffeomorphism with the following properties:

- $\widetilde{H}\big|_{\{x=0\}} = \mathrm{id}_{\{x=0\}}$ and $\widetilde{H}\big|_{T'_j} = \mathrm{id}_{T'_j}$.

- It preserves the trivial foliation $\{x = c\}_{c\in(\mathbb{R},0)}$.
- It maps the foliation $\{x + t_1(u) = c\}_{c\in(\mathbb{R},0)}$ to $\{x + t_2(u) = c\}_{c\in(\mathbb{R},0)}$.

As an immediate consequence of these properties, there exists a real analytic function $\widetilde{h}$ defined around $(0, q_j)$ satisfying

$$\widetilde{H}(x, u) = (x, u + x(u - q_j)\widetilde{h}(x, u)) . \tag{7.6}$$

We shall conclude the uniqueness proving that $\widetilde{h}$ vanishes identically.

First we will show that $t_1 = t_2$. The diffeomorphism $\widetilde{H}$ maps the curve $\{x + t_1(u) = c\}$ to the curve $\{x + t_2(u) = c\}$, since the line $\{x = c\}$ is mapped into itself and $t_1(q_j) = t_2(q_j) = 0$. Let $I$ and $J$ be open neighborhoods of 0 and $q_j$, respectively, satisfying that the product $I \times J$ is contained in the domain of $\widetilde{H}$. Then for every $u_0 \in J$,

$$(0, u_0) = \widetilde{H}(0, u_0) \in \widetilde{H}\ \big(\{x + t_1(u) = t_1(u_0)\}\big) = \big\{x + t_2(u) = t_1(u_0)\big\} ,$$

that is, $t_1(u_0) = t_2(u_0)$ for every $u_0 \in J$. By the expression (7.6), given $c \in (\mathbb{R}, 0)$,

$$\widetilde{H}(c - t_1(u), u) = \big(c - t_1(u), u + (c - t_1(u))(u - q_j)\widetilde{h}(c - t_1(u), u)\big)$$

coincides with $\{x + t_1(u) = c\}$, that is,

$$t_1(u) = t_1\big(\Gamma(c, u)\big) , \tag{7.7}$$

where

$$\Gamma(c, u) = u + (c - t_1(u))(u - q_j)\widetilde{h}(c - t_1(u), u)$$

is defined on a neighborhood of $(0, q_j)$. Now we will use the previous expression to prove that $\widetilde{h}$ vanishes identically.

Differentiating with respect to $c$ the equality (7.7), we obtain the following equality in a neighborhood of $(c, u) = (0, q_j)$

$$0 = t_1'\big(\Gamma(c, u)\big)\cdot(u - q_j) \cdot \widehat{h}(c, u) , \tag{7.8}$$

where $\widehat{h}(c, u) := \widetilde{h}(c - t_1(u), u) + (c - t_1(u))\widetilde{h}_x(c - t_1(u), u)$.

We will observe that the equality (7.8) is equivalent to $\widehat{h} \equiv 0$. Since $t_1$ is a nonconstant function, $q_j$ is an isolated zero of $t_1'$. On the other hand, $\Gamma$ is a submersion around $(0, q_j)$ since $\nabla\Gamma(0, q_j) = (0, 1)$; in particular the level curve $\{\Gamma = q_j\}$ is the line $\{u = q_j\}$. Therefore, there exists a neighborhood of $(c, u) = (0, q_j)$ where the equality (7.8) takes place if and only if $\widehat{h}$ vanishes identically.

Since the function $(c, u) \mapsto (c - t_1(u), u)$ is a real analytic diffeomorphism around $(0, q_j)$, $\widehat{h} \equiv 0$ if and only if $\widetilde{h}(x, u) = -x\, \widetilde{h}_x(x, u)$ around $(x, u) = (0, q_j)$. We will prove from this last equality that $\widetilde{h}$ is the identically zero function in a neighborhood of $(0, q_j)$.

We consider $\widetilde{I}$ and $\widetilde{J}$ open intervals centered at 0 and $q_j$ respectively, satisfying $\widetilde{I} \times \widetilde{J}$ is contained in the domain of the function $\widetilde{h}$. One can verify by induction that $(k+1)\widetilde{h}_{x^k} = -x\widetilde{h}_{x^{k+1}}$ in $\widetilde{I} \times \widetilde{J}$, for every $k \geqslant 1$. Then, for every $u_0 \in \widetilde{J}$, $\widetilde{h}(0, u_0) = \big(-x\widetilde{h}^1_x(x, u)\big)\big|_{(0,u_0)} = 0$ and for every $k \geq 1$, $\widetilde{h}_{x^k}(0, u_0) = \big(-\frac{x}{k+1}\widetilde{h}_{x^{k+1}}(x, u)\big)\big|_{(0,u_0)} = 0$.

Because of $\widetilde{h}$ is real analytic, for every $u \in \widetilde{J}$ there is an open interval $\widetilde{I}_u \subseteq \widetilde{I}$ centered at 0 such that for every $x \in \widetilde{I}_u$,

$$\widetilde{h}(x, u) = \sum_{k \geq 0} \frac{1}{k!}\, \widetilde{h}_{x^k}(0, u)\, x^k = 0\,.$$

Since $\widetilde{I}$ is connected, $\widetilde{h}(x, u) = 0$ for every $x \in \widetilde{I}$. The previous property is obtained by all $u \in \widetilde{J}$, therefore $\widetilde{h}\big|_{\widetilde{I} \times \widetilde{J}} \equiv 0$.

■

**Remark 7.0.10.** The proof of the Lemma 7.0.8 allows us to conclude the following result:

*Let $\widetilde{H} : (\mathbb{R}^2, (0, q_j)) \longrightarrow (\mathbb{R}^2, (0, q_j))$ be a real analytic diffeomorphism which is the identity on the lines $\{x = 0\}$ and $\{u = q_j\}$, preserves the trivial foliation $\{x = c\}_{c \in (\mathbb{R},0)}$ and sends the foliation $\{x + t_1(u) = c\}_{c \in (\mathbb{R},0)}$ to the foliation $\{x + t_2(u) = c\}_{c \in (\mathbb{R},0)}$, where $t_i$ is a nonconstant real analytic function and $t_i(q_j) = t_i'(q_j) = 0$ for $i = 1, 2$. Then $t_1$ is equal to $t_2$ and moreover, $\widetilde{H}$ is the identity function.*

In addition, we consider the germ $\upsilon \in \Sigma_n^{\mathbb{R}}$ satisfying the properties indicated in the following lemma.

***Lemma* 7.0.11.** *We suppose that $\mathfrak{q}_j \in \mathbb{D}_{\mathbb{R}}$ is a nonsingular point of the foliation $\widetilde{\mathscr{F}_\upsilon}$ whose tangency points with the auxiliar foliation $\widetilde{\mathscr{F}}$ define the curve $T_j$. Then there exists a unique real analytic diffeomorphism on a neighborhood of $\mathfrak{q}_j$ in $\mathbb{M}_{\mathbb{R}}$, sending the pair of foliations $(\widetilde{\mathscr{F}_\vartheta}, \widetilde{\mathscr{F}})$ to the pair $(\widetilde{\mathscr{F}_\upsilon}, \widetilde{\mathscr{F}})$ locally near $\mathfrak{q}_j$ and being the identity map on $\mathbb{D}_{\mathbb{R}}$ and $T_j$.*

**Proof**. Since $T_j$ is transversal to $\{x = 0\}$, we can construct a real analytic diffeomorphism $\widehat{H} : (\mathbb{R}^2, (0, q_j)) \longrightarrow (\mathbb{R}^2, (0, q_j))$ being the identity map on $\{x = 0\}$ and sending the curve $T_j$ into the line $T'_j = \{u = q_j\}$. In this way, considering the construction of Lemma 7.0.8, we can find real analytic diffeomorphisms $\widehat{H}_1, \widehat{H}_2 : (\mathbb{R}^2, (0, q_j)) \longrightarrow (\mathbb{R}^2, (0, q_j))$ being the identity map on $\{x = 0\}$, coinciding with $\widehat{H}$ in $T_j$ and mapping the pairs of foliations $(\widetilde{\mathscr{F}_\vartheta}, \widetilde{\mathscr{F}})$ and $(\widetilde{\mathscr{F}_\upsilon}, \widetilde{\mathscr{F}})$ to the pairs

$$\begin{aligned}&(\{x = c\}_{c\in(\mathbb{R},0)}, \{x + t_1(u) = c\}_{c\in(\mathbb{R},0)}) \quad \text{and}\\ &(\{x = c\}_{c\in(\mathbb{R},0)}, \{x + t_2(u) = c\}_{c\in(\mathbb{R},0)})\,,\end{aligned}$$

respectively, where $t_i$ is a nonconstant real analytic function satisfying $t_i(q_j) = t'_i(q_j) = 0$, for $i = 1, 2$.

We shall check that $t_1 = t_2$. Let $\mathcal{J}_j$ be the first integral of the foliation $\widetilde{\mathscr{F}}$ described in Lemma 7.0.7. Since $\mathcal{J}_{j,x}(0, q_j) \neq 0$, $\mathcal{J}_{j,u}(0, q_j) = 0$ and the curve $T_j$ is transversal to the divisor $\mathbb{D}_\mathbb{R}$, $\mathcal{J}_j\big|_{T_j}$ is invertible in a neighborhood of $(0, q_j) \in T_j$. Thus without loss of generality we can assume that $\widehat{H}_1$ and $\widehat{H}_2$ are defined on $I \times J$, where $I$ and $J$ are open intervals around 0 and $q_j$ respectively, satisfying $\big(I \times J\big)\cap T_j$ is an open set of $T_j$ where $\mathcal{J}_j$ is invertible. Also we shall suppose that $t_1$ and $t_2$ are defined on an interval $\widehat{J} \subseteq J$ satisfying $(t_i(u), q_j) \in \widehat{H}_i\big(I \times J\big)$, for every $u \in \widehat{J}$.

We consider $\widehat{u} \in \widehat{J}$ and denote by $\mathscr{L}_{(0,\widehat{u})}$ the leaf of the foliation $\widetilde{\mathscr{F}}$ passing through $(0, \widehat{u})$. For $i = 1, 2$ $\widehat{H}_i(0, \widehat{u}) = (0, \widehat{u})$, and as a consequence

$$\widehat{H}_i\big(\mathscr{L}_{(0,\widehat{u})}\big)= \{x + t_i(u) = t_i(\widehat{u})\}\,.$$

Then $\big(t_i(\widehat{u}), q_j\big)\in \widehat{H}_i\big(\mathscr{L}_{(0,\widehat{u})}\big)$, that is,

$$\widehat{H}_i^{-1}\big(t_i(\widehat{u}), q_j\big)\in \mathscr{L}_{(0,\widehat{u})} \cap T_j \cap \big(I \times J\big)\,.$$

Since a leaf of the foliation $\widetilde{\mathscr{F}}$ cannot intersect $T_j \cap \big(I \times J\big)$ in two different points (since the first integral $\mathcal{J}_j$ is invertible in this set), then

$$\widehat{H}^{-1}\big(t_1(\widehat{u}), q_j\big)= \widehat{H}_1^{-1}\big(t_1(\widehat{u}), q_j\big)= \widehat{H}_2^{-1}\big(t_2(\widehat{u}), q_j\big)= \widehat{H}^{-1}\big(t_2(\widehat{u}), q_j\big)\,,$$

thus $t_1(\widehat{u}) = t_2(\widehat{u})$. Therefore the diffeomorphism $\widehat{H}_2^{-1} \circ \widehat{H}_1$ satisfies the required properties.

Finally, we will check the uniqueness. Given a real analytic diffeomorphism $H$ which satisfies the required conditions, then by Remark 7.0.10, $\widehat{H}_2 \circ H \circ \widehat{H}_1^{-1}$ must be the identity map.

■

The proof of the following corollary is obtained by the complexification of the real analytic diffeomorphism of Lemma 7.0.11.

***Corollary* 7.0.12.** *We suppose that* $\mathfrak{q}_j \in \mathbb{D}_{\mathbb{R}}$ *is a nonsingular point of the foliation* $\widetilde{\mathscr{F}_{\upsilon}^{\mathbb{C}}}$ *whose tangency points with the auxiliar foliation* $\widetilde{\mathscr{F}^{\mathbb{C}}}$ *constitute the curve* $T_j^{\mathbb{C}}$*, the complexification of the curve* $T_j$*. Then there exists a unique real analytic diffeomorphism defined on a neighborhood of* $\mathfrak{q}_j$ *in* $\mathbb{M}_{\mathbb{R}}$ *whose complexification sends the pair of foliations* $(\widetilde{\mathscr{F}_{\vartheta}^{\mathbb{C}}}, \widetilde{\mathscr{F}^{\mathbb{C}}})$ *to the pair* $(\widetilde{\mathscr{F}_{\upsilon}^{\mathbb{C}}}, \widetilde{\mathscr{F}^{\mathbb{C}}})$ *locally near* $\mathfrak{q}_j$ *and is the identity map on* $\mathbb{D}_{\mathbb{C}}$ *and* $T_j^{\mathbb{C}}$*.*

# Chapter 8

# Extension around the Singular Points

In Chapter 7 we have proved that the biholomorphism $\boldsymbol{\mathcal{H}}$ defined at the end of Chapter 6 (see the equation 7) can be extended to neighborhoods of the polar curves $T_1^{\mathbb{C}}, \dots, T_{n-2}^{\mathbb{C}}$. To conclude the proof of the real-formal orbital rigidity theorem it remains only to check that this biholomorphism can be extended to neighborhoods of the singular points (Section 8.1) and finally, to show that the real Möbius band is invariant by the biholomorphism $\boldsymbol{\mathcal{H}}$ (Section 8.2).

## 8.1 Extension of $\boldsymbol{\mathcal{H}}$ around the Singular Points

Let $\mathfrak{p}_i$ be a fixed singular point, which is written as $(0, p_i)$ with respect to the coordinates $(x, u)$.

Since the holonomy maps are moduli for the orbital analytic classification of germs of complex saddles (see [EI84] or [IY08]), whenever $\lambda_i$, the characteristic number of $\mathfrak{p}_i$, is a negative number, the transformation $\boldsymbol{\mathcal{H}}$ extends holomorphically at $\mathfrak{p}_i$ preserving the auxiliary foliation $\widetilde{\mathscr{F}^{\mathbb{C}}}$. If $\lambda_i \in \mathbb{C} \smallsetminus \big(\mathbb{R}_- \cup \mathbb{Q}_+ \cup \{\widehat{0}\}\big)$, by Lemma 8.3.1 (Section 8.3), there exist $f_i, g_i : \big(\mathbb{M}_{\mathbb{C}}, \mathfrak{p}_i\big) \longrightarrow \big(\mathbb{C}^2, (0, p_i)\big)$ two biholomorphisms mapping the foliation $\widetilde{\mathscr{F}^{\mathbb{C}}}$ to the trivial foliation $\{u = c\}_{c \in (\mathbb{C}, 0)}$ and sending the foliations $\widetilde{\mathscr{F}_{\nu}^{\mathbb{C}}}$ and $\widetilde{\mathscr{F}_{\omega}^{\mathbb{C}}}$ into the foliation generated by the linear vector field $\vartheta(x, u) = (\lambda_i x, u - q_i)$, respectively. Moreover, in the coordinate chart $(x, u)$, $f_i\big|_{\{x=0\}} = g_i\big|_{\{x=0\}} = \mathrm{id}_{\{x=0\}}$, and if $p_i \in \mathbb{R}$ then $f_i$ and $g_i$ are complexifications of real analytic diffeomorphisms (that is, if $\mathfrak{p}_i \in \mathbb{M}_{\mathbb{R}}$ then $f_i$ and $g_i$ send $\mathbb{M}_{\mathbb{R}}$ into $\mathbb{R}^2$).

To conclude that $\boldsymbol{\mathcal{H}}$ can be extended to a neighborhood of $\mathfrak{p}_i$, it suffices to show that $g_i \circ \boldsymbol{\mathcal{H}} \circ f_i^{-1}$ can be extended to a neighborhood of $(0, p_i)$. In fact, we shall prove that the last map is extended by a transformation $(x, u) \mapsto (\alpha_i x, u)$ for some $\alpha_i \in \mathbb{C} \smallsetminus \{0\}$.

Without loss of generality we suppose that $f_i$ and $g_i$ have the same domain $U_i$, and $D_i := U_i \cap \mathbb{D}_{\mathbb{C}}$ is written with respect to the coordinates $(x, u)$ as an open disc centered at $(0, p_i)$, namely, $\widetilde{D}_i = \{(0, u) \,|\, |u - p_i| < r_i\}$, where $r_i > 0$.

We consider $\widetilde{\mathfrak{q}}_i \in U_i \setminus \{\mathfrak{p}_i\}$ being $(0, \widetilde{q}_i)$ with respect to the coordinates $(x, u)$. As before, $\mathscr{L}_{\widetilde{\mathfrak{q}}_i}$ is the leaf of the foliation $\widetilde{\mathscr{F}^{\mathbb{C}}}$ passing through $\widetilde{\mathfrak{q}}_i$. We take a path $\delta_i$ whose image is contained in $\mathbb{D}_{\mathbb{C}} \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}$, beginning at $\mathfrak{q}_0$ and ending at $\widetilde{\mathfrak{q}}_i$ (see Figure 8.1). We define

$$h_i := \boldsymbol{\mathcal{H}}\big|_{\mathscr{L}_{\widetilde{\mathfrak{q}}_i}} = \Delta^{\omega}_{\delta_i} \circ \Delta^{\nu}_{\delta_i}{}^{-1}.$$

The following remark will be useful to conclude the extension of $\boldsymbol{\mathcal{H}}$ around $\mathfrak{p}_i$.

**Remark 8.1.1.** We consider $A_i \subseteq \mathbb{D}_{\mathbb{C}}$ written with respect to coordinates $(x, u)$ as the annulus $\widetilde{A}_i = \left\{(0, u) \,\middle|\, \frac{|\widetilde{q}_i - p_i|}{2} < |u - p_i| < r_i\right\}$ (see Figure 8.1). There exists $V_i \subseteq \mathbb{M}_{\mathbb{C}}$ a neighborhood of $A_i$ in $\mathbb{M}_{\mathbb{C}}$ such that $V_i$ and $\boldsymbol{\mathcal{H}}(V_i)$ are contained in $U_i$ and it satisfies the following property:

Let $\gamma$ be a path beginning at $(0, q)$, ending at $(0, p)$ and whose image is contained in $\widetilde{D}_i \setminus \{(0, p_i)\}$. We shall denote the holonomy map for the linear vector field $\vartheta$ over $\gamma$ by $\Delta_\gamma : \{u = q\} \to \{u = p\}$. Then for every $(x, u) \in f_i(V_i)$

$$g_i \circ \boldsymbol{\mathcal{H}} \circ f_i^{-1}(x, u) = \Delta_{\gamma_{(0,u)}} \circ \left(g_i \circ h_i \circ f_i^{-1}\right) \circ \Delta_{\gamma^{-1}_{(0,u)}}, \tag{8.1}$$

where $\gamma_{(0,u)}$ is a path beginning at $(0, \widetilde{q}_i)$, ending at $(0, u)$ and whose image is contained in $\widetilde{A}_i$.

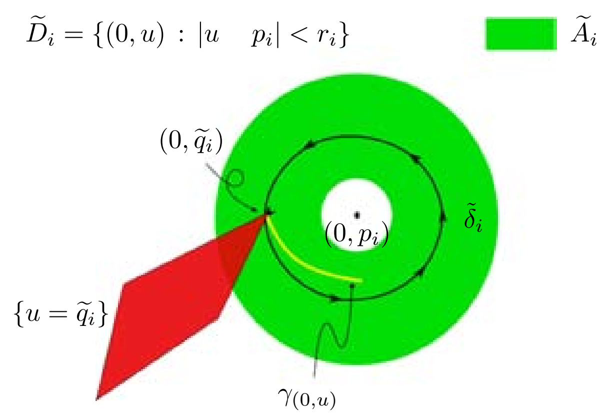


Figure 8.1: Construction in the coordinates $(x, u)$.

Now we will analyze the function $\widetilde{h}_i := g_i \circ h_i \circ f_i^{-1}$. Since this is an automorphisms of $\left(\{u = \widetilde{q}_i\}, (0, \widetilde{q}_i)\right)$, then there exists $\widehat{h}_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ a biholomorphism such that $\widetilde{h}_i(x, \widetilde{q}_i) = \left(\widehat{h}_i(x), \widetilde{q}_i\right)$.

Consider the holonomy maps defined in the above Remark. By the properties of $f_i$ and $g_i$ we have

$$\widetilde{h}_i \circ \Delta_\tau \circ \widetilde{h}_i^{-1} = \Delta_\tau\,, \tag{8.2}$$

for every loop $\tau$ with base point $(0,\widetilde{q}_i)$ contained in $\widetilde{D}_i \smallsetminus \{(0,p_i)\}$. Let $\widetilde{\delta}_i$ be a simple positive loop defined as

$$t \mapsto \big(0, p_i + (\widetilde{q}_i - p_i)\exp(2\pi \mathrm{i} t)\big)\,, \quad t \in [0,1]$$

around $(0,p_i)$ (see Figure 8.1). With respect to such a loop the equality (8.2) is equivalent to

$$\begin{aligned}\big(\widehat{h}_i(\exp(2\pi\mathrm{i}\lambda_i)x), \widetilde{q}_i\big) &= \widetilde{h}_i(\exp(2\pi\mathrm{i}\lambda_i)x, \widetilde{q}_i) \\ &= \Delta_{\widetilde{\delta}_i} \circ \widetilde{h}_i(x,\widetilde{q}_i) = \big(\exp(2\pi\mathrm{i}\lambda_i)\widehat{h}_i(x), \widetilde{q}_i\big)\,,\end{aligned}$$

for all $x \in (\mathbb{C},0)$. Since $\exp(2\pi\mathrm{i}\lambda_i)$ is not a root of unity, then $\widehat{h}_i$ is a linear map[1], that is, there exists $\alpha_i \in \mathbb{C}\smallsetminus\{0\}$ such that $\widehat{h}_i(x) = \alpha_i x$.

In what follows we shall check that the transformation $(x,u) \mapsto (\alpha_i x, u)$ extends to $g_i \circ \mathcal{H} \circ f_i^{-1}$ on a neighborhood of $(0,p_i)$. To conclude this property it suffices to prove that for every $(x,u) \in f_i(V_i)$

$$\Delta_{\gamma_{(0,u)}} \circ \widetilde{h}_i \circ \Delta_{\gamma_{(0,u)}^{-1}}(x,u) = (\alpha_i x, u)\,, \tag{8.3}$$

where $\gamma_{(0,u)}$ is any path beginning at $(0,\widetilde{q}_i)$, ending at $(0,u)$ and whose image is contained in $\widetilde{D}_i \smallsetminus \{(0,p_i)\}$.

If $u = \widetilde{q}_i$, the equality (8.3) follows directly. Now we suppose that $u \neq \widetilde{q}_i$. Then we consider a branch of complex logarithm such that $\frac{u-p_i}{\widetilde{q}_i - p_i}$ belongs to its domain. We choose the path defined as

$$t \mapsto \Big(0, \exp\big(t\ln\big(\tfrac{u-p_i}{\widetilde{q}_i-p_i}\big)\big)(\widetilde{q}_i - p_i) + p_i\Big)\,, \qquad t \in [0,1].$$

The image of this path is contained in $\widetilde{D}_i \smallsetminus \{(0,p_i)\}$ (see Figure 8.1). Then

$$\begin{aligned}\Delta_{\gamma_{(0,u)}}(x,\widetilde{q}_i) &= \Big(\exp\big(\lambda_i \ln\big(\tfrac{u-p_i}{\widetilde{q}_i-p_i}\big)\big)x, u\Big)\,, \\ \Delta_{\gamma_{(0,u)}^{-1}}(x,u) &= \Big(\exp\big(-\lambda_i \ln\big(\tfrac{u-p_i}{\widetilde{q}_i-p_i}\big)\big)x, \widetilde{q}_i\Big)\,.\end{aligned}$$

[1] If $f : (\mathbb{C},0) \longrightarrow (\mathbb{C},0)$ is a holomorphic function and there exists $a \in \mathbb{C}\smallsetminus\{0\}$ which is not a root of unity and satisfies $af(x) = f(ax)$ then $f$ must be a linear map, as is proved: it is satisfied $f(x) = b_1x + b_2x^2 + \cdots$ in a neighborhood of 0, then $af(x) = f(ax)$ if and only if for all $k \geqslant 2$, $ab_k = a^k b_k$. Since $a \neq 0$ and it is not a root of unity the $b_k = 0$ for all $k \geqslant 2$.

Therefore

$$\Delta_{\gamma_{(0,u)}} \circ \widetilde{h}_i \circ \Delta_{\gamma^{-1}_{(0,u)}}(x,u) = \Delta_{\gamma_{(0,u)}}\Big(\alpha_i \exp\big(-\lambda_i \ln\big(\tfrac{u-p_i}{\widetilde{q}_i-p_i}\big)\big)x, \widetilde{q}_i\Big)$$
$$= \Big(\exp\big(\lambda_i \ln\big(\tfrac{u-p_i}{\widetilde{q}_i-p_i}\big)\big)\Big(\alpha_i \exp\big(-\lambda_i \ln\big(\tfrac{u-p_i}{\widetilde{q}_i-p_i}\big)\big)x\Big), u\Big) = (\alpha_i x, u)\,,$$

In this way, we conclude (8.3) and, as a consequence the extension of $\boldsymbol{\mathcal{H}}$ on a neighborhood of $\mathfrak{p}_i$. Now we have $\boldsymbol{\mathcal{H}}$ is a biholomorphism defined on a neighborhood $(\mathbb{M}_{\mathbb{C}}, \mathbb{D}_{\mathbb{C}})$.

**Remark 8.1.2.** The extension of $\boldsymbol{\mathcal{H}}$ does not depend on the choice of $\widetilde{\mathfrak{q}}_i$.

## 8.2 Invariance of the Real Möbius Band $\mathbb{M}_{\mathbb{R}}$

The biholomorphism $\boldsymbol{\mathcal{H}}$ is defined in a neighborhood of the complex exceptional divisor $\mathbb{D}_{\mathbb{C}}$ in the complex Möbius band $\mathbb{M}_{\mathbb{C}}$. It preserves the auxiliar foliation $\widetilde{\mathscr{F}}^{\mathbb{C}}$ and its restriction on $\mathbb{D}_{\mathbb{C}}$ is the identity map.

If there are no singular points in $\mathbb{D}_{\mathbb{R}}$, it is possible to extend $\boldsymbol{\mathcal{H}}$ around the polar curves $T_j^{\mathbb{C}}$ leaving invariant $\mathbb{M}_{\mathbb{R}}$ (Lemma 7.0.11). For every point $\mathfrak{q} \in \mathbb{D}_{\mathbb{R}} \smallsetminus \{\mathfrak{q}_1, \ldots, \mathfrak{q}_{n-2}\}$ we can choose $\delta_{\mathfrak{q}}$ a path beginning at $\mathfrak{q}_0$, ending at $\mathfrak{q}$ and contained in $\mathbb{D}_{\mathbb{R}}$. Since $\Delta^{\nu}_{\delta_{\mathfrak{q}}}$ and $\Delta^{\omega}_{\delta_{\mathfrak{q}}}$ send $\mathscr{L}_{\mathfrak{q}_0} \cap \mathbb{M}_{\mathbb{R}}$ into $\mathscr{L}_{\mathfrak{q}} \cap \mathbb{M}_{\mathbb{R}}$, then $\boldsymbol{\mathcal{H}}\big|_{\mathscr{L}_{\mathfrak{q}}} = \Delta^{\omega}_{\delta_{\mathfrak{q}}} \circ \Delta^{\nu}_{\delta^{-1}_{\mathfrak{q}}}$ leaves invariant $\mathscr{L}_{\mathfrak{q}} \cap \mathbb{M}_{\mathbb{R}}$.

Let us now look to the case where there is at least one singular point in $\mathbb{D}_{\mathbb{R}}$. Assume that $\mathfrak{q}_0 \in \mathbb{D}_{\mathbb{R}}$ is as before. We shall stress that it is enough to prove the existence of a neigborhood of $\mathfrak{q}_0$ where the diffeomorphism $\boldsymbol{\mathcal{H}}$ leaves invariant the real Möbius band $\mathbb{M}_{\mathbb{R}}$. To prove this assertion we first observe that in the coordinate $(x,u)$ this is equivalent to the condition $\boldsymbol{\mathcal{H}}(x_0,u_0) = \overline{\boldsymbol{\mathcal{H}}(\bar{x}_0,\bar{u}_0)}$ for $(x_0,u_0)$ in a neighborhood of $(0,\mathfrak{q}_0)$.

Without loss of generality we may assume that $\boldsymbol{\mathcal{H}}$ is defined in $U \subseteq \mathbb{C}^2$ a connected open neighborhood of the axis $\{x=0\}$ which is invariant by complex conjugation. Since the map $(x,u) \mapsto \overline{\boldsymbol{\mathcal{H}}(\bar{x},\bar{u})}$ is a well defined holomorphic map in $U$, then the equality $\boldsymbol{\mathcal{H}}(x_0,u_0) = \overline{\boldsymbol{\mathcal{H}}(\bar{x}_0,\bar{u}_0)}$ for $(x_0,u_0)$ in a neighborhood of $(0,\mathfrak{q}_0)$ implies that the same happens all over $U$ (see Identity Theorem [Gun90]). Hence, for $(a,b) \in \mathbb{R}^2 \cap U$, $\boldsymbol{\mathcal{H}}(a,b) = \overline{\boldsymbol{\mathcal{H}}(\bar{a},\bar{b})} = \overline{\boldsymbol{\mathcal{H}}(a,b)}$. Therefore for any $\mathfrak{q} \in \mathbb{D}_{\mathbb{R}}$ which belongs to the coordinate chart $(x,u)$, $\mathscr{L}_{\mathfrak{q}} \cap \mathbb{M}_{\mathbb{R}}$ is invariant by $\boldsymbol{\mathcal{H}}$. In the coordinate chart $(v = x/y, y)$ we denote by $\mathscr{L}_{\infty}$ the leaf of the auxiliary foliation passing through $(v,y) = (0,0)$. For $\mathfrak{p} \in \mathscr{L}_{\infty} \cap \mathbb{M}_{\mathbb{R}}$ we take a sequence of points $(\widetilde{\mathfrak{p}}_i)_{i\in\mathbb{N}}$ tending to $\mathfrak{p}$, such that $\widetilde{\mathfrak{p}}_i \in \mathscr{L}_{\widetilde{\mathfrak{q}}_i} \cap \mathbb{M}_{\mathbb{R}}$ for $\widetilde{\mathfrak{q}}_i \in \mathbb{D}_{\mathbb{R}}$ belonging to the chart $(x,u)$. The continuity of $\boldsymbol{\mathcal{H}}$ implies that $\big(\boldsymbol{\mathcal{H}}(\widetilde{\mathfrak{p}}_i)\big)_{i\in\mathbb{N}} = \boldsymbol{\mathcal{H}}(\mathfrak{p})$. Since in $(v,y)$ this is a Cauchy sequence in $\mathbb{R}^2$, then $\boldsymbol{\mathcal{H}}(\mathfrak{p}) \in \mathbb{R}^2$. Thus $\mathscr{L}_{\infty} \cap \mathbb{M}_{\mathbb{R}}$ is invariant by $\boldsymbol{\mathcal{H}}$.

It remain to prove the existence of a neighborhood of $\mathfrak{q}_0$ in $\mathbb{M}_{\mathbb{C}}$ in which $\boldsymbol{\mathcal{H}}$ leaves invariant the real Möbius band $\mathbb{M}_{\mathbb{R}}$. Let $\mathfrak{q}$ be a point in $\mathbb{D}_{\mathbb{R}}$ (in the chart $(x,u)$), $u(\mathfrak{q}) = q$, such that the path $\delta_{\mathfrak{q}}$ defined in $(x,u)$ as $t \mapsto (0, tq + (1-t)q_0)$ does not pass through any singular point. Then $\Delta^{\nu}_{\delta_{\mathfrak{q}}}$ and $\Delta^{\omega}_{\delta_{\mathfrak{q}}}$ take the intersection $\mathscr{L}_{\mathfrak{q}_0} \cap \mathbb{M}_{\mathbb{R}}$ into $\mathscr{L}_{\mathfrak{q}} \cap \mathbb{M}_{\mathbb{R}}$, and so, $\boldsymbol{\mathcal{H}}\big|_{\mathscr{L}_{\mathfrak{q}}} = \Delta^{\omega}_{\delta_{\mathfrak{q}}} \circ \Delta^{\nu}_{\delta_{\mathfrak{q}}^{-1}}$ leaves invariant $\mathscr{L}_{\mathfrak{q}} \cap \mathbb{M}_{\mathbb{R}}$. Since $\mathfrak{q}_0$ is a nonsingular point, there is a sufficiently small connected neighborhood of $\mathfrak{q}_0$ in $\mathbb{M}_{\mathbb{C}}$ such that any point in the intersection of this neighborhood with $\mathbb{M}_{\mathbb{R}}$ satisfies the previous condition. Therefore in this neighborhood $\boldsymbol{\mathcal{H}}$ leaves invariant the real Möbius band $\mathbb{M}_{\mathbb{R}}$.

## 8.3 Special Local Linearization of Holomorphic Foliations

We consider the foliation $\mathscr{F}_v$ on $(\mathbb{C}^2, 0)$ described by the vector field

$$(\dot{x}, \dot{u}) = v(x,u) = \big(x(a + f_1(x,u)), cx + bu + f_2(x,u)\big)\,,$$

satisfying $f_1 = \mathrm{O}(\|(x,u)\|)$, $f_2 = \mathrm{O}(\|(x,u)\|^2)$ are holomorphic functions, the coefficients $a$, $b$ are nonzero and $a/b, b/a \in \big(\mathbb{C} \smallsetminus \mathbb{N}\big) \cap \big(\mathbb{C} \smallsetminus \mathbb{R}^-\big)$, or $a/b \in \mathbb{R}^- \smallsetminus \mathbb{Q}$ and in such a case $a, b$ satisfy the Brjuno condition.

We take another foliation $\mathscr{F}$ on $(\mathbb{C}^2, 0)$ described by

$$(\dot{x}, \dot{u}) = w(x,u) = \big(d + g_1(x,u), e + g_2(x,u)\big)\,,$$

where $d, e \in \mathbb{C}$, with $d \neq 0$, and $g_1, g_2 = \mathrm{O}(\|(x,u)\|)$ are holomorphic functions. Besides, the leaf of $\mathscr{F}$ passing through the origin coincides with one of the separatrices of $\mathscr{F}_v$.

***Lemma* 8.3.1.** *There exists a biholomorphism $F : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ whose restriction on $\{x = 0\}$ is the identity and which maps the leaves of the foliation $\mathscr{F}$ to the leaves of the trivial foliation $\{u = \varepsilon\}_{\varepsilon \in (\mathbb{C},0)}$, and the leaves of the foliation $\mathscr{F}_v$ to the leaves of the foliation induced by the linear equation*

$$(\dot{x}, \dot{u}) = (\lambda\, x, u)\,,$$

*where $\lambda = a/b$.*

*Moreover, if $v$ and $w$ are complexifications of real analytic vector fields on $(\mathbb{R}^2, 0)$, then $F$ is the complexification of a real analytic diffeomorphism.*

**Proof**. First we will construct a biholomorphism $\mathcal{G}$ mapping the foliation $\mathscr{F}$ to the trivial foliation $\{u = \varepsilon\}_{\varepsilon \in (\mathbb{C},0)}$ whose restriction on $\{x = 0\}$ is the identity map. After that we will linearize the foliation $\mathcal{G}\big(\mathscr{F}_v\big)$ by another biholomorphism $\mathfrak{G}$, which leaves invariant the trivial foliation and whose restriction on

$\{x=0\}$ is the identity map. If $v$ and $w$ are complexifications of real analytic vector fields, we shall prove that $\mathcal{G}$ and $\mathfrak{G}$ are complexifications of real analytic diffeomorphisms. As a consequence $F := \mathfrak{G} \circ \mathcal{G}$ will be the required function.

Let $\mathrm{B}_1 \subseteq \mathbb{C}$ and $\mathrm{B}_2 \subseteq \mathbb{C}^2$ be open discs centered at 0 and at the origin of $\mathbb{C}^2$ respectively, such that it is defined $\phi_w : \mathrm{B}_1 \times \mathrm{B}_2 \longrightarrow \mathbb{C}^2$ the flow of $w$, with $\phi_w(0,(x,u)) = (x,u)$. By the inverse function theorem, the holomorphic function $(x,u) \mapsto \phi_w\big(x,(0,u)\big)$ is invertible in a neighborhood of $\widehat{0} \in \mathbb{C}^2$. Its inverse function, denoted by $\mathcal{G}$, maps $\mathscr{F}$ to the trivial foliation $\{u=\varepsilon\}_{\varepsilon\in(\mathbb{C},0)}$ and its restriction on $\{x=0\}$ is the identity map. Moreover, if $w$ is the complexification of a real analytic vector field, $\mathcal{G}$ is the complexification of a real analytic diffeomorphism.

The foliation $\mathcal{G}\big(\mathscr{F}_v\big)$ is described by the vector field

$$(\dot{x},\dot{u}) = \widehat{v}(x,u) = \big(x(\lambda + f(x,u)), u\big)\,,$$

being $f(x,u) = \mathrm{O}(\|(x,u)\|)$ a holomorphic function. By Poincaré's and Brjuno's linearization theorems, there exists $\mathfrak{G} = (\mathfrak{G}_1,\mathfrak{G}_2) : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^2,0)$ a biholomorphism sending the vector field $\widehat{v}$ into its linear part. This biholomorphism is the complexification of a real analytic diffeomorphism when $v$ and $w$ are complexifications of real analytic vector fields. Since $\mathrm{D}\,\mathfrak{G}_0$ and the linear part of $\widehat{v}$ commute, without loss of generality we can assume $\mathrm{D}\,\mathfrak{G}_0$ is the identity matrix.

Now we shall prove that the biholomorphism $\mathfrak{G}$ leaves invariant the trivial foliation $\{u=\varepsilon\}_{\varepsilon\in(\mathbb{C},0)}$ and $\mathfrak{G}\big|_{\{x=0\}} = \mathrm{id}_{\{x=0\}}$, that is, $\mathfrak{G}_1(x,u) = x\big(1+\widetilde{\mathfrak{G}}_1\big)$, where $\widetilde{\mathfrak{G}}_1$ is a holomorphic function on $(\mathbb{C}^2,0)$ mapping the origin to 0, and $\mathfrak{G}_2(x,u) = u$.

We will use the font $\mathbf{k}$ to refer to a order pair of natural numbers $(k_1,k_2)$, and define $|\mathbf{k}| := k_1+k_2$. We consider the Taylor series of the above functions:

$$f(x,u) = \sum_{|\mathbf{k}|\geqslant 1} \gamma_k x^{k_1}u^{k_2}\,, \qquad \mathfrak{G}_1(x,u) = x + \sum_{|\mathbf{k}|\geqslant 2} \zeta_{1,\mathbf{k}} x^{k_1}u^{k_2}\,,$$

$$\mathfrak{G}_2(x,u) = u + \sum_{|\mathbf{k}|\geqslant 2} \zeta_{2,\mathbf{k}} x^{k_1}u^{k_2}\,.$$

The fact that $\mathfrak{G}$ maps the vector field $\widehat{v}$ to its linear part, is equivalent to the following equality

$$\begin{pmatrix} \mathfrak{G}_{1,x}(x,u) & \mathfrak{G}_{1,u}(x,u) \\ \mathfrak{G}_{2,x}(x,u) & \mathfrak{G}_{2,u}(x,u) \end{pmatrix} \begin{pmatrix} x(\lambda+f(x,u)) \\ u \end{pmatrix} = \begin{pmatrix} \lambda\mathfrak{G}_1(x,u) \\ \mathfrak{G}_2(x,u) \end{pmatrix}\,. \tag{8.4}$$

The fist line of the equation (8.4) is equivalent to

$$\sum_{|\mathbf{k}|\geqslant 2, |\mathbf{l}|\geqslant 1} \gamma_{\mathbf{l}}\, \zeta_{1,\mathbf{k}} x^{k_1+l_1} u^{k_2+l_2} + x \sum_{|\mathbf{l}|\geqslant 1} \gamma_{\mathbf{l}} x^{l_1} u^{l_2} = \sum_{|\mathbf{k}|\geqslant 2} \zeta_{1,\mathbf{k}} \big(\lambda - (\lambda k_1 + k_2)\big) x^{k_1} u^{k_2} .$$

Given $\widetilde{\mathbf{k}} = (\widetilde{k}_1, \widetilde{k}_2) \in \mathbb{N}^2$ such that $|\widetilde{\mathbf{k}}| \geqslant 2$, by the previous equality we can obtain $\zeta_{1,\widetilde{\mathbf{k}}}$ in terms of the coefficients $\zeta_{1,\mathbf{k}}$ satisfying $|\mathbf{k}| < |\widetilde{\mathbf{k}}|$, namely,

$$\zeta_{1,\widetilde{\mathbf{k}}} = \frac{1}{\lambda - \big(\lambda \widetilde{k}_1 + \widetilde{k}_2\big)} \left[ \gamma_{\widetilde{\mathbf{k}}-(1,0)} + \sum_{\substack{2\leqslant |\mathbf{k}|<|\widetilde{\mathbf{k}}|,\\ 1\leqslant|\mathbf{l}|\,,\, \mathbf{k}+\mathbf{l}=\widetilde{\mathbf{k}}}} \gamma_{\mathbf{l}}\, \zeta_{1,\mathbf{k}} \right] .$$

It follows immediately that $\zeta_{1,(0,j)} = 0$ for all $j \geqslant 2$. The second line of the equation (8.4) is equivalent to

$$\sum_{|\mathbf{k}|\geqslant 2, |\mathbf{l}|\geqslant 1} \gamma_{\mathbf{l}}\, \zeta_{2,\mathbf{k}} x^{k_1+l_1} u^{k_2+l_2} = \sum_{|\mathbf{k}|\geqslant 2} \zeta_{2,\mathbf{k}} \big(1 - (\lambda k_1 + k_2)\big) x^{k_1} u^{k_2} .$$

As before, considering $\widetilde{\mathbf{k}} = (\widetilde{k}_1, \widetilde{k}_2) \in \mathbb{N}^2$ such that $|\widetilde{\mathbf{k}}| \geqslant 2$, by the previous equality we obtain $\zeta_{1,\widetilde{\mathbf{k}}}$ in terms of the coefficients $\zeta_{1,\mathbf{k}}$ satisfying $|\mathbf{k}| < |\widetilde{\mathbf{k}}|$, namely,

$$\zeta_{2,\widetilde{\mathbf{k}}} = \frac{1}{1 - \big(\lambda \widetilde{k}_1 + \widetilde{k}_2\big)} \left[ \sum_{\substack{2\leqslant |\mathbf{k}|<|\widetilde{\mathbf{k}}|,\\ 1\leqslant|\mathbf{l}|\,,\, \mathbf{k}+\mathbf{l}=\widetilde{\mathbf{k}}}} \gamma_{\mathbf{l}}\, \zeta_{2,\mathbf{k}} \right] .$$

From this equality it follows immediately that $\zeta_{2,(2,0)} = \zeta_{2,(1,1)} = \zeta_{2,(0,2)} = 0$. By induction we obtain $\zeta_{2,\mathbf{k}} = 0$ for all $\mathbf{k} \in \mathbb{N}^2$ such that $|\mathbf{k}| \geqslant 2$. Therefore $\mathfrak{G}_2(x, u) = u$.

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