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"LAS CONJUNCIONES COMO MARCADORES DE RELACIONES DISCURSIVAS"

T E S I S

QUE PARA OBTENER EL GRADO DE

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INTRODUCCION

En este trabajo abarcamos el estudio de las conjunciones en el marco del análisis del discurso. Se eligieron cuatro niveles: expresión, presuposición, implicación e implicatura. Con esta elección se logra delimitar adecuadamente el problema. Las relaciones en dichos niveles dan cuenta de la presencia de una conjunción determinada en un contexto determinado. De hecho, lo que se plantea es que la conjunción marca las relaciones.

El enfoque adoptado no implica, por supuesto, que los niveles elegidos constituyan el discurso. Para otros estudios, con otros propósitos, en los que estén o no involucradas las conjunciones, podrían requerirse otros niveles.

Proponemos que las conjunciones son elementos que marcan relaciones tanto implícitas como explícitas en el discurso, mas no establecen tales relaciones, como se expresa en la gramática tradicional. En otras palabras, el trabajo gira en torno a la siguiente preocupación: ¿establecen las conjunciones relaciones entre elementos lingüísticos o solamente marcan esas relaciones?

Para aproximarnos al problema recurrimos a los criterios y definiciones que delimitan a la conjunción. Es posible hablar de dos criterios para explicar las conjunciones: como relacionadores (desde el punto de vista sintáctico) y como palabras que indican cierta función (desde el punto de vista semántico). Podríamos hablar inclusive de un solo criterio de conjunción y sería el de 'función', siendo ésta la de relacionar elementos linguísticos. Desde esta perspectiva general se ubican las conjunciones en un sistema de relaciones lingüísticas. Aquí nos preguntamos: ¿qué relaciones?, pues el mismo lenguaje se define como un sistema de relaciones.

La gramática tradicional explica las conjunciones en las relaciones de conexión. Si nuestros elementos relacionan palabras u oraciones del mismo nivel serán coordinadores, y son subordinadores cuando las unidades que relacionan se encuentren en diferente nivel. En la coordinación la relación puede tener más de dos miembros; en la relación de subordinación sólo dos cláusulas interactúan. La subordinación permite una organización de cláusulas múltiples; cada cláusula puede a su vez tener otras cláusulas formando una jerarquía de cláusulas una dentro de la otra. Cabría señalar al margen que las conjunciones forman parte de estas relaciones, pero no son los únicos elementos responsables de la conexión (1).

Las conjunciones también intervienen en las relaciones de significado. En este nivel denotan relaciones entre eventos o estados del mundo expresados por las proposiciones que unen, tales como causal, adversativa, temporal, aditiva, etc. De aquí han surgido clasificaciones en las cuales, al compararlas, el número y clasificación de las conjunciones varía. Si las conjunciones establecen relaciones, entonces cada una estará encargada de un número determinado de esas relaciones.

En cuanto a la designación y definición vemos que en la literatura no aparece un término único para referirse a nuestros elementos. A saber, el criterio para su designación depende en ocasiones de su función en las oraciones o proposiciones, del tipo de relación que establecen y de la disciplina que la está explicando. Por ejemplo, en la gramática tradicional se usa el término conjunción derivado del latín 'conjunctus': junto con. El término 'conectiva' se usa en ámbitos más especializados, como la lógica y la lógica proposicional, en donde también se emplea, de una manera más genérica, 'nexos lógicos'. Al estudiar unidades más grandes que la oración, a las conjunciones se les llama conectores o 'linking words', aunque estos últimos términos pueden incluir otros tipos de expresión. Nuestros elementos se llaman coordinadores y subordinadores, cuando se hace referencia a relaciones sintácticas entre palabras u oraciones. En lógica semántica se llaman 'operadores' derivados, por similitud, de los operadores de adición y substracción en matemáticas.

Lo anterior nos hizo pensar que sería pertinente realizar un estudio de las conjunciones en un número más amplio de relaciones discursivas que las tradicionalmente estudiadas por la gramática. Se consideró que esto era factible y podría ser productivo dado que ya se han realizado estudios de las conjunciones tomando en cuenta las diferencias y representaciones que el receptor hace del discurso (Caron 1987; Levelt 1978; Noodman 1979) (2).

De Beaugrande (1981) opina que por el uso de las conjunciones los productores de textos pueden trabajar en el control sobre las relaciones que serán recuperadas y reconstruídas por los receptores. Ellas demuestran cómo la interacción comunicativa determina qué forma sintáctica (no sólo reglas gramaticales obligatorias) usar ayudando a que la recepción de un texto sea eficiente. Auxilian al productor del texto durante la organización y presentación de un modo textual, así como pueden implicar o imponer una interpretación particular. En consecuencia, se seleccionan sólo después de que las palabras de contenido han sido seleccionadas.

A partir de la opinión de De Beaugrande es que nosotros derivamos nuestra propuesta en este trabajo acerca de las conjunciones. La idea estriba en que las relaciones entre proposiciones se dan por los significados que las mismas proposiciones expresan y por las relaciones implícitas entre ellas, y no necesariamente por la presencia de alguna conjunción. Por lo tanto, al hablar de conjunción estaremos considerando tanto aquello que la precede como aquello que la sucede.

Las conjunciones pueden considerarse como señales de advertencia para el lector, que le dicen qué presuposiciones, implicaciones e implicaturas,considerar o tomar en cuenta como válidas (ver también Caron 1987, donde propone que las conectivas conllevan rasgos pragmáticos).

Si trasladamos las oraciones a esquemas lógicos, veremos que las proposiciones se traducen en operaciones cuyas relaciones se indican por conectores o conectivas. Entonces, si las relaciones lógicas parecen ser obvias en el análisis proposicional y de alguna manera éstas se manifiestan en el lenguaje natural, los conectores lógicos, pensamos deberán expresarse en conjunciones (o por otros medios, como la puntuación y la entonación). Es decir las palabras que funcionan como conjunciones podrían, en un sentido lógico, traducirse al número de conectivas que existen en lógica. Pero el hecho de usar una u otra expresión conjuntiva y la diferencia entre ellas, suponemos, depende de otras funciones no lógicas; por ejemplo, nos parece que depende de consideraciones pragmáticas que tienen que ver con la organización y procesamiento del discurso. Por lo tanto, proponemos que estas diferencias pueden dilucidarse a través de un análisis de proposiciones en diferentes niveles del discurso:

a) de las proposiciones expresadas
b) de las presuposiciones
c) de las implicaciones lógicas
d) de las implicaturas.

Las conjunciones no crean las relaciones entre las proposiciones: simplemente las marcan, las hacen explícitas. Desde este punto de vista las oraciones complejas con conjunciones y las formadas por simple yuxtaposición serían equivalentes en su significado, aunque no en su procesamiento. Estas hipótesis son generales; pensamos podrán aplicarse a toda aquella expresión que, bajo nuestra caracterización pudiera considerarse como conjunción.

Usamos el término conjunción para referirnos a la expresión discursiva que indica o marca relaciones expresadas o implícitas entre los enunciados, verbales o no verbales (matemáticos, en el sentido estricto del término).

Nos limitamos al discurso escrito y solamente tomamos las relaciones (niveles del discurso) ya mencionados arriba. El estudio se enfoca en las conjunciones 'and', "but', 'if', y 'then', porque pensamos son las más recurrentes y podrían quedar implícitas en otras expresiones conjuntivas. Esta idea la tomamos de Halliday y Hasan (1976) cuando consideran que muchas expresiones conjuntivas aparecen en forma más o menos sinónimas.

Como dato se tomaron los usos de conjunciones en un artículo del área de matemáticas del corpus seleccionado en la Maestría en Linguística Aplicada (Marron 1987) (ver anexo 1).

El capítulo primero versa brevemente sobre la metodología que incluye las definiciones, las hipótesis y los procedimientos que nos sirvieron de base en el análisis. El segundo capítulo del trabajo presenta la descripción, en sus características y funciones, de las conjunciones estudiadas.

Las conclusiones incluyen los comentarios pertinentes sobre las hipótesis y algunos otros datos e interrogativas que surgen en el curso del trabajo.

Anexamos una sección donde presentamos algunas reflexiones acerca de las conjunciones en relación con el tema de los lenguajes especializados.

1. METODOLOGIA.

En este capítulo presentamos los conceptos principales que forman el marco de referencia de la tesis y los procedimientos y criterios de análisis como ya se dijo en la introducción.

Dos principios que orientan el desarrollo de la investigación objeto de esta tesis son:

1) Las conjunciones tienen una determinación múltiple; su ocurrencia está condicionada conjuntamente por las diferentes relaciones entre lo que se dice, explícita e implícitamente, en los enunciados con los que aparece;

2) tanto lo que se dice, explícita e implícitamente, como las relaciones entre lo que se dice en dos enunciados contiguos, operan con referencia a un marco de reglas.

Estos principios que permitirán, en una sección posterior, formular las hipótesis de la investigación y que son la base para definir criterios y análisis, son afines a supuestos de distintas convenciones del lenguaje que son ampliamente aceptadas. Además se corroboran al hacer posible el trabajo que aquí se presenta.

Ahora bien, plantearse una orientación por dichos principios implica comprometerse a identificar lo más claramente posible lo que se dice explícita e implícitamente, así como los tipos de relaciones entre lo que se dice. Por lo tanto en la próxima sección se presentan las definiciones empleadas para tratar lo uno y las otras, que son las de: proposición expresada, implicación, presuposición e implicatura. Se eligió este orden por ser conveniente a la exposición.

Dichas definiciones están precedidas de otras, requeridas para un análisis sintáctico, de los siguientes términos: oración compuesta, yuxtaposición, coordinación y subordinación. Se eligió presentarlas antes de las mencionadas en el párrafo anterior porque, como se verá en la sección correspondiente a procedimientos, el análisis sintáctico se efectuó en forma preliminar al análisis lógico-pragmático que es nuestro interés principal.

1.1. Definiciones.

Oración Compuesta. Yuxtaposición. Coordinación. Subordinación.

La oración compuesta o período es una unidad lingüística que contiene oraciones que guardan ciertas relaciones para expresar cierto contenido unitario. No se trata de la simple agrupación de oraciones simples, sino de la relación de estas oraciones simples subordinadas a la intención subjetiva que las produce. Debido a esto se puede establecer toda clase de conexiones expresivas con o sin signo gramatical que las designe. Las oraciones yuxtapuestas coordinadas y subordinadas pueden llevar o no signo expresivo de la relación existente entre los componentes.

La yuxtaposición es la relación de oraciones asindéticas que forman una oración compuesta, por ejemplo,

(1) Some of these axioms make assertions primarily about space itself; others pertain to figures in space.
(Kline: 113)

La coordinación o parataxis relaciona elementos de la misma clase, por ejemplo,

(2) From the standpoint of pure geometry the methodologies of analytic geometry and differencial geometry were far too successful.
(Kline: 117)

En la subordinación o hipotaxis los componentes no pueden separarse nunca sin mutilación de lo expresado, puesto que ninguno de ellos tiene sentido más que dentro del periodo.

(3) The closer the area of the triangle is to zero, the closer the angle sum is to 180 degrees.
(Kline: 118)

La Proposición Expresada.

Se hará uso en este nivel del cálculo predicativo que es una rama desarrollada de la lógica formal, ya que se presenta como un modelo para la descripción lingüística. Se asume que en las estructuras profundas del lenguaje natural contiene cuantificadores y variables del cálculo predicativo, con el alcance definido en el cálculo predicativo. Es un sistema usado para la representación de la estructura lógica de las proposiciones simples, y complejas estas últimas mediante el uso de las conectivas lógicas. Nos ilustra la forma lógica subyacente de las oraciones de las lenguas. Se manejan dos clases de términos: nombres y predicados.

Como convención común se usan las primeras letras del alfabeto para representar los nombres, $\{a, b, c...\}$, y letras mayúsculas para los predicados, N para 'number', LS para 'line segment', etc. Entonces, en el nivel de la expresión, diremos que (4) se expresa como en (5),

(4) the answer was not a number but a line segment.

(5) $\sim N(a)\ \&\ LS(a)$.

Implicación Lógica.

Cuando dos oraciones son tales que su condicional es lógicamente verdadero, decimos que 'p' implica lógicamente 'q'. Un condicional es lógicamente verdadero, cuando no deja de ser verdadero para cualquier substitución uniforme de sus términos no lógicos (entendemos por términos lógicos 'Si... entonces'), por ejemplo podemos expresar la implicación de nuestro enunciado (6) sobre un esquema de taxonomía científica, en (7),

(6) His first point was somewhat technical but essential.
(Kline: 118)

(7) Si todo lo científico es esencial, entonces lo técnico no es esencial.

La Presuposición.

La presuposición es una inferencia pragmática que se basa en la expresión linguística. Se origina especialmente en debates acerca de la naturaleza de las expresiones de referencia; de cómo estas expresiones debían trasladarse a lenguajes lógicos, de cómo esta expresión nos da información de referencia en el momento de su uso.

Strawson (1950) propone la distinción entre oraciones y el uso de esas oraciones, nos dice que hay un tipo de relación entre (8) y (9),

(8) The King of France is wise

(9) There is a present King of France.

(9) es una precondición para (8) y Strawson llama a esta relación presuposición, una especie de inferencia pragmática diferente de la implicación lógica.

Strawson hace la distinción entre una oración (sentence), el uso de una oración (use) y la emisión de una oración (utterance), y otro tanto por lo que toca a las expresiones que son parte de una oración. Así, una oración o una expresión puede ser emitida en diversas ocasiones o por diferentes personas, y tendremos entonces otras tantas emisiones de la expresión u oración de que se trate; pero distintas personas, o la misma en distintas ocasiones, pueden referirse con una expresión al mismo objeto, en cuyo caso habrá que decir que hacen de ella el mismo uso, y que hacen un uso diferente sólo cuando se refieran a un distinto objeto o entidad. La idea de Strawson es que la verdad o la falsedad han de predicarse del uso de las oraciones, pero no de las oraciones mismas, así como la referencia ha de predicarse del uso de las expresiones y no de las expresiones mismas; (8) no se refiere por sí sola a nadie. Esta expresión adquiere referencia al ser usada para decir algo sobre alguien y adquirirá un valor de verdad al ser usada en una ocasión determinada, lo que es verdadero o falso no es la oración sino la aserción o enunciado hechos al usarla.

La verdad y la falsedad no son propiedades de las oraciones, sino de usos de oraciones. Referirse no es algo que haga una expresión, sino algo que hace un hablante cuando usa esa expresión

(Hierro 1982: 145-46). Así diremos que (10) presupone (11),

(10) The Classical Greek civilization that gave rise to Euclidean geometry was (was not) destroyed by Alexander the Great.

(11) The Classical Greek civilization was destroyed.
(Kline: 113)

La Implicatura.

La idea la propuso Grice en 1987 y la publicó en 1975.

La teoría en la que desarrolla el concepto de implicatura es una teoría de cómo la gente usa la lengua.

La implicatura nos proporciona consideraciones explícitas sobre cómo es posible significar 'más de lo que en realidad se dice'.

Volvamos ahora al enunciado (6) donde notamos la presencia de una implicatura.

(6) His first point was somewhat technical but essential.

El autor, al citar el término técnico, también nos hizo pensar en un esquema taxonómico donde sabemos que el término ciencia llevaría una categoría superior a técnico, es decir implicaríamos que lo técnico no sería esencial en el esquema.

Pero el esquema es lo primero que aparecería en nuestra inferencia como lectores. Esta inferencia es la que prevé el autor pero él mismo desea dar al término técnico la categoría de esencial; esta relación se hace explícita por la presencia de 'but' como explijaremos más adelante. Lo importante es que el esquema que llama el enunciado es una implicatura.

Según Grice, las implicaturas conversacionales están esencialmente conectadas con ciertos rasgos generales del discurso que derivan de que la comunicación linguística es una forma de conducta cooperativa, que sirve a un propósito común de los hablantes (principio de cooperación). Esta conducta Grice la formula así: 'haz que tu contribución a la conversación se requiera, en el momento en que tiene lugar, por el propósito de dirección aceptado del intercambio linguístico en el que tomas parte' (Hierro: 192). Este principio se particulariza en cuatro máximas de conversación que son: cantidad, calidad, relevancia (relevance), y modo de la comunicación (manner), (Grice 1975: 45-47).

La razón del interés linguístico en las máximas es que ellas generan inferencia más allá del contenido semántico de las oraciones expresadas. Tales inferencias son implicaciones conversacionales. El término implicatura se usa para contrastar con los términos de implicación lógica e implicación analítica, que se usan por lo general para referirse a inferencias que se derivan solamente del contenido lógico o semántico.

Las implicaturas no son inferencias semánticas; sino inferencias que se basan en el contenido de lo que se ha dicho y en algunas características específicas de la naturaleza cooperativa de la interacción verbal ordinaria.

1.2. Hipótesis.

Las hipótesis de esta investigación son:

1. Las conjunciones no establecen las relaciones entre las proposiciones, simplemente las marcan.

2. Las conjunciones marcan relaciones entre las proposiciones explícitas e implícitas.

3.- Las conjunciones dan instrucciones al lector sobre lo que deben derivar o no.

4. Cada una de las conjunciones tiene ciertas características propias que limitan su uso.

Por lo planteado en las secciones anteriores y atendiendo a la naturaleza del corpus que hemos elegido, entendemos por conjunción la expresión discursiva que indica o marca relaciones expresadas o implícitas entre los enunciados, verbales o no verbales (matemáticos, en el sentido restringido del término). Ellas indican que implícitos tomar en cuenta y cuáles no se habrán de considerar.

Definimos como implícito toda aquella información que pueda derivarse como presuposición, implicación o implicatura de un enunciado, (ver 1.1.).

1.3. Procedimientos.

Como ya se indicó, el análisis se limitó al discurso escrito. Se tomó un artículo del corpus ya establecido en la Maestría en Linguística Aplicada (Texto 1: "Geometry", de Kline 1964).

No se tomó en cuenta el aspecto fonético propiamente, aunque las pausas que nos obligaba a hacer la puntuación nos ayudaron en gran medida a distinguir las relaciones de yuxtaposición.

Como tratamos con diferentes niveles, el vocabulario que ocupamos obedeció a las diferentes unidades de análisis. En la descripción hablamos de oraciones, cláusulas, proposiciones, inferencias, derivaciones, implicaciones, etc.

A continuación se describen los pasos seguidos en el estudio y se presentan los criterios de identificación seguidos en cada uno, así como las convenciones notacionales empleadas.

En primer lugar nos interesó aislar las oraciones donde aparecían las conjunciones 'and' 'or' 'but' 'if...then'. En seguida, como análisis preliminar, se identificaron las relaciones de yuxtaposición, coordinación y subordinación entre dichas oraciones. Posteriormente, se hizo una descripción de la aparición de las relaciones en los párrafos. Aquí se identificaron las presuposiciones, implicaciones e implicaturas.

Más adelante se observó la posición de las conjunciones en las relaciones y se aisló un número suficiente de ejemplos.

En cada uno de los ejemplos se realizó un estudio de los diferentes tipos de relación: en las proposiciones expresadas, en las presuposiciones, en las implicaciones, en las implicaturas. A estos tipos de relación los llamamos 'niveles', refiriéndolos como niveles del discurso en nuestro análisis, mismos que quedaron expresados así: Ex, P, I, e Im.

Se reunió la información en cada uno de los ejemplos en cuadros. Esto nos auxilió para ver la diferencia entre conjunciones y poder así corroborar la relación que cada conjunción marcaba.

Al hacer el análisis del enunciado completo señalamos sus componentes de la siguiente manera: enunciado A, enunciado B, enunciado C, etc.

Posteriormente estos enunciados se analizaron en los diferentes niveles para indicar las relaciones expresadas e implícitas entre ellas. En este esquema pudimos determinar a qué nivel las conjunciones marcaban determinada relación; esto se registró así:

A —— B	A ——→ B	A - - -→ B	A = B
presupone	implica	implicatura	equivale

En ocasiones nos encontramos que algunos implícitos contenían a su vez otros elementos intermedios entre el implícito y la expresión siguiente sin el cual la relación no podría entenderse con claridad, por ejemplo en presencia de negación. De aquí que en nuestras configuraciones incluímos otros elementos; éstos fueron representados como, x, y, z.

El signo que usamos para indicar negación es el convencional que se usa en lógica, es decir '∼'.

En cuanto a los criterios sobre los cuales nos guiamos para determinar con qué tipos de relaciones estábamos trabajando decidimos adoptar los siguientes de manera sistemática:

a) Los elementos que entran en los distintos tipos de relación constituyen niveles: tenemos entonces nivel de expresión, nivel de presuposición, etc. La descripción de cada enunciado se analizó por niveles.

b) Al registrar las proposiciones expresadas, los predicados de las proposiciones se presentan de manera sencilla y en ocasiones simbólicas.

c) Para verificar las presuposiciones que identificamos seguimos el criterio general de aplicar la prueba de la negación a las expresiones linguísticas: si la información de una expresión seguía presente después de negarla tomamos esta información como una presuposición. Si el contenido se modificaba, entonces se descartaba que fuera una presuposición (Levinson, 1983: 191). Esto último lo señalamos con el signo de 0. Cuando esto sucedía entonces nos enfocamos de inmediato a otro nivel de relación.

d) Para determinar las implicaciones aplicamos la prueba del "si ... entonces", tomando como primera parte de la argumentación el primer enunciado analizado y derivando nosotros su consecuencia "lógica", sin tomar en cuenta el enunciado que seguía a la conjunción. También aquí se presentó el caso de no existir implicación, por ejemplo en el uso de una metáfora. En estos casos se desechó la implicación indicándola también con el signo de

0.

e) Para determinar las implicaturas, nos preguntamos ¿qué mas quiso decirnos el autor, aparte de lo ya expresado y con relación al tema tratado? Aunque inevitablemente éste es un nivel menos estricto, sin él no es posible dar cuenta del discurso como un todo coherente.

f) En el primer análisis del funcionamiento de una conclusión, siempre supusimos que un implícito de A quedaría expresado en B. Cuando esto no se comprobaba, formulábamos un enunciado que expresara dicho implícito. A éste le llamamos enunciado intermedio. A continuación verificamos si uno de los implícitos de ese intermedio estaba expresado por B.

2.- CARACTERISTICAS Y FUNCIONES DE LAS CONJUNCIONES 'AND', 'BUT', 'OR', 'IF ... THEN'; ANALISIS EN LOS NIVELES DEL DIRCURSO DE EXPRESION, PRESUPOSICION, IMPLICACION E IMPLICATURA.

Retomando el enfoque y los principios adoptados, podemos introducir este capítulo diciendo que las conjunciones hacen explícitas ciertas relaciones implícitas en las expresiones. Digamos que son los representantes linguísticos de una relación de probables derivaciones, indican al lector qué derivación seleccional entre todo aquello que las expresiones conllevan. Así, indican en qué sentido debemos considerar lo que se encuentra antes de ellas con respecto a lo que viene después. En otras palabras, tienen cierto dominio sobre el procesamiento de lo que ya se dijo, señalando qué información se habrá de mantener para complementar el mensaje de lo nuevo que aparece después de ella (3).

2.1. 'And'

Es la conjunción más generalizada. Aparte de que su uso es el más frecuente, ésta puede usarse como equivalente a algunas otras expresiones conjuntivas.

Estas equivalencias son posibles cuando las otras expresiones conjuntivas comparten características de 'and'.

En relación de coordinación encontramos que 'and' marca el último elemento en sucesión (1),

(1) "On the other hand, a circle, a figure eight and a trefoil were not interchangeable curves..."
(Kline: 120)

Por lo general marca la coordinación entre dos elementos.

En los dos casos pueden tratarse desde elementos linguísticos como palabras u oraciones completas hasta en algunos casos elemenos no linguísticos (2),

(2) "...for the point P, x=3 and y=4"

(Kline: 117)

Lo único que tenemos que cuidar para su identificación como conjunción que marca coordinación, es que los elementos están al mismo nivel, es decir en relación paratáctica. De no ser así estaremos frente a un 'and' que indica una relación de subordinación.

En relación de subordinación 'and' marca las relaciones de implicación e implicatura de un enunciado hacia otro en la secuencia de la oración compleja. Así en las oraciones (3) y (4) vemos una relación de implicación,

(3) "Thus congruence, similarity and equivalence are major themes of Euclidean geometry, and the majority of theoremes deal with these question."

(Kline: 115)

A ⟶ B

and

	A	B
Ex.	Themes (c, s, e) of E-G.	deal whith (majority of themes) (c, s, e)
P.	Congruence, similariyy and equivalence are themes of E-G.	there are theoremes that deal with these questions.
I.	C, s and e, are included in any part of Euclidean Geometry.	The themes are part of most theoremes.
Im.	These themes are important to be considered in the E.G.	The majority of themes take these themes as part of their content.

(4) "Riemann was one of Gauss's students and undoubtely acquired from him an interest in the study of the physical world."

A -------> B

and

	A	B
Ex.	GS(R)	interested in the physical world (Riemann)
P.	Riemann was a student Gauss was a teacher	Riemann acquired an interest in the study of physical world.
I.	Riemann learned from Gauss	he would have studied physical fenomena
Im.	Gauss as a teacher, would have interested his students in his field (physics).	Gauss's interest in the physical world was transmitted to Riemann.

En el ejemplo (5) también encontramos una relación de implicación con un implícito previo a la implicación expresada en D.

(5) "Geometry up to this time had been essentialy Euclidean geometry; analytic and differential geometry were merely alternative technical methodologies and although projective geometry dealt with new concepts and new themes, they were enterely in accord with Euclidean geometry"

(Kline: 118)

A ⟶ (x) ⟶ D

and although

	A,	B	C	D
Ex.	E-G (geometry) (new concepts)	technical (a&d methodo. geo)	P-G dealt new con&themes	were in accord with E-G.
P.	0	0	P-G dealt with something	the new concepts were in accord with E-G.
I.	E-G is a basis for geometry con&themes	a&d geo. are kinds of geo.methodologies.	P-G was new	the new concepts & themes were not different from E-G.
Im.	E-G is valid up to this time	a&d geometry are not different geometries	P-G is a completely diff. geom.	P-G is not different from E-G.

(x) Geometry has precepts valid for other geometries.

lo mismo sucede en relación de implicatura, encontramos un implícito intermedio para llegar a la implicatura expresada en el siguiente enunciado, así en (6),

(6) "Moreover, by invalidating classic Greek mechanics which presupposed a stationary earth, the heliocentric theory necessitated a completely new science of motion and therefore the study of curves along which subjects move"

(Kline: 415)

A ----→ (x) ----→ B

and therefore

	A	B
Ex.	necessitated a new science (HT)	necessitated the study of curves (HT)
P.	0	0
I.	There is a need of a science that describes a non stationary earth	There are curves that have not been studied.
Im.	One knows the earth moves, this is the reason theory necessitated a new science.	the study of curves along which subjects move had to be expanded.

(x) The heliocentric theory is not in accord with the old theory.

La relación de implicatura también la vemos en el ejemplo (7) y en el ejemplo (8); tanto la implicatura de A como el enunciado B son equivalentes:

(7) "Thus one might be interested in studying functions such as x^2, $3x^2$ and x^2 2x and be interested in the value of these functions as x varies from 0 to 1"

(Kline: 121)

A ----→ B

	A	B
Ex.	I(one, s)	= I (one,v)
P.	There is a possibility that one would be interested in studying functions. There are such as x^2, $3x^2$...	There is a possibility that one would be interested in the values of these functions. There are values of functions.
I.	The functions have values	these values are numbers.
Im.	These functions and its values can be taken to be studied.	The value may give information for scientific purposes

(8) " Projective geometry flourished rather briefly and then was pushed aside by a rival geometry that appeared on the scene"

(Kline: 115)

A -------→ B

	A	B
Ex.	F. briefly (P-G)	= App (rival) → Pushed (P-G) aside
P.	P-G Flourished	= P-G was pushed aside by a rival geometry.
I.	0	P-G was not pushed aside all the time.
Im.	methafor x is like p- P-G did something which is like something that flourishes.	there is another new geometry.

Briefly = replaced by other.

En otro ejemplo encontramos la relación de enunciado de presuposición-implicación así (A— B) ⟶ C como en (9)

(9) "A circle like any other curve is just a particular collection of points. And if the circle is placed on a coordinate system then each point on the circle has a pair of coordinates."

(Kline: 116)

(A— B) ⟶ C

		and	if them	
	A		B	C
Ex.	Collect of points	(circle)	Place a circle on a coord. system	has a pair (each) of coord.
P.	A circle is a collection of points		There is a circle the circle is parallel somewhere.	There are points on a circle
I.	A circle is a curve		a coord. system has two coord. axes.	Each point has a different position.
Im.	0		suppose that...	0

En otras relaciones encontramos que tanto la implicación como la implicatura de A se encontraron expresadas en el iniciado C, ver ejemplo (10) y (11),

(10) "if we change the relevant axiom of Euclid accordingly and if we assume that there are no parallel lines, we have another set of axioms from which we can deduce still another non-Euclidean geometry"

(Kline: 119)

(A&B) ⟶ C

	if	and/coord. if	
	A	B	C
Ex.	change (axiom)	Assume (no parallel lines)	another set (axioms) → D(non-E-G)
P.	There is a relevant axioms	there are parallel lines	0
I.	there would be a different axiom	0	the new set of axioms prove the existence of a non-E-G.
Im.	Suppose one changes the relevant axiom one needs to describe the changes.	Suppose there are no parallel lines we exclude the possibility of existence of parallel lines in the scheme	a non-E-G exists

(11)"Two hundred years ago this subject was an extension of coordinative geometry and was devoted to the study of curves that are more complicated than conic section..."

(Kline: 121)

A ----→ B

and

	A	B
Ex.	Ex of coord. geometry (subj.)	was devoted to the study of compl. curves (subject)
P.	There is a subject There is a coord. geometry	The subject studied curves
I.	The subject was dedicated to some area of coord. geometry.	The only area of the subject was "complicated curves"
Im.	The fact took place in the past.	the fact took place in the past.

En el ejemplo (12) encontramos equivalencia entre implícitos.

(12) "One can see at left a plane intersecting a cone to produce a curve, at center the resulting curve and at right the corresponding surface"

(Kline: 114)

Donde ABC son implicaturas.

	A	B	C
Ex.	CS (one) ⟶ (intersect a cone (p) ⟶ produce a curve)	CS (one) (curve)	CS (one) (surface)
		=	=
P	There is a plane intersecting a cone	There is a resulting curve	There is a surface
	=	=	
I	A plane can produce a curve	a curve on a plane has a surface	0
Im	this is the description of a drawing or plane intersecting a cone	⟶	

Cuando nos encontramos con expresiones como 'and so' la implicación se hace enfática en el nivel de expresión, con la presencia de 'so'. Sin este elemento también tendríamos la misma relación que en (13).

(13) "The character of the surface changes from place to place and so the distance formula that determines the geometry must change from place to place...

(Kline: 119)

A ⟶ B

and so

	A	B
Ex.	Ch. from place to (surface) place	must Ch. from place to (the distance) place
P.	There is a character of the surface	There is a distance that determines the geometry
I.	There must be also changes in other area of geometry	The distance varies
Im.	Consider the surface	Consider the distance

2.2. <u>'But'</u>

Aparece marcando una relación entre dos elementos linguísticos que pueden ser palabras u oraciones a nivel coordinación.

Su característica principal es que conlleva un elemento 'not' implícito, mismo que puede hacerse explícito en los enunciados. Cuando esta negación no se encuentra explícita, entonces puede actuar negando un implícito ya sea que se encuentre en la expresión anterior a 'but' o en la que le sucede. Esta conjunción marca una relación de implicación o posibles implicaturas negándola. Cuando la negación no es muy clara o factible en los enunciados, el papel de 'but' en la facilitación de la lectura es de primordial importancia; aquí parece como si la conjunción fuera compuesta por dos miembros: 'not...but', 'no longer... but', 'not only...but'.

Esta negación no necesariamente nos indica una relación de contradicción entre los elementos contiguos a ella. Sólo nos marca qué implícito habrá que negar, por ejemplo en (1), donde técnico no quiere decir que sea contrario a esencial, sólo que el esquema de lo científico donde se ubica lo técnico nos haría pensar que lo técnico no es importante. Vemos la negación de esta implicatura, dando lugar en B a la derivación deseada. En esta descripción (x) señala el implícito

(1) "His first point was somewhat technical but essential"

(Kline: 118)

A ---> ~ (x)

but

A	B
Ex. technical (1st point)	Essential (1st. point)
P. There is a 1st. point	There is a 1st. point
I. 0	1st. point is essential
Im. The point is not ~~so~~ important within a scientific scheme, (x) technical is of a lower rank than scientific	Place 'his' technical in the cathegory of essential in a scientific scheme.

Cuando 'not' está explícito se debe, como ya dijimos, a que la negación del implícito no queda clara, como en el siguiente ejemplo, donde nuevamente se niega la implicatura (2),

(2) "For example, the solution of second degree equations in one unknown (x-8x +7 is such an equation) was carried out geometrically and the answer given by Euclid was not a number but a line segment"

(Kline: 113)

B -------→ ∼ (x)

	A	and B	but C
Ex.	carried (solution) out geom.	∼ N (answer)	LS (answer)
P.	the solution was not carried out geometrically	0	0
I.	The solution was not carried out algebraically	The solution something else instead of a number	a line segment equals the solution of an equation translated into a geometrical language
Im.	it is possible to get a solution of 2nd degree eqs. by geometrical means Algebra and Geometry are alternative methods	the solution ∼ is not a number as we may have expected. (x)	This was Euclid's solution

(x) the method of solution one might have expected would have been algebraic.

En ocasiones la negación se torna enfática mediante otras expresiones, por ejemplo en 'but...instead' como en (3),

(3) "The significance of this and other theoremes of projective geometry is that this geometry no longer discusses congruence, similarity, equivalence and other concepts of Euclidean geometry, but instead deals with colinearity... concurrency... and other notion stemming from projection and section."

(Kline: 115)

A ----→ B
B~ ——→ A (enfático)

but instead

A	B
Ex. sign. (theoremes) $\xrightarrow{\simeq}$	Deals (P. geom.) with ...
discusses congruence, (P. geom.) similarity equivalence	
P. Projective geom. does not discuss congruence...	Projective geom. deals with something
I. Congruence, similarity... ~ and other concepts are not topic of discussion of projective geom.	Collinearity, concurrency are topics of discussion of projective geom.
Im. Geometry deals with congruence, similarity	There are specific topics in projective geom.

Otra forma en que encontramos 'but' fue en la expresión 'not only...but also', como en el ejemplo (4). Se trata de una conjunción bimembre. Su primera parte expresa una proposición de segundo orden (4). Nos dice que una proposición que se expresa en el mismo enunciado no es la única verdadera. Esto implica que hay al menos otra proposición diferente de ella que es verdadera, lo cual es, a su vez, afirmado por la segunda parte de la conjunción 'but also'.

(4) "All these problem not only increased the need for knowledge of properties of familiar curves but also introduced new curves".

(Kline: 116)

A —— (x) ---→ B

A ——→ B

but also

	A	B
Ex.	increased (probls. need for knowledge)	Introduced (probls.) new curves
P.	it is true that there is another proposition which is true. (x)	The problems introduced new curves.
I.	New knowledge about curves was needed. (not only familiar)	New curves appeared.
Im.	Besides of increasing knowledge, the problems led to something else.	____________

Pero ¿por qué es necesario poner estos predicados de segundo orden? ¿Por qué es necesario afirmar que lo que se va a decir es verdadero? ¿Por qué es necesario afirmar que lo segundo que se diga será diferente de lo primero? Es normal asumir que se dicen cosas verdaderas y que no se está repitiendo lo que ya se dijo.

Al predicar acerca de las preposiciones que se enumeran y no sólo enunciarlas, el autor está advirtiendo explícitamente que lo que sigue será diferente de lo que se expresaría; está indicando que la segunda proposición de primer orden será uno que no ha sido enunciado antes, aunque está relacionado con los que ya se mencionaron. Esto es, hay una implicatura importante: la segunda expresión será sobre algo distinto que 'familiar curves'. Podríamos esquematizar lo que se dijo de la siguiente manera:

Ex. A

p existe y es verdadera

P.p no es la única expresión que existe y es verdadera.
(p: p(a,b), donde b está expresado por medio de una descripción que indica 'familiar curves')

I. Existe r ≠ p y r es verdadera.

Im. r: R(c) y c es diferente de b y de 'familiar curves'
p: P(a,b), b está expresado por medio de una descripción que involucra a 'familiar curves".

Este último implícito, la implicatura coincide con lo expresado en B. En forma diagramática, resumimos:

p — (existe r y r es verdadera) --→ R(c) y c es distinto de b y distinto de 'familiar curves'.

A continuación encontramos otro uso de 'but' donde la negación se da sobre una supuesta generalización, y esta negación actúa o se expresa en el enunciado después de 'but' como en los ejemplos (5) y (6), haciendo notar que lo que se expresa en el enunciado posterior a 'but' es la excepción a la generalización:

(5) "A Great circle cuts the sphere in half; the Equator is a Great circle but a circle of Latitude is not."

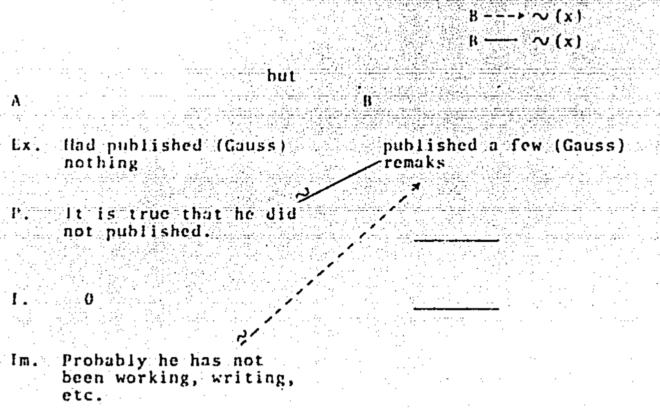
(Kline: 117)

$$A \longrightarrow B$$
$$A \sim \longrightarrow C$$

	;	but	
	A	B	C
Ex.	cuts the sphere in half (GC)	GC (Equator)	~GC (c Latitude)
P.	there is a G C	there is a Equator	there is a circle of Lat.
I.	a GC divides the sphere in two parts	The Eq. cuts the sphere in half	a circle of Lat. does not cut the sphere in half.
Im.	a generalization definition of GC.	Comparison of the Eq. with a GC and a definition	It is not possible to define a circle of Lat. as a GC.

(6) "Gauss had published nothing but a few cryptic remarks on this topic..."

(LE Corbeiller 1954: 128) (Anexo 2)



Otro ejemplo en este sentido lo tenemos en (7), donde se hace evidente una excepción en la generalización de la primera proposición. La forma 'neither' enfatiza la negación de la primera proposición para introducir mediante la proposición con 'if' el elemento de excepción o el elemento que no entra en la generalización. La negación de 'but' actúa sobre la implicación de B.

(7) "Experience, he pointed out does not assume us of the infinitude of the physical strainght line. Experience tells us only that in following a straight line we do not come to an end. But neither would one come to an end if one followed the Equator of the earth"

(Kline: 119)

B∼ ---→ A

but

	A	B
Ex.	F (SL) ⟶ ∼ end	F (E) ⟶ ∼ (end)
P.	Experience tells us that following a straight line we do not come to an end.	The Equator of the earth is not a straight line.
I.	0	The Equator is a straight line
Im.	A straight line does not come to an end (es una generalización) There is more than experience.	There is a contrast in the concept of a straight line and a curve line.

En ocasiones encontramos que la negación que representa 'but' se encuentra implícita en el significado del elemento que la antecede, por ejemplo en (8). En este ejemplo existe una metáfora, misma que analizamos en dos niveles para llegar a la implicación que se da sólo pasando primero por el nivel de implicatura:

(8) "The geometry suffers but the algebra flourishes"

(Kline: 121)

A ∼----→ (x)

but

	A	B
Ex.	S (geometry)	F. (algebra)
P.	There is geometry	There is algebra
I.	0 (sufrir no es una propiedad de la geometría)	0 (florecer no es una propiedad del álgebra)
I.2	Geometry gets complicates geometry can not be represented.	Algebra develops, expands
Im.	(Metáfora: (x) es como p) The geometry is doing something which is like something suffering	(Metáfora: (x) es como p) The algebra is doing something which is like something flourishing.
Im.2	Geometrical representations of the values can hardly be imagined (en un párrafo posterior en el texto)	Algebra takes complex. values (en el texto)

2.3 'Or'

Es una conjunción que tiene predominio en la expresión. Marca la relación a nivel coordinación de elementos (1).

(1) "The type of curve drawn on the sphere at left does not bound on an area on the torus at center or on the double torus at right"

(Kline: 121)

A nivel lógico nos señala que los valores de verdad de los elementos que coordina no están determinados. Podríamos encontrar un equivalente como en 'if not' (2).

(2) "Consequently even the simple equation $x^2 + y^2 = 25$, which when x and y have real values represents the circle discussed previously, can represent a Rieman's surface or (if not) a structure so unconventional that it can hardly be imaginen"

(Kline: 121)

Es esta misma indeterminación la que podríamos pensar nos lleva a una elección de probables opciones que al mismo tiempo podríamos trasladar a un nivel de implicatura donde las interpretaciones pueden ser variadas. Lo que deseamos destacar es que tanto en los niveles lógicos y de implicatura se traslapan.

Habría en este sentido que hacer un análisis en otras relaciones, por ejemplo en el semántico (Lyons 1979), en el lógico de tres valores (Kempson 1982), y pragmático (Caron 1987).

Otra equivalencia que podemos encontrar con 'or' es un 'and' en relación de coordinación (3),

(3) "It is possible to characterize closed surface in terms of curves that do or do not bound on the surface..."

(Kline: 120)

En este ejemplo los elementos que marcan 'or' son oraciones que no indican precisamente una opción sino una coordinación de elementos con valores de verdad determinados, es decir los dos elementos son verdaderos, lo que aquí influye en esta 'opción' es la expresión de 'it is possible' y de la presuposición de la existencia de 'curves'. Pero es el contenido de la expresión lo que nos lleva a las sutiles diferencias entre uno y otro 'or' (ver también Van Dijk 1977: 43-46).

Otra expresión de esta coordinación la encontramos con 'either... or' (4), (ver también nota (5))

(4) "Thus it is evident that we can recognize that a sphere is round either by observing it from a distance or if we stand on id, by observing objects far away."

(LE Corbeiller 1954: 128)

Una forma menos sutil de 'or' en su interpretación de 'elección de opciones' es la expresión 'whether...or', como en el ejemplo (5)

(5) "The study of abstract space is, surprisingly, part of topology because the properties of these structures that are important, whether the structures are regarded as actual space or as collection of functions, are preserved..."

(Kline: 121)

En este ejemplo uno de los elementos que marca 'or' es falso y el otro es verdadero. (Van Dijk: 44)

2.4. '<u>if...then</u>'

La relación que 'if' marca será entre un implícito de la proposición que encabeza que puede ser una implicación (1) o una implicatura (2) con otra expresión que por lo general es la afirmación de lo que se plantea en el implícito de la cláusula 'if'. Consideremos por ejemplo,

(1) if a straight line n cuts the line l and m so as to make corresponding angles with each line that total less than 180 degrees, then l and m will meet on that side on the line n on which the angle lie"

(Kline:118)

A ——→ B

if then

A	B
Ex. cuts (sl)→Make l&m (corresponding angles...)	* Will meet on that side or the line (l&m)
P. There are lines m&l&n The line n makes corresponding angles with each line	0

I.	The lines form angles* that total less than 180 degrees	l & m meet on line n.
Im.	0	0

(2) "If one takes the surfaces of the earth to be a sphere, the answer is simple".

(Kline: 117)

A ----→ B

	if	
	A	B
Ex.	takes the surface (one) → of the earth (*) sphere	simple (answer)
P.	(el valor de verdad de p no está determinado)	there is an answer
I.	(implicamos que los paralelos de la tierra son paralelos)	0
Im.	(las implicaciones de p son triviales)	One is searching for answers.

A (x) B

(*) Para efectos de la discusión el valor de p es verdadero.

En el ejemplo (2) lo que se expresa en la cláusula 'if' nos lleva a pensar en una implicación que no es la que se expresa en la cláusula siguiente. En este caso el autor nos llevó a una implicatura haciéndonos entender que las posibles implicaciones son conocidas, resultando familiares.

Es interesante destacar que 'if' tiene dominio sobre la cláusula que encabeza.

Vemos que en relación de yuxtaposición en la expresión la relación de implicación se entiende (3),

> (3) "if however (*) one tears a figure or contracts it in such a way as to make points coalesce, the new figure is not topologically equivalent to the old one"
>
> (Kline: 120)

A ⟶ B

if

	A	B
Ex.	T or (figure) ⟶ C coalesce (points)	~Ēquivalent (new fig) & (olf fig)
P.	There is a figure	There is a new figure there is an old figure
I.	we have another figure a different figure	They do not share topological characteristics
Im.	we deform the figure	0

(*) (however) es diferente de 'if' aunque se encuentre en la misma cláusula)

Es posible alterar, otra vez a nivel expresión, el orden sin que haya error de interpretación como en (4),

(4) The new figure is not topologically equivalent to the old one, if one tears a figure or contracts it in such a way as to make points coalesce.

Pero no es posible alterar el orden sin la presencia de 'if' (5) ya que la relación no sería clara. Pensamos que 'if' sirve de indicador para saber qué implicación o implicatura tomar en cuenta para efectos de la afirmación que tendrá que considerarse en las expresiones anteriores o subsiguientes a la cláusula 'if'.

(5) The new figure is not topologically equivalent to the old one, one tears a figure or contracts it in such a way as to make points coalesce.

Esto quiere decir que en relación de yuxtaposición el orden de las expresiones tiene importancia. Vemos en el ejemplo (5) que no es posible hacer las mismas derivaciones que en el ejemplo (3). Por otra parte la alteración del orden es posible con la presencia de 'if', misma que enmarca la existencia de una posible implicación.

La relación de implicación o implicatura que 'if' marca es evidente en nuestro análisis. En otros niveles como en la expresión, el valor de verdad de la cláusula 'if' se toma como verdadero. En cambio, a nivel presuposición, el valor de verdad es indeterminado. En esta correlación de niveles podemos distinguir la función 'if' como marcador de implicaciones o de implicaturas, pero también tiene la función de indicar la indeterminación de los valores de verdad.

El siguiente ejemplo (6) nos relaciona dos expresiones y es 'then' el elemento que redondea la relación de implicación. La primera se asemeja a una cláusula 'if', pero en este caso son otros elementos como 'it is also possible' e 'imagining' los que se encargan del'juego' de los valores de verdad.

> (6) "It is also possible to describe topologically equivalent figure by imagining them to be of rubber. Then any figure that can be obtained by stretching, bending or contracting, but not tearing, the rubber would be topologically equivalent to the initial one"
>
> (Kline: 120)

A ----> B

then

	A	B
Ex.	D(top. eq. fig.) & imagining (Rubber (fig))	Obtained (any)→equiv. fig (initial one)
P.	The true value of p is not determined made (rubber (fig))	The figures are obtained by stretching, bending or contracting.
I.	Topological figures are flexible.	The new figure are topologically equiv. to the initial one.
Im.	A figure equivalent to another can be obtained by modifying or deforming it.	Figures made of rubber would be topologically equivalent

En cuanto a 'then', está marcando una afirmación que viene siendo una derivación o inferencia de algo expreso o se va a expresar, y puede estar implícito o expresado.

En relación de yuxtaposición la relación que marcaría 'then' se entiende debido a la presencia de la cláusula 'if'. Cuando 'if' no se encuentra, 'then' se hace necesario. También 'then' marca una implicación o una implicatura que se afirma en la expresión que precede aunque no haya una cláusula 'if'. En el siguiente ejemplo (7) se pone como indeterminado el valor de verdad de la expresión anterior a 'then', pero contiene una implicatura que nos explica la relación con la siguiente expresión.

(7) There he reconstructed without the aid of any book all he had learned from Monge; he then proceeded to create new results in the projective geometry".

(Klined: 117)

A ----→ B

; then

	A	B
Ex.	Recons. (he)→all...	Proceeded (he) ⟶ created (new results)
P.	He had learned from Monge	0
I.	0	0
Im.	He had an outsanding talent.	He contributed with new results to the projective geometry.

Nuestra explicación sobre los implícitos no iría en contra de ninguna explicación del tipo semántico de temporalidad, adición, condicional, etc.; sólo proponemos que la explicación de otras relaciones tendrían que partir de un punto de vista pragmático. Los siguientes ejemplos (8) y (9) ilustran este comentario.

(8) "The history of these investigations would be worth nothing if for no other reason than to see how persistent and critical mathematicians can be"

(Kline: 118)

A ---→ B

if

	A	B
Ex.	worth (the history...) nothing	reason ⟶ persistent critical (mathematicians)
P.	The history is worthy	there is a reason mathematicians can be persistent and critical
I	0	0
Im.	There is something that make the history be worthy.	____________

(9) "if two figures posess these properties, they must be topologically equivalent as the congruence of two triangles is guaranteed, if two sides and the included angle of one triangle are equal to the perspective parts of the other"

(Kline: 119)

A ⟶ C

	,	if	
	A	B	C
Ex.	Posess (2 figs) properties	must be (2 figs) top. eq.	equal (2 sides & the angle of a triangle) → perspective parts of the other.
P.	There are two figs there are properties	0	There are two sides & the included angle of the triangle there are perspective parts.
I.	The figs. share characteristics	Taken congruence into account	the triangle and the perspective parts are equal.
Im.	"one seeks to characterize equivalent figure by some definitive properties" (en el texto) if it is the case that two figs...	0	that is to say... for example...

3. CONCLUSIONES

Nuestro punto de vista de estudiar las conjunciones como elementos que hacen expresar ciertas relaciones implícitas en las expresiones, nos ha llevado a consideraciones que más que conclusiones nos sugieren el planteamiento de otras posibles investigaciones que nos puedan ayudar a esclarecer de qué manera se dan las relaciones discursivas.

Nuestro trabajo se limitó a las relaciones de: las proposiciones expresadas de coordinación, subordinación, yuxtaposición; la presuposición; la implicación e implicatura. Pero nuestra propuesta quedaría abierta para incluir otras relaciones (semánticas, lógicas, de tres valores, inductivo-deductivo, etc.)

Consideramos que las conclusiones indican la relación a partir de lo que viene antes y después de ellas teniendo control de inferencia sobre la proposición colocada después de ellas; debido a las características de cada una con respecto a la relación en que intervienen.

La conjunción'and' interviene a nivel expresión y en relación de coordinación como marcador del último elemento en sucesión, así como la coordinación entre dos elementos que pueden ser lingüísticos o no lingüísticos. En relación de subordinación 'and' marca las relaciones de implicación o bien de implicatura. Podría asimismo marcar otras relaciones por ejemplo de presuposición-implicación la cual pensamos se deba a la influencia de otras expresiones conjuntivas que se encuentran contiguas a 'and' (como en el ejemplo (9) de 'and'; 'and if...then'), mismas que conllevan características propias.

'And' podría tener por su extensión importantes consecuencias a nivel semántico si tomamos en cuenta su característica como encabezador de implicaciones e implicaturas. En esto último es donde pensamos se podría discutir sus características de'adversativa'o de'condicional'.

'But' conlleva un elemento 'not' implícito que puede hacerse explícito en la expresión, teniendo influencia sobre algún implícito de las proposiciones expresadas. Pensamos sea ésta la razón por la cual se le considera como 'adversativa' en otras clasificaciones, aunque para nosotros no necesariamente indica una relación de contradicción. Marca una relación de implicación o posibles implicaturas negándolas. Puede aparecer sola o compuesta por dos miembros 'not...but', 'no longer...but'. 'not only... but' y con otros elementos que sirven como enfatizadores de la relación correspondiente ('but instead', 'but also' acompañado de 'not only', 'but neither'). Nos pareció interesante que el 'not' puede actuar implícito en los mismos significados de las palabras que ya connotan una negación en cierta relación semántica, nos referimos al ejemplo (8) de 'but'. Se trata de una relación compleja que merecería especial atención.

La conjunción 'or' tiene predominio en la expresión marcando que los elementos se encuentran coordinados. Proponemos que señala a nivel implicatura una 'elección de posibles opciones'. Pero con nuestro análisis no pudimos constatarlo con claridad ya que nos percatamos del traslape de este nivel de implicatura con el nivel lógico de valores de verdad. Necesitaríamos otro análisis que incluyera por ejemplo un valor de verdad 'indeterminado' además de una relación semántica; estudiar también las posibles equivalencias con 'and' e 'if not', las relaciones que marcan y revisar los conceptos de 'inclusivo' y 'exclusivo' de 'or'. En nuestro estudio sólo pudimos confirmar que se trata de una conjunción que marca una relación de coordinación. En realidad nada nuevo, pero las interrogativas que plantea no dejan de ser interesantes.

En cuanto a 'if...then' vemos que 'if' tiene dominio sobre la cláusula que encabeza. La relación que marca será entre un ímplícito de la proposición que encabeza (este puede ser de implicación o implicatura) con otra expresión que por lo general es la afirmación de lo que se plantea en el implícito de la cláusula 'if'. A nivel presuposición tiene la función de indicar la indeterminación de los valores de verdad. 'Then' es el elemento que redondea la relación de implicación. Marca una afirmación de una inferencia de algo expreso o se va a expresar (y esto último puede encontrarse implícito o expreso). No es necesaria su presencia mientras exista una cláusula 'if', pues la relación de implicación se entiende en la expresión en relación de yuxtaposición.

Retomando nuestras hipótesis, y de acuerdo a nuestras consideraciones en este trabajo, destacamos que el análisis por niveles nos permite aceptar que la función de las conjunciones no es establecer sino marcar o indicar relaciones engre proposiciones explícitas e implícitas (hipótesis 1, y 2,). En el análisis constatamos la posibilidad de definir características de las conjunciones (hipótesis 4,), a partir de la correspondencia o no entre implícitos en los diferentes niveles y el nivel de la expresión de las proposiciones que se encuentran antes y después de ellas.

Después de haber descrito las características de cada una de las conjunciones que estudiamos en nuestro análisis podemos decir que ellas, al estar encabezando, marcando o haciendo explícita alguna relación se vuelven predecibles. El que emite el discurso se vale de ellas para introducir las diferentes inferencias que él hace y que desea que los lectores hagan. En este sentido diremos que ellas dan instrucciones, es decir, que también su función podría ser pragmática (hipótesis 3,).

En la red de relaciones las oraciones complejas yuxtapuestas y las que contienen conjunciones son equivalentes, es decir, las relaciones se dan en diferentes niveles de las proposiciones; entonces su uso no resulta indispensable. Podemos pensar que existen otros recursos para indicar las inferencias además de las conjunciones.

Pero también es cierto que en su papel de marcadores de relación y sus características de 'dirigir' las inferencias, su uso sí resultaría indispensable; auxilian en la simplificación de la expresión linguística haciendo explícitas con su presencia, las relaciones implícitas de las proposiciones afirmando o rechazando la información implícita.

Es interesante encontrar que entre unas y otras conjunciones existen equivalencias y también rasgos diferenciadores que limitan su uso (por ejemplo pueden ser semejantes o sustituirse entre ellas al indicar determinado nivel que comparten en la relación de expresiones, al mismo tiempo estaría presente otro implícito que los diferenciaría, característica, pues, de cada una). Esta idea de las equivalencias no la estudiamos en nuestro análisis. En nuestras reflexiones proponemos que sea este el punto de la especialización de las conjunciones, por ejemplo, cuando observamos la posibilidad de extender el número de ellas con expresiones que conllevan una función conjuntiva, y por otra parte pensamos que en las posibles clasificaciones se tendrían que tomar en cuenta estas coincidencias y diferencias de niveles.

Cuando nos acercamos al cálculo predicativo para trasladar las conjunciones que estudiamos a las conectivas lógicas nos sirvió para darnos cuenta, hasta cierto punto, de esas equivalencias de las que hablamos arriba, a nivel de la expresión; no atendimos ninguna clasificación o agrupación pues también nos dimos cuenta que se trataba del traslado de un lenguaje a otro, bajo las condiciones de verdad del lenguaje lógico y no del natural.

El nivel implicatura nos puso de manifiesto que el autor hace supuestos sobre lo que el lector ya conoce, haciendo un llamado a los esquemas generales de conocimiento, pero a su vez ubicándolos en el esquema del tema tratado en el contexto general del discurso. Este nivel nos permitió un margen más amplio de interpretación. La implicatura se refiere a proposiciones asociadas con entidades o hechos que se mencionan conformando un esquema o patrón de comportamiento. Nos dimos cuenta que una implicatura se puede derivar de la mera presencia de un predicado o un argumento solos, o de la forma de expresarlos; a diferencia de la implicación, no necesariamente es una consecuencia de una proposición completa. En este sentido, la implicatura es similar a la presuposición.

Esta investigación nos ha expuesto la posibilidad de estudiar las conjunciones desde una perspectiva diferente. Sabemos se trata de una etapa inicial que necesita de precisiones importantes así como de resolver otras preguntas; por ejemplo, si estamos tratando con distintos tipos de relaciones al mismo tiempo ¿cuál domina y cómo se refleja en el enunciado? ¿Qué está señalando esas relaciones? ¿Cómo coactuan las relaciones?

Esta forma de acercarnos a las conjunciones nos ha permitido visualizar una nueva perspectiva y nos abre una serie de posibilidades para interpretar la tarea de estas partes del lenguaje natural. En nuestra opinión las explicaciones que la gramática tradicional nos proporciona son suficientes para un nivel de expresiones, pero ahondando en otros niveles nos damos cuenta que aún adolecemos de elementos para reconocer en la red de relaciones linguísticas la función específica que desempeñan cada una de ellas y las derivaciones en expresiones conjuntivas, así como sus posibles equivalencias.

Reiteramos por último, que este punto de vista de acercarnos a las conjunciones puede ampliarse incluyendo más tipos de relaciones discursivas. Nos podría ayudar a determinar el alcance de las conjunciones en su uso y función.

A continuación presentamos anexo a este trabajo algunos comentarios sobre los lenguajes especializados y su relación con nuestro punto de vista del estudio de las conjunciones.

4. LA ESPECIALIZACION DE LAS CONJUNCIONES.

Nos gustaría en esta parte tocar el tema de los lenguajes especializados y el lenguaje natural en relación con las conjunciones.

Tres son los aspectos hasta ahora discutidos que distinguen a un lenguaje natural de un lenguaje especializado:

1) Hacer precisos y definir los significados de las palabras.

2) Estipular reglas precisas de sintaxis lógica.

3) Crear nuevas palabras.

El lenguaje natural es el que vincula a todos los que hablan una misma lengua y sobre el cual se basa cualquier otro lenguaje (Sartori 1981).

La primera pregunta que nos formulamos sería ¿qué tan especializadas son las conjunciones en estos tres aspectos? Consideremos que las conjunciones son palabras de 'función' y tomando en cuenta que el lenguaje especializado ha sido creado por la necesidad de comunicar el descubrimiento de nuevos aspectos del universo, desde nuestra perspectiva de que la función de las conjunciones es la de marcar relaciones, entonces diremos que la creación de un número cada vez más amplio de expresiones conjuntivas obedece a esta experiencia de precisión en la exposición de información.

Las teorías científicas son cada vez más precisas y especializadas por lo que se expresan en términos cada vez más especializados. Las conjunciones no estarían expresando conceptos, estarían auxiliando a expresar esos nuevos conceptos mediante el rechazo y afirmación de las aseveraciones que se hacen de ellos. Este hecho hace necesaria una gama de expresiones conjuntivas que indiquen con precisión qué relaciones, tanto implícitas como explícitas considerar de la información. Podríamos pensar que se han creado como auxiliares para evitar la ambiguedad.

Pasa por nuestra mente que las formas $\wedge$, $\vee$, $\sim$, $\rightarrow$, $\Leftrightarrow$, que se usan en lógica o los signos $+$. $-$, $\div$, de las operaciones matemáticas, por ejemplo, serían formas análogas a las conjunciones, o bien formas especializadas de expresiones conjuntivas del lenguaje natural. Pero estas formas que indican relaciones también tienen una función específica; por ejemplo, en lógica el que conllevan valores de verdad, y el de las operaciones matemáticas indican algo que se debe hacer con los elementos que relacionan. Pero son de naturaleza diferente, es decir, ellas pertenecen a otros lenguajes que no son el natural; a otro sistema de signos que expresan el universo.

La diferencia la encontraríamos en el mismo lenguaje natural, lo especializado de las conjunciones serían otras expresiones equivalentes a las conjunciones 'and', 'or', 'if... then', etc.. Serían formas nuevas que compartirían las funciones de ellas pero con ciertos rasgos diferenciadores. Estos últimos irían desde el orden sintáctico hasta el pragmático pero con rasgos probablemente más precisos, por ejemplo expresiones como 'although', 'therefore' 'nevertheless', etc. Nos preguntamos ¿por qué razón, por ejemplo, los niños no usan las conjunciones más especializadas, o bien por ejemplo, cuando iniciamos un curso de lectura de comprensión no podemos deducir estos elementos del contexto? Para éstas se hace necesaria una instrucción para su comprensión o bien la traducción para hacer posible las relaciones, las derivaciones de la información.

Se habla también de lo polisémico de las conjunciones (Caron 1987). Este aspecto tendría que replantearse considerando que las implicaciones, presuposiciones e implicaturas dependen de la información que contienen las expresiones. Aunque no quedarían exentas de marcar relaciones semánticas; en este rubro sugerimos que derivarían directamente del nivel de implicatura de las expresiones.

Otro punto de la discusión es la noción de registro. Un registro es un conjunto de funciones únicas del lenguaje asociadas cada una con una forma única. Al respecto Castaños opina que "la asociación entre forma y función no es necesaria. Es decir que una misma forma tendría diferentes formas, y a su vez una misma forma tendría diferentes funciones, y que los lenguajes de dos disciplinas comparten formas y funciones al menos en un par forma-función" (1986:37). La función de marcar relaciones conjuntivamente se expresa de diferentes formas, para esto sólo tendríamos que revisar alguna lista de estas expresiones para ver la gama que existe, pero además como ya vimos en las relaciones de yuxtaposición, aparecen otras formas que podrían asimismo marcar las relaciones por ejemplo, la puntuación, las pausas y hasta la entonación.

En cuanto a que una misma forma tendría diferentes funciones en nuestro análisis, la diferencia estriba en los tipos de relaciones que marcan las conjunciones, es decir, en presuposiciones, implicaciones, implicaturas, etc.

La compartición de dos disciplinas en un par forma-función estaría especificada en el tipo de relación que estuvieran marcando las expresiones conjuntivas.

Si por otra parte tomamos las palabras de Halliday que "un registro es junto con las palabras y las estructuras que lo expresan, una serie de significados adecuada para una función particular del lenguaje. Lo que constituye un registro son los significados, incluso los estilos de significación y los modos de argumentación más que las palabras y las estructuras como tales" (1986:254), entonces habríamos de concentrarnos en la parte prágmática, en nuestro caso en el nivel de implicatura, que proponemos replantearla y especificarla en las relaciones de implícito y expresión. Y también especificar de qué manera las conjunciones constituyen en la significación y los modos de argumentación para tomar parte precisamente de un registro.

Con nuestro enfoque sobre el estudio de las conjunciones en el análisis por niveles de relación del discurso, cabría la posibilidad, en trabajos posteriores, de replantear y responder, desde una perspectiva diferente, cuestionamientos sobre la especialización de las conjunciones.

NOTAS

(1) Cuando hablamos de conexión de unidades(dentro de la oración y entre cláusulas) estaremos abarcando una serie más amplia de elementos que desempeñan esta función además de los elementos sintácticos, a saber: la situación de comunicación, el medio de comunicación, la relación entre participantes y los propósitos de la comunicación.

(2) En este trabajo no abarcaremos estos dos últimos aspectos mas que como referencia.

(3) Como se verá en la descripción, esta opinión es contraria a la que señalan Halliday y Hasan (1976) quienes proponen que las conclusiones tienen dominio sobre lo que se dice después de ellas.

(4) Es una proposición cuyo argumento es otra proposición, es decir, otra proposición sobre una proposición.

(5) De acuerdo a Quirk et al. el agregar 'either' en la primera cláusula ayuda a hacer explícita la exclusión de una de las opciones en que 'or' interviene.
Comentan también que 'either' se usa como pronombre o determinante para referirse a sólo 'dos' elementos, y puede relacionarse a la expresión 'both...and' que se encuentra más limitado a 'dos' elementos. Los autores observan que en 'either' se usa también con una tercera cláusula que permite las dos alternativas anteriores de manera explícita:

> "You can either boil yourself an egg, or you can make some cheese sandwiches, or you can do both". (1972:563)

Lo que deseamos destacar en nuestra discusión es que 'either...or' en esta combinación aparece como indicador de una relación de coordinación, que bien nos sería util para un análisis de 'or' como 'elección de probables opciones', así como la posibilidad de considerar esta combinación como expresión conjuntiva.

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16 GEOMETRY

MORRIS KLINE • September 1964

The evolution of mathematics depends on advances in both number and geometry. It cannot be said, however, that these key elements of mathematics have always advanced side by side. Frequently they have competed and the advance of one has been at the expense of the other. The history of this sometimes strained relation between two disciplines that actually have a common purpose is reminiscent of controversial themes in music.

The first genuine stride of mathematics was taken by geometry. Some primitive mathematics was created by Egyptian and Babylonian carpenters and surveyors in the 4,000 years preceding the Christian era. [redacted] it was the classical Greek philosophers who, between 600 B.C. and 300 B.C., gave mathematics its definitive architecture of abstraction and deductive proof, erected the vast structure of Euclidean geometry and dedicated the subject to the understanding of the universe.

Of the several forces that turned the Greeks toward geometry, perhaps the most important was the difficulty Greek scholars had with the concept of the irrational number, a number that is neither a whole number nor a ratio of whole numbers. The difficulty arose in connection with the famous Pythagorean theorem that the length of the hypotenuse of a right triangle is the square root of the sum of the squares of the two sides. In a right triangle with sides of one unit each the hypotenuse must then be $\sqrt{2}$, an irrational number. Such a concept was beyond the Greeks; a number to them had always meant whole number or ratio of whole numbers. They resolved the difficulty by banishing it, producing a geometry that affirmed theorems and offered proofs without reference to number. Today this geometry is known as pure geometry or synthetic geometry, the latter an unfortunate term that has only historical justification.

Since the mathematics of the classical Greeks was devoted to deducing truths of nature, it had to be founded on truths. Fortunately there were some seemingly self-evident truths at hand, among them the following: two points determine a line, a straight line extends indefinitely far in either direction, all right angles are equal, equals added to equals yield equals, figures that can be made to coincide are congruent. Some of these axioms make assertions primarily about space itself; others pertain to figures in space.

From these axioms Euclid, in his *Elements*, deduced almost 500 theorems. In other works he and his successors, notably Archimedes and Apollonius, deduced many hundreds more. Because the Greeks chose to work purely in geometry, many of the theorems stated results now regarded as algebraic. For example, the solution of second-degree equations in one unknown ($x^2 - 5x + 7 = 0$ is such an equation) was carried out geometrically and the answer given by Euclid was not a number [redacted] a line segment. Thus Euclidean geometry embraced the algebra known at that time.

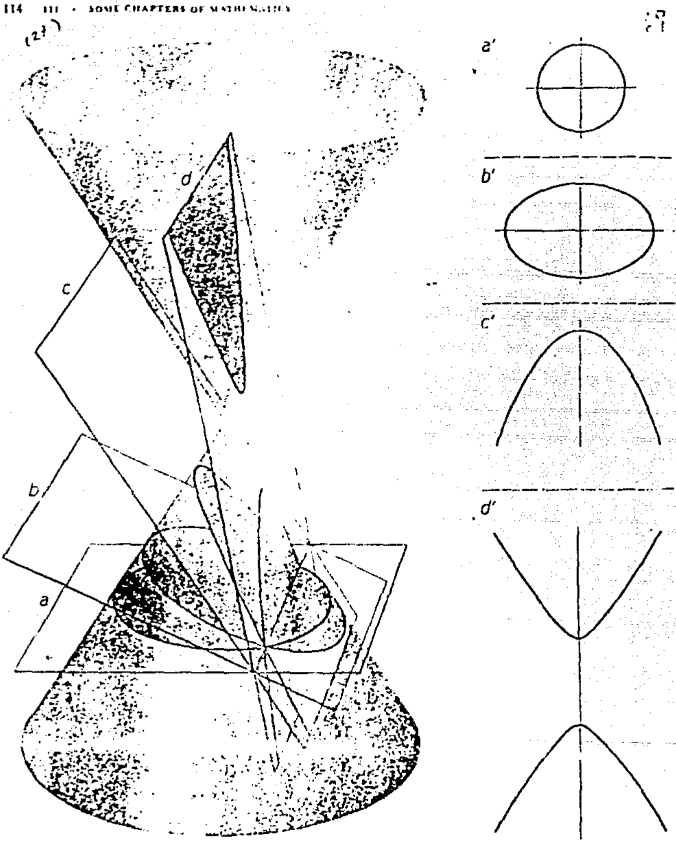
[illegible] suggest that the Greeks drifted from topic to topic. That would be a false impression. The figures they chose were basically lines and curves in one category and surfaces in another. In the first category are such figures as the triangle and the conic sections, circle, parabola, ellipse and hyperbola. In the second category are such figures as the cube, sphere, paraboloid, ellipsoid and hyperboloid [redacted] (see illustration on page 114). With these the Greek geometers tackled basic problems concerning these figures. For instance, what must one know about two figures to assert that they are congruent (identical except for position in space), similar (having the same shape but not the same size) or equivalent (having the same area). Thus congruence, similarity and equivalence are major themes of Euclidean geometry, and the majority of the theorems deal with these questions.

The classical Greek civilization that gave rise to Euclidean geometry was destroyed by Alexander the Great and rebuilt along new lines in Egypt. Alexander moved the center of his empire from Athens to the city he modestly named Alexandria, and he proclaimed the goal of fusing Greek and Near Eastern civilizations. This objective was ably executed by his successors, the Ptolemys, who ruled Egypt from 323 B.C. until the last member of the family, Cleopatra, was seduced by the Romans. Under the influence of the Near Eastern civilizations, notably the Egyptian and the Persian, the culture of the Alexandrian Greek civilization became more engineering-minded and more practically oriented. The mathematicians responded to the new interests.

Applied science and engineering must in large part be quantitative. What the Alexandrians appended to Euclid's geometry in order to obtain quantitative results was number, arithmetic and algebra. The disturbing fact about these

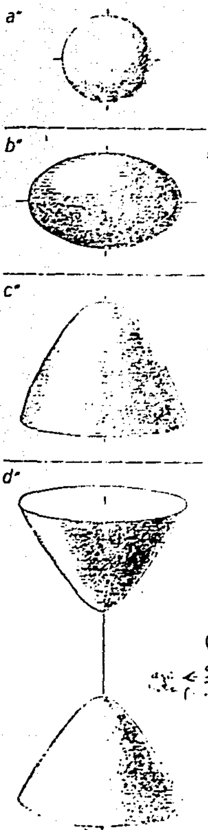
RAPHAEL'S "SPOSALIZIO," or "Marriage of the Virgin," part of which is reproduced on the opposite page, indicates how Renaissance painters solved problems of perspective and so contributed to the evolution of projective geometry. The superimposed white lines show how the artist depicted as converging on a "principal vanishing point" lines that in actuality were horizontal, parallel and receding directly from the viewer.

(27)



(28)

CONIC SECTIONS provide the basic curves with which geometry deals. By following each series of letters, such as a, a' and a", one can see at left a plane intersecting a cone to produce a curve, at center the resulting curve and at right the corresponding surface. Thus a' is a circle and a" a sphere, b' an ellipse and b" an ellipsoid, c' a parabola and c" a paraboloid, d' a hyperbola and d"



a hyperboloid. Definitions and properties of conic sections were worked out by ancient Greek scholars, notably Apollonius.

...elements was that they did not have logical foundations; the Alexandrians merely built over the empirically based arithmetical knowledge built up by the Egyptians and Babylonians. Because Euclidean geometry offered the security of proof, it continued for centuries to dominate mathematics. Not until late in the 19th century did mathematicians solve the problem of providing an axiomatic basis for arithmetic and algebra.

Actually geometry consists of several geometries. The first break in the direction of a new geometry was made by Renaissance painters who sought to solve the problem of depicting exactly what the eyes see. Because real scenes are three-dimensional, whereas a painting is flat, it would appear to be impossible to paint realistically. The painters solved their problem by recognizing a fundamental fact about vision. Suppose a man, using one eye, looks through a window at some real scene. He sees the scene because light rays from various points in it travel to his eye. This collection of light rays is called a projection. Since the rays pass through the window, it is possible to mark a point on the window where each light ray pierces it. This collection of points is called a section. What the painters discovered is that the section creates the same impression on the eye as the scene itself does. This is physically understandable [see top illustration on first page]. Whether the light rays emanate from particles in the real scene or from points on the window, the same light rays reach the eye. Hence the canvas could contain what appears on the window. Even though this is a one-eye scheme and sight involves two eyes, the painters compensated for the restriction by using diminution of light intensity with distance and by using shadows. How well they succeeded in solving the problems of perspective can be judged by the painting reproduced on page 113.

The use of projection and section raised a basic geometrical question, first voiced by the painters and later taken up by mathematicians: What geometrical properties do an original figure and its section have in common that enable them to create the same impression on the eye? The answer to this question led to new concepts and theorems that ultimately constituted a new branch of geometry called projective geometry [see the article by Morris Kline, "Projective Geometry," page 122 in this volume]. Some of the concepts and theorems are as follows. It is apparent from the top illustration on the next page that the section of the projection of a line is a line and that, if two lines intersect, the section of the projection of these two intersecting lines will also be two intersecting lines, although the angle between the two lines of the section will generally not be the same as the angle between the two lines in the original figure. It follows that a triangle will give rise to a triangular section and a quadrilateral will give rise to a quadrilateral section.

A more significant example of the properties common to a figure and a section was furnished in the 17th century by the self-educated French architect and engineer Gérard Desargues. In what is now known as Desargues's theorem he showed that for any triangle and any section of any projection of that triangle any pair of corresponding sides will meet at a point and the three points of intersection of the three pairs of corresponding sides lie on one straight line [see bottom illustration on first page]. The significance of this and other theorems of projective geometry is that the geometry no longer discusses congruence, similarity, equivalence and other concepts of Euclidean geometry but instead deals with collinearity (points that lie on a line), concurrency (lines that go through a point) and other notions stemming from projection and section.

Projective geometry flourished rather briefly and then was pushed aside temporarily by a rival geometry that appeared on the scene. The rival, which embodied an algebraic approach to geometry, is now called analytic geometry or coordinate geometry. It was motivated by a series of events and discoveries that in the 16th and 17th centuries launched the scientific age in western Europe and brought to the fore the problem of deriving and using the properties of curves and surfaces.

For one thing, the creation by Nicolaus Copernicus and Johannes Kepler of the heliocentric theory of planetary motion made manifest the need for effective methods of working with the conic sections; these curves are the paths of the celestial bodies in such a system. Moreover, by invalidating classical Greek mechanics, which presupposed a stationary earth, the heliocentric theory necessitated a completely new science of motion and therefore the study of curves along which objects move.

Several other forces pushed geometry

by algebraic processes applied to the equations. With this development the relation between number and geometry had come full circle. The classical Greeks had buried algebra in geometry, but now geometry was eclipsed by algebra. As the mathematicians put it, geometry was arithmetized.

Descartes and Fermat were not entirely correct in expecting that algebraic techniques would supply the effective methodology for working with curves. For instance, those techniques could not cope with slope and curvature, which are fundamental properties of curves. Slope is the rate at which a curve rises or falls per horizontal unit; curvature is the rate at which the direction of the curve changes per unit along the curve. Both rates vary from point to point along all curves except the straight line and the circle. To calculate rates of change that vary from point to point the purely algebraic techniques of Descartes and Fermat are not adequate; the calculus, particularly the differential calculus, must be employed. Indeed, the distinguishing feature of the calculus is its power to yield such rates.

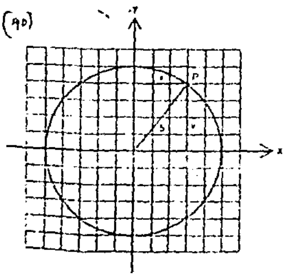
With the aid of the differential calculus the study of curves and surfaces was expedited so much that a new term, differential geometry, was introduced to designate this study. Differential geometry considers a variety of problems beyond the calculation of slope and curvature. It considers in particular the all-important problem of geodesics, or the shortest distance between two points on a surface. Given a surface such as the surface of the earth, what curve joining two given points P and Q on the surface is the shortest distance from P to Q along the surface? If one takes the surface of the earth to be a sphere, the answer is simple: The geodesics are arcs of great circles. (A great circle cuts the sphere in half; the Equator is a great circle and a circle of latitude is not.) If one more accurately takes the surface of the earth to be an ellipsoid, however, the geodesics are more complicated curves and depend on which points P and Q one chooses. The concerns of differential geometry include the curvature of surfaces, map making and surfaces of least area bounded by curves in space, the last of which are so handsomely realized by soap films (see bottom illustration on page 119).

[illegible] the methodologies of analytic geometry and differential geometry were far too successful. Although these subjects treated geometry, the representations of curves were equations and the methods of proof were algebraic or analytic (that is, they involved the use of the calculus). The beautiful geometrical reasoning was abandoned and geometry was submerged in a sea of formulas. The spirit of geometry was banished.

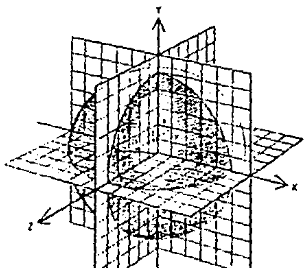
For 150 years the pure geometers remained in the shadows. In the 19th century, however, they found the courage and the vitality to reassert themselves. The revival of geometry was launched by Gaspard Monge (1746–1818), a leading French mathematician and adviser to Napoleon. Monge thought the analysts had sold geometry short and had even handicapped themselves by failing to interpret their analysis geometrically and to use geometrical pictures to help them think. Monge was such an inspiring teacher that he gathered about him a number of very bright pupils, among them L. N. M. Carnot (1753–1823), Charles J. Brianchon (1785–1864) and Jean Victor Poncelet (1788–1867). These men, imbued by Monge with a fervor for geometry, went beyond the intent of their master and sought to show that geometric methods could accomplish as much and more than the algebraic and analytic methods. To defeat Descartes or, as Carnot put it, "to free geometry from the hieroglyphics of analysis," became the goal.

The geometers, led by Poncelet, turned back to projective geometry, which had been so ruthlessly abandoned in the 17th century. Poncelet, serving as an officer in Napoleon's army, was captured by the Russians and spent the year 1813–1814 in a Russian prison. There he reconstructed without the aid of any books all he had learned from Monge; he then proceeded to create new results in projective geometry.

Projective geometry was actively pursued throughout the 19th century. Curiously an algebraic method, essentially an extension of the method of coordinate geometry, was developed to prove

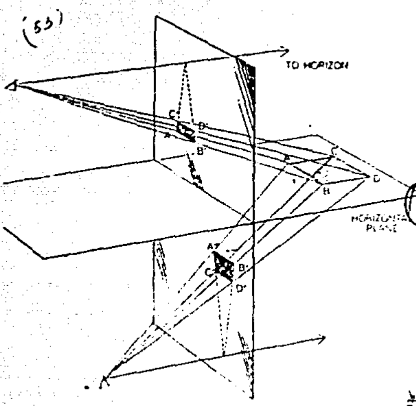


$x^2 + y^2 = 25$

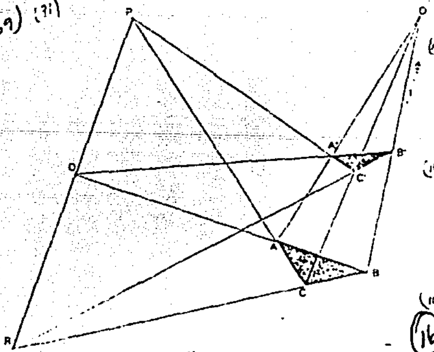


$x^2 + y^2 + z^2 = 25$

CARTESIAN COORDINATE SYSTEM made it possible to express any shape as an equation. For the circle at left, with a radius of five units, the equation is $x^2 + y^2 = 25$. Any values of x and y that produced 25 in the equation would represent a point on this circle: for the point P, $x = 3$ and $y = 4$. At right is a visualization of the sphere represented by the equation $x^2 + y^2 + z^2 = 25$.



PROJECTION AND SECTION were concepts that arose from the work of artists and helped lead to projective geometry. Projections of a square, such as *ABCD*, to two observers form sections (*color*) on an intersecting plane. In a drawing the square must be represented as a section in order to appear realistic to an observer looking at the drawing.



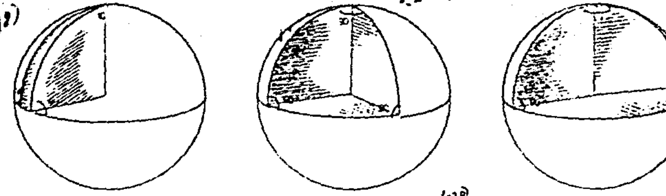
DESARGUES'S THEOREM illustrates the concern of projective geometry with properties common to a figure and its sections. The theorem states that any pair of corresponding sides of a triangle (*ABC*) and a section (*color*) will meet in a point—as, for example, the sides *BC* and *B'C'* meet in point *R*—and that the three points *P*, *Q* and *R* will lie on a line.

in the same direction. The gradually increasing use of gunpowder raised problems of projectile paths. The discovery of the telescope and the microscope motivated the study of lenses. Geographical exploration called for maps and in particular for the correlation of paths on the globe with paths on flat maps. All these problems not only increased the need for knowledge of properties of familiar curves but also introduced new curves. As René Descartes and Pierre de Fermat realized, the Euclidean synthetic methods were too limited to deal with these problems.

Descartes and Fermat, both major contributors to the fast growing discipline of algebra, saw the potentialities in that subject for supplying methodology to geometry. The analytic geometry they developed replaced curves by equations through the device of a coordinate system. Such a system locates points in a plane or in space by numbers. In a plane the system uses two numbers, an abscissa and an ordinate (*see illustration on opposite page*). The abscissa expresses the distance of a point from a fixed vertical line, called the *Y* axis; the ordinate expresses the distance of the point from a fixed horizontal line, called the *X* axis. Distances to the right of the *Y* axis or above the *X* axis are positive, distances in the opposite directions are negative.

How does this device enable one to represent curves algebraically? Consider a circle with a radius of five units. A circle, like any other curve, is just a particular collection of points. And if the circle is placed on a coordinate system, then each point on the circle has a pair of coordinates. Since the circle is a particular collection of points, the coordinates of these points are special in some way. The specialized nature is expressed by the equation $x^2 + y^2 = 25$. What this equation states is that if one takes the abscissa of any point on the curve and substitutes that for x, and if one takes the ordinate of that same point and substitutes it for y, then the number obtained for $x^2 + y^2$ will be 25. One says that the coordinates of any point on the curve satisfy the equation. Moreover, the coordinates of only those points that do lie on the curve satisfy the equation. In the case of surfaces an equation in three coordinates serves. For example, the equation of a sphere with a five-unit radius is $x^2 + y^2 + z^2 = 25$.

Thus under the Descartes-Fermat scheme points became pairs of numbers, and curves became collections of pairs of numbers subsumed in equations. The properties of curves could be deduced



SPHERICAL TRIANGLES can have angles that sum to more than 180 degrees. On the sphere at left the triangle has angles summing to 180 degrees. In the succeeding spheres the angles of the triangles sum respectively to 270 degrees, 350 degrees and 510 degrees.

its theorems, and to this extent the interests of the pure geometers who launched the revival were subverted. But projective geometry was again put in the shade by another development as dramatic and as weighty as the creation of mathematics by the classical Greeks: the creation of non-Euclidean geometry.

Throughout the long reign of Euclidean geometry many mathematicians were troubled by a slight blemish that seemed to mar the collection of axioms. Apropos of parallel lines, by which is meant two lines in the same plane that do not contain any points in common, Euclid formulated an axiom that reads as follows: If the straight line n cuts the lines l and m so as to make corresponding angles with each line that total less than 180 degrees, then l and m will meet on that side of the line n on which the angles lie. This axiom is essential to the derivation of the most important theorems, among them the theorem that the sum of the angles of a triangle is 180 degrees. The axiom is a bit involved, and there are reasons to believe Euclid himself was not too happy about it. Neither he nor any of the later mathematicians up to about 1800 really doubted the truth of the statement, that is, the [illegible] reci idealization of the [illegible] of actual [illegible] Euclid and his successors was that the axiom was not quite so self-evident as, say, the axiom that any two right angles are equal.

From Greek times on mathematicians sought to replace the axiom on parallels by an equivalent one, an axiom that, together with the other nine axioms of Euclid, would make it possible to deduce the same body of theorems Euclid deduced. Some equivalent axioms were proposed. One of these, which was suggested by the mathematician John Playfair (1748–1819) and is the one usually taught in high schools, states that given a line l and a point P not on l, there is only one line m in the plane of P and l that passes through P and does not meet l.

Playfair's axiom is not only equivalent to Euclid's axiom, but it is also simpler and appears to be intuitively convincing that it does seem to state an unquestionably or self-evident property of lines in physical space. Later mathematicians, however, were not satisfied with Playfair's axiom or any of the other proposed equivalents of Euclid's axiom. The reason they were not satisfied was that every proposed substitute directly or indirectly involved an assertion about what happens far out in space. Thus Playfair's axiom asserts that l and m will not meet, no matter how far out these lines are extended. As a matter of fact, Euclid's axiom is superior in this respect because all it asserts is a condition under which lines will meet at some finite distance.

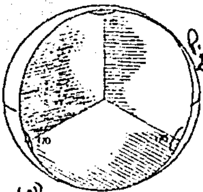
What is objectionable about axioms that assert what happens far out in space? The answer is that they transcend experience. The axioms of Euclidean geometry were supposed to be unquestionable truths about the real [illegible] be sure that [illegible] straight lines will extend indefinitely far [illegible] physical space without [illegible] being forced to meet. The problem the mathematicians faced was that Euclid's parallel axiom was not quite self-evident, and that the equivalent axioms, which were seemingly more self-evident, proved on closer examination to be somewhat suspect also.

The problem of the parallel axiom or, as the French mathematician Jean Le Rond d'Alembert put it, "the scandal of geometry," engaged the mathematicians of every period from Greek times up to 1800. The history of these investigations would be worth noting if for no other reason than to see how persistent and critical mathematicians can be. It is necessary here to forgo the history and jump to the result. The truth that destroyed truth was seen clearly by the greatest of all 19th-century mathematicians, Karl Friedrich Gauss (1777–1855). His first point was somewhat technical but essential, namely, that the parallel axiom is independent of the other nine axioms, that is, it is logically possible to choose a contradictory axiom and use it in conjunction with the other nine Euclidean axioms to deduce theorems of a new geometry. Thus one might assume that given a line l and a point P not on l, there is an infinite number of lines through P and in the plane of P and l that do not meet l. Gauss adopted this very axiom and from it and the other nine axioms deduced a number of theorems. Gauss called his new geometry non-Euclidean geometry.

As might be expected, many theorems of the new geometry contradict theorems of Euclidean geometry. The sum of the angles of a triangle in this geometry is always less than 180 degrees. Moreover, the sum varies with the size of the triangle; the closer the area of the triangle is to zero, the closer the angle sum is to 180 degrees.

[illegible] of a logical alternative to Euclidean geometry was in itself a startling fact. Geometry up to this time had been essentially Euclidean geometry; analytic and differential geometry were merely alternative technical methodologies, and although projective geometry dealt with new concepts and new themes, they were entirely in accord with Euclidean geometry. Non-Euclidean geometry was in conflict with Euclidean geometry.

Gauss's second conclusion was even more disturbing. It was that non-Eu-



grees. Such triangles typify concepts of Bernhard Riemann's non-Euclidean geometry.

clidean geometry could be used to represent physical space just as well as Euclidean geometry does. This assertion seems at first to be downright nonsense. If the sum of the angles of a triangle is 180 degrees, how could it also be less than 180 degrees? The answer to this seeming impossibility is that the non-Euclidean geometry calls for an angle sum arbitrarily close to 180 degrees when the size of the triangle is small enough. The triangles man usually deals with are small; therefore the angle sums of these triangles might be so close to 180 degrees that measurement of the sum, in view of the inevitable errors of measurement, would not exclude either possibility.

The implications of non-Euclidean geometry are drastic. If both Euclidean and non-Euclidean geometry can represent physical space equally well, which is the truth about space and figures in space? One cannot say. In fact, the choice might not be limited to just these two. This doleful possibility was soon to be realized. The fact gradually forced on the mathematicians is that geometry is not the truth about physical space [redacted] the study of possible spaces. Several of these mathematically constructed spaces, differing sharply from one another, could fit physical space equally well as far as experience could decide.

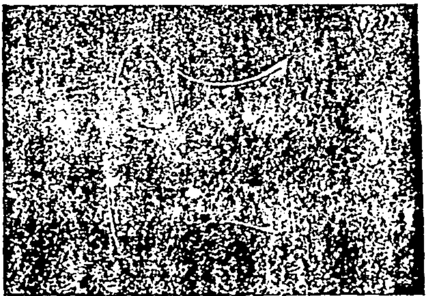
The concept of geometry had then to be revised, [redacted] the same was true for the concept of mathematics itself. Since for more than 2,000 years mathematics had been the bastion of truth, non-Euclidean geometry, the triumph of reason, proved to be an intellectual disaster. This new geometry drove home the idea that mathematics, for all its usefulness in organizing thought and advancing the works of man, does not offer truths but is a man-made fable having the semblance of fact.

The new vista opening up in geometry was widened immeasurably by the work of Georg Friedrich Bernhard Riemann (1826–1866). Riemann was one of Gauss's students and undoubtedly acquired from him an interest in the study of the physical world. Riemann's first observation in the field of geometry was that the mathematicians had been deceived into believing the Euclidean parallel axiom was necessarily true. Perhaps they were equally deceived in accepting one or more of the other axioms of Euclid. Riemann fastened at once on the axiom that a straight line is infinite. Experience, he pointed out, does not assure us of the infinitude of the physical straight line. Experience tells us only that in following a straight line we do not come to an end [redacted] neither would one come to an end if one followed the Equator of the earth. In other words, experience tells us only that the straight line is endless or unbounded. If we change the relevant axiom of Euclid accordingly, and if we assume that there are *no* parallel lines, we have another set of axioms from which we can deduce still another non-Euclidean geometry.

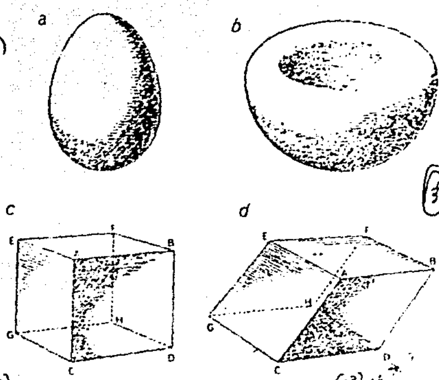
In a paper of 1854 entitled "On the Hypotheses Which Underlie Geometry" Riemann launched an even deeper investigation of possible spaces, utilizing only the surest facts about physical space. He constructed a new branch of geometry, now known as Riemannian geometry, that opened up the variety of mathematical spaces a thousandfold [see "The Curvature of Space," by P. Le Corbeiller, on page 125 of this volume].

To appreciate Riemannian geometry one must first perceive that what is chosen as the distance between two points determines the geometry that results. This can be readily seen. Consider three points on the surface of the earth. One can take as the distance between any two the length of the ordinary straight-line segment that joins them through the earth. In this case one obtains a triangle that has all the properties of a Euclidean triangle. In particular, the sum of the angles of this triangle is 180 degrees. One could, however, take as the distance between any two points the distance along the surface of the earth, meaning the distance along the great circle through these points. In this case the three points determine what is called a spherical triangle. Such triangles possess quite different properties. For example, the sum of the angles in them can be any number between 180 degrees and 540 degrees [*see illustration at the top of these two pages*]. This is a fact of spherical geometry.

What Riemann had in mind was a geometry for changing configurations. Suppose one were to try to design a geometry that would fit the surface of a mountainous region. In some places the surface might be flat, in others there might be conical hills and in still others hemispherical hills. The character of the surface changes from place to place,



SOAP FILMS, which always assume a shape with the least possible area, illustrate a concern of differential geometry: surfaces of least area bounded by curves in space. Differential geometry is also applicable to problems of map making and curvature of surfaces.



TOPOLOGICAL DEFORMATIONS of familiar shapes are portrayed. A sphere can be deformed into an egg shape (a), a squashed-ball shape (b), a cube (c) and a deformation of the cube (d). Each deformation is topologically equivalent to the others and to the sphere.

and so the distance formula that determines the geometry must change from place to place and possibly even from point to point. Riemann proposed, in other words, nonhomogeneous spaces—spaces whose characteristics vary from point to point or spaces with varying curvature.

Riemann died at the age of 40 and was therefore able to do little more than sketch the broad outline of his conception of space. The further development of Riemannian geometry became the task of many men and is still under way. Early in this century the Italian mathematicians Gregorio Ricci and Tullio Levi-Civita made significant contributions. Ricci introduced the tensor calculus, a formalism that enables one to express geometric relations independently of the coordinate system. Levi-Civita brought a concept of parallelism to Riemannian geometry: it provided a way of expressing the Euclidean notion of parallelism for more general spaces.

The creation of the general theory of relativity by Albert Einstein not only stimulated further work in Riemannian geometry [redacted] also suggested the problem of unifying gravitation and electromagnetism in one mathematical framework. Toward this end Hermann Weyl in 1918 introduced what he called affinely connected spaces, a concept that uses Levi-Civita's notion of parallelism rather than the notion of distance to relate the points of a space to one another. An expression of distance even more generalized than Riemann's produced the spaces called Finsler spaces.

Riemann was also the founder of topology, another branch of geometry in which research is most active today. During the 1850's he was working with what are now called functions of a complex variable, and he introduced a class of surfaces, called Riemann surfaces, to represent such functions. The properties of the functions proved to be intimately connected with the geometric properties of the surfaces. For any given function, however, the precise shape of the surface was not critical, and so he found it desirable to classify surfaces in accordance with a new principle.

Given two similar figures, for example a large and a small triangle having the same shape, either can be regarded as a deformation or transformation of the other, the change being a uniform expansion of the smaller to obtain the larger, or a uniform contraction of the larger to obtain the smaller. Under projection and section the deformation of one figure into another is more radical. Yet even in these deformations a quadrilateral, say, remains a quadrilateral. It is possible to make still more radical deformations. For instance, a circle can be deformed by being bent into an ellipse or into an even more complicated shape, and a sphere can be stretched to assume the shape of an egg. For Riemann's purposes the circle could be replaced by the ellipse and the sphere by the egg shape. On the other hand, a circle, a figure eight and a trefoil were not interchangeable curves, and the sphere, the doughnut-shaped torus and the pretzel-shaped double torus were not interchangeable surfaces.

Hence Riemann was led to consider deformations that permit stretching, bending, contracting and even twisting. Figures that can be obtained from one another by such deformations are said to be homeomorphic, or topologically equivalent. If, however, one tears a figure or connects it in such a way as to make points coalesce, the new figure is not topologically equivalent to the old one. Thus one can pinch a circle top and bottom and form a figure eight, [redacted] the latter is not topologically the same as the original. It is also possible to describe topologically equivalent figures by imagining them to be made of rubber. Then any figure that can be obtained by stretching, bending or contracting but not tearing the rubber would be topologically equivalent to the initial one.

The major problem of topology is to know when two figures are topologically equivalent. This may be difficult to see by looking at the figures, particularly since topology considers three-dimensional and even higher-dimensional figures. For this reason and others one seeks to characterize equivalent figures by some definitive properties so that if two figures possess these properties, they must be topologically equivalent, just as the congruence of two triangles is guaranteed if two sides and the included angle of one triangle are equal to the respective parts of the other. For example, if one draws any closed curve on the surface of a sphere or an ellipsoid, the curve bounds a region on the surface. This is not true on the torus [see illustration on opposite page]. The sphere and the torus are therefore not topologically equivalent. It is possible to characterize closed surfaces in terms of curves that do or do not bound on the surface, but this criterion will not suffice for more complicated surfaces or for higher-dimensional figures.

Although many basic problems of topology remain unsolved, mathematicians make progress where they can, and in the past 10 years they have turned to the branch called differential topology. In this endeavor they combine the methods of topology and of differential geometry in the hope that two tools will be better than one.

Another enormously active field today is algebraic geometry. Two hundred years ago this subject was an extension of coordinate geometry and was devoted to the study of curves that are more complicated than the conic sections and are represented by equations of degree higher than the second. Since the latter part of the 19th century, however, the proper domain of algebraic geometry has been regarded as the study of the properties of curves, surfaces and higher-dimensional structures defined by algebraic equations and invariant under rational transformations. Such transformations distort a figure more than projective transformations and less than topological transformations do.

Mathematicians, yielding to their propensity to complicate and to algebraicize, have allowed the coordinates in the equations of algebraic geometry to take on complex values and even values in algebraic fields [see "Number," page 89, and "Algebra," page 102]. Consequently even the simple equation $x^2 + y^2 = 25$, which when x and y have real values represents the circle discussed previously, can represent a complicated Riemann surface or a structure so unconventional that it can hardly be imagined. The geometry suffers, [redacted] the algebra flourishes.

This discussion of geometry as the study of the properties of space and of figures in space may have exhibited the growth, variety and vitality of geometry and the interconnections of the branches with each other and with other divisions of mathematics. [redacted] it does not present the full nature of modern geometry. It is often said that algebra is a language but is geometry.

Today mathematicians pursue the subject of abstract spaces, and one might infer from the term that the pursuit involves some highly idealized, esoteric spaces. This is true, [redacted] the major use of the theory of abstract spaces—indeed, historically the motivation for its study—is to expedite the use of classes of functions in analysis. The "points" of an abstract space are usually functions, and the distance between two points is some significant measure of a difference between two functions. Thus one might be interested in studying functions such as x^2, $3x^3$ and $x^3 - 2x$ and be interested in the values of these functions as x varies from 0 to 1. One could define the distance between any two of these functions as the largest numerical difference between the two for all values of x between 0 and 1. Such function spaces prove to be infinite-dimensional. The Hilbert spaces and Banach spaces about which one hears much today are function spaces. On the mathematical side these are important in the subject known as functional analysis, which is now the chief tool in quantum mechanics.

Why talk about spaces when one is really dealing with functions? It is because the geometrical mode of thinking is helpful and even suggestive of theorems about functions. What may be complicated and obscure when formulated analytically may in the geometrical interpretation be intuitively obvious.

The study of abstract spaces is, surprisingly, part of topology because the properties of these structures that are important, whether the structures are regarded as actual spaces or as collections of functions, are preserved, or invariant, under topological transformations.

The subject of abstract spaces clearly exhibits the abstractness of modern mathematics. Geometry supplies models not only of physical space [redacted] also of any structure whose concepts and properties fit the geometric framework.

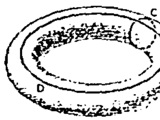
In still another vital respect geometry proves to be far more than the receptacle for matter. The present century is witnessing the realization of an assertion by Descartes that physics could be geometrized. In the theory of relativity, one of the two most notable scientific advances of this century (quantum theory is the other), the gravitational effect of gross matter has been reduced to geometry. Just as the geometry of a mountainous region requires a distance formula that varies from place to place to represent the varying shape of the land, so Einstein's geometry has a variable distance formula to represent the different masses in space. Matter determines the geometry, and the geometry as a result accounts for phenomena previously ascribed to gravitation.

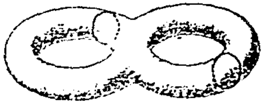
Geometry has ingested part of reality and may have to ingest all of it. Today in quantum mechanics physicists are striving to resolve the seemingly contradictory wave and particle properties of subatomic matter, and they may have to generate both from quanta of space. Perhaps matter itself will also dissolve into pure space.

If one assesses today the competition between number and geometry, one must admit that insofar as methodology of proof is concerned, geometry has largely given way to algebra and analysis. The geometric treatment of complicated structures and of course of higher-dimensional spaces can, as Descartes complained of Euclidean geometry, "exercise the understanding only on condition of greatly fatiguing the imagination." Moreover, the quantitative needs of science can be met only by ultimate recourse to number.

Geometry, however, supplies substance and meaning to bare formulas. Geometry remains the major source of rich and fruitful intuitions, which in turn supply creative power to mathematics. Most mathematicians think in terms of geometric schemes, even though they leave no trace of that scaffolding when they present the complicated analytical structures. One can still believe Plato's statement that "geometry draws the soul toward truth."

TOPOLOGICAL EQUIVALENCE of surfaces can be determined by drawing closed curves on the figures. If each curve bounds an area on a surface, the surface is topologically equivalent to a sphere.





The type of curve drawn on the sphere at left does not bound an area on the torus at center or on the double torus at right; thus the latter figures are not topologically equivalent to a sphere.

18 THE CURVATURE OF SPACE

P. LE CORBEILLER • November 1954

In the spring of 1854 a young German mathematician named Bernhard Riemann was greatly worried about his future and about a test he faced immediately. He was already 28, and still not earning—he was living meagerly on a few thalers sent each month by his father, a Protestant minister in a small Hanover town. He wrote modestly to his father and brother that the most famous university professors, in Berlin and in Göttingen, had unaccountably been extraordinarily kind to him. He had his doctor's degree, now, to obtain an appointment as a lecturer (without stipend), he had to give a satisfactory lecture before the whole Faculty of Philosophy at Göttingen. He had offered three subjects. "The two first ones I had well prepared," Bernhard wrote his brother, "but Gauss chose the third one, and now I'm in trouble."

Karl Friedrich Gauss was the dean of German mathematicians and the glory of his university. In Bernhard's picture of Heaven, Gauss's professorial armchair was not very far from the Lord's throne. (This is still the general view in Göttingen today.) The subject Gauss had chosen for young Riemann's lecture was [illegible] tions of Geometry." Gauss had published nothing but a few cryptic remarks on this topic, but he selected it in preference to the two others proposed by Riemann because he was curious to find out what the young man would have to say on such a deep and novel subject—a subject to which Gauss himself had given much thought and had already made a great, though as yet not widely appreciated, contribution.

The day of Riemann's public lecture was Saturday, June 10, 1854. Most of his auditors were classicists, historians, philosophers—anyway, not mathematicians. Riemann had decided that he would discourse about the curvature of n-dimensional spaces without writing any equations. Was that a courteous gesture on his part, or a mildly Machiavellian scheme? We shall never know. What is sure is that without equations Gauss understood him very well, for walking home after the lecture he told his colleague Wilhelm Weber, with unwonted warmth, of his utmost admiration for the ideas presented by Riemann.

Gauss's enthusiasm was justified. The young man had reached into realms of thought so new that few scientists then could follow him. But his abstract ideas were to make contact with experimental reality half a century later through the work of Albert Einstein, who saw that Riemann's speculations were directly applicable to the problem of the interaction between light and gravitation, and made them the basis of his Generalized Relativity Theory, which today controls our view of the universe.

Let us then go back 100 years and acquaint ourselves with the thoughts which Riemann made public on that June day of 1854. Before [illegible] Rie- [illegible] some rather elementary background.

Everybody is familiar with the elements of plane geometry. A straight line is the shortest way between two points; parallel lines never meet; the sum of the three angles of a triangle equals two right angles, or 180 degrees, and so on and so forth. Also familiar is the geometry of figures drawn on the surface of a sphere, which obey somewhat different rules. The shortest route between two points on a sphere is called a "great circle"; this is the curve made by a cut through the points and the center of the sphere, splitting the sphere into equal halves. Two great circles always meet in two points; for instance, any two meridians of the earth always meet at the North and South poles. When segments of three great circles (for instance, one quarter of the earth's equator and the northern halves of two meridians) intersect to form a "spherical triangle," the three angles of 90 degrees add up to 270 degrees, or three right angles. The difference between this triangle and one in a plane derives from the fact that the sides of the former are drawn on a curved surface instead of on a flat one.

Now how do we know that the surface of a table is flat and that of the earth is spherical? All early civilizations imagined the earth as a flat disk, with mountains heaped upon it like food on the king's table. Not being able to go to the moon to look at the earth, men could not see its true shape. How, then, did Greek astronomers come to the conclusion that the earth was round? By observing that the North Star was higher in the sky in Greece than in Egypt. Thus it is evident that we can recognize that a sphere is round either by observing it from a distance or, if we stand on it, by observing objects far away.

Man also could, and did, discover that the earth was round in two entirely different ways. One way was his circumnavigation of the earth. He found that while the surface of the earth had no "edge," no boundaries, its area was nevertheless limited. This is a most remarkable fact: the surface of the earth is boundless and yet it is finite. Obviously that situation rules out the possibility that the earth could be a plane. The surface of a plane is boundless and also infinite. (In common speech we consider these two words strictly synonymous—one of the many instances which prove that the sphericity of the world has not yet really taken hold of our consciousness.)

Thus mankind would have discovered that the earth is round even if it were constantly covered with a canopy of thick clouds. But suppose that he had somehow been prevented from exploring the whole planet. There is still another way in which he could have found out he was living on a globe, and that is by using the spherical geometry we have been talking about. If we look at a small triangle on the earth's surface, say one with sides about 30 feet long, it is indistinguishable from a flat triangle; the sum of its three angles exceeds 180 degrees by an amount so small that it cannot be measured. As we consider larger and larger triangles on the earth's spherical surface, however, their curvature will become more and more significant, and it will show up in the excess of the sum of their angles over 180 degrees. Thus by developing more and more precise methods of surveying and of making maps men eventually could prove the sphericity of the earth, and from their measurements they could find out the globe's radius. We shall return presently to this matter.

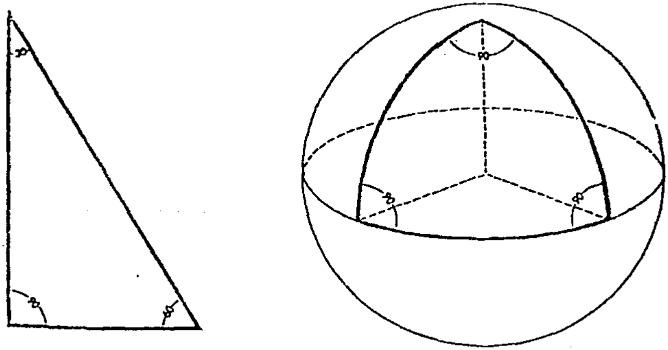
There are many types of surfaces besides those of a plane and a sphere. Consider an egg. It has a large end and a small end. A round piece of shell from the large end looks as if it were cut from a sphere; a round piece from the small end looks as if it belonged to a sphere with a smaller radius than the first. The piece from the small end looks more curved than that from the large end. Geometers define the curvature of a sphere as the inverse of its radius squared. So the smaller the radius, the larger the curvature, and vice versa.

If we were given a piece of shell from the middle zone of the egg, could we define its curvature? That is a little difficult, because such a piece cannot be identified with a portion of a simple sphere. The problem has been solved as follows. Suppose we lay the piece, which has the shape of a more or less elongated oval, on a table. It forms a rather flat dome. Any vertical cross section of that dome will be a curve concave downward. Every vertical cross section will look approximately like a portion of a circle, but not all will have the same radius. The section through the narrowest part of the base will have the smallest radius; the section through the elongated part, the largest. Let us call the first radius R_1 and the second R_2. Geometers then take a sort of average, and define the curvature of that small portion of eggshell as the inverse of the product R_1R_2. You can see that if the eggshell were a perfect sphere, we would be brought back to the previous definition.

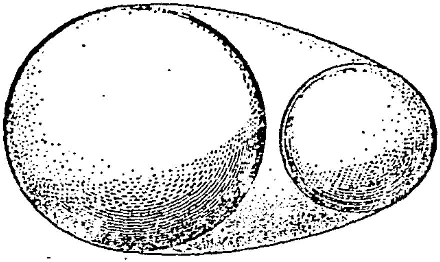
On the basis of these definitions one finds that the curvature of a small piece of an eggshell changes as we travel on the surface of the egg. It would make no sense to talk about the curvature of the whole egg; we can only talk about the curvature of a small piece.

Consider next the surface of a saddle. A crosswise vertical section cut through a saddle forms a curve which is concave downward, whereas a lengthwise vertical cut forms a curve concave upward. This makes even a small piece of the surface of a saddle something radically different from a small piece of an eggshell. Geometers say that the eggshell has everywhere positive curvature, and the saddle has everywhere negative curvature. The curvature of a small portion of a saddle-shaped surface can again be defined as the inverse of the product R_1R_2, but this time it must be given a negative sign.

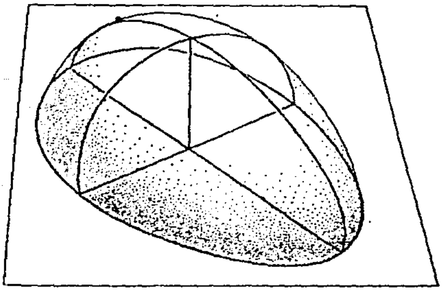
And here is still something else. Consider a doughnut. If you compare the inner half of the surface (facing the center of the hole in the doughnut) with the outer half, you will recognize that any small portion of the outer half has positive curvature, while any small por-



TRIANGLES drawn on a plane and on a sphere obey somewhat different rules. On a plane the sum of the angles of a triangle always is equal to 180 degrees. The intersection of three great circles on the surface of a sphere forms three angles adding up to 270 degrees.



AN EGG has a curved surface which looks as if the surface of the large end belonged to one sphere and the surface of the small end to another. The middle has a different curvature.



HALF AN EGG, laid on a table and cut into vertical cross sections, will yield sections with concave curves downward. These curves look like portions of circles of different radii.

tion of the inner half has negative curvature, as in the case of a saddle. Thus we must not think that the curvature need be positive or negative all over a given surface; as we travel from point to point on a surface the curvature not only can become greater or smaller, it can also change its sign.

Remember that we are engaged in taking a bird's-eye view of what was known about the curvature of surfaces before Riemann's time. What we have seen so far had been recognized in the 18th century by Leonhard Euler, a Swiss mathematician of considerable imagination and output, and had been developed by a group of French geometers at the newly founded Ecole Polytechnique. Then in 1827 Gauss, Riemann's senior examiner, added much generality and precision to the topic. He published a memoir on curved surfaces which is so jewel-perfect that one can still use it today in a college course.

Gauss started from the fact that geographers specify the location of a city on the globe by giving its longitude and latitude. They draw meridians of longitude (such as the one which unites all the points on the globe 85 degrees west of the north-and-south great circle through Greenwich) and also parallels of latitude. We may speak of the "family" of meridians and the "family" of parallels. In order to specify the location of a point on any mathematically given surface, Gauss imagined that we draw on that surface two families of curves, called p-curves and q-curves. We take suitable precautions so that any point on the surface will be pinpointed if we specify its p-coordinate and its q-coordinate.

Gauss's great insight was this. On an absolutely flat surface, if we travel three miles in one direction, then turn left and travel four miles in the perpendicular direction, we know from Pythagoras' theorem that we are at a point five miles from home. But Gauss reasoned that on a curved surface, whether egg, or saddle or what have you, the distance will be different. To begin with, the p-curves and q-curves will not intersect everywhere at right angles, and this adds a third term to the sum of the two squares in the Pythagorean equation $a^2+b^2=c^2$. Moreover, if we visualize the two families of curves as a kind of fish net drawn tight all over the surface, the angles and sides of the small meshes will change slowly as we travel from one region of the surface to another where the curvature is different.

Gauss expressed his reasoning in a famous mathematical equation. One p-curve and one q-curve pass through a given point M on a curved surface. The "quasi longitude" p and the "quasi latitude" q of point M have specific numerical values. We wish to move from point M to a neighboring point P on the sur face. We first increase the value of p by a small quantity, letting q remain the same. Gauss used dp as the symbol fo an arbitrarily small increase of p. We thus get to a point N, of longitude $p+dp$ and latitude q. We next increase the value of q by a small quantity, dq, letting $p+dp$ remain the same. We thu reach a point P, of longitude $p+dp$ an latitude $q+dq$. We wish to know th distance from point M to point P. Sin this distance is arbitrarily small, Gau used for it the symbol ds. In Gauss's n tation, the square of the distance ds w be expressed by the sum of three term

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,d$$

This equation is one of the high poi in the whole of mathematics and phys —a mountain-top where we should claim in awe, like Faust suddenly p ceiving the symbol of the macrocos "Was he a god, whoever wrote th signs?" It needed only two steps, taken by Riemann and the other by Einstein, to carry us from Gauss's equation into the land of general relativity.

At any point M on our arbitrary surface, this equation is not different from a Euclidean theorem about the square of the third side, ds, of any triangle, the first two being dp, dq. That is because in the immediate neighborhood of a point the surface is very nearly a plane. But here is the novelty: Gauss introduced the functions E, F and G, whose numerical values change continuously as we move from point to point on the surface. Gauss saw that each of the quantities E, F, G was a function of the two arbitrary quantities p and q, the quasi longitude and the quasi latitude of point M. On a plane we can draw p-lines and q-lines dividing the plane into small equal squares, as on a checkerboard; we have then $ds^2 = dp^2 + dq^2$, so that E is constantly equal to 1, F to zero and G to 1 all over the plane. But on a curved surface E, F and G vary in a way which expresses, in an abstract but precise manner, just those variations in the curvature of a surface that make every point different from every other.

Gauss now proved this remarkable theorem: that the curvature of the surface at any point can be found as soon as one knows the values of E, F and G at the point, and how they vary in its immediate neighborhood. Why is this theorem so remarkable? Because if we return to our fictional humanity living on some beclouded globe, not a spherical one this time but of arbitrary shape, the surveyors of any particular nation on that globe, knowing the theorem, could obtain all the information about E, F and G without seeing the stars and without going to the moon. Thus from measurements taken on the surface itself they would be able to calculate the curvature of their globe at various points and to find out whether the surface of their country was curved like a portion of an egg, saddle or doughnut, as the case might be.

Now of all the ridiculous and useless puzzles scientists like to solve, this one, you may think, surely takes the prize. Why should mathematicians find it important to describe the behavior of imaginary people in a nonexistent world? For a very good reason: *These people are ourselves*. Only it takes some little explanation to make you realize I have been talking about you and me.

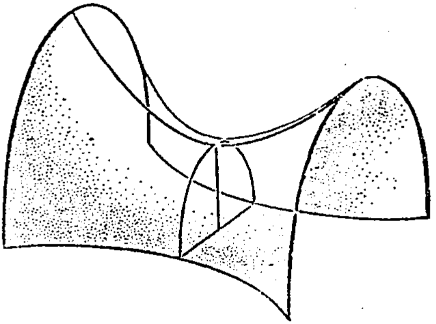
Let us imagine small bits of paper of various irregular shapes on a large, smooth sphere. These bits of paper are alive and moving: they are the people of that world, only their bodies are not volumes enclosed by surfaces but surfaces enclosed by curves. These people, having absolutely flat bodies without thickness, can form no conception of the space above or below them. They are themselves only portions of surfaces, two-dimensional beings. Their senses are adapted to give them information about the surroundings in their two-dimensional world. But they have no experience whatsoever of anything outside that world; so they cannot conceive of a third dimension.

However, they are intelligent; they have discovered mathematics and physics. Their geometry consists of two parts—line geometry and plane geometry. In physics they illustrate problems in one variable by diagrams on a line; problems in two variables, by surface diagrams. Problems in three, four or more variables they solve by algebra: "It's too bad," they say, "that for these we can't have the help of diagrams."

In the first half of the 19th century (*their* 19th century) an idea dawned upon several of them. "We cannot," they said, "imagine a third dimension, but we do handle physical problems in three variables, x, y, z. Why couldn't we *talk* about a space of three dimensions? Even if we cannot visualize it, it might be helpful to be able to talk about points, lines and areas located in that space. Maybe something might come of it; anyway, there's no harm in trying." And so they tried.

We need not carry this fable any farther, its meaning is clear enough. We are just like these people, only our bodies have three dimensions and are moving about in a three-dimensional world. Neither you nor I can visualize a fourth dimension, yet we handle problems about a particle moving in space, and this is a problem in four variables: x, y, z for space and t for time. We also handle problems about electromagnetic fields. Well, the electric field vector E at any point (x, y, z) has three projections, E_x, E_y, E_z, and it changes in space and time; that makes seven variables. Add three more for its twin brother, the magnetic field B, and we have 10. It looks as if the mathematical physicist could well use spaces of four or 10 or any number of dimensions.

Riemann in his dissertation assumed at the outset a space of an arbitrary number of dimensions. Now a lesser geometer would have found it very straightforward to define the distance of two neighboring points in that space. Don't we know from Pythagoras' theorem that in a plane the square of that distance, ds^2, is equal to the sum of two squares: $ds^2 = dx^2 + dy^2$? Well then obviously in an n-dimensional space ds^2 must be the sum of n squares, the sum of all the terms similar to dx^2 which we can find. A very convenient shorthand for the expression "the sum of all the terms similar to" is the Greek capital Σ. Thus a simple minded geometer would have written $ds^2 = \Sigma\, dx^2$. But Riemann saw far-



A SADDLE cut into lengthwise cross sections forms curves upward, while crosswise sections curve downward with shorter radii. A saddle is described as having negative curvature.

ther than that. He had given [illegible] thought to the 1827 memoir of [illegible] Gauss. He reasoned that, if we assumed that $ds^2 = \Sigma\, dx^2$, we were beaten at the outset. For Pythagoras' theorem is valid only in a plane, divided into equal little squares like a checkerboard. Actually what we need to generalize is Gauss's equation, which works for any curved surface whatsoever, including a plane as a very special case. Gauss had added two things to Pythagoras' formula: (1) to the squares of dp and dq he had added the product $dp\,dq$ of these two quantities; (2) he had multiplied each of these three terms by a coefficient of its own, and assumed that these coefficients E, F, G varied from point to point over the surface.

Let us do the same thing, then, for a "supersurface" of three dimensions, whatever that may be. We shall stretch over this supersurface three families of surfaces p, q, r or, as they are more conveniently designated, x_1, x_2, x_3. The square of the distance between two neighboring points, ds^2, should be built not only from the squares of dx_1, dx_2, dx_3, but also from their products two by two, and there are three such products: dx_2dx_3, dx_3dx_1 and dx_1dx_2. This makes a total of six terms, and we must give them six coefficients. Let us represent these coefficients by the letter g, with suitable subscripts. We must then write $ds^2 = g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + 2g_{23}dx_2dx_3 + 2g_{31}dx_3dx_1 + 2g_{12}dx_1dx_2$. (The factor 2 is not indispensable, but it is esthetically satisfying to the algebraist, and Gauss had taken a fit when a young Berlin professor, Dirichlet, had committed the faux pas of writing a memoir which dispensed with the factor 2.) This, then, is the correct form of the ds^2 for a supersurface of three dimensions, and the six coefficients will in general vary from point to point over the supersurface.

Riemann, as we have said, [illegible] at the outset [illegible] with, not a specific number such as three or four. He needed a name for the kind of geometrical objects he was thinking about. He noticed two things. First, a particle is free (in theory) to move smoothly and continuously from one point of a line or curve to another; it may also move continuously from one point to another on a surface or in space. Second, while studying plane geometry we think of nothing but figures drawn on a plane; that plane is for the time being our whole "universe of discourse," as logicians say. Yet the next year, as we study solid geometry, we imagine planes of any orientation in space. Any one of these planes might well be *the* plane of plane geometry which was last year's universe of discourse. It makes no difference in the geometry of a plane whether this plane exists all by itself or whether it is "embedded," as we now say, in three-dimensional space.

Putting these remarks together, Riemann coined the name "continuum" for any geometrical object, of any number of dimensions, upon which a point can continuously roam about. A straight line, for instance, is a continuum in one dimension—and it makes no difference to the geometry of points and segments on that line whether this one-dimensional continuum exists all by itself or is embedded in a plane, in three-dimensional space or for that matter in a space of any number of dimensions. The surface of a sphere or of a saddle is, as we have seen, a two-dimensional continuum; again it makes no difference whether we consider it by itself or embedded in a space of any number of dimensions.

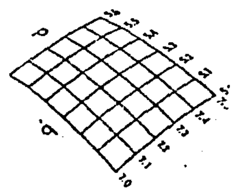
Now our space is a three-dimensional continuum. And we are bound to add that geometry in our space will be the same whether we consider that space by itself or assume it is embedded in a space of four, five or any number of dimensions. We cannot visualize what this means. Just the same, we might follow up this trail and see where logic leads us.

Such must have been young Riemann's thoughts about the year 1850. We must now try to say in a few words how far he progressed from there, and what, mainly, his dissertation of 1854 contained.

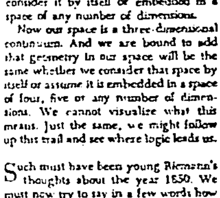
At a first reading, the outstanding result of Riemann's efforts *seems* to be that he succeeded in defining the curvature of a continuum of more than two dimensions. A two-dimensional continuum is a surface, and we have seen that its curvature is defined, for a small region surrounding any point of the surface, by a single number—positive on an egg-shaped surface, negative on a saddle-shaped surface. If the curvature is zero at every point, the surface is a plane, and vice versa. Riemann showed that the concept of curvature can be generalized for the case of a continuum of *n* dimensions. Only it will not be a single number any more; a set of three numbers will be needed to define the curvature of a continuum of three dimensions, a set of six numbers for one of four dimensions, and so forth. Riemann only stated these results and made them seem mathematically plausible; their proof and elaboration would have filled a long memoir or occupied several weeks of lectures.

These considerations seem purely abstract—a completely vacuous game of mathematics running wild. However, Riemann's main object in his dissertation was to convince us that he was talking not about abstract mathematical concepts but about a question of physics which could be settled by the experimental method.

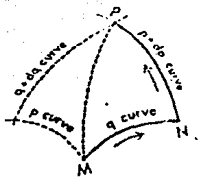
Let us return to those perfectly flat beings that live on a huge surface. Gauss's "remarkable theorem" proves that the two-dimensional inhabitants of this two-dimensional universe, provided they understood enough mathematics, could find the curvature of any small region of their universe. How could these people conceive of a curved surface, if they could not visualize a space of three dimensions? The answer is that such is precisely the power of mathematics. These people would be familiar with the concept of a curved road, contrasting with a "straight" road which would be the shortest route between two points. If then some Riemann among them had generalized this notion, in *a purely algebraic way*, into a theory of the curvature



LOCATION of a point on any mathematically given surface may be specified by giving one coordinate from the family of p-curves and one from the intersecting family of q-curves. On any surface but a sphere these curves will not intersect at right angles.



A DOUGHNUT'S SURFACE shows positive curvature in its outer half, while the inner half has negative curvature (black).



DISTANCE from one point P to a point M on a surface of any curvature cannot be determined by the Pythagorean rule. Gauss defined it as a function of the intersecting coordinates locating points and curvature varying on the surface from point to point.

of a continuum of n dimensions, their surveyors would be able to calculate from a formula given by Riemann a certain number which, they would find, would change slightly from country to country. Thus they would have measured the curvature of their two-dimensional universe without being able in any way to visualize what that could be.

Such, of *course*, is exactly our situation regarding the curvature of our own universe, and we must return to Riemann's work to form some idea of how he came to define it.

Riemann suggested that if all the numbers which defined the curvature of an n-dimensional space were zero, this space should be called flat, for that is what we call a surface whose curvature is zero. Now if we divide a three-dimensional space into equal little cubes, as a chessboard is divided into equal little squares, then dr^2 is simply the sum $dx^2 + dy^2 + dz^2$, with dx, dy, dz representing the three sides of each little cube. That space is a "flat" space, just as a plane is a flat surface. In other words, what our intuition tells us is that space is flat—in the sense given to that word by Riemann.

Is it really so? That the small portion of space in our neighborhood should appear flat is only to be expected. It may well be that space is actually flat, not only in our vicinity but away into the realm of the farthest nebulae. On the other hand, it is equally possible that space is ever so slightly curved. How could we ever find out? Riemann's answer was: *from experience*. That is the revolutionary message which, very quietly but very firmly, he brought to the scientific world.

Euclid and Kant had unconsciously accepted the intuitive notion of space as flat. Riemann declared that this proposition should not be asserted without proof, as self-evident; it was only a hypothesis, subject to test by experiment. To start with we could make three hypotheses about our space: that it had constant positive curvature, or constant negative curvature or no curvature at all (i.e., that it was flat, or Euclidean, as we now say). Which of these hypotheses was correct was for astronomers and physicists to find out. Such was the meaning of Riemann's cryptic title, "On the Hypotheses That Are the Foundations of Geometry," which had, how very rightly, aroused the curiosity of Gauss.

There are many other important things in this dissertation of Riemann's, such as a very clear-sighted appreciation of the possibility that we may have to adopt eventually [illegible] theory of space—so [illegible] physicists are just now rather [illegible] trying [illegible]. But the point we [illegible] presented here—the appeal to experiment in order to find out a possible curvature of space—is, we believe, the most important one.

Riemann wisely made no attempt to suggest what specific experiments should be made. Looking back from the vantage point of our post-Einsteinian knowledge, we realize they were very difficult to discover. One might have expected them to be in the domain of classical astronomy, of measurement of angles between stars, but that doesn't cut deep enough. Einstein showed that gravitation had a great deal to do with the matter and that Riemann's provisional hypothesis of a space of constant curvature had to be abandoned in favor of local variations (e.g., the curvature in the neighborhood of the sun or of Sirius was greater than in empty interstellar space). He also showed that time had to be brought in, in other words, a four-dimensional space-time was what had to be investigated experimentally. And thus it came about that in the three experimental checks on Einstein's theory obtained in 1920, space, time and gravitation were seen to be indissolubly mixed.

Riemann's contention that the geometry of the universe was just a chapter of physics to be advanced like any other by the close cooperation of theory and experiment, was thereby fully justified. So also was Riemann's faith in his master, Gauss. The more we gaze upon Riemann's and Einstein's truly gigantic pyramids of thought, the more we admire how much was invisibly contained in the short, unassuming formula written by Gauss in 1827.