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Gráficas CPR y Politopos abstractos regulares

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Prólogo

Debido a que el presente texto está escrito en inglés anexamos este prólogo en español en donde describimos brevemente el trabajo.

Los politopos regulares abstractos son una generalización sin geometría intrínseca de los politopos regulares convexos, que son, a su vez, una generalización multidimensional de los sólidos platónicos. En el Capítulo 1 nos extendemos en el contexto histórico de los politopos regulares y de los problemas que aquí tratamos.

En el Capítulo 2 se da la definición formal así como ciertos resultados de los politopos regulares abstratos. El principal afirma la existenca de una correspondencia biyectiva entre los politopos regulares abstractos y ciertos grupos cuyo nombre en inglés es "string C-groups". La letra "C" es debido a su cercanía con los grupos de Coxeter.

El presente trabajo aborda algunos problemas acerca de los politopos abstractos regulares. La técnica utilizada consiste en representar en ciertas gráficas las acciones de cada "string C-group" en diferentes conjuntos. Dichas gráficas reciben el nombre de "gráficas CPR". Si bien ciertas representaciones de este tipo se han desarrollado con anterioridad, por primera vez se están utilizando como herramienta para trabajar con politopos abstractos y se muestran como un método útil para atacar problemas de los politopos abstractos. En el Capítulo 3 se describen las gráficas CPR y sus principales propiedades.

En los Capítulos 4, 5 y 6 se exponen los resultados obtenidos mediante esta técnica.

En el Capítulo 4 se habla de los poliedros (politopos de rango tres) y de su relación con las gráficas CPR. Se da solución al problema de decidir qué grupos alternantes son"string C-groups" asociados a politopos regulares (teorema 4.3.4). Este es un problema propuesto por M. Hartley en 2005. La solución que aquí mostramos aparecerá publicada en [20]. También se muestran familias infinitas de ciertos poliedros que no se habían podido describir explícitamente con anterioridad a pesar de conocerse su existencia.

Decimios que un politopo $\mathcal{P}$ es extensión de un politopo regular $\mathcal{K}$ si las facetas de $\mathcal{P}$ son isomorfas a $\mathcal{K}$. El problema de existencia de extensiones de politopos regulares fue estudiado en la década de los 80's con los siguientes resultados. Se encontró una extensión universal (infinita y con último número de Schläfli infinito) para cada politopo de modo que cualquier extensión es cociente de ésta. Se encontraron además otras dos extensiones finitas descritas por L. Danzer en [7] (1984) y E. Schulte en [24] (1983). Estas dos extensiones tienen últimos números de Schläfli 4 y 6 respectivamente, quedando abierta la pregunta de si existen tales extensiones para cualquier número de Schläfli. M. Hartley probó en [13] que en general no existen extensiones con último número de Schläfli impar quedando abierta la pregunta para los números pares.

En los Capítulos 5 y 6 se da solución al problema de existencia de extensiones de politopos abstractos con último número de Schläfli n para todo n par (teoremas 5.3.1 y 6.2.2), esto se hace mediante la construcción de gráficas CPR. Como consecuencia se da una solución parcial a una conjetura publicada en [28] (teorema 5.8.2) acerca de extensiones autoduales de rango $d+1$ de politopos autoduales de rango $d-1$ con primero y último números de Schläfli predeterminados. Las extensiones descritas en estos dos capítulos forman parte de los artículos en preparación [21] y [22].

Finalmente, en el capítulo 7 se exponen preguntas abiertas que surgen del trabajo expuesto en los capítulos anteriores.

Chapter 1

Introduction

The convex polytopes have been studied since antiquity starting with the convex regular polygons and continuing with the platonic solids. The pentagram shown in Figure 1.1 was also studied before Christ even if it was not considered a regular polygon.

Later, in the fourteenth century, Bredwardin studied all the star polygons. It was Kepler in the fifteenth century who investigated the star polyhedra, although he only found two of them, the small and great stellated dodecahedra. The two other star polyhedra, the great dodecahedron and the great icosahedron were found by Poinsot in the nineteenth century.

A convex polytope is the convex hull of a finite set of points in an euclidean space, and it is said to be regular if it has all the possible symmetries. In the twentieth century Coxeter investigated the regular convex polytopes in higher dimensions including the star polytopes, continuing a line of investigation begun by Schläfli and others in the nineteenth century. The technique to determine the convex ones was to investigate whether or not it is possible

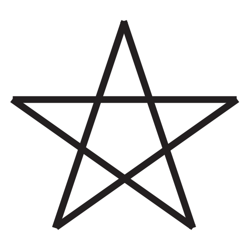


Figure 1.1: Pentagram

to construct a regular polytope in $\mathbb{R}^4$ with tetrahedra, octahedra, cubes, dodecahedra or icosahedra as facets. An affirmative answer was found for all except the icosahedron. For higher dimensions $d \geq 5$ there are only three regular convex polytopes, the d-simplex, the d-cube and the d-cross polytope, and only the first two of them appear as facets of a higher dimensional one (see H. S. M. Coxeter [5] for further details).

Also in the twentieth century, Grünbaum worked with polyhedra with *skew polygons* (non-planar finite polygons) as faces. He also considered polyhedra as maps on surfaces conjecturing that for any p and q there are regular maps on compact closed surfaces such that their faces are regular p-gons, q of them at each vertex. In [31] A. Vince solves the existence problem, moreover he proves that for almost any given p and q there are infinitely many such maps.

In [8] L. Danzer and E. Schulte introduce a definition of abstract regular polytopes (see Chapter 2). Since then, several authors have worked in this concept. The encyclopedic book of P. McMullen and E. Schulte [19] includes most of the known results about abstract regular polytopes. One of these results gives a one-to-one correspondence between the abstract regular polytopes and the so-called string C-groups that allows us to work with groups rather than with the polytope. The "C" stands for Coxeter due to the fact that the string C-groups are related to the well known Coxeter groups

The action on a set V of a group with distinguished generators gives rise naturally to an edge-labeled graph with vertex set V. This is not new, for instance, a Cayley graph is an example with the group acting on itself (see White [32]). In Chapter 3 we introduce the CPR graphs, which are Cayley-type diagrams of string C-groups arising from their effective actions. In this way we encode all the properties of the group on an edge-labeled graph and translate some of them to graph-theoretic terms. In this work the CPR graphs prove to be a useful tool for solving problems about abstract polytopes.

Using these graphs we are able to construct string C-groups with particular properties like being isomorphic to alternating or symmetric groups, or to correspond to polytopes with a certain local structure. In Chapter 4 we solve a question of existence of polyhedra with alternating automorphism group asked by M. Hartley in 2005 and published in the article by Schulte and Weiss [30]. We also construct some infinite families of polyhedra (maps on compact closed surfaces) with p-gons as faces, q of them at each vertex, that were known to exist but had not been explicitly described before.

It is a natural question to determine if for a given rank d polytope $\mathcal{K}$ there exist rank $d+1$ polytopes that contain $\mathcal{K}$ as a face (as Coxeter did for the convex ones). Such rank $d+1$ polytopes are called extensions of $\mathcal{K}$. In Chapter 5 we construct an extension of regular polytopes that generalizes and gives insight to a previous example given by L. Danzer (see [7]) and solves partially a conjecture of Schulte stated in [28]. In Chapter 6 we construct another extension related to the extension described by E. Schulte [8] and generalizes it for a certain class of polytopes.

In Chapter 7 we post some questions that arise naturally from the results of the previous chapters and from the technique used to obtain them.

In Appendices A and B we include some necessary algebraic and combinatorial technical results that are not properly about abstract regular polytopes or CPR graphs. Finally, in Appendix C we give a list of examples of CPR graphs of well-known polyhedra.

Chapter 2

Regular Polytopes

The purpose of this chapter is to introduce the preliminary concepts that will be used in the remaining chapters. The main source for all this topics is the encyclopedic book of McMullen and Schulte [19].

2.1 Definitions and Examples

In this section we introduce the notion of abstract regular polytopes as well as their connection with the so called string C-groups. We also give some results about regular polyhedra and show some examples.

The idea of the abstract polytopes is to generalize the concept of convex polytopes preserving their basic combinatorial properties.

A partially ordered set (poset) X with a rank function

$$rank : X \mapsto \{-1, \ldots, d\}$$

such that it has a greatest element F_d of rank d and a least element F_{-1} of rank -1 will be called a *flagged poset*. The *flags* of the poset are the maximal totally ordered subsets of X and they are required to have exactly $d+2$ elements including F_d and F_{-1}. Two elements $x, y \in X$ are called *incident* if $x \leq y$ or $y \leq x$.

We will say that a flagged poset satisfies the *diamond condition* if for any two incident elements x, y such that $rank(x) - rank(y) = 2$ there exist exactly two elements w_1 and w_2 such that $y < w_i < x$.

If a flagged poset satisfies the diamond condition then it implies that for any flag f and any $0 \leq i \leq d-1$ there exists a unique flag f^i such that f

and f^i differ only in the element of rank i. We will say that f and f^i are *adjacent flags* and f^i will be called the *i-adjacent flag* of f.

We will say that a flagged poset satisfying the diamond condition is *strongly flag connected* if for any two flags f and g there exists a sequence of flags $f = f_1, f_2, \dots, f_m = g$ such that f_i is adjacent to f_{i+1} and $f \cap g \subseteq f_i$ for all i. If we do not require that $f \cap g \subseteq f_i$ for all i we will simply say that the poset is *flag connected*.

Now we are ready to define an abstract polytope.

Definition 2.1.1 *An* abstract polytope *of rank d (or d-*polytope*) is a flagged poset with rank function valued in $\{-1, \dots, d\}$ satisfying the diamond condition and being strongly flag connected.*

Since there is little possibility of confusion we will refer to the abstract polytopes simply by "polytopes".

Aiming to preserve some terminology of the theory of convex polytopes we will say that a polyhedron is a rank 3 polytope, the elements of any polytope $\mathcal{K}$ are called *faces*, the elements of rank i are called *i-faces*; the 0-faces are called *vertices*, the 1-faces *edges*, and the $(d-1)$-faces *facets*. The set of i-faces of $\mathcal{K}$ is denoted by $\mathcal{K}_i$. When working with polyhedra the 2-faces are simply called faces.

Observe that any *section* $\{H \,|\, G \leq H \leq F\}$ for a given $G \leq F$ is again a polytope. Any face F may be identified with the polytope section $\{G \,|\, G \leq F\}$. The section $\{G \,|\, G \geq F\}$ is called the *co-face* of F, or, if F is a vertex, the *vertex figure* of F.

Given a polytope $\mathcal{K}$ we define its *dual* $\mathcal{K}^*$ as the same set of faces with the partial order reversed. It is easy to see that the dual of any polytope is again a polytope. We say that a polyope is *self-dual* if it is isomorphic (as a partially ordered set) to its dual.

An *automorphism* of a polytope is an order preserving permutation of its faces. The set of automorphisms of a polytope forms a group with the composition, denoted by $\Gamma(\mathcal{K})$. The automorphism group of abstract polytopes plays an important role in their study.

A polytope K is said to be *regular* if $\Gamma(\mathcal{K})$ is transitive on the flags of $\mathcal{K}$. Some classes of non-regular polytopes have been studied, however we will only work with regular polytopes.

The following proposition, which follows from flag connectivity, shows an equivalence of regularity for polytopes (see [19] Chapter 2B).

Proposition 2.1.2 *The following statements are equivalent for an abstract d-polytope $\mathcal{K}$.*

- *$\mathcal{K}$ is regular.*
- *For some flag f and for each $i \in \{0, \dots, d-1\}$ there is an involutory automorphism ϕ such that $\phi(f) = f^i$.*

From now on we choose a reference, or *base*, flag f for $\mathcal{K}$. For $i = 0 \dots, d-1$ we denote the element ϕ defined in Proposition 2.1.2 by ρ_i. Every regular polytope $\mathcal{K}$ satisfies that $\Gamma(\mathcal{K}) = \langle \rho_0, \dots, \rho_{d-1} \rangle$ with the additional relations $(\rho_i \rho_j)^2 = \varepsilon$ for $|i - j| \geq 2$, and the so-called *intersection property* described next. This motivates the following definition (see [19] Chapter 2B).

Definition 2.1.3 *A* string C-group *is a group generated by involutions ρ_0, $\dots$, ρ_{d-1}, such that the generators satisfy*

- *$(\rho_i \rho_j)^2 = \varepsilon$ if $|i - j| \geq 2$,*
- *$\langle \rho_k \,|\, k \in I \rangle \cap \langle \rho_k \,|\, k \in J \rangle = \langle \rho_k \,|\, k \in I \cap J \rangle$ (intersection property).*

It is also true that every string C-group is the automorphism group of a regular polytope. The proof of this statement can be found in [19] Chapter 2E. This establishes a one-to-one correspondence between the regular polytopes and the string C-groups. It implies that any definition, result or example for $\Gamma(\mathcal{K})$ has an equivalent definition, result or example respectively for $\mathcal{K}$. In the present work we will mostly work with the automorphism groups of polytopes rather than with the polytope itself.

The next proposition gives us a useful way to determine that a group generated by involutions with the suitable commutativity relations satisfies the intersection property (see [19] Chapter 2E).

Proposition 2.1.4 *Let $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ be a group such that ρ_k is an involution for all k, and $(\rho_i \rho_j)^2 = \varepsilon$ for $|i - j| \geq 2$. If $\langle \rho_0, \dots, \rho_{n-2} \rangle$ is a string C-group, and*

$$\langle \rho_0, \dots, \rho_{n-2} \rangle \cap \langle \rho_k, \dots, \rho_{n-1} \rangle = \langle \rho_k, \dots, \rho_{n-2} \rangle$$

for $k = 1, \dots, n-1$, then Γ is also a string C-group.

The sections between two incident faces of ranks $i-2$ and $i+1$ of a regular polytope $\mathcal{K}$ are isomorphic to p_i-gons, that are determined by the relations $(\rho_{i-1}\rho_i)^{p_i} = \varepsilon$. The number p_i then indicates how many i-faces (or $(i-1)$-faces) are glued together around an $(i-2)$-face inside a single $(i+1)$-face. We say that the *Schläfli type* or *Schläfli symbol* of the polytope is $\{p_1, \ldots, p_{d-1}\}$.

If $p_j = 2$ then the automorphism group is the direct product

$$\langle \rho_0, \ldots, \rho_{j-1} \rangle \times \langle \rho_j \ldots, \rho_{d-1} \rangle,$$

and for many purposes it suffices to analyze the polytopes with automorphism groups $\langle \rho_0, \ldots, \rho_{j-1} \rangle$ and $\langle \rho_j \ldots, \rho_{d-1} \rangle$. In general we will assume that no entry of the Schläfli symbol is 2.

It is easy to see that the dual of a regular polytope $\mathcal{K}$ with Schläfli type $\{p_1, \ldots, p_{d-1}\}$ is a regular polytope with Schläfli type $\{p_{d-1}, \ldots, p_1\}$.

Whenever $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle$ is the group determined only by the relations $(\rho_i\rho_j)^2 = \varepsilon$ for $|i-j| \geq 2$, and $(\rho_{i-1}\rho_i)^{p_i} = \varepsilon$, we denote $\mathcal{K}$ simply by $\{p_1, \ldots, p_{d-1}\}$. The automorphism group of any other polytope with the same Schläfli symbol will have extra generating relations. An exponent n in an entry of the Schläfli symbol indicates n equal entries in it, for example, the polytopes

$$\{3^n\} \text{ and } \{4, 3^{n-1}\} \tag{2.1}$$

have n entries in their Schläfli symbols, all of them equal to 3 except the 4 in the first entry of the second polytope. The polytopes in (2.1) are respectively the $n+1$-simplex and the $n+1$-cube.

The *even subgroup* of a group $\Gamma = \langle g_1, \ldots, g_s \rangle$ is the subgroup of the words of even length in terms of the generators g_i's

$$\Gamma^+ = \langle g_i g_j \,|\, i, j \in \{1, \ldots, s\} \rangle.$$

It is clear that this is a subgroup of index at most two, but in some cases it is the whole group Γ.

For each $p \in \mathbb{N}$ there exists one and only one abstract regular polygon with Schläfli symbol $\{p\}$. This is the regular p-gon $\{p\}$ and has all the combinatorial properties of the regular convex p-gon in the plane.

The platonic solids are examples of regular polyhedra. Modifying the shape of the edges and faces they can be embedded on the sphere in such a way that the i-faces are mapped onto i-cells and the group of isometries of the sphere preserving the embedding is isomorphic to the automorphism group of the corresponding polyhedron. This idea is generalized for any compact surface without boundary in the following way.

An embedding of a connected (multi)graph G into a compact surface without boundary S is called a *map on* S when the components of $S \setminus G$ are topological open disks. The vertex and edge sets of the map are the vertex and edge set of G, while the faces of the map are the closure of the 2-cells in $S \setminus G$. A flag of a map is a triple including a vertex, an edge including it, and a face including the edge. We say that a map is regular if the group of homeomorphisms of the surface preserving the embedding is transitive on the flags (these are sometimes called *reflexive* maps as in M. Conder [4]).

There is an injection from the set of abstract regular polyhedra to the set of regular maps on a compact surface without boundary in such a way that the partial order of the polyhedron is represented by the inclusion relation on the set of vertices, edges and faces of the corresponding map (see [19] Chapter 6B). We say that a polyhedron is *orientable* if its corresponding map is embedded on an orientable surface. The even subgroup of an orientable polyhedron $\mathcal{K}$ has index 2 in $\Gamma(\mathcal{K})$, and if $\mathcal{K}$ is non-orientable then the even subgroup of $\Gamma(\mathcal{K})$ is again $\Gamma(\mathcal{K})$. The *genus* of a polyhedron is defined as the genus of the surface where its corresponding map is embedded. Note that the genus of a polyhedron $\mathcal{K}$ can be derived from the Euler characteristic in the following way.

$$g(\mathcal{K}) = \begin{cases} (2-\chi(\mathcal{K}))/2 & \text{if } \mathcal{K} \text{ is orientable,} \\ 1-\chi(\mathcal{K}) & \text{if } \mathcal{K} \text{ is non-orientable,} \end{cases}$$

where $g(\mathcal{K})$ denotes the genus of $\mathcal{K}$ and $\chi(\mathcal{K})$ denotes the Euler characteristic of the surface where we embed the map corresponding to $\mathcal{K}$. We recall that the Euler characteristic can be computed by $\chi(\mathcal{K}) = v + f - e$ where v is the number of vertices, f the number of faces and e the number of edges of $\mathcal{K}$ See [4] for further details about these concepts.

A map is said to be a *polyhedral map* if the partial order associated to it induces a polyhedron.

Example 2.1.5 *The cube, octahedron, dodecahedron and icosahedron in* $\mathbb{R}^3$ *are symmetric with respect to the center. If we identify antipodal points in the embedding of them in the sphere we get new maps on the projective plane. The corresponding non-orientable polyhedra are called hemicube, hemioctahedron,*

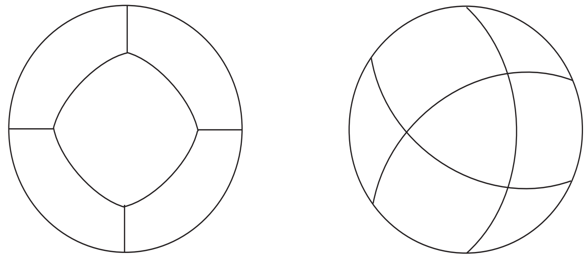


Figure 2.1: Maps of the hemicube and the hemioctahedron on the projective plane

hemidodecahedron and hemiicosahedron. Figure 2.1 shows the corresponding maps of the hemicube and the hemioctahedron.

Example 2.1.6 *The d-cube and the d-cross polytope are convex regular polytopes in $\mathbb{R}^d$ for every $d \geq 3$. They have symmetry with respect to the center. Hence we can construct the d-hemicube and the d-hemicross polytope in a way analogous to that described in Example 2.1.5.*

Example 2.1.7 *Starting with the tessellation by squares of the plane and taking quotient by two orthogonal vectors of the same length in such a way that they are parallel to the edges of the squares or to their diagonals we obtain the regular toroidal maps of types $\{4,4\}_{(t,0)}$ and $\{4,4\}_{(t,t)}$. They are regular polyhedral maps for $t \geq 2$ (see [6]). Figure 2.2 shows the maps corresponding to $\{4,4\}_{(2,0)}$ and $\{4,4\}_{(2,2)}$. The subscript indicates one of the orthogonal vectors.*

Example 2.1.8 *The great dodecahedron has the vertex and edge sets of the icosahedron as vertex and edge sets, while its faces are the regular convex pentagons determined by the five vertices surrounded any vertex of it (see figure 2.3). The great dodecahedron has 12 vertices, 30 edges and 12 faces. It can be proved that it lies on an orientable surface. By the Euler characteristic we know that its corresponding map lies on an orientable surface of genus 4 (see [4]).*

In this work we concentrate mainly on combinatorial properties of polytopes. For questions about geometric realizations of abstract polytopes in Euclidean spaces we refer to, for example, [1], [3], [17] and [18].

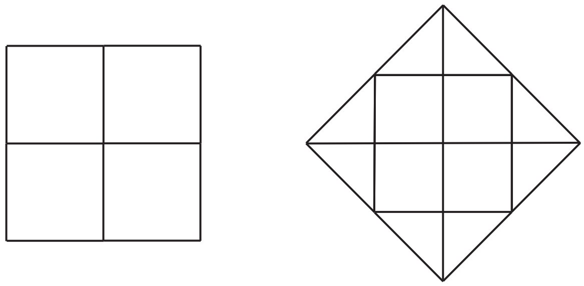


Figure 2.2: The toroids $\{4,4\}_{(2,0)}$ and $\{4,4\}_{(2,2)}$

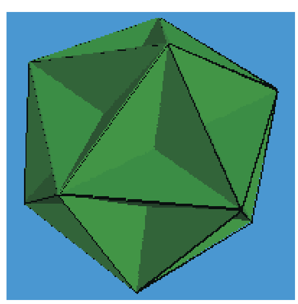
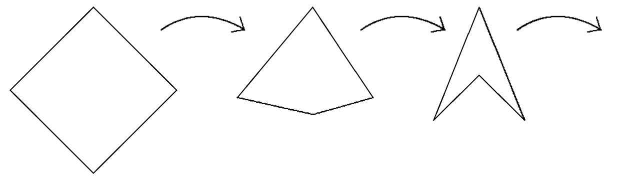


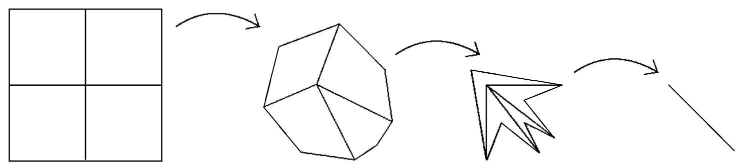
Figure 2.3: The great dodecahedron

2.2 Flat Amalgamation Property

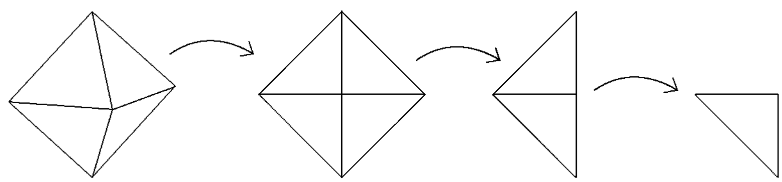
Some polytopes can be "folded" onto one of their i-faces in such a way that each j-face is sent into a j-face for all $j \leq i$. For example, the square (actually any $2p$-gon) for $i = 1$,



the toroid $\{4, 4\}_{(2,0)}$ for $i = 1$,



and the octahedron for $i = 2$



can be "folded" into its i-face in the way the figures show.

However there is no way of "folding" a cube into one square in the way described above.

This motivates the definition of the *flat amalgamation property (FAP).* We say that the square and the toroid $\{4, 4\}_{(2,0)}$ satisfy the FAP with respect to their 1-faces (a line segment), and the octahedron satisfies the FAP with respect to its 2-faces.

Before giving the formal definition of FAP we introduce some concepts and results.

Notation 2.2.1 *Given a regular polytope* $\mathcal{K}$ *we denote the normal closure of* $\{\rho_k, \ldots, \rho_{d-1}\}$ *as*

$$N_k^+(\mathcal{K}) = N_k^+ := \langle \phi^{-1}\rho_i\phi \,|\, i \geq k, \phi \in \Gamma(\mathcal{K})\rangle.$$

and dually, the normal closure of $\{\rho_0, \ldots, \rho_k\}$ *as*

$$N_k^-(\mathcal{K}) = N_k^- := \langle \phi^{-1}\rho_i\phi \,|\, i \leq k, \phi \in \Gamma(\mathcal{K})\rangle.$$

In [19] Chapter 4E they prove the following result.

Lemma 2.2.2 *Let N_k^- and N_k^+ be as previously described. Then, for $0 \leq k \leq d-1$,*

a) $N_k^- := \langle \phi^{-1}\rho_i\phi \,|\, i \leq k, \phi \in \langle \rho_{k+1}, \ldots, \rho_{d-1} \rangle \rangle$,

b) $N_k^+ := \langle \phi^{-1}\rho_i\phi \,|\, i \geq k, \phi \in \langle \rho_0, \ldots, \rho_{k-1} \rangle \rangle$,

c) $\Gamma(\mathcal{K}) = N_k^- \langle \rho_{k+1}, \ldots, \rho_{d-1} \rangle = \langle \rho_{k+1}, \ldots, \rho_{d-1} \rangle N_k^-$,

d) $\Gamma(\mathcal{K}) = N_k^+ \langle \rho_0, \ldots, \rho_{k-1} \rangle = \langle \rho_0, \ldots, \rho_{k-1} \rangle N_k^+$,

Definition 2.2.3 *We say that a regular polytope $\mathcal{K}$ satisfies the* flat amalgamation property (FAP) with respect to its k-faces *if the products in Lemma 2.2.2 d) are semi-direct. It satisfies the* FAP with respect to the co-k-faces *if the products in Lemma 2.2.2 c) are semi-direct. This is the DAP in [28].*

The next proposition gives an equivalence of the FAP (see [19] Chapter 4E).

Proposition 2.2.4 *Let $\mathcal{K}$ be the regular polytope with automorphism group presentation $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{d-1} \,|\, \mathcal{R} \rangle$, where $\mathcal{R}$ is a set of defining relations in terms of $\rho_0, \ldots, \rho_{d-1}$.*

The polytope $\mathcal{K}$ has the FAP with respect to its k-faces if and only if $\langle \rho_0, \ldots, \rho_{d-1} \,|\, \mathcal{R}$ and $\rho_i = \varepsilon$ for $i \geq k \rangle$ is a group presentation of the automorphism group of the k-faces. Dually, the polytope $\mathcal{K}$ has the FAP with respect to its co-k-faces if and only if $\langle \rho_0, \ldots, \rho_{d-1} \,|\, \mathcal{R}$ and $\rho_i = \varepsilon$ for $i \leq k \rangle$ is a group presentation of the automorphism group of the co-k-faces.

For example, a group presentation for the automorphism groups of the square, toroid $\{4,4\}_{(2,0)}$ and the octahedron are

$$\begin{aligned}
\Gamma(\{4\}) &= \langle \rho_0, \rho_1 \,|\, \rho_0^2 = \rho_1^2 = (\rho_0\rho_1)^4 = \varepsilon \rangle, \\
\Gamma(\{4,4\}_{(2,0)}) &= \langle \rho_0, \rho_1, \rho_2 \,|\, \rho_0^2 = \rho_1^2 = \rho_2^2 = \\
&\qquad (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_0\rho_2)^2 = (\rho_0\rho_1\rho_2\rho_1)^2 = \varepsilon \rangle, \\
\Gamma(\{3,4\}) &= \langle \rho_0, \rho_1, \rho_2 \,|\, \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^4 = (\rho_0\rho_2)^2 = \varepsilon \rangle.
\end{aligned}$$

respectively. It is easy to see that if we add the additional relation $\rho_1 = \varepsilon$ to the automorphism groups of the square, and the relations $\rho_1 = \rho_2 = \varepsilon$ to the automorphism group of the toroid $\{4,4\}_{(2,0)}$ we get the automorphism group of the 1-faces (an edge with group $\langle \rho_0 \mid \rho_0^2 = \varepsilon \rangle$), while if we add the additional relation $\rho_2 = \varepsilon$ to the automorphism group of the octahedron we get the automorphism group of the 2-faces (triangles with group $\langle \rho_0, \rho_1 \mid \rho_0^2 = \rho_1^2 = (\rho_0\rho_1)^3 = \varepsilon \rangle$).

On the other hand, if we add to the automorphism group of the cube given by

$$\langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^3 = (\rho_0\rho_2)^2 = \varepsilon \rangle$$

the additional relation $\rho_2 = \varepsilon$ the group collapses into the automorphism group of its 1-face (a line segment). However, the cube has the FAP with respect to its vertex figures (this is the dual statement to the octahedron satisfying the FAP with respect to its faces), and it can be checked adding the relation $\rho_0 = \varepsilon$ and obtaining the automorphism group of the triangle with generators ρ_1 and ρ_2 rather than ρ_0 and ρ_1.

For further details about the FAP see [19] Chapter 4.

2.3 Mixing Operations

Some regular polytopes are related to others. For example, the dual of a polytope can always be obtained by reversing the partial order of the original polytope. Another example is the great dodecahedron, that can be obtained from the icosahedron in the way described in Section 2.1 (see Example 2.1.8).

This motivates the definition of mixing operations of regular polytopes.

Definition 2.3.1 *Let $\mathcal{K}$ be a regular polytope with automorphism group $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle$. A* mixing operation *$\mu$ is a choice of new generators $\sigma_0, \ldots, \sigma_{m-1}$ of a subgroup Δ of $\Gamma(\mathcal{K})$ in terms of $\rho_0, \ldots, \rho_{d-1}$ and it is denoted by*

$$\mu : (\rho_0, \ldots, \rho_{d-1}) \mapsto (\sigma_0, \ldots, \sigma_{m-1}).$$

We are interested in the cases where Δ is a string C-group in terms of the generators $\sigma_0, \ldots, \sigma_{m-1}$ so we will assume that the σ_i's are involutions such that $(\sigma_i\sigma_j)^2 = \varepsilon$ if $|i-j| \geq 2$.

The mixing operations below will be used in this work. Unfortunately, among these mixing operations, only the dual operation guarantees that we obtain a polytope. For an example of two polytopes such that their mix is not a polytope see [19] Chapter 7A, and for an example of a polytope such that its petrial is not a polytope see [19] Chapter 7B.

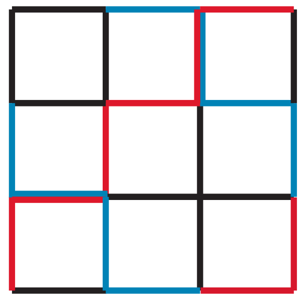


Figure 2.4: Petrie polygons

2.3.1 Dual operation

The dual $\mathcal{K}^*$ of a polytope $\mathcal{K}$ can be obtained by the dual operation

$$\delta : (\rho_0, \ldots, \rho_{d-1}) \mapsto (\rho_{d-1}, \ldots, \rho_0).$$

It can be proved easily using the correspondence between string C-groups and regular polytopes that the dual of a regular polytope as a poset is the same as the one obtained by the mixing operation δ.

2.3.2 Petrie operation

This involutory operation can be applied only to polyhedra and is defined by

$$\pi : (\rho_0, \rho_1, \rho_2) \mapsto (\rho_0\rho_2, \rho_1, \rho_2).$$

If we obtain a polyhedron by applying this operation to the polytope $\mathcal{K}$, the new polyhedron is known as the *petrial* of $\mathcal{K}$ and $\Gamma(\mathcal{K}) = \Gamma(\pi(\mathcal{K}))$. The faces are the so-called *petrie polygons* of $\mathcal{K}$. These are zigzags such that any two consecutive edges are in the same face (of $\mathcal{K}$), but any three consecutive edges are not in the same face. For example, in the toroid $\{4, 4\}_{(3,0)}$ of Figure 2.4 two petrie polygons appear, one of them in red and the other in blue.

The following lemma shows how the dual and petrie operations interact (see [19] Chapter 7B).

Lemma 2.3.2 *The mixing operation $\delta\pi$ has order* 3*, that is,* $(\delta\pi)^3(\mathcal{K}) = \mathcal{K}$ *for any regular polytope $\mathcal{K}$.*

A consequence of Lemma 2.3.2 is that the families of polyhedra generated by the dual and petrie operations starting with a polyhedron $\mathcal{K}$ contain in general six polyhedra. They contain three polyhedra if $\mathcal{K}$ is self-dual or self-petrial, and one polyhedron if $\mathcal{K}$ is self-dual and self-petrial. See Section 4.2 for examples.

Example 2.3.3 *The tetrahedron is a self-dual polyhedron. Its petrial is isomorphic to the hemicube, and the dual of the hemicube is the hemioctahedron. It can be seen from Figure 2.1 that the hemioctahedron is a self-petrie polyhedron. Hence the family of the tetrahedron obtained by duality and petrials includes only the three polyhedra mentioned here.*

2.3.3 Facetting operation

Similarly to the petrie operation, the k-th facetting operation can be applied only to polyhedra, and it consists in replacing the faces of a polyhedron with its so-called k-holes. A k-hole of a polyhedron $\mathcal{K}$ is a sequence of vertices and edges such that "between" two consecutive edges there are $k-1$ edges in the polyhedral map associated to $\mathcal{K}$. The k-th facetting operation is formally defined by

$$\varphi_k : (\rho_0, \rho_1, \rho_2) \mapsto (\rho_0, \rho_1(\rho_2\rho_1)^{k-1}, \rho_2).$$

Example 2.3.4 *The great dodecahedron is obtained by applying the* 2*-facetting operation to the icosahedron.*

It is clear that $\varphi_1(\mathcal{K}) = \mathcal{K}$. Moreover, if $\mathcal{K}$ is a polyhedron with Schläfli type $\{p, q\}$, then $\varphi_k(\mathcal{K}) = \varphi_{q-k}(\mathcal{K})$, and if $k = q/2$ then $\varphi_k(\mathcal{K})$ is a polyhedron of type $\{r, 2\}$. In the following we consider φ_k only for $2 \leq k \leq \lfloor (q-1)/2 \rfloor$.

Given k and a polyhedron $\mathcal{K}$ with Schläfli type $\{p, q\}$ such that the greatest common divisor $(q, k) = 1$, we have that $\Gamma(\mathcal{K}) = \Gamma(\varphi_k(\mathcal{K}))$. If $(q, k) > 1$ then the polyhedron $\varphi_k(\mathcal{K})$ might have, as vertex and edge sets, proper subsets of the vertex and edge sets of $\mathcal{K}$. In this case the k-holes of $\mathcal{K}$ form a disconnected partially ordered set and $\varphi_k(\mathcal{K})$ is only one connected component of it. This reduces the automorphism group. In Section 4.2 we show an example where $\Gamma(\mathcal{K}) = \Gamma(\varphi_k(\mathcal{K}))$ with $(q, k) = 2$.

Of particular interest is the composition of the petrie and facetting operations (see [19] Chapter 7B).

Lemma 2.3.5 *The petrie and the k-facetting operation commute for all k.*

The faces of the polyhedron $\varphi_k \pi(\mathcal{K})$ are the so-called *k-zigzags*. They can be seen as k-holes such that, in each vertex, the local orientation used to skip $k-1$ edges changes.

2.3.4 Mix of two polytopes

The *mix* operation is related to the concept of *blending*, used in realizations of polytopes (see [19] Chapters 5A and 7A). The idea is to construct a polytope such that two different projections (quotients) of it are isomorphic to two given polytopes $\mathcal{K}$ and $\mathcal{P}$. However it is a purely combinatorial operation. We define it in terms of its automorphism group. For a definition as a mixing operation see [19] Chapter 7A.

Let $m \geq n$, and let $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and $\Gamma(\mathcal{P}) = \langle \sigma_0, \ldots, \sigma_{m-1} \rangle$ be the automorphism groups of the polytopes $\mathcal{K}$ and $\mathcal{P}$. We consider the subgroup Λ of $\Gamma(\mathcal{K}) \times \Gamma(\mathcal{P})$ generated by $\tau_0, \ldots, \tau_{m-1}$ where $\tau_i = (\rho_i, \sigma_i)$ (we define $\rho_i = \varepsilon$ if $i > n-1$). If Λ is a string C-group with respect to the generators $\tau_0, \ldots, \tau_{m-1}$, then the regular polytope $\mathcal{Q}$ associated to it is said to be the *mix* of $\mathcal{K}$ and $\mathcal{P}$, denoted by $\mathcal{K} \Diamond \mathcal{P}$.

The following results talk about certain cases when we know that the mix of two polytopes is a polytope (see [19] Chapter 7A).

Remark 2.3.6 *For every regular polytope $\mathcal{K}$, the mix $\mathcal{K} \Diamond \mathcal{K}$ is isomorphic to $\mathcal{K}$.*

Theorem 2.3.7 *Let $\mathcal{K}$ be a regular polytope and $\mathcal{P}$ be the polytope of rank* 1 *(that is, an edge). Then $\Gamma(\mathcal{K} \Diamond \mathcal{P}) \cong \Gamma(\mathcal{K})$ if the* 1*-skeleton of $\mathcal{K}$ (graph with vertex and edge sets equal to those or $\mathcal{K}$) is a bipartite graph; otherwise $\Gamma(\mathcal{K} \Diamond \mathcal{P}) \cong \Gamma(\mathcal{K}) \times \mathbb{Z}_2$.*

For further details about these mixing operations see [19] Chapter 7.

Chapter 3

CPR Graphs

In this chapter we introduce the concept of CPR graphs as well as many results about them that are necessary for the main results in the remaining chapters of this work.

3.1 Definitions

The term "CPR" graph comes from "C-group Permutation Representation" graph and is explained next.

Given a regular d-polytope $\mathcal{K}$ we can embed its automorphism group $\Gamma(\mathcal{K})$ on a symmetric group S_n. Since the generators of $\Gamma(\mathcal{K})$ are involutions, their images under the embedding are also involutions, or equivalently, products of disjoint transpositions in S_n.

Now we can construct a (multi)graph G (we allow multiple edges but not loops) with a labeling on its edges representing the (generators of the) group $\Gamma(\mathcal{K})$ by defining the vertex set

$$V(G) = \{v_1, \ldots, v_n\}$$

and allowing an edge of label k between the vertices v_i and v_j if and only if the image under the embedding of ρ_k interchanges i and j.

Since we are going to use graphs labeled only on the edges we will refer to them simply as "labeled graphs", and if the label set has d elements we will refer to them as "d-labeled graphs".

Definition 3.1.1 *Let $\mathcal{K}$ be a regular d-polytope, and π be an embedding of $\Gamma(\mathcal{K})$ in S_n for some n. The* CPR graph *G of $\mathcal{K}$ given by π is a d-labeled*

Figure 3.1: Tetrahedron

(multi)graph with vertex set $V(G) = \{1, \ldots, n\}$ *and such that there is an edge of label* k *between the vertices* i *and* j *if and only if* $(\pi\rho_k)i = j$.

The loops make no contribution in the representation, so they will be ignored.

A CPR graph G is a way of encoding the information of the automorphism group $\Gamma(\mathcal{K})$ in G. Moreover we can recover $\Gamma(\mathcal{K})$ from G as a group of permutations on $V(G)$ by taking the group generated by the permutations ρ_k, where ρ_k is obtained by the product of the transpositions of each pair of vertices with an edge of label k between them.

Remark 3.1.2 *The edges of each color in any CPR graph of a polytope form a matching (set of edges such that no two of them are incident to the same vertex).*

Proof

Since ρ_k is an involution, it follows that $(\pi\rho_k)^2 = Id$ and each connected component of the subgraph induced by the edges of label k has at most two vertices.

□

Example 3.1.3 *The tetrahedron's group is* S_4*, and can be seen as the group of the permutations of its vertices (or faces). If we label the vertices* $1, 2, 3, 4$*, and we consider the base flag to be the one containing the vertex* 1*, the edge* 12 *and the face* 123*, the canonical generators of* S_4 *will be*

$$(\rho_0, \rho_1, \rho_2) = ((12), (23), (34));$$

and the CPR graph given by the natural embedding described above will be the one of Figure (3.1).

When working with CPR graphs of polyhedra, the color *black* will correspond to label 0, the color *red* to label 1, and the color *blue* to label 2.

In some cases, the embedding π is not relevant, so we will say only "a CPR graph of the polytope $\mathcal{K}$".

In general, there is not a unique representation of the group of a polytope as a permutation group, so each polytope has a family of graphs associated to it. We mention some examples.

Example 3.1.4 *The tetrahedron's group can be embedded in S_6, S_8, S_{12} and S_{24} in the ways shown in Figure (3.2).*

Example 3.1.5 *If $\Gamma(\mathcal{K})$ acts faithfully on $\mathcal{K}_j$, then it can be embedded in S_m, where m is the number of j-faces of $\mathcal{K}$. In this case, the graph will be called the j-face CPR graph of $\mathcal{K}$. Figure 3.1 shows the vertex (and facet) CPR graph of the tetrahedron, while Figure 3.2 A shows the edge CPR graph of the same polyhedron. For more examples see appendix C.*

Example 3.1.6 *If we consider $\Gamma(\mathcal{K})$ as a group of permutations on the flags of $\mathcal{K}$, then the CPR graph obtained will be the Cayley graph of $\Gamma(\mathcal{K})$ (see Appendix B) with generators $\{\rho_0, \dots, \rho_{d-1}\}$. In [31] Vince calls this graph "the combinatorial map of the regular polytope".*

A similar notion of a permutation group given by the action on the flags of a polytope determined by the i-adjacency of flags, has been used for non-regular polytopes by M. Hartley in [10] and [11], and by Hubard, Orbanic and Weiss in [14]. This group is called the *monodromy* group and for regular polytopes it is isomorphic to the automorphism group.

From now on, we will use the following notation.

- $G_{0,\dots,d-1}$ will be a d-labeled graph with colors $0, \dots, d-1$.
- Given a graph $G_{0,\dots,d-1}$ and a subset $I = \{i_1, \dots, i_m\}$ of $\{0, \dots, d-1\}$, $G_{i_1,\dots,i_m}$ will be the spanning subgraph (with all the vertices of G) including only the edges of labels $i \in I$.

3.2 Action of Groups on the CPR Graphs

An embedding of a group into a symmetric group S_n may be seen as an action of the group on the set $\{1, \dots, n\}$. The action will be faithful (in the sense that two different elements of the group act in a different way on $\{1, \dots, n\}$)

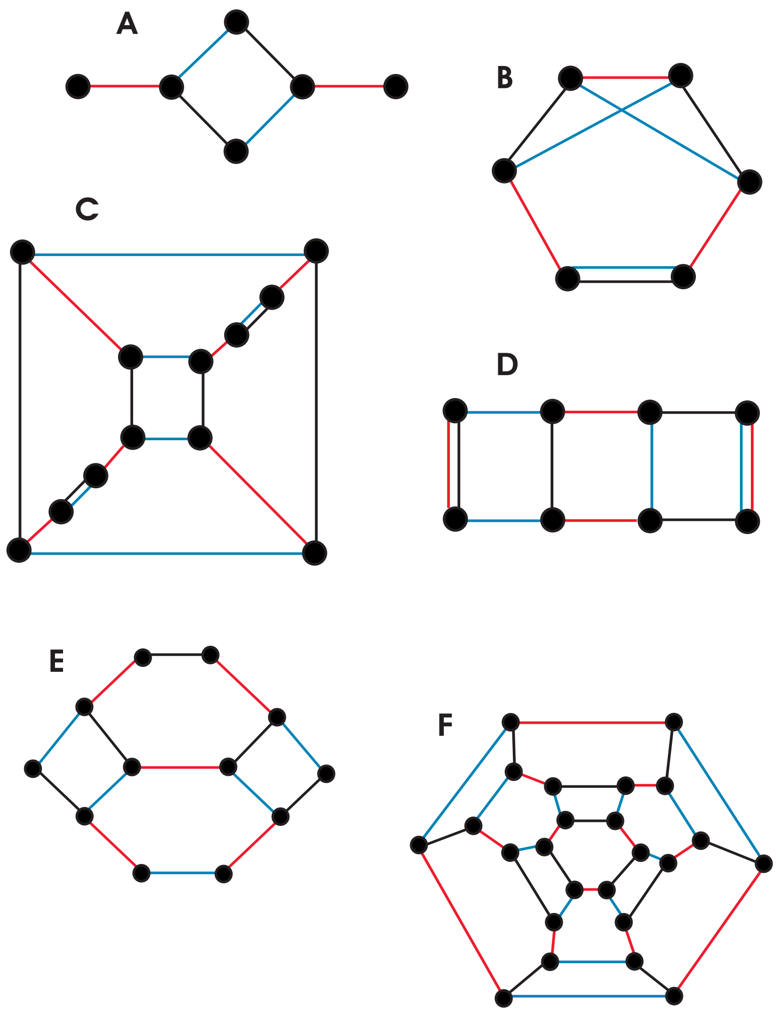


Figure 3.2: Tetrahedron CPR-graphs

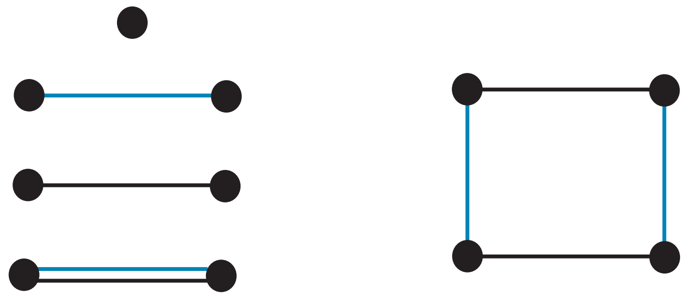


Figure 3.3: Connected components of $G_{i,j}$

if and only if the embedding is injective. In this setting, a CPR graph G of the polytope $\mathcal{K}$ also represents a faithful action of $\Gamma(\mathcal{K})$ on the vertex set of G. Now we proceed to the formal definition.

The following results give information about the CPR graphs of a given polytope, as well as information of any polytope given one of its CPR graphs, in terms of the action of the automorphism group of the polytope on the vertex set of the graph.

Proposition 3.2.1 *Let $G = G_{0,\ldots,d-1}$ be a CPR graph of a polytope $\mathcal{K}$. Then, every connected component of $G_{i,j}$ with $|i-j| \geq 2$ is either a single vertex, a single edge, a double edge, or an alternating square (see Figure 3.3).*

Proof

The set of edges of $G_{i,j}$ is the union of two matchings so its connected components are either alternating paths (including single vertices), or alternating cycles (including double edges).

The single vertices correspond to fixed points of both, ρ_i and ρ_j; the paths of length 1 correspond to vertices interchanged by one generator and fixed by the other; the double edges correspond to vertices interchanged by both generators; and the alternating squares correspond to 4 vertices such that ρ_i and ρ_j act in them like $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It is easy to see that an alternating path of length at least 2, or an alternating cycle of length greater than 4 correspond to the action of non-commutative involutions. Since ρ_i and ρ_j commute for $|i-j| \geq 2$, the proof is complete.

□

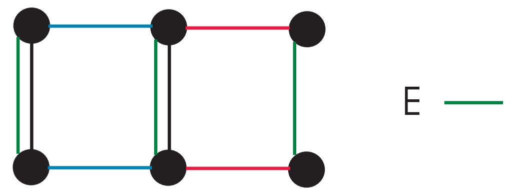


Figure 3.4: Centralizing involution represented by E

Corollary 3.2.2 *If $G_{0,\ldots,d-1}$ is a CPR graph of a polytope $\mathcal{K}$ of Schläfli type $\{p_1, \ldots, p_{d-1}\}$, then $G_{i-1,i}$ has at least one alternating path of length at least 2, or one alternating cycle of length greater than 4 as a connected component for every i such that $p_i > 2$.*

The central involutions (proper involutions in the automorphism group that fix no vertex of the polytope and commute with every element in the group) play an important role in the polytope theory. The following result relates them to the CPR graphs.

Proposition 3.2.3 *A central involution in the automorphism group of a polytope can be recognized in any of its CPR graphs $G = G_{0,\ldots,d-1}$ as a perfect matching (matching that covers all vertices) E in each connected component of G on which it does not act trivially. Moreover, if E is such a perfect matching, then, for any $i \in \{0, \ldots, d-1\}$, the spanning subgraph of G containing E and all edges of G labeled i has connected components as in Proposition 3.2.1.*

Proof

If a central involution ϕ has a fixed vertex on a connected component of a CPR graph, then, by conjugacy, every vertex of that connected component is fixed by ϕ, and that involution acts like ε in that component. Hence any central involution can be seen as a perfect matching in some connected components of the CPR graphs. The commutativity with every element of the group can be seen in the generators, so it is enough to check the conditions described in Proposition 3.2.1 in the spanning subgraphs determined by E and each of the generators ρ_i's.

□

However, such a matching in a CPR graph does not guarantee a central involution in the automorphism group of the polytope. It is also necessary to prove that the corresponding permutation of the vertices of the CPR graph can be generated by $\rho_0, \ldots, \rho_{d-1}$. For example, in the edges CPR graph of the hemicube we can draw a matching E with the commutativity properties (see Figure 3.4), but the hemicube does not have central involutions. Thus the involution represented by E cannot be generated by ρ_0, ρ_1 and ρ_2.

The automorphism group of a CPR graph G as an edge-labeled graph may be trivial. As an example, the one in Figure 3.1 is, even though the automorphism group of the tetrahedron is not. The following results relate the automorphism group of a polytope with the automorphism group of its CPR graphs as labeled graphs.

Lemma 3.2.4 *Let G be a CPR graph of a polytope $\mathcal{K}$, let Λ be any group of automorphisms of G as a labeled graph, and let O_v be the orbit under Λ for each vertex v. Then, the group*

$$N = \{\phi \in \Gamma(\mathcal{K}) \,|\, \phi(v) \in O_v \textit{ for all } v \in V(G)\}$$

is a normal subgroup of $\Gamma(\mathcal{K})$.

Proof

First note that, if $\lambda \in \Lambda$ then for all $v \in V$ and for all i, λ maps the edge $\{v, \rho_i(v)\}$ with label i onto the edge $\{\lambda(v), \lambda\rho_i(v)\}$ with label i. This implies that $\lambda\rho_i(v) = \rho_i\lambda(v)$.

Let $\phi \in N$. Given a vertex v, $\phi\rho_i(v) = \lambda_v\rho_i(v) = \rho_i\lambda_v(v)$ for some $\lambda_v \in \Lambda$. Then $\rho_i\phi\rho_i(v) = \lambda_v(v) \in O_v$. Since $\rho_i N \rho_i = N$ for all i, N is a normal subgroup of $\Gamma(\mathcal{K})$.

□

Proposition 3.2.5 *Let G, $\mathcal{K}$, Λ and N be as in lemma 3.2.4, let G' be the graph with vertex set*

$$V(G') = \{O_v : v \in V(G)\},$$

such that O_vO_w is an edge labeled i of G' if and only if $v'w'$ is an edge labeled i of G for some $v' \in O_v$ and $w' \in O_w$. If G' is a CPR graph of a polytope $\mathcal{K}'$, then $\mathcal{K}'$ is the quotient of $\mathcal{K}$ determined by the subgroup N of $\Gamma(\mathcal{K})$.

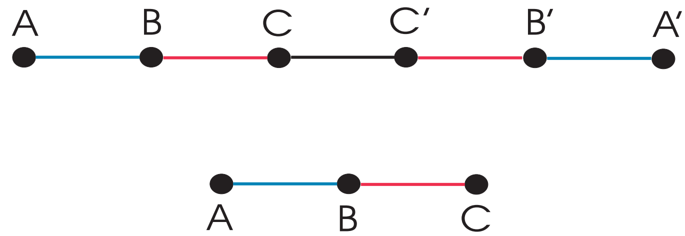


Figure 3.5: Facet CPR graphs of cube and triangle

Proof

First note that the group $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle$ acts on G' in the following way,

$$\phi(O_v) = O_{\phi(v)}.$$

In particular, the involution ρ'_j represented by the edges of label j on G' acts like ρ_j, so we have the epimorphism from $\Gamma(\mathcal{K})$ into $\Gamma(\mathcal{K}')$ defined in the generators by $\rho_j \mapsto \rho'_j$. Now, by definition,

$$N = \bigcap_{v \in V(G)} St_v$$

where St_v is the stabilizer of O_v in $\Gamma(\mathcal{K})$, but this is the kernel of the morphism. Hence,

$$\Gamma(\mathcal{K}') \cong \Gamma(\mathcal{K})/N.$$

Since this isomorphism maps ρ_i to ρ'_i, $\Gamma(\mathcal{K})/N = \langle \rho'_0, \ldots, \rho_{d-1} \rangle$ is the string C-group describing $\mathcal{K}'$.

□

Note that any generator ρ_i contained in N vanishes, so the quotient may be a polytope of rank less than the one of $\mathcal{K}$. For example, the 2-face CPR graph of the cube shows that the triangle is a quotient of the cube (see Figure 3.5). In Chapter 5 another example of this occurs.

Proposition 3.2.5 does not guarantee proper quotients of a polytope given a CPR graph with nontrivial automorphism group as a labeled graph. The subgroup N plays an important role, and it may be trivial even for CPR graphs with nontrivial automorphism groups as labeled graphs. For example,

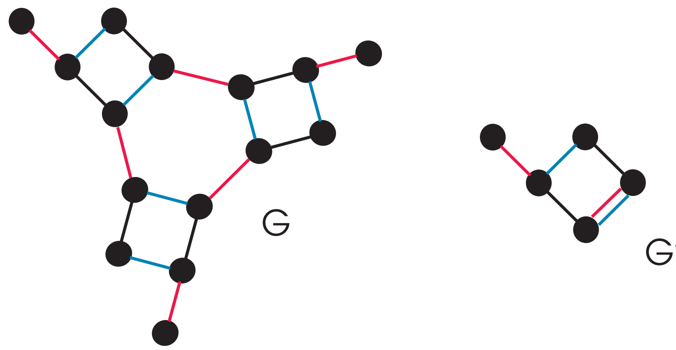


Figure 3.6: Hemidodecaedron CPR graphs

the edge CPR graph of the hemidodecahedron (see Figure 3.6) has automorphism group isomorphic to $\mathbb{Z}_3$, but in this case, $N = \{\varepsilon\}$ and G' is another CPR graph of the same polyhedron.

Now we will describe the way in which the group $\langle \rho_i, \rho_{i+1} \rangle$ acts in each connected component of $G_{i,i+1}$. We recall that $\langle \rho_i, \rho_{i+1} \rangle$ is isomorphic to the dihedral group $D_{p_{i+1}}$, where p_{i+1} is the (i+1)-th entry of the Schläfli symbol of $\Gamma(\mathcal{K})$.

Let C be a connected component of $G_{i,i+1}$. If C is a path, then it is easy to see that $\langle \rho_i, \rho_{i+1} \rangle$ acts as isometries (rotations and reflections) of the polygon formed as Figure 3.7 shows. The element $(\rho_i \rho_{i+1})^k$ acts as the k-th power of the rotation by an angle of $2\pi/m$ where m is the number of vertices of the polygon (or the number of vertices of C); and the element $(\rho_i \rho_{i+1})^k \rho_i$ acts as the composition of the reflection determined by ρ_i and the rotation determined by $(\rho_i \rho_{i+1})^k$. Note that m divides p_{i+1}.

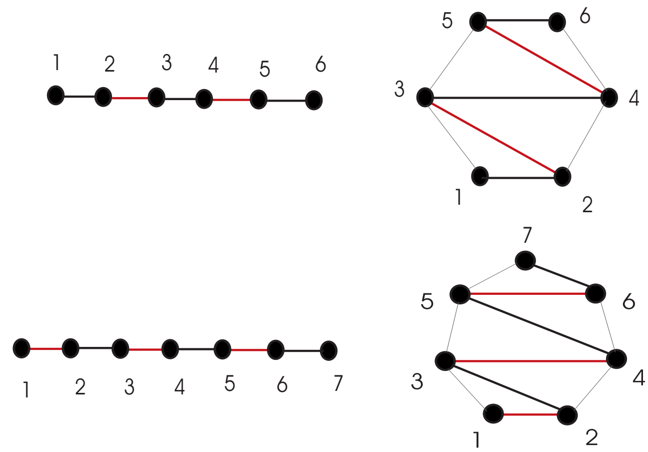


Figure 3.7: D_m acts in the polygon

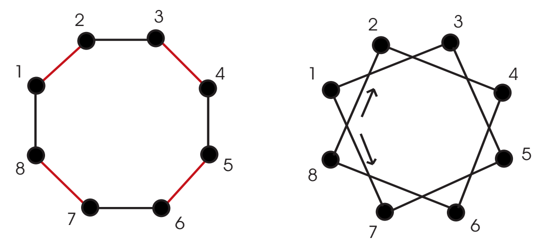


Figure 3.8: D_4 acts in both squares

If C is an alternating cycle of length $2s \geq 4$, then, the element $(\rho_i\rho_{i+1})^k$ acts like rotations by $2k\pi/s$ and $-2k\pi/s$ respectively on the two polygons formed by alternating the cycle's vertices, as Figure 3.8 shows. The element $(\rho_i\rho_{i+1})^k\rho_i$ acts like the composition of ρ_i and the rotations given by $(\rho_i\rho_{i+1})^k$, that is, like a rotation of the s-gons along the $2s$-gon such that it interchanges both polygons. Now s divides p_{i+1}.

Since this action is used several times in this work we introduce the following definition.

Definition 3.2.6 *Let G be a CPR graph. The action of $\phi \in \langle \rho_i, \rho_{i+1} \rangle$ in any connected component of $G_{i,i+1}$ described in Figures 3.7 and 3.8 will be called the* **polygonal action** *in that component.*

From the polygonal action of the elements in $\langle \rho_i, \rho_{i+1} \rangle$, it follows that

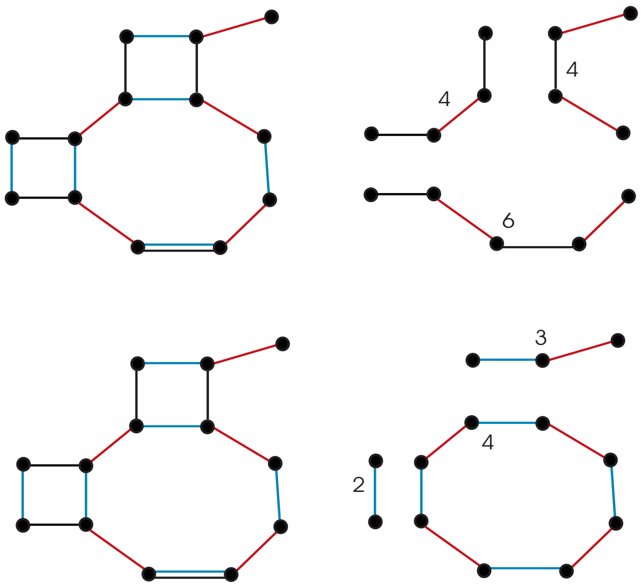


Figure 3.9: $\{20, 12\}$

each connected component of $G_{i,i+1}$ represented by a path with m vertices, or by a cycle with $2m$ vertices, induces a group isomorphic to D_m. Since the automorphisms ρ_i, ρ_{i+1} represented in $G_{i,i+1}$ generate a group isomorphic to $D_{p_{i+1}}$ it follows that p_{i+1} is the least common multiple of the number of vertices of its path components and of the halves of the number of vertices of its cycle components. This fact allows us to know the Schläfli type of a polytope given any of its CPR graphs, for example, the Schläfli type of the graph in the picture 3.9 (we will prove in Chapter 4 that it is indeed a CPR graph) is $\{20, 12\}$.

As we have seen, the elements of $\langle \rho_i, \rho_{i+1} \rangle$ of the form $(\rho_i\rho_{i+1})^k$ act differently on the connected components of $G_{i,i+1}$ than those of the form $(\rho_i\rho_{i+1})^k\rho_i$. We introduce the next definition.

Definition 3.2.7 *An element ϕ of $\langle \rho_i, \rho_{i+1} \rangle$ of the form $(\rho_i\rho_{i+1})^k$ will be called $\rho_i\rho_{i+1}$-**even**, and the remaining will be called $\rho_i\rho_{i+1}$-**odd**.*

Remark 3.2.8 *The set of $\rho_i\rho_{i+1}$-even elements of $\langle \rho_i, \rho_{i+1} \rangle$ form the even subgroup of $\langle \rho_i, \rho_{i+1} \rangle$.*

Remark 3.2.9 *The polygonal action of a $\rho_i\rho_{i+1}$-even ($\rho_i\rho_{i+1}$-odd) element $\phi \in \langle \rho_i, \rho_{i+1} \rangle$ in any connected component of $G_{i,i+1}$ is totally determined by its action on one of the vertices of the component.*

Note that the $\rho_i\rho_{i+1}$-even elements are those of the rotation group in the polygonal action of the elements of $\langle \rho_i, \rho_{i+1} \rangle$ in the path connected components of $G_{i,i+1}$, while the $\rho_i\rho_{i+1}$-odd elements are those of the reflections.

Remark 3.2.10 *Let G be a CPR graph, $\phi \in \langle \rho_i, \rho_{i+1} \rangle \setminus \{\varepsilon\}$ be $\rho_i\rho_{i+1}$-even and let C be a connected component of $G_{i,i+1}$ where ϕ does not act like ε, then $\phi(v) \neq v$ for every vertex v of C (in other words, ϕ does not have fixed points in the components where it does not act like identity).*

Proof

It is easy to see from Figures 3.7 and 3.8 that the $\rho_i\rho_{i+1}$-even elements are powers of rotations in the polygonal action of $\langle \rho_i, \rho_{i+1} \rangle$, hence they have no fixed points if they are distinct from ε.

□

Lemma 3.2.11 *Let C be a connected component of $G_{i,i+1}$ with at least three vertices, and let $\phi \in \langle \rho_i, \rho_{i+1} \rangle$ such that $\phi_{|C} = (\rho_i)_{|C}$. Then ϕ is $\rho_i\rho_{i+1}$-odd.*

Proof

The polygonal action of a $\rho_i\rho_{i+1}$-even element of $\langle \rho_i, \rho_{i+1} \rangle$ is the same than the polygonal action of ρ_i in at most two vertices.

□

Given a graph $G = G_{0,\dots,d-1}$, it is hard to say if the intersection property holds for the generators represented by the edges of each color. We will show some results for $d = 3$ and connected CPR graphs in chapter 4. In general we prove the intersection property in a case by case way.

3.3 Connected CPR Graphs

We can construct disconnected CPR graphs for any polytope, for example, two copies of one of its connected CPR graphs. Actually, if G is a disconnected CPR graph for $\mathcal{P}$ with components $C_1, \dots, C_m$, and C_i is a CPR

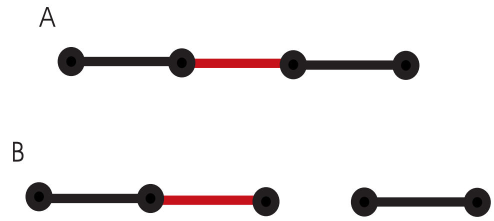


Figure 3.10:

graph of a polytope $\mathcal{D}_i$, then $\mathcal{P}$ is the mix $\mathcal{P} = \mathcal{D}_1 \diamondsuit \mathcal{D}_2 \diamondsuit \ldots \diamondsuit \mathcal{D}_m$ (see Chapter 2). In this section we will show that, in general, regular polytopes also have several connected CPR graphs and we will give a way of constructing them.

A CPR graph of a given polytope, with the smallest number of vertices, can be either connected or disconnected. For example, the square's smallest CPR graph is connected (a path of length 3, see Figure 3.10 A) while the hexagon's smallest CPR graph is the disjoint union of two paths of lengths 1 and 2 respectively (see Figure 3.10 B). From the automorphism groups and the polygonal action we can see that no disconnected graph with four or less vertices will be the CPR graph of the square, and no connected graph with five or less vertices will be the CPR graph of the hexagon.

Examples 3.1.5 and 3.1.6 show some connected CPR graphs of any polytope.

The following results link the connected CPR graphs of a polytope $\mathcal{K}$ with the structure as a group of $\Gamma(\mathcal{K})$.

Proposition 3.3.1 *Any connected CPR graph $G = G_{0,\ldots,d-1}$ with s vertices of a polytope $\mathcal{K}$ can be constructed by the embedding of $\Gamma(\mathcal{K})$ into the symmetric group $S_{\mathcal{B}}$, where $\mathcal{B} = \{B_1, \ldots, B_s\}$ and B_i is a suitable set of flags of $\mathcal{K}$ such that*

a) $\bigcup_i B_i$ *is the full set of flags of the polytope,*

b) $B_i \cap B_j = \emptyset$ *if* $i \neq j$, *and*

c) For any $i \in \{0, \ldots, d-1\}$ *and* $j \in \{1, \ldots, s\}$ *there exists* $k \in \{1, \ldots, s\}$ *such that* $\rho_i(B_j) = B_k$.

Proof

Let f be the base flag of $\mathcal{K}$, u_0 be a vertex of the graph, and H be the stabilizer of u_0 in $\Gamma(\mathcal{K})$.

Now we associate the set of flags $H(f)$ with u_0, and, for a vertex v of G such that $\psi(u_0) = v$ with $\psi \in \Gamma(\mathcal{K})$, we associate $\psi H(f)$ to v (actually, we are making $\Gamma(\mathcal{K})$ act on the left cosets of H). Lemma A.1.1 implies that $\Gamma(\mathcal{K})$ acts faithfully on these sets of flags, and that the graph constructed by them is the original graph G.

Items a) and b) are consequence of the construction with the cosets ψH, and c) is consequence of the action of $\Gamma(\mathcal{K})$ on these cosets.

□

Proposition 3.3.2 *Let H be a subgroup of the automorphism group $\Gamma(\mathcal{K})$ of a polytope $\mathcal{K}$ such that H does not contain as a subgroup any normal subgroup of $\Gamma(\mathcal{K})$ distinct from $\{\varepsilon\}$. Then, H determines a connected CPR graph for $\mathcal{K}$. Conversely, we can associate such a subgroup H to any connected CPR graph G of $\mathcal{K}$.*

Proof

To prove the first part of this proposition let $\Gamma(\mathcal{K})$ act on the left cosets of H by left multiplication. Then we construct a graph G_H whose vertices are the left cosets of H and whose edges are determined by the action of the generators of $\Gamma(\mathcal{K})$. This graph will be a CPR graph for $\mathcal{K}$ if and only if $\Gamma(\mathcal{K})$ can be recovered from its action on the left cosets of H in $\Gamma(\mathcal{K})$; in other words, if and only if $\Gamma(\mathcal{K})$ acts faithfully on the set of these left cosets. This part of the proof is implied by the purely algebraic and general Lemmas A.1.1 and A.1.2.

The converse of the proposition is a consequence of the proof of Proposition 3.3.1. It remains to prove that the subgroup H does not have as a subgroup any normal subgroup of $\Gamma(\mathcal{K})$ different from $\{\varepsilon\}$, but this follows from the fact that the stabilizer of any vertex v of G is $\phi H \phi^{-1}$, where H is the stabilizer of a fixed vertex u_0, and $\phi(u_0) = v$. If $W \leq H$ is a normal subgroup of $\Gamma(\mathcal{K})$, then W stabilizes all the vertices of G, but this implies that $W = \{\varepsilon\}$.

□

Propositions 3.2.5 and 3.3.2 imply the following result.

Corollary 3.3.3 *Any connected CPR graph of a polytope $\mathcal{K}$ is a quotient in the sense of Proposition 3.2.5 of the Cayley CPR graph of $\mathcal{K}$.*

As a consequence of proposition 3.3.2 and the subgroup lattice of S_4, we have that all the connected graphs for the tetrahedron are those in Figures 3.1 and 3.2

3.4 Mixing Operations

In this section we explain how to work with the mixing operations using CPR graphs.

Let G be a CPR graph of the polytope $\mathcal{P}$, and ξ a mixing operation given by

$$(\rho_0, \dots, \rho_{d-1}) \mapsto (\sigma_0, \dots, \sigma_{m-1}).$$

Assume that $\mathcal{P}^\xi$ is a polytope $\mathcal{Q}$. Then we can construct a CPR graph G' for $\mathcal{Q}$ in the following way. Let G' have the same vertex set as G and introduce an edge of color j between the vertices u and v whenever $\sigma_j(u) = v$, $j = 0, \dots, m-1$ (recall the action of $\Gamma(\mathcal{P})$ on the vertices of G).

Example 3.4.1 *Given a CPR graph $G_{0,\dots,d-1}$ of the polytope $\mathcal{P}$, a CPR graph for its dual $\mathcal{P}^*$ is $G_{d-1,\dots,0}$ and is obtained from the duality mixing operation.*

Example 3.4.2 *To construct a graph of the petrial of a polyhedron P with CPR graph $G_{0,1,2}$, we only have to modify some components of $G_{0,2}$ in the following way:*

- *Those edges with color 2 will have now double edges of colors 0 and 2.*
- *The double edges will change into a single edge of color 2.*
- *The squares will remain being squares, but their edges of color 0 will be replaced by the diagonals of the old squares and again be colored 0.*

In Figure 3.11 we show an example of how the dual and petrie operation work in the CPR graphs.

We recall that the k-facetting operation is defined by

$$(\rho_0, \rho_1, \rho_2) \mapsto (\rho_0, (\rho_1\rho_2)^k\rho_1, \rho_2).$$

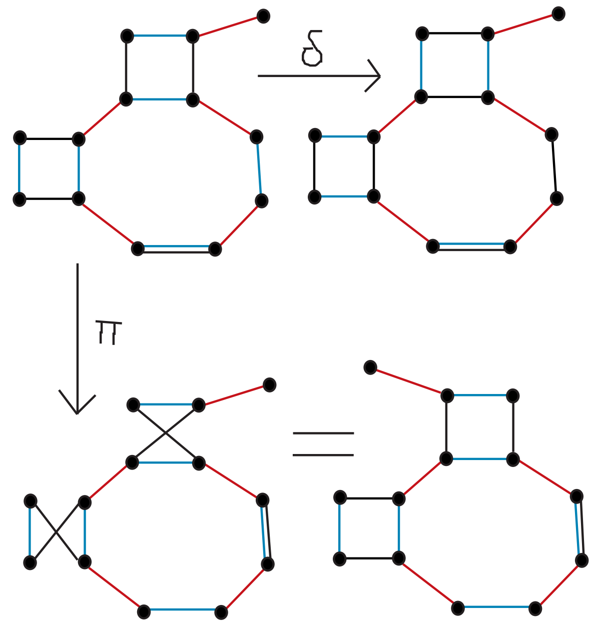


Figure 3.11: Dual and petrie operation

Therefore to construct a CPR graph for the k-facetting operation $\phi_k(\mathcal{P})$ from a CPR graph G of the polyhedron $\mathcal{P}$ we only have to change the edges of color 1 of G to suitable ones. The following example illustrates the case $k = 2$.

Example 3.4.3 *The 2-facetting operation applied to the icosahedron gives the great dodecahedron. The first graph in Figure 3.12 is the vertex CPR graph of the icosahedron, while the remaining graphs are isomorphic to the vertex (face) CPR graph of the great dodecahedron. They were obtained from the first one by deleting the red edges and adding a red edge between two vertices u and v if $\rho_1\rho_2\rho_1(u) = v$.*

3.5 Finding the automorphism group

In general it is hard to determine $\Gamma(\mathcal{K})$ (or even $|\Gamma(\mathcal{K})|$) given a CPR graph $\mathcal{K}$ of a polytope $\mathcal{K}$. However there are cases where something can be said about it. In this section we discuss some ways to determine subgroups of the

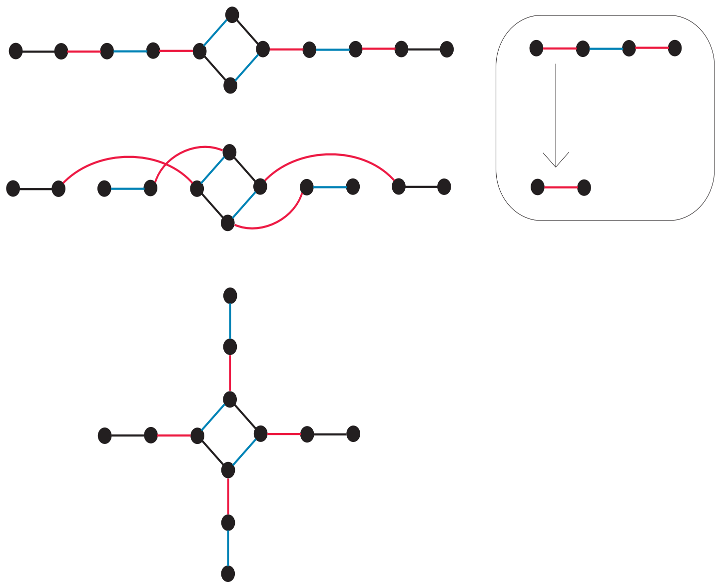


Figure 3.12: Facetting operation

automorphism group, or the automorphism group itself, of some polytopes given CPR graphs of them.

Lemmas A.2.1 and A.2.3 describe when an alternating or symmetric group is a subgroup of the automorphism group. They are used in CPR graphs where we are able to get a transposition or a 3-cycle from the generators of the automorphism group.

Lemma 3.2.4 gives a (normal) subgroup N of the automorphism group of a polytope $\mathcal{K}$ given a CPR graph with nontrivial automorphism group as labeled graph. This is a useful criterion to determine that the automorphism group of a polytope with such a graph G is isomorphic to neither the alternating group nor the symmetric group on the vertex set of G. Sometimes we are able to find a subgroup $M \leq \Gamma(\mathcal{K})$ such that $\Gamma(\mathcal{K}) = MN$ (or NM). In this cases $\Gamma(\mathcal{K}) \cong M \ltimes N$, with the action of M on N determined by the permutation group of the orbits O_v induced by M. In chapter 5 an example of this appears.

The next proposition is also used several times in the rest of this work.

Proposition 3.5.1 *Let G be a CPR graph of a polytope $\mathcal{K}$ such that $V(G)$ can be divided in two sets U and V satisfying that*

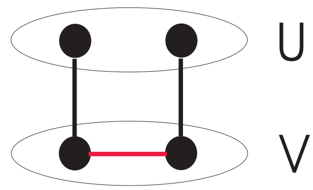


Figure 3.13: CPR graph of the square

- *The edges of one color i form a perfect matching between U and V, and*
- *the edges of any color $j \neq i$ join either two vertices on U or two vertices on V.*

Then $\Gamma(\mathcal{K}) \cong \langle\rho_i\rangle \ltimes \Lambda$ where Λ is a subgroup of $S_U \times S_V$, and ρ_i acts on Λ by interchanging the first and second entries of the elements.

Proof
Note that if $\phi \in \Gamma(\mathcal{K})$ maps a vertex in U into a vertex in U, then ϕ preserves the sets U and V; while if ϕ maps a vertex in U into a vertex in V, then it interchanges both sets. Let $\Lambda \leq \Gamma(\mathcal{K})$ be the subgroup preserving the sets U and V (or abusing notation, let $\Lambda = \Gamma(\mathcal{K}) \cap (S_U \times S_V)$). Clearly Λ has index 2 in $\Gamma(\mathcal{K})$, therefore $\Lambda \lhd \Gamma(\mathcal{K})$. Finally, since $\rho_i \notin \Lambda$ it follows that $\Gamma(\mathcal{K}) \cong \langle\rho_i\rangle \ltimes \Lambda$. Since ρ_i interchanges the sets U and V, ρ_i acts on Λ in the expected way.

□

Example 3.5.2 *Figure 3.13 shows the vertex CPR graph of the square. This graph satisfies the conditions of Proposition 3.5.1 with $i = 0$; and ρ_1 induces the symmetric group on V, while $\rho_0\rho_1\rho_0$ induces the symmetric group on U. Hence, the automorphism group of the square is isomorphic to*

$$S_2 \ltimes (S_2 \times S_2) \cong \langle\rho_0\rangle \ltimes (\langle\rho_1\rangle \times \langle\rho_0\rho_1\rho_0\rangle).$$

Chapter 4

CPR Graphs of Polyhedra

Given a natural number n, the only connected CPR graphs of the polygon $\{n\}$ are its Cayley graph (an alternating $2n$-gon) and an alternating path of length $n-1$ (see proposition 3.3.2). The latter is the vertex CPR graph if one of the vertices of degree 1 of the graph has an edge of label 1 incident to it, and is the edge CPR graph if one of the vertices of degree 1 of the graph has an edge of label 0 incident to it. Note that if n is odd the vertex CPR graph and the edge CPR graph are the same graph.

On the other hand, any group obtained from a nontrivial 2-labeled graph such that the edges of the two labels form different matchings, satisfies the intersection property; hence such a graph is a CPR graph of a polygon.

It is more difficult to determine all the CPR graphs of any polyhedron because they have more complicated automorphism groups. It is also harder to determine whether a 3-labeled graph is a CPR graph of a polyhedron. However, those graphs where we are able to decide that the groups associated satisfy the intersection property will prove useful.

In Section 4.1 we give some criteria to determine that a 3-labeled graph is a CPR graph. In Section 4.2 we give a complete list of polyhedra with automorphism group S_7. In Section 4.3 we show a method to construct polyhedra with automorphism group A_n for $n \geq 21$. Finally, in Section 4.4 we give some infinite families of polyhedra with preassigned Schläfli type $\{p, q\}$ for some p and q.

We recall that while working with CPR graphs of polyhedra the color black will be identified with label 0, color red with label 1, and color blue with label 2.

4.1 Intersection Property

In Chapter 3 we explained that a CPR graph of a polyhedron is a labeled (multi)graph G with label set $\{0, 1, 2\}$ such that the edges of labels $i = 0, 1, 2$ induce three different matchings of the vertices of G. In this section we give some sufficient conditions for such a graph to be a CPR graph of a polyhedron. First we introduce the following definition.

Definition 4.1.1 *A d-labeled multigraph G satisfying the conditions of Proposition 3.2.1 with the properties that the set of edges of each label $i \in \{0, 1, \ldots, d-1\}$ forms a matching M_i on G, and that M_i represents a different pairing of the vertices of G from M_j for $i \neq j$, will be called a* proper d-labeled *graph.*

Remark 4.1.2 *Every CPR graph is a proper d-labeled graph.*

A proper 3-labeled graph G is a CPR graph of a polyhedron if the group generated by the involutions ρ_0, ρ_1 and ρ_2 determined by the edges of labels $0, 1$ and 2 is a string C-group with respect to ρ_0, ρ_1 and ρ_2. The only remaining condition for G to be a CPR graph of a polyhedron is the intersection property.

By Proposition 2.1.4, the only necessary equality for a polyhedron $\mathcal{K}$ with automorphism group $\Gamma(\mathcal{K}) = \langle \rho_0, \rho_1, \rho_2 \rangle$ to satisfy the intersection property is

$$\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle = \langle \rho_1 \rangle.$$

Let $\Gamma(\mathcal{K}) = \langle \rho_0, \rho_1, \rho_2 \rangle$ be a polyhedron, let $\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$ be a $\rho_0\rho_1$-even element of $\Gamma(\mathcal{K})$, and let C be a connected component of $G_{0,1}$, where $G = G_{0,1,2}$ is a CPR graph of $\mathcal{K}$. If ϕ does not act like ε in C, then ϕ acts as a power of the rotation on the vertices of C described by the polygonal action of $\rho_0\rho_1$ on C, so it is a product of disjoint cycles of the same length d. Moreover, if $d \geq 3$, then ϕ is also $\rho_1\rho_2$-even ($\rho_i\rho_j$-odd permutations are involutions). This leads us to the first criteria to determine that a 3-labeled graph is a CPR graph.

Theorem 4.1.3 *Let $G_{0,1,2}$ be a connected, proper 3-labeled graph. If C_1, C_2 are two connected components of $G_{0,1}$ (or $G_{1,2}$) with n and m vertices, $n, m \geq 2$, such that $(n, m) = 1$, then $G_{0,1,2}$ is a CPR graph.*

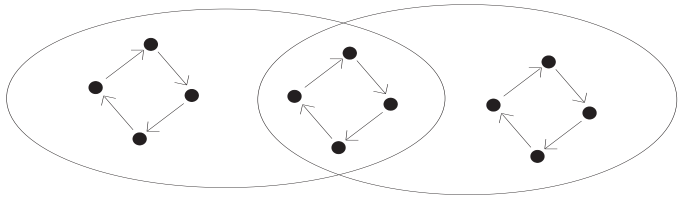


Figure 4.1:

Proof

It only remains to prove the intersection property.

Suppose to the contrary that $\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$ and $\phi \notin \langle \rho_1 \rangle$. If ϕ is $\rho_0\rho_1$-odd then $\phi\rho_1$ will be $\rho_0\rho_1$-even, $\phi\rho_1 \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$, and $\phi\rho_1 \notin \langle \rho_1 \rangle$. Thus we can suppose that ϕ is $\rho_0\rho_1$-even.

Suppose that n is odd (otherwise m is odd). Then $n \geq 3$ and ϕ forms cycles of length d in a connected component C with n vertices of $G_{0,1}$, leaving no vertex fixed in this component. Therefore d is a divisor of n, and $d \geq 3$, so ϕ is also $\rho_1\rho_2$-even. This implies that ϕ forms cycles of length d in the components of $G_{1,2}$ intersecting C moving all the vertices of such components (see Figure 4.1). It follows from the connectedness of G that ϕ is a product of disjoint cycles of length d without fixed points, so d is a divisor of both, m and n, but that is a contradiction to the hypothesis.

□

The proposition above is a powerful criterion to determine that many connected, proper 3-labeled graphs are CPR graphs, for example, that of Figure 3.9. We will give another two useful criteria in Theorems 4.1.6 and 4.1.7, but first we prove some lemmas concerning the action of automorphisms in $\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$ on the vertices of a given CPR graph.

Lemma 4.1.4 *Let $\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$ such that, $\phi_{|C} = (\rho_1)_{|C}$ and $\phi_{|D} \neq (\rho_1)_{|D}$ for C and D connected components of $G_{0,1}$ and $G_{1,2}$ respectively, with at least one edge of label 1 in their intersection. Then, every vertex of C is incident to an edge of label 2.*

Proof

If $C \cap D$ has no vertex incident to edges of both labels, 1 and 2, then D is only the edge of label 1 with its two vertices, and $\phi_{|D} = (\rho_1)_{|D}$, so we can

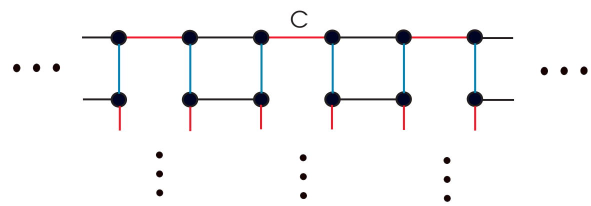


Figure 4.2:

assume that $C \cap D$ has at least one vertex incident to edges of both labels, 1 and 2.

Let v_0 be a vertex of $C \cap D$ incident to edges of labels 1 and 2. We claim that ϕ is not $\rho_1\rho_2$-odd because the only reflection interchanging v_0 and $\phi(v_0) = \rho_1(v_0)$ in the polygonal action of $\langle \rho_1, \rho_2 \rangle$ on D, is ρ_1 (see Figures 3.7 and 3.8). Therefore ϕ must be $\rho_1\rho_2$-even, but the only $\rho_1\rho_2$-even permutation that interchanges two vertices that are also interchanged by ρ_1 is $(\rho_1\rho_2)^{(n/2)}$, with n the number of vertices of D, n even. In this case, $\phi_{|D} = (\rho_1\rho_2)^{(n/2)}$, D is a path of odd length $n-1$, and the edge of label 1 incident to v_0, say u_0v_0, is the central edge of it. This implies that u_0 is also incident to an edge of label 2. Because ϕ is $\rho_1\rho_2$-even, it cannot act as ρ_1 in any connected component of $G_{1,2}$ with at least three vertices (see Lemma 3.2.11), and any edge of label 1 in C is either adjacent to 2 edges of label 2, or it is adjacent to none of them.

If xw is an edge of label 0 and x is incident to an edge of label 2, then, by Proposition 3.2.1, w is also incident to an edge of label 2.

The last two paragraphs and the existence of v_0 imply that every vertex of the component C is incident to an edge of label 2 (see Figure 4.2).

□

Note that if C and D are connected components of $G_{0,1}$ and $G_{1,2}$ respectively and their intersection does not contain an edge of label 1, then C and D are paths and $C \cap D$ contains at most two vertices (the first one and the last one of each path). Moreover, if it has two vertices u and v, then C and D are paths of odd length. The following lemma generalizes Lemma 4.1.4.

Lemma 4.1.5 *Let $\phi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$ such that, for C and D connected components of $G_{0,1}$ and $G_{1,2}$ respectively with $C \cap D \neq \emptyset$, $\phi_{|C} = (\rho_1)_{|C}$ and $\phi_{|D} \neq (\rho_1)_{|D}$. Then, every vertex of C is incident to an edge of label*

2. *Moreover, if* C *and* D *have at least* 3 *vertices, then* ϕ *is* $\rho_0\rho_1$*-odd and* $\rho_1\rho_2$*-even.*

Proof

It was proved in Lemma 4.1.4 that if $C \cap D$ contains an edge of label 1 the statement is satisfied. The remainder case is when $C \cap D$ is either a single vertex or a couple of isolated vertices.

The lemma holds trivially if C is a single vertex, and Proposition 3.2.1 shows that the lemma also holds when C is a single edge of label 0 with its two vertices, or C consists of two isolated vertices. Note that D has at least two vertices, because $\phi_{|D} \neq (\rho_1)_{|D}$, but, if D has exactly two vertices u and v, then D consists only of u, v and a single edge of label 2 between them (here we use that there is no edge of label 1 in $C \cap D$). Then $\phi(u) = v$, otherwise $\phi_{|D}$ would be equal to $(\rho_1)_{|D}$. Since $\phi_{|C} = (\rho_1)_{|C}$, then there is an edge of label 1 between u and v, but that contradicts our hypothesis.

We now assume that both, C and D have at least one edge of label 1 and at least 3 vertices.

Let w_0 be a vertex in $C \cap D$. Since there is no edge of color 1 in $C \cap D$, it follows that w_0 is a vertex of an alternating square of colors 0 and 2 in G with no edge of label 1 between two of its vertices. Let w_1 and w_2 be the vertices adjacent to w_0 of C and D respectively, and let w_3 be the other vertex of the square (see Figure 4.3). Let D' be the connected component containing w_3 of $G_{1,2}$. We know that ϕ is $\rho_0\rho_1$-odd because it fixes w_0 and moves w_1 (see remark 3.2.10). This also says that $\phi_{|D}$ is either Id_D or $(\rho_1)_{|D}$ because they are the only two elements of $\langle \rho_1, \rho_2 \rangle$ whose polygonal action on D fix w_0 (see Remark 3.2.9). So we can assume that $\phi_{|D} = Id_{|D}$ and ϕ is $\rho_1\rho_2$-even (this is a consequence of the fact that D has at least 3 vertices). This implies that ϕ does not act as ρ_1 on D' (note that w_1, w_3 and $\rho_1(w_1)$ are three different vertices of D'). Since the labeled-1 edge between w_1 and $\rho_1(w_1)$ is in $C \cap D'$, we apply Lemma 4.1.4 to C and D' to get the desired result.

□

Theorem 4.1.6 *Let* $G = G_{0,1,2}$ *be a connected, proper* 3*-labeled graph. If* G *has a vertex* v_0 *such that no edge of labels* $1, 2$ *(or dually* $0, 1$*) are incident to it (in other words,* $\langle \rho_1, \rho_2 \rangle$ *has a fixed point* v_0 *in* G*), then* $G_{0,1,2}$ *is a CPR graph.*

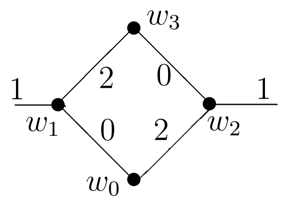


Figure 4.3:

Proof

Let C_0 be the connected component on $G_{0,1}$ of the vertex v_0, and let $\phi \in \langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle$. We can assume that ϕ is $\rho_0\rho_1$-odd, otherwise we would take $\rho_1\phi$. Note that $\phi(v_0) = v_0$ because $\{v_0\}$ is the only vertex of the connected component of v_0 in $G_{1,2}$. Then Remark 3.2.9 implies that $\phi_{|C_0} = (\rho_1)_{|C_0}$.

Suppose to the contrary, that $\phi \neq \rho_1$. Then, there exist adjacent vertices u and w such that $\phi(u) = \rho_1(u)$ but $\phi(w) \neq \rho_1(w)$. The vertices u and w are in the same connected component D_0 of either $G_{0,1}$ or $G_{1,2}$, but since ϕ is $\rho_0\rho_1$-odd and $\phi(u) = \rho_1(u)$, it follows from Remark 3.2.9 that D_0 is a connected component of $G_{1,2}$ and that ϕ is $\rho_1\rho_2$-even.

Lemma 4.1.5 shows that ϕ acts as ρ_1 on every connected component D of $G_{1,2}$ such that $C_0 \cap D \neq \emptyset$; here we are using that v_0 is not contained in an edge of label 2. While ϕ is $\rho_1\rho_2$-even and $\phi_{|D} = (\rho_1)_{|D}$, then D has at most two vertices (see Lemma 3.2.11). Then G has no edge of color 2, but that is not possible. So we have that ϕ acts like ρ_1 in every connected component of $G_{0,1}$ and $G_{1,2}$.

Hence, the only $\rho_0\rho_1$-odd element of $\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle$ is ρ_1.

□

Theorem 4.1.7 *Let $G = G_{0,1,2}$ be a connected, proper 3-labeled graph. If G has an edge v_0u_0 of label 1, such that no edge of label 2 (or dually, 0) is incident to either v_0 or u_0; and v_0u_0 is not the central edge of an odd-path connected component in $G_{0,1}$ (resp. $G_{1,2}$) then $G_{0,1,2}$ is a CPR graph.*

Proof

Let C_0 be the connected component of v_0u_0 in $G_{0,1}$, and let $\phi \in \langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle$. We can assume that ϕ is $\rho_0\rho_1$-odd, otherwise we would take $\rho_1\phi$. Note that the vertices of the connected component of u_0 in $G_{1,2}$ are only u_0 and v_0. We know then that $\phi_{|C_0} = (\rho_1)_{|C_0}$ because the polygonal action of ρ_1

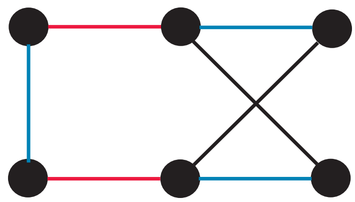


Figure 4.4: Intersection property fails

on C_0 is the only one fixing $\{v_0, u_0\}$ as a set once we ruled out the possibility of C_0 being an odd-path connected component in $G_{0,1}$ having u_0v_0 as its central edge.

Again, lemma 4.1.5 shows that ϕ acts as ρ_1 on every connected component of $G_{1,2}$ intersecting C_0. We suppose that $\phi \neq \rho_1$ and continue the proof in the same way as in Theorem 4.1.6.

□

Remark 4.1.8 *The graph of Figure 4.4 shows that theorem 4.1.7 fails if we allow v_0u_0 to be the central edge of an odd-path connected component in $G_{0,1}$ or, as in this case, of $G_{1,2}$. Note that $\rho_0 \in \langle \rho_1, \rho_2 \rangle$.*

4.2 Polyhedra with automorphism group S_7

Michael Hartley built the atlas [12] of regular polyopes with small automorphism groups. This atlas was constructed by means of an exhaustive computer search and considers the groups of order n that are automorphism groups of polytopes for almost every $n \leq 2000$. We describe next the polytopes with symmetric automorphism group that appear in this atlas.

The triangle is the only polytope with automorphism group isomorphic to S_3. The tetrahedron, hemicube and hemioctahedron are the only polytopes with automorphism group isomorphic to S_4. The 4-simplex has automorphism group isomorphic to S_5, and the remaining seven polytopes with this automorphism group are polyhedra. Finally, there are eleven polytopes with that automorphism group isomorphic to S_6; one of them is the 5-simplex, there are seven 4-polytopes and the remaining three are polyhedra.

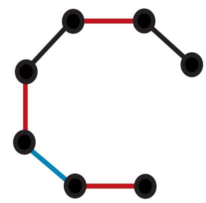


Figure 4.5: $\{10, 4\}$

In this section we will find all the polyhedra with automorphism group isomorphic to S_7. In order to do that, we construct proper 3-labeled CPR graphs on seven vertices such that

a) The graph is connected (otherwise it would not generate the whole symmetric group).

b) The group generated by the involutions satisfy the intersection property.

c) The group generated by the involutions is S_7.

In most of the cases, the intersection property can be verified easily with the criteria of the last section; the ones where they do not apply can be verified separately looking at the subgroups explicitly.

We will list the graphs by families closed under dual and petrie operations, hence each family has 1, 3 or 6 polyhedra.

All the 3-labeled graphs that satisfy conditions a) and b) will also satisfy c). To see this it is enough to verify it in just one of the graphs of each of the families, given the fact that the dual and petrie operations are invertible.

In order to determine that the automorphism group generated is isomorphic to S_7, we can find a transposition of two vertices of the graph and a 6-cycle not including one of the vertices of the transposition and apply Lemma A.2.1, or find a 3-cycle and a 5-cycle not including two of the vertices of the 3-cycle and apply Lemma A.2.3; this is because at least one of the involutions ρ_0, ρ_1 and ρ_2 will be an odd permutation. Other polyhedra need a little more work. For example, let us check that the polytope of Schläfli type $\{10, 4\}$ in Figure 4.7 C has automorphism group isomorphic to S_7 (see Figure 4.5).

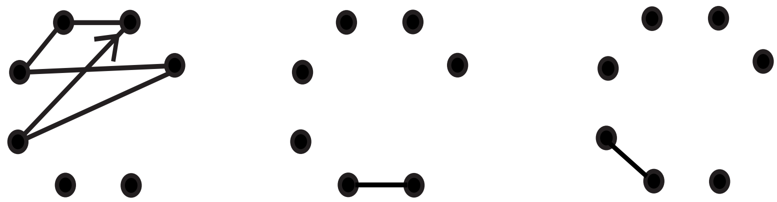


Figure 4.6: $(\rho_0\rho_1)^2, (\rho_0\rho_1)^5$ and ρ_2

First we see that $(\rho_0\rho_1)^2$ induces a cycle of length 5 in the vertices of the graph, leaving the other two fixed, while $(\rho_0\rho_1)^5$ leaves those 5 vertices fixed and interchanges the other two (see Figure 4.6).

With $(\rho_0\rho_1)^2$ and ρ_2 we generate a group isomorphic to S_6 by Lemma A.2.1, and this group with $(\rho_0\rho_1)^5$ generates the desired group isomorphic to S_7.

Now we give the list of the graphs of the polyhedra with automorphism group S_7. In Figure 4.7 we list the polyhedra with acyclic CPR graphs (the only cycles allowed are the double edges). It is easy to see that those 24 graphs include all the possibilities of polyhedra without squares or other cycles. It is important to remark that the polyhedron of Schläfli type $\{6, 6\}$ of Figure 4.7 A (left, upper corner) is not self dual because in this polyhedron $(\rho_0\rho_1\rho_0\rho_1\rho_2)^6$ is not the identity, but in its dual it is.

In Figures 4.8 and 4.9, we list the rest of the graphs of the polyhedra with automorphism group S_7. We can see that they are all the possible graphs noting that a graph with at least a cycle that generates S_7 needs a square of labels 0 and 2; then we have to look at all the possibilities of graphs with $1, 2$ and 3 edges of label 1 incident to this square and to an outside vertex. Note that some of the graphs of Figure 4.9 belong to self petrie or self dual polyhedra, while the last graph belongs to a self dual and self petrie polyhedron of Schläfli type $\{12, 12\}$.

Some of these families are related by facetting operations. We can see here some examples of the 2-facetting operation acting on a polyhedron of Schläfli type $\{p, 2q\}$ giving as a result another polyhedron with the same automorphism group. For instance, the 2-facetting operation applied to the polyhedron of Schläfli type $\{6, 12\}$ at the upper right corner of Figure 4.8 H, gives rise to the polyhedron with Schläfli type $\{10, 6\}$ at the left of Figure 4.9 K (recall that in order to see the 2-hole of a polyhedron given its graph

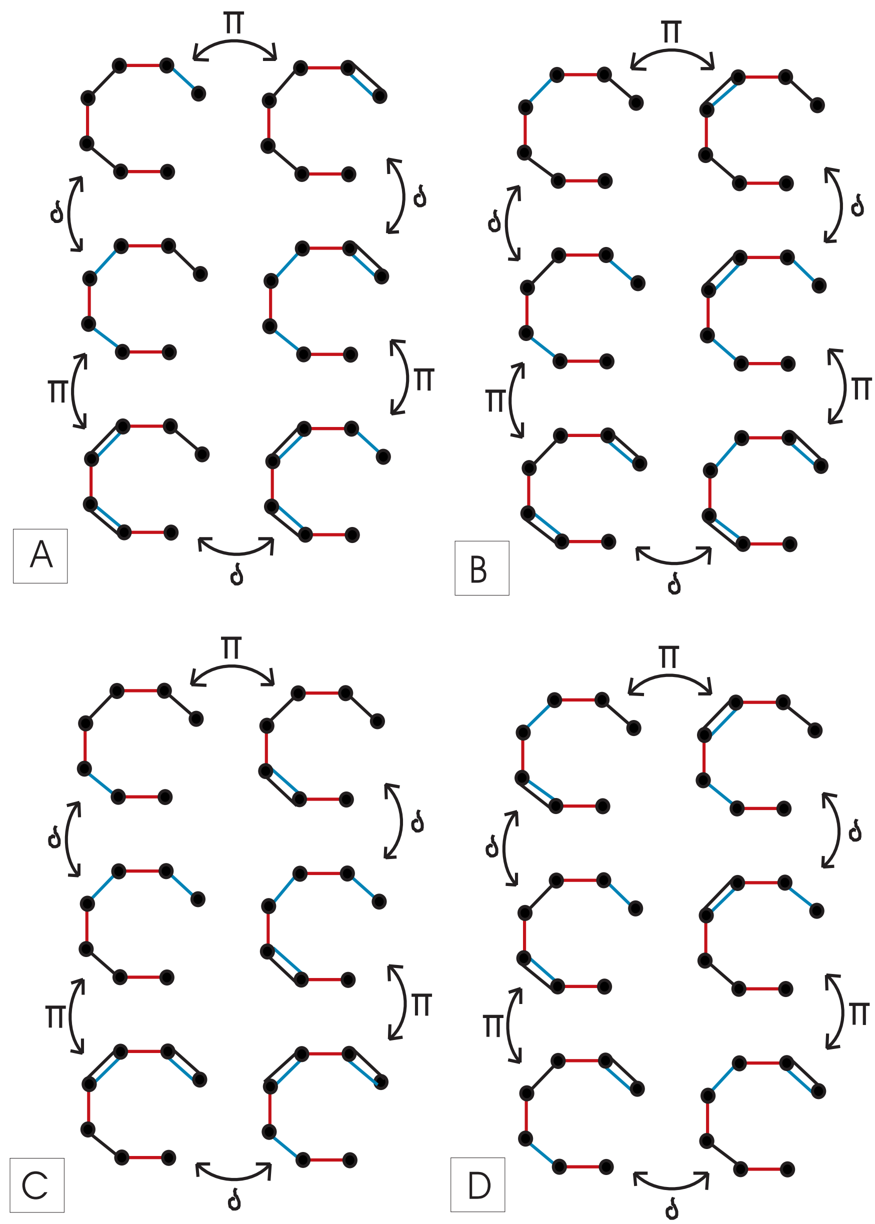


Figure 4.7: Acyclic graphs of S_7

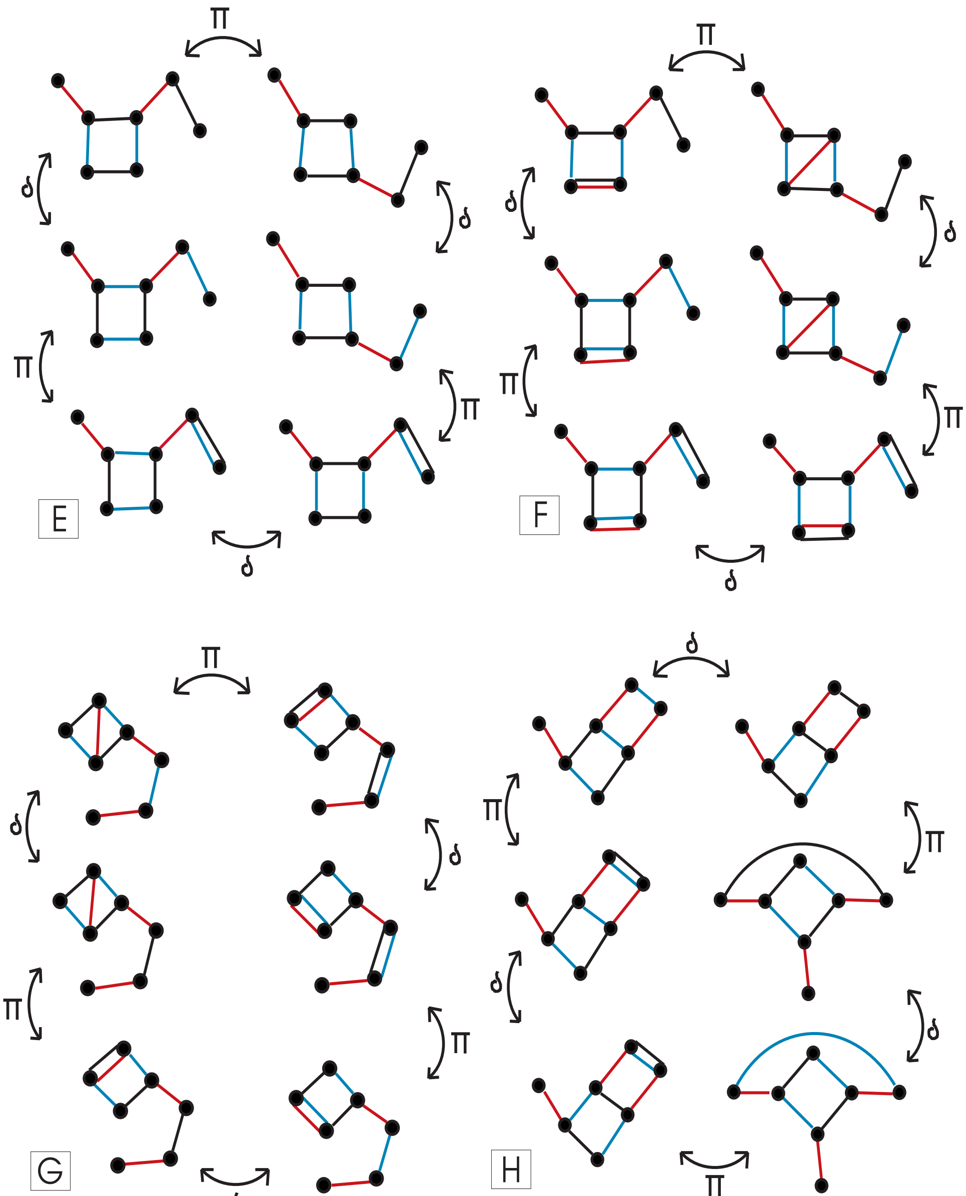


Figure 4.8: Remaining graphs of S_7

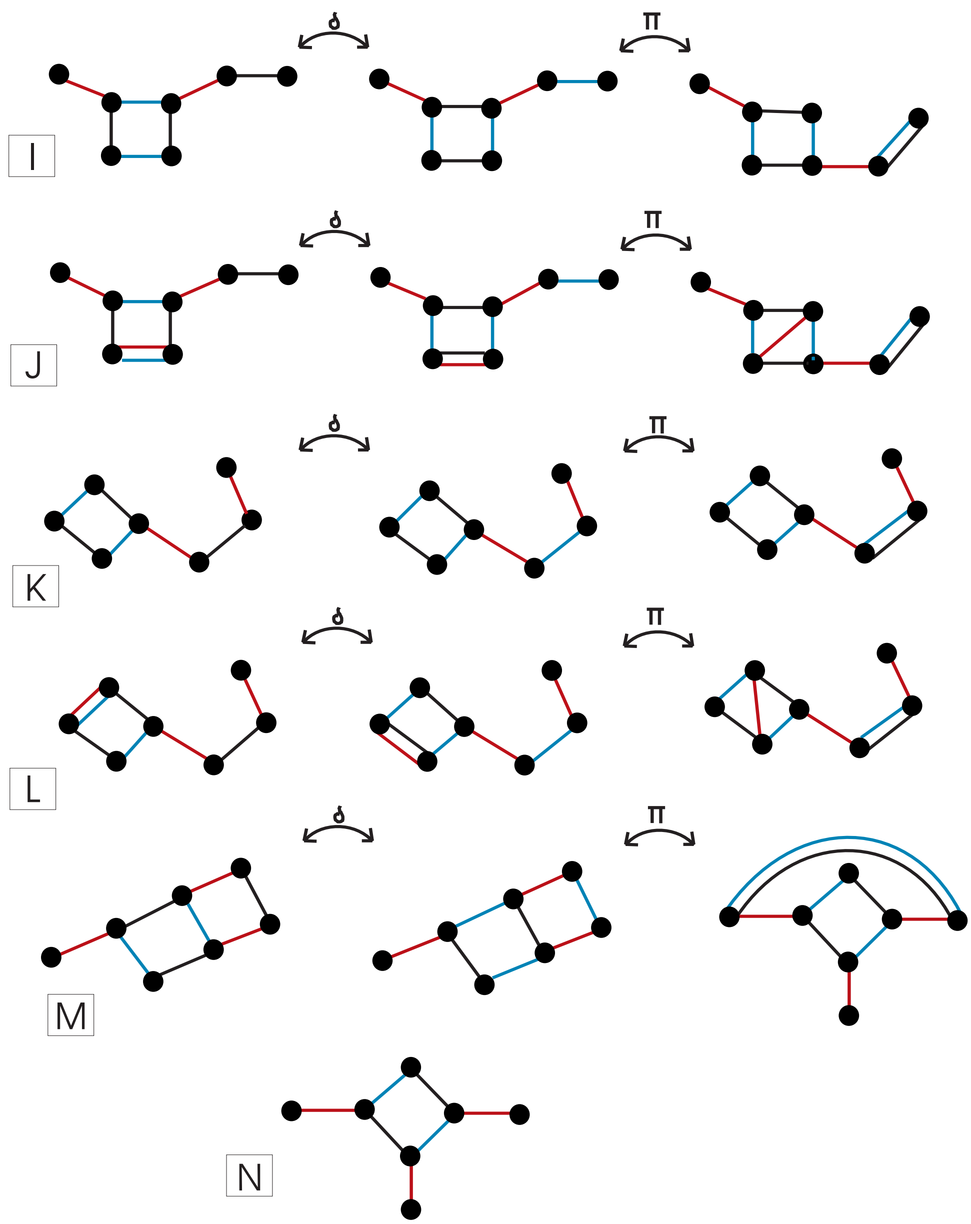


Figure 4.9: Remaining graphs of S_7

G we have to change ρ_1 to $\rho_1\rho_2\rho_1$).

In Table 4.1 we give the Schläfli symbols of the polyhedra with automorphism group isomorphic to S_7. In each family the first polyhedron will be the upper left one in the correspondent figure. In [16] D. Leemans displays the same table up to duality, but these results are based on a computer search, not the CPR graph technique.

Group	**First**	δ	$\pi\delta$	$\delta\pi\delta$	$\delta\pi$	π
A	{6, 6}	{6, 6}	{7, 6}	{6, 7}	{6, 7}	{7, 6}
B	{12, 4}	{4, 12}	{7, 12}	{12, 7}	{4, 7}	{7, 4}
C	{10, 4}	{4, 10}	{7, 10}	{10, 7}	{4, 7}	{7, 4}
D	{12, 6}	{6, 12}	{10, 12}	{12, 10}	{6, 10}	{10, 6}
E	{10, 3}	{3, 10}	{12, 10}	{10, 12}	{3, 12}	{12, 3}
F	{5, 6}	{6, 5}	{7, 5}	{5, 7}	{6, 7}	{7, 6}
G	{10, 7}	{7, 10}	{5, 10}	{10, 5}	{7, 5}	{5, 7}
H	{12, 6}	{6, 12}	{7, 12}	{12, 7}	{6, 7}	{7, 6}
I	{12, 4}	{4, 12}	{12, 12}			
J	{7, 4}	{4, 7}	{7, 7}			
K	{10, 6}	{6, 10}	{10, 10}			
L	{7, 6}	{6, 7}	{7, 7}			
M	{7, 12}	{12, 7}	{7, 7}			
N	{12, 12}					

Table 4.1: Polyhedra with automorphism group A_7

4.3 Polyhedra with automorphism group A_n

In May 2005, in the Conference "Convex and Abstract Polytopes" in Banff, Canada, M. Hartley proposed in the open problems session the following question. For which alternating groups A_n does there exist an abstract regular polytope with automorphism group isomorphic to A_n? This question can be found in [30]. Hartley already knew an affirmative answer for $n = 5, 9$ and a negative answer for $n = 3, 4, 6, 7$ and 8 obtained with the aid of the computer (see [12]). The question can be rephrased in the following way. What alternating groups are generated by three involutions (ρ_0, ρ_1 and ρ_2) two of which commute ($(\rho_0\rho_2)^2 = \varepsilon$), and such that the intersection property is satisfied ($\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle = \langle\rho_1\rangle$)? This question already appears in [15], problem 7.30 for all the finite simple groups, without taking on account the intersection property. In this section we will construct (CPR graphs of) polyhedra with automorphism groups isomorphic to A_n for $n \geq 9$, giving an affirmative answer for the remaining alternating groups.

It is easy to see that A_3 and A_4 cannot be the automorphism group of any polyhedra because they do not have enough involutions, while A_5 is the automorphism group of the hemi-dodecahedron, the hemi-icosahedron and the hemi-great dodecahedron.

Now we show that A_6, A_7 and A_8 cannot be the automorphism group of any polyhedra.

Proposition 4.3.1 *No polyhedron has automorphism group isomorphic to A_6.*

Proof

Let $G = G_{0,1,2}$ be a connected CPR graph with 6 vertices of a polyhedron $\mathcal{K}$. Suppose to the contrary that $\Gamma(\mathcal{K}) \cong A_6$, then G has exactly two edges of each label 0, 1 and 2 in order to have only even permutations. By Proposition 3.2.1, $G_{0,2}$ is either an alternating square together with two isolated vertices, or it contains one isolated edge of label 0, one of label 2 and an isolated double edge (see Figure 4.10).

If $G_{0,2}$ is an alternating square together with two isolated vertices, then the two edges of label 1 join the square with each of these vertices. We have three cases (actually it is only one modulo petrie and dual operations). If the vertices of the square having an edge of label 1 incident to them are adjacent by an edge of label 0, then we have the edge CPR graph of a hemicube. If

Figure 4.10:

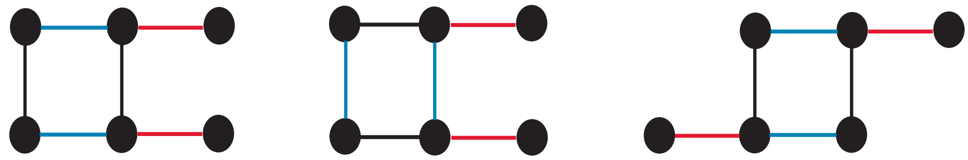


Figure 4.11: CPR graphs of hemicube, hemioctahedron and tetrahedron

these vertices are joined by an edge of label 2, then we have the edge CPR graph of a hemioctahedron. Finally, if these two vertices are opposite in the square, we have the edge CPR graph of the tetrahedron (see 4.11). These three polyhedra have automorphism group S_4.

If $G_{0,2}$ contains one isolated edge of each label 0 and 2 and one double edge of both labels, then the two edges of label 1 join these three connected components forming a path with a double edge. Now we have three cases (again there is only one modulo dual and petrie operation). If the central edge of the path is the one of label 0, then we have the face CPR graph of the hemidodecahedron. If the central edge of the path is the one of label 2, then we have the vertex CPR graph of the hemiicosahedron. Finally, if the central edge of the path is the double edge, we have the vertex (or face) CPR graph of the hemi-great dodecahedron (see Figure 4.12). These three polyhedra have automorphism group isomorphic to A_5.

Hence, no polyhedron has automorphism group A_6.

□

Proposition 4.3.2 *No polyhedron has automorphism group isomorphic to A_7.*

Proof

Let $G = G_{0,1,2}$ be a connected CPR graph with 7 vertices of a polyhedron $\mathcal{K}$.

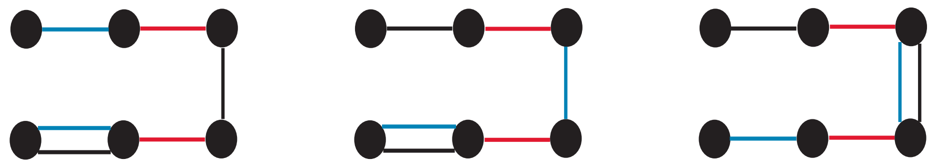


Figure 4.12: CPR graphs of hemidodecahedron, hemiicosahedron and hemi-great dodecahedron

Suppose to the contrary that $\Gamma(\mathcal{K}) \cong A_7$, then G has exactly two edges of each label 0, 1 and 2. Again, by Proposition 3.2.1, $G_{0,2}$ is either an alternating square together with three isolated vertices; or it contains one isolated vertex, one isolated edge of label 0, one of label 2 and an isolated double edge. In any of these two cases we have 4 connected components of $G_{0,2}$ that have to be connected by only two edges of label 1, but that is impossible.

□

Proposition 4.3.3 *No polyhedron has automorphism group isomorphic to* A_8.

The proof of this last proposition involves the cases when the CPR graph has 2 or 4 edges of each label. We do not include the proof because of its length, but it follows from the same case by case arguments.

For $n \geq 9$ we can find polyhedra with automorphism group A_n. Let us take a look first at the graph G_k consisting in k squares of labels 0 and 2 joined by edges of label 1 as Figure 4.13 shows.

We know that these graphs are CPR graphs because of Proposition 4.1.3, and we can determine that A_n is a subset of the automorphism groups of these polyhedra in the following way.

Note that $(G_k)_{0,1}$ and $(G_k)_{1,2}$ have only one connected component C_0 with three vertices. The remaining connected components have 4 or 2 vertices. From the polygonal action of $\rho_0\rho_1$ we know that $\phi = (\rho_0\rho_1)^4$ is a cycle $\mathcal{C}$ that includes only the three vertices of C_0.

By a similar reasoning we can see that each of the automorphisms $\psi_1 = (\rho_0\rho_1)^3$ and $\psi_2 = (\rho_1\rho_2)^3$ acts on the vertex set of G_k as a product of disjoint cycles of length 4 and a transposition. It follows that the group generated by ψ_1 and ψ_2 is transitive on all the vertices of the graph G_k with the exception of two vertices of $\mathcal{C}$.

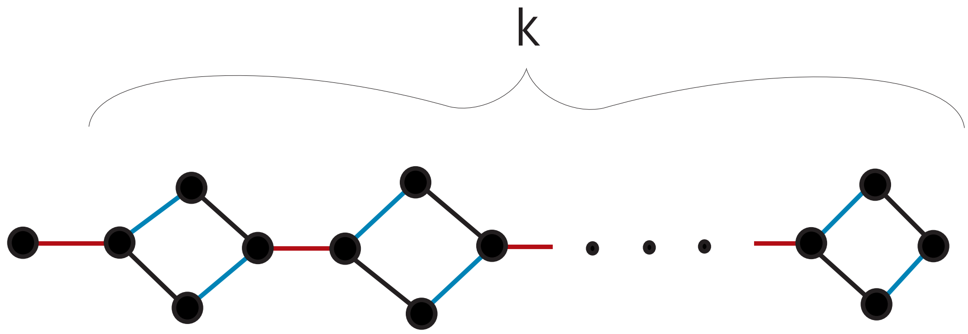


Figure 4.13: The graph G_k

By Lemma A.2.3 we have that A_n is included in the automorphism group of the polyhedra of these graphs. Since G_k has always an even number of edges of labels 0 and 2, to determine whether the automorphism group is A_n or S_n it suffices to see whether the number of edges of label 1 is even or odd, respectively.

Therefore we can construct a polyhedron with automorphism group A_n for $n = 8k+1$ and S_n for $n = 8k+5$ with k an integer. Now we modify these graphs to obtain polyhedra with automorphism groups A_n for the rest of the n's. There are many ways to do this, but it will be enough with a couple of them.

We can add an edge of label 1 as a diagonal of any square of the graph with the exception of the first one and the last one (see Figure 4.14). In this situation, $\phi = (\rho_0\rho_1)^8$ will be the cycle $\mathcal{C}$ of length 3 while the subgroup generated by $\psi_1 = (\rho_0\rho_1)^3$ and $\psi_2 = (\rho_1\rho_2)^3$ is transitive on the rest of the vertices along with one of $\mathcal{C}$. This helps to construct graphs with automorphism groups A_n from those with automorphism groups S_n. Note that two diagonals of label 1 in consecutive squares would not work in the same way because, in this case, no power of $\rho_0\rho_1$ will be a cycle of length three.

In order to add more vertices to the graphs of the type of Figure 4.13, we can attach them to any square except the one at the beginning and the one at the end, with an edge of label 1 (see Figure 4.15). In this situation, $\phi = (\rho_0\rho_1)^{20}$ gives us the cycle $\mathcal{C}$ of length 3, while ψ_1, ψ_2 as before still work as generators of a group transitive in the rest of the vertices along with one of $\mathcal{C}$. Actually, we can add two vertices to the same square, but we cannot add vertices to consecutive squares in opposite sides of the graph, or add

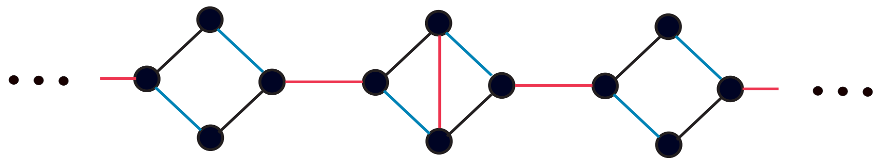


Figure 4.14: Obtaining A_n from S_n

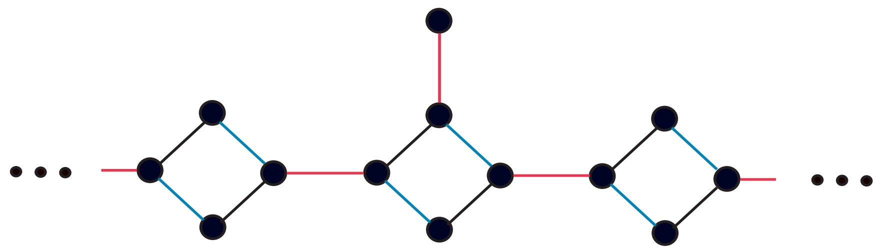


Figure 4.15: Obtaining A_{n+1} from A_n

a vertex to a square next to another square with diagonal of label 1, again because no power of $\rho_0\rho_1$ will be a cycle of length three.

With these two modifications, we are ready to generate families of polyhedra with automorphism groups isomorphic to A_n for $n \geq 21$ as Figure 4.16 shows. Now we are ready to state the following theorem.

Theorem 4.3.4 *For any $n \geq 9$ there is a polyhedron with automorphism group isomorphic to A_n.*

The Schläfli type of the polyhedra for the above graphs is $\{p, p\}$ where p is 12 if we added no diagonal or vertex to G_k, 24 if we added one diagonal, 60 if we added vertices but no diagonals, and 120 if we added vertices and diagonals. Note that these graphs will correspond to self-dual polyhedra if they have $4k + 1$ vertices, or if they have $4k + 3$ vertices and the two extra vertices are attached to the same square. In the remaining cases it is not immediate to determine whether the polyhedra they represent is self-dual or not.

The graphs of the polyhedra with automorphism groups isomorphic to A_n, $9 \leq n \leq 20$ have to be constructed separately. In Figure 4.17 we give a list of graphs of polyhedra with automorphism groups $A_9, A_{10}, \ldots, A_{20}$. These results will be published in [20].

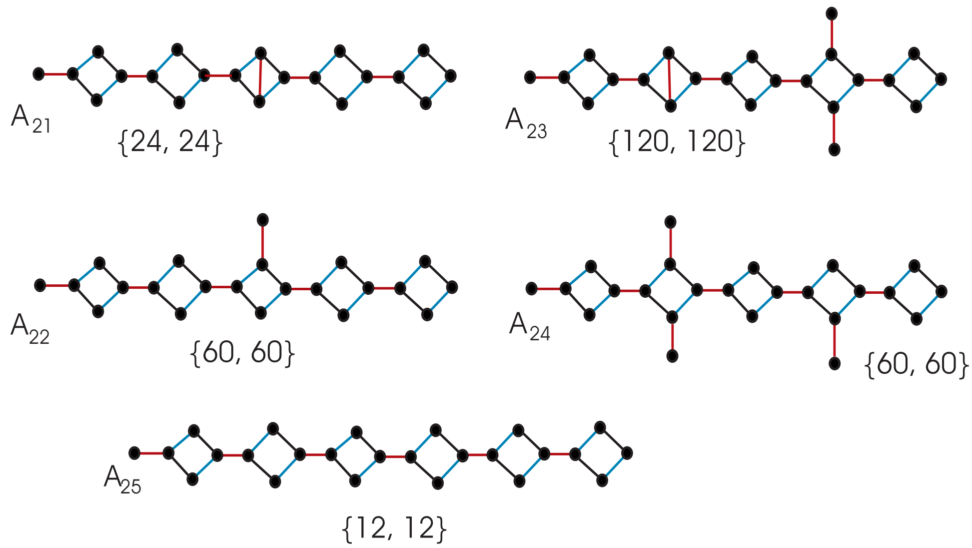


Figure 4.16: Graphs with automorphism group A_n

4.4 Infinite families of finite polyhedra

Vince proves in [31] (see also [19] Chapter 4C) that there exist infinitely many finite polyhedra with Schläfli type $\{p, q\}$ for any given pair (p, q) such that $p, q \geq 3$ and

$$(p, q) \notin \{(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\}.$$

However, the proof is non-constructive.

In [33] and [34] Wilson gives an algorithm to build infinitely many regular maps for each Schläfli symbol, but he does not give explicitly an infinite family including the automorphism groups of the maps; he does not mention either if this algorithm provides infinitely many polyhedral maps.

In this section we construct CPR graphs for infinite families of finite polyhedra for some pairs (p, q).

4.4.1 The Polyhedron $P^k_{12|2s}$

For $s \geq 2$ and $k \geq 1$, let $G^k_{12|2s}$ be the graph such that

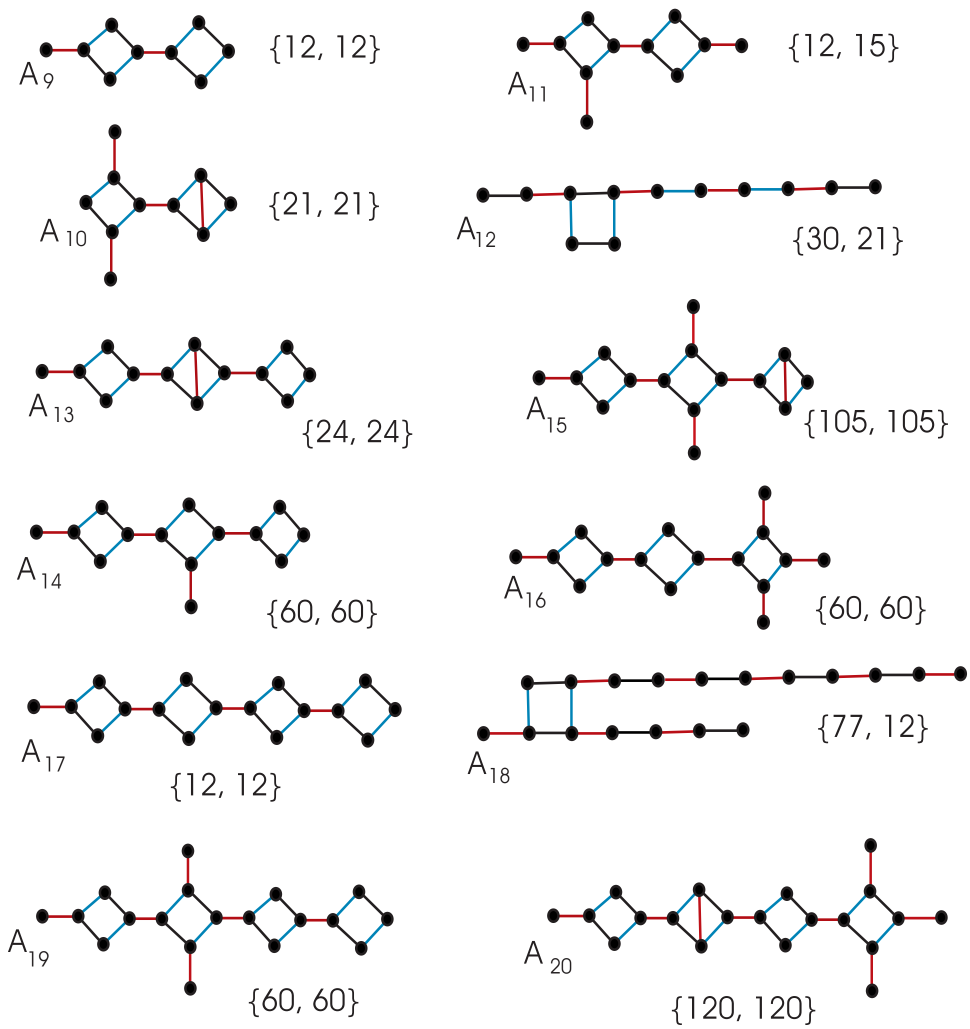


Figure 4.17: Graphs with automorphism group A_n

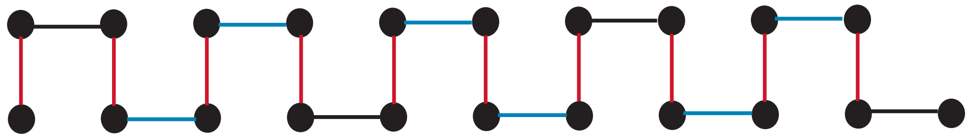


Figure 4.18: The graph $P^3_{12|6}$

- it is a simple (with no multiple edges) path of length $2ks+2$ with its edges ordered $e_1, \dots, e_{2ks+2}$,
- all the edges with odd index are labeled 1,
- all the edges with index congruent to 2 modulo $2s$ are labeled 0, and
- the remaining edges are labeled 2.

Note that the subgraph induced by edges of labels 1 and 2 has $k+2$ connected components; the first one is a single edge, the last one is an isolated vertex, and the remaining ones are alternating paths of length $2s-1$. The connected components of the subgraph induced by edges of labels 0 and 1 are single edges, a path of length 2, and k paths of length 3 (see Figure 4.18 for an example).

Since $G^k_{12|2s}$ is a connected, proper 3-labeled graph, Theorem 4.1.3 imply that it is a CPR graph of a polyhedron $P^k_{12|2s}$, and from the polygonal action of $\langle\rho_0,\rho_1\rangle$ and $\langle\rho_1,\rho_2\rangle$ on the connected components of the corresponding subgraphs of $G^k_{12|2s}$ we can see that the Schläfli type of the latter is $\{12, 2s\}$.

The polygonal action of $\langle\rho_0,\rho_1\rangle$ on the connected components of the subgraph induced by edges of labels 0 and 1 tells us that $(\rho_0\rho_1)^4$ is a 3-cycle involving the connected component of $(G^k_{12|2s})_{0,1}$ with three vertices. Moreover the subgroup generated by ρ_2 and $(\rho_0\rho_1)^3$ is transitive on all the vertices except by two of the 3-cycle. Lemma A.2.3 implies that $A_V \le \Gamma(P^k_{12|2s}) \le S_V$, where V is the vertex set of $P^k_{12|2s}$. Since $|V| = 2ks+3$ and there are $k+1$ edges of label 0, $ks+1$ edges of label 1 and $k(s-1)$ edges of label 2,

$$\Gamma(P^k_{12|2s}) \cong \begin{cases} A_{2ks+3} & \text{if } k \text{ and } s \text{ are odd,} \\ S_{2ks+3} & \text{if } k \text{ or } s \text{ is even.} \end{cases}$$

We have constructed an infinite family of finite polyhedra $\{P^k_{12|2s}\}_{k\ge1}$ with Schläfli type $\{12, 2s\}$.

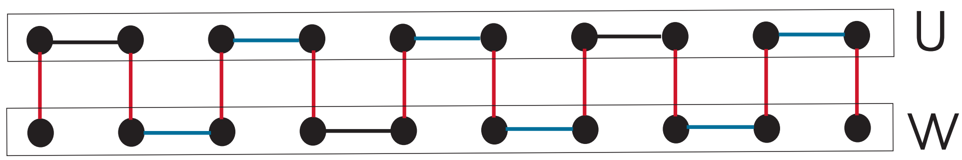


Figure 4.19: The bipartition of the graph $P^3_{4|6}$

4.4.2 The Polyhedron $P^k_{4|2s}$

For $s \geq 3$ let $G^k_{4|2s}$ be $G^k_{12|2s} - v$, where v is the vertex with only one edge (of label 0) incident to it. Proposition 3.2.1 and Theorem 4.1.7 imply that $G^k_{4|2s}$ is a CPR graph of a polyhedron $P^k_{4|2s}$ with Schläfli type $\{4, 2s\}$.

Now, by Proposition 3.5.1, $\Gamma(P^k_{4|2s}) \cong \langle \rho_1 \rangle \ltimes \Lambda$ for some $\Lambda \leq S_U \times S_W$, where U and W are the sets of vertices described in Figure 4.19, and $\langle \rho_1 \rangle$ acts on Λ by interchanging the entries of elements (as elements in $S_U \times S_V$).

It can be checked easily from $G^k_{4|2s}$ that $(\rho_0\rho_1\rho_2\rho_1)^4$ is a 3-cycle in U, and $\rho_1(\rho_0\rho_1\rho_2\rho_1)^4\rho_1$ is a 3-cycle in W. The subgroup generated by ρ_2 and $(\rho_0\rho_1\rho_2\rho_1)^3$ is transitive in U minus two of the vertices of the 3-cycle. The same group works analogously in W. Hence $A_U \times A_W \leq \Lambda \leq S_U \times S_W$. By Proposition A.3.2 we have that Λ can only be isomorphic to $A_n \times A_n$, $(S_n \times S_n)](A_n \times A_n)$, or $S_n \times S_n$, where $n = sk + 1$ is the number of vertices of U (or W).

Since there are k edges of label 0, $\lceil k/2 \rceil$ of them in U; $ks + 1$ edges of label 1; and $k(s-1)$ edges of label 2, $\lfloor k(s-1)/2 \rfloor$ of them in U, we conclude that

$$\Gamma(P^k_{4|2s}) \cong \begin{cases} A_n \times A_n & \text{if } k \text{ and } k/2 \text{ are even} \\ (S_n \times S_n)](A_n \times A_n) & \text{if } k \text{ is even and } k/2 \text{ is odd,} \\ S_n \times S_n & \text{if } k \text{ is odd.} \end{cases}$$

We have constructed an infinite family of finite polyhedra $\{P^k_{4|2s}\}_{k\geq 1}$ with Schläfli type $\{4, 2s\}$. Note that every polyhedra in this family has 2-holes of length 12.

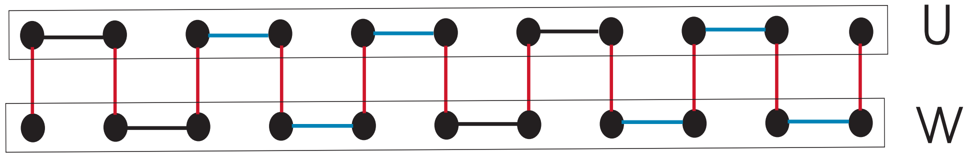


Figure 4.20: The bipartition of the graph $P^2_{6|8}$

4.4.3 The Polyhedron $P^k_{6|2s}$

For $s \geq 3$ let $G^k_{6|2s}$ be the graph such that

- it is a simple path of length $(2s+2)k+1$ with its edges ordered $e_1, \ldots, e_{(2s+2)k+1}$,
- all the edges with odd index are labeled 1,
- all the edges with index congruent to 2 or 4 modulo $2s+2$ are labeled 0, and
- the remaining edes are labeled 2.

The subgraph induced by edges of labels 1 and 2 has single edges and alternating paths of length $2s-1$ as connected components. The connected components of the subgraph induced by edges of labels 0 and 1 are single edges, and k paths of length 5.

Proposition 3.2.1 and Theorem 4.1.7 imply that $G^k_{6|2s}$ is a CPR graph of a polyhedron $P^k_{6|2s}$ with Schläfli type $\{6, 2s\}$.

Again by Proposition 3.5.1, $\Gamma(P^k_{6|2s}) \cong \langle \rho_1 \rangle \ltimes \Lambda$ for some $\Lambda \leq S_U \times S_W$, where U and W are the sets of vertices described in Figure 4.20, and $\langle \rho_1 \rangle$ acts on Λ by interchanging the entries.

Now $(\rho_0\rho_1\rho_0\rho_1\rho_2)^4$ is a 3-cycle in W, and $\rho_1(\rho_0\rho_1\rho_2\rho_1)^4\rho_1$ is a 3-cycle in V. The subgroup generated by $\rho_1\rho_2\rho_1$ and $(\rho_0\rho_1\rho_2\rho_1)^3$ is transitive in W minus two of the vertices of the 3-cycle. The same group works analogously in V. Hence $A_U \times A_W \leq \Lambda \leq S_U \times S_W$. Again Proposition A.3.2 implies that Λ can only be isomorphic to $A_n \times A_n$, $(S_n \times S_n)](A_n \times A_n)$, or $S_n \times S_n$, where $n = (s+1)k+1$ is the number of vertices of U (or W).

Since there are $2k$ edges of label 0, k of them in U; $k(s+1)+1$ edges of label 1; and $k(s-1)$ edges of label 2, $\lfloor k(s-1)/2 \rfloor$ of them in U, we conclude

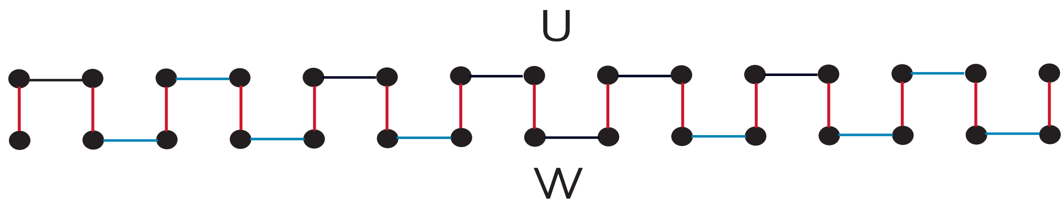


Figure 4.21: The graph $P^1_{8,8}$

that

$$\Gamma(P^k_{6|2s}) \cong \begin{cases} A_n \times A_n & \text{if } k \text{ and } k(s-1)/2 \text{ are even,} \\ (S_n \times S_n)](A_n \times A_n) & \text{if } k \text{ is even and } k(s-1)/2 \text{ is odd,} \\ & \text{or if } k \text{ and } s \text{ are odd,} \\ S_n \times S_n & \text{if } k \text{ is odd and } s \text{ is even.} \end{cases}$$

We have constructed an infinite family of finite polyhedra $\{P^k_{6|2s}\}_{k\geq 1}$ with Schläfli type $\{6, 2s\}$ for $s \geq 3$.

4.4.4 The Polyhedron $P^k_{4p,4q}$

For $p \geq 2$, $q \geq 2$ and $k \geq 1$, let $G^k_{4p,4q}$ be the graph such that

- it is a simple path of length $m = 4k(p + q + 1) + 4q + 1$ with its edges ordered $e_1, \ldots, e_m$,
- all the edges with odd index are labeled 1,
- all the edges with index congruent to $2 + 4q$ or $2 + 4q + 4 + 2s$ modulo $4(p + q) + 4$ for $s = 1, \ldots, 2p$ are labeled 0, and
- the remaining edges are labeled 2.

Now the subgraph induced by all edges of labels 1 and 2 has single edges, $2k$ paths of length 4 and $k + 1$ paths of length $4q$ as connected components. The connected components of the subgraph induced by the edges of labels 0 and 1 are single edges, $2k + 1$ paths of length 4 and k paths of length $4p$ (see Figure 4.21 for an example).

The graph $G^k_{4p,4q}$ can be considered as an alternating sequence of paths with edges labeled 0 and 1 and paths with edges labeled 1 and 2, such that the intersection of two consecutive paths contains only a single edge labeled 1. The sequence of the lengths of the paths is shown in Table 4.2. The first two rows of Table 4.2 appear only once in $G^k_{4p,4q}$, while the last six rows repeat k times. In Figure 4.21 the first path appears at the left and the last at the right.

Path number	**length**	**labels**
1	4-1	0, 1
2	4q-1	1, 2
3	4-1	0, 1
4	4-1	1, 2
5	4p-1	0, 1
6	4-1	1, 2
7	4-1	0, 1
8	4q-1	1, 2.

Table 4.2: Structure of $P^k_{4p,4q}$

Proposition 3.2.1 and Theorem 4.1.7 imply that $G^k_{4p,4q}$ is a CPR graph of a polyhedron $P^k_{4p,4q}$ with Schläfli type $\{4p, 4q\}$. Now we find $\Gamma(P^k_{4p,4q})$ in a similar way as $\Gamma(P^k_{4|2s})$.

By Proposition 3.5.1, $\Gamma(P^k_{4p,4q}) \cong \langle\rho_1\rangle \ltimes \Lambda$ for some $\Lambda \leq S_U \times S_W$, where U and W are the sets of vertices as in Figure 4.19 (see Figure 4.21), and $\langle\rho_1\rangle$ acts on Λ again by interchanging the entries.

Now $(\rho_0\rho_1\rho_2\rho_1)^{10}$ is a 3-cycle in U, and $\rho_1(\rho_0\rho_1\rho_2\rho_1)^{10}\rho_1$ is a 3-cycle in W. The subgroup generated by ρ_2 and $(\rho_0\rho_1\rho_2\rho_1)^3$ restricted to U is again transitive in U minus two of the vertices of the 3-cycle. The same group works analogously in W. Hence $A_U \times A_W \leq \Lambda \leq S_U \times S_W$. By Proposition A.3.2 we have that Λ can only be isomorphic to either $A_n \times A_n$, $(S_n \times S_n)](A_n \times A_n)$, or $S_n \times S_n$, where $n = 2k(p+q+1) + 2q + 1$ is the number of vertices of U (or W).

Since there are $k(2p+1)+1$ edges of label 0, $k(p-1)$ of them in W; $2k(p+q+1)+2q+1$ edges of label 1; and $2k+(k+1)(2q-1)$ edges of label 2, $(k+1)(q-1)$ of them in U, we conclude that

$$\Gamma(P^k_{4p,4q}) \cong \begin{cases} A_n \times A_n & \text{if } k \text{ and } p \text{ are odd} \\ (S_n \times S_n)](A_n \times A_n) & \text{if } k \text{ is odd and } p \text{ is even,} \\ S_n \times S_n & \text{if } k \text{ is even.} \end{cases}$$

We have constructed an infinite family of finite polyhedra $\{P^k_{4p,4q}\}_{k\geq 1}$ with Schläfli type $\{4p, 4q\}$ for $p, q \geq 2$. Note that every polyhedra of this family has 2-holes of length 30.

4.4.5 Remarks

We have constructed infinite families of finite polyhedra for the Schläfli types listed in Table 4.3. They all have groups related to alternating or symmetric groups.

The graphs $G^k_{4p,4q}$ can be modified in several ways in order to get different polyhedra with the same Schläfli type. For example, we can reduce the length of some paths of length $4p$ to $2p$, or $4q$ to $2q$, for p and q odd numbers. Since there is still a connected component with 4 vertices, the Schläfli symbol would not change, but the group would be smaller.

With the help of some similar techniques to those used to construct the polyhedra $P^k_{12|2s}$, $P^k_{4|2s}$, ,$\mathrm{P}^k_{6|2s}$ and $P^k_{4p,4q}$ it might be possible to construct infinite families of finite polyhedra with Schläfli type $\{2p, 2q\}$ for any $p, q \geq 3$. Moreover, it might also be possible to preassign the length of a k-hole, or of a k-zigzag.

Note that if the automorphism $\rho_i\rho_{i+1}$ has order $2p+1$, then

$$\rho_i = (\rho_{i+1}\rho_i)^p \rho_{i+1}[(\rho_{i+1}\rho_i)^p]^{-1}.$$

Polyhedron	**Schläfli s.**	**n**	**parameters**	**Automorphism group**
$P^k_{12\|2s}$	$\{12, 2s\}$	$2ks + 3$	k, s odd	A_n
			k or s even	S_n
$P^k_{4\|2s}$	$\{4, 2s\}$	$sk + 1$	k, $k/2$ even	$A_n \times A_n$
			k even, $k/2$ odd	$(S_n \times S_n)](A_n \times A_n)$
			k odd	$S_n \times S_n$
$P^k_{6\|2s}$	$\{6, 2s\}$	$(s + 1)k + 1$	k, $k(s - 1)/2$ even	$A_n \times A_n$
			k even, $k(s - 1)/2$ odd,	
			or k, s odd	$(S_n \times S_n)](A_n \times A_n)$
			k odd, s even	$S_n \times S_n$
$P^k_{4p,4q}$	$\{4p, 4q\}$	$2k(p + q + 1)+$	k, p odd	$A_n \times A_n$
		+2q+1	k odd, p even	$(S_n \times S_n)](A_n \times A_n)$
			k even	$S_n \times S_n$

Table 4.3: Infinite families of polyhedra

Since ρ_i is conjugate of ρ_{i+1} they have the same cyclic structure as permutations when embedded into any permutation group. This means that there are the same number of edges of labels i and $i + 1$ in any CPR graph of the polytope. This is highly restrictive and makes working with CPR graphs of polyhedra (and of polytopes in general) with odd entries in their Schläfli symbols more complicated than working with CPR graphs of those with even numbers in all the entries of the Schläfli symbol.

Chapter 5

The Polytope $2s^{\mathcal{K}-1}$

The *amalgamation* problem asks if, given two regular d-polytopes $\mathcal{K}$ and $\mathcal{P}$, there are any regular $(d+1)$-polytopes such that their facets are isomorphic to $\mathcal{K}$ and their vertex figures are isomorphic to $\mathcal{P}$ (see [19] Chapter 4A). If such a polytope exists it is said to be an amalgamation of $\mathcal{K}$ and $\mathcal{P}$. An obvious condition for an amalgmation of $\mathcal{K}$ and $\mathcal{P}$ to exist is to require the vertex figure of $\mathcal{K}$ to be isomorphic to the facet of $\mathcal{P}$, but that condition is not enough. For instance, there is no 4-polytope with facets isomorphic to a hemicube and vertex figures isomorphic to a tetrahedron (see M. Hartley [13]). In general it is hard to give an answer for other choices of $\mathcal{K}$ and $\mathcal{P}$.

We can relax the problem in the following way. Given a regular d-polytope $\mathcal{K}$, are there $(d+1)$-polytopes with facets (or dually vertex figures) isomorphic to $\mathcal{K}$? Any such a polytope will be called an *extension* of $\mathcal{K}$.

In [19] Chapter 4D it is proved that any polytope has a universal extension in the sense that any other extension is a quotient of it (in the sense of groups; for a formal definition see [19] Chapter 2D). This new polytope is infinite, moreover its last entry of the Schläfli symbol is infinity.

The next step is to ask if any d-polytope $\mathcal{K}$ has an extension that has a finite last entry of the Schläfi symbol. Additionally we can ask the extension to be finite if $\mathcal{K}$ is finite (*finiteness property*), and to be a lattice if $\mathcal{K}$ is a lattice (*lattice property*). In [19] Chapters 8B-D the polytope $2^{\mathcal{K}}$ is described. It solves the dual of these questions by finding a $(d+1)$-polytope with vertex figure isomorphic to $\mathcal{K}$ and first entry of the Schläfli symbol equal to 4 (see also Danzer [7] and Schulte [29]). This extension satisfies the finiteness and lattice properties.

In [24], [26] and [27] E. Schulte gives another approach to this problem.

It will be described in Chapter 6.

The question if any regular d-polytope $\mathcal{K}$ has an extension with a preassigned last entry of the Schläfli symbol equal to n for $n \geq 3$ had remained open. In this Chapter we construct a generalization of the polytope $2^{\mathcal{K}}$ that gives an affirmative answer to (the dual of) this problem for n even. Moreover, the extension also satisfies the finiteness and lattice properties.

5.1 First Construction

We start with a regular d-polytope $\mathcal{K}$ of Schläfli type $\{p_1, \ldots, p_{d-1}\}$ with the task of constructing a regular $(d+1)$-polytope $\mathcal{P}$ of Schläfli type $\{p, p_1, \ldots, p_{d-1}\}$, for a preassigned number p, with vertex figure isomorphic to $\mathcal{K}$. The procedure is to construct a CPR graph of $\mathcal{P}$ from CPR graphs of $\mathcal{K}$ by increasing the label of each edge by 1 and adding edges of a new label 0. From the polygonal action of $\langle \rho_0, \rho_1 \rangle$ on the connected components of the subgraph induced by edges of labels 0 and 1 of the new CPR graph we know that its connected components have to be paths with their number of vertices dividing p, or cycles with half the number of vertices dividing p. The attempt here is to build a CPR graph having an alternating cycle of edges 0 and 1 of length $2p$ and leaving the remaining connected components of the subgraph induced by the edges of those labels to be squares or single edges. We are assuming, then, that p is an even number, $p = 2s$ (say).

Proposition 3.2.1 tells how the connected components of the subgraphs induced by the edges of labels 0 and k have to be for $k \geq 2$. Since this is somehow restrictive we select a particular CPR graph G with a particular condition that will be discussed in Section 5.7. The first approach will be to take a CPR graph of $\mathcal{K}$ containing a vertex with only one edge (of label 0) incident to it. Then we can build the cycle using this edge and completing the new CPR graph in such a way that each connected component of the new graph obtained by erasing the edges of label 0 will be isomorphic to G. Before giving the formal description we show an example of how it is done. In Figure 5.1 we show the vertex CPR graph of a triangle (black = 0, red = 1) and how we construct the new graph for $s = 2$, that turns out to be a CPR graph of the cube (black = 0, red = 1, blue = 2).

The construction of the extension is done by taking $2s$ copies of the vertex CPR graph of the polytope $\mathcal{K}$ (we are assuming here that $\Gamma(\mathcal{K})$ acts faithfully on the vertices), arranging them in s pairs in a cycle, and joining

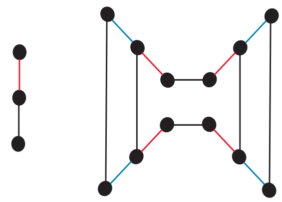


Figure 5.1: Triangle and its extension

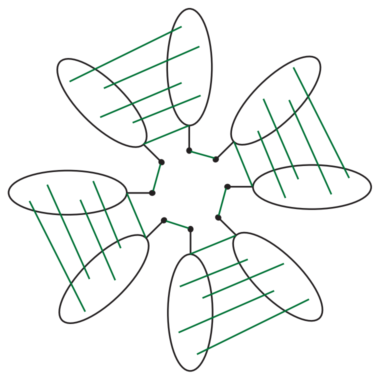


Figure 5.2: G_{2s}

corresponding vertices in each pair by an edge of label 0 except for those corresponding to the base vertex of $\mathcal{K}$; the latter have to be joined among themselves in such a way that they belong to the new cycle of length $4s$ of labels 0 and 1. In Figure 5.2 we show how it looks like for $s = 4$. Each elipse altogether with the vertex joined to it by a black edge represents a copy of the vertex CPR graph of $\mathcal{K}$, while the green edges are the new edges of label 0. Now we describe the construction formally.

Let $\mathcal{K}$ be a d-polytope such that its automorphism group acts faithfully on the vertices, and let G be its vertex CPR graph, with $x_1, \dots, x_n$ the vertex set of the latter, x_1 corresponding to the base vertex of $\mathcal{K}$. We assume $n \geq 2$, otherwise $\mathcal{K}$ is the 1-polytope and the extension will give as a result the regular $2s$-gon. Now we construct a CPR graph G_{2s} defining its vertex set by

$$V(G_{2s}) = \{x_{i,j} : i = 1, \dots, n; j \in \mathbb{Z}_{2s}\}$$

Two vertices $x_{i,j}$ and $x_{k,l}$ are adjacent if and only if one of the following conditions is satisfied.

1) $j = l$ and x_i is adjacent to x_k in G. The label of this edge will be 1 plus the label of the edge between x_i and x_k in G.

2) $i = k = 1$, $j = l + 1$, j even, with label 0.

3) $i = k \neq 1, j = l + 1$, j odd, with label 0.

The following results show that the CPR graph we just defined is the one we were looking for.

Theorem 5.1.1 *For any regular polytope $\mathcal{K}$ of Schläfli type $\{p_1, \ldots, p_{d-1}\}$ the graph G_{2s} described above is a CPR graph of a polytope $2s^{\mathcal{K}-1}$ with vertex figure isomorphic to $\mathcal{K}$ and Schläfli type $\{2s, p_1, \ldots, p_{d-1}\}$.*

The name of the polytope $2s^{\mathcal{K}-1}$ comes from its number of vertices. In Section 5.5 we prove that if $\mathcal{K}$ has n vertices, then $2s^{\mathcal{K}-1}$ has $2s^{n-1}$ vertices. Moreover, the notation is consistent with that of the polytope $2^{\mathcal{K}}$.

Proof

The first entry of the Schläfli symbol can be seen from the polygonal action of $\langle \rho_0, \rho_1 \rangle$ in the graph $(G_{2s})_{0,1}$ since the connected components of the latter are one cycle of length $4s$, alternating squares and single edges of label 0.

In order to check the intersection property let $\phi \in \langle \rho_1, \ldots, \rho_d \rangle \cap \langle \rho_0, \ldots, \rho_j \rangle$ and let

$$\phi = \rho_{i_1} \rho_{i_2} \cdots \rho_{i_m}, \tag{5.1}$$

with $i_k \in \{0, \ldots, j\}$.

We can obtain an element $\psi \in \langle \rho_1, \ldots, \rho_j \rangle$ by eliminating all factors ρ_0 in equation (5.1). We know that ϕ and ψ fix the second subindex of each vertex of G_{2s} because $\phi, \psi \in \langle \rho_1, \ldots, \rho_d \rangle$; and they act in the same way in the first subindex because they have the same factors in $\{\rho_1, \ldots, \rho_d\}$ in the same order. Hence, $\phi = \psi \in \langle \rho_1, \ldots, \rho_d \rangle$, and by the dual of Proposition 2.1.4 the intersection property holds. Since G_{2s} is a proper $(d+1)$-labeled graph, it is a CPR graph of a regular polytope.

Since each connected component of G_{2s} after deleting all the edges of label 0 is isomorphic to a copy of G obtained by adding 1 to the label of each edge, it follows from Remark 2.3.6 that the vertex figure of $2s^{\mathcal{K}-1}$ is isomorphic to $\mathcal{K}$.

□

Proposition 5.1.2 *The polytope $2s^{\mathcal{K}-1}$ has the FAP with respect to its vertex figures.*

Proof

From the proof or Theorem 5.1.1 it follows that the generating relations for $\Gamma(\mathcal{K})$ can be obtained from those for $\Gamma(2s^{\mathcal{K}-1})$ by adding the relation $\rho_0 = \varepsilon$. By Proposition 2.2.4 we conclude that $2s^{\mathcal{K}-1}$ satisfies the FAP with respect to its vertex figures.

□

Proposition 5.1.2 can also be proved using Lemma 2.2.2 and Proposition 3.2.5.

5.2 The automorphism group

In this section we describe the automorphism group of the polytope $2s^{\mathcal{K}-1}$.

Definition 2.2.3 and Proposition 5.1.2 imply that

$$\Gamma(2s^{\mathcal{K}-1}) \cong N_0^-(2s^{\mathcal{K}-1}) \rtimes \langle \rho_1, \dots, \rho_d \rangle.$$

It remains to say which group is $N_0^- = N_0^-(2s^{\mathcal{K}-1})$ and how does $\langle \rho_1, \dots, \rho_d \rangle$ act on it.

First, note that every γ in N_0^- preserves the first subindex of the vertices of G_{2s} because the generators $\psi^{-1}\rho_0\psi, \psi \in \langle \rho_1, \dots, \rho_d \rangle$, of N_0^- act in this way (see Lemma 2.2.2). Actually

$$\psi^{-1}\rho_0\psi(x_{i,j}) = \begin{cases} x_{i,j-1} & \text{for } j \text{ even, } \psi(x_i) = x_1 \text{ or } j \text{ odd, } \psi(x_i) \neq x_1, \\ x_{i,j+1} & \text{for } j \text{ odd, } \psi(x_i) = x_1 \text{ of } j \text{ even, } \psi(x_i) \neq x_1. \end{cases} \tag{5.2}$$

The next step is to find the even subgroup $(N_0^-)^+$ of N_0^-. Since its generators are products of two generators of N_0^- we will analyze the action of elements of the form $\psi^{-1}\rho_0\psi\phi^{-1}\rho_0\phi$, with $\psi, \phi \in \langle \rho_1, \dots, \rho_d \rangle$ on the vertices of G_{2s}. From (5.2) it follows that for j even

$$\psi^{-1}\rho_0\psi\phi^{-1}\rho_0\phi(x_{i,j}) = \begin{cases} x_{i,j} & \text{if } \phi(x_i) = x_1,\ \psi(x_i) = x_1; \\ & \text{or } \phi(x_i) \neq x_1,\ \psi(x_i) \neq x_1; \\ x_{i,j+2} & \text{if } \phi(x_i) \neq x_1,\ \psi(x_i) = x_1; \\ x_{i,j-2} & \text{if } \phi(x_i) = x_1,\ \psi(x_i) \neq x_1; \end{cases} \tag{5.3}$$

while for j odd the subindices $j+2$ and $j-2$ in the right side of 5.3 are interchanged. It follows from (5.3) that any two generators of $(N_0^-)^+$ commute.

Now we introduce the mapping Φ from the generators of $(N_0^-)^+$ into $\mathbb{Z}_s^n$ given by $\Phi(\psi^{-1}\rho_0\psi\phi^{-1}\rho_0\phi) = (y_1, \dots, y_n)$, where

$$y_i = \begin{cases} 0 & \text{if } \phi(x_i) = x_1,\ \psi(x_i) = x_1; \\ & \text{or } \phi(x_i) \neq x_1,\ \psi(x_i) \neq x_1; \\ 1 & \text{if } \phi(x_i) \neq x_1,\ \psi(x_i) = x_1; \\ -1 & \text{if } \phi(x_i) = x_1,\ \psi(x_i) \neq x_1. \end{cases}$$

Note that if $(y_1, \dots, y_n) \neq (0, \dots, 0)$, then $(y_1, \dots, y_n)$ has exactly one entry equal to 1 and one equal to -1.

The mapping Φ can be extended to a group morphism Ψ from $(N_0^-)^+$ to $\mathbb{Z}_s^n$ because the relations of the generators of $(N_0^-)^+$ (commutativity and the order of each generator) are also satisfied by their images in $\mathbb{Z}_s^n$. It is clear that if two automorhisms of $2s^{\mathcal{K}-1}$ have the same image under Ψ then they have the same action on the vertex set of G_{2s} (and thus they are the same automorphism), hence Ψ is a monomorphism. Moreover, the image under Ψ of the generators of $(N_0^-)^+$ is a generating set of

$$\{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\}.$$

Hence

$$(N_0^-)^+ \cong \{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\}.$$

It follows from (5.3) that $\rho_0 \notin (N_0^-)^+$. Hence $(N_0^-)^+$ is a subgroup of index 2 of N_0^- and we can rewrite

$$N_0^- \cong (N_0^-)^+ \rtimes \langle \rho_0 \rangle \cong \{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\} \rtimes \mathbb{Z}_2.$$

Since ρ_0 interchanges even and odd second subindices of the vertices of G_{2s}, it follows from Equation 5.3 and its analogue for j odd that $\mathbb{Z}_2$ acts on $\{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\}$ by sending each element to its inverse.

We recall that the first subindex of the vertices of G_{2s} represents the vertices of $\mathcal{K}$. It follows from the definition of G_{2s} that $\langle \rho_1, \dots, \rho_d \rangle$ acts on $\{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\} \rtimes \mathbb{Z}_2$ by permuting the entries in the first factor. Now we can state the following theorem.

Theorem 5.2.1 *The automorphism group of* $2s^{\mathcal{K}-1}$ *is*

$$(H \rtimes \langle \rho_0 \rangle) \rtimes \langle \rho_0, \dots, \rho_{d-1} \rangle, \tag{5.4}$$

where $H \cong \{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \,|\, \sum x_i = 0\}$, $\langle \rho_0 \rangle$ *acts in* H *as the automorphism that sends each element to its inverse, and* $\langle \rho_0, \dots, \rho_{d-1} \rangle$ *acts on* $H \rtimes \langle \rho_0 \rangle$ *permuting the coordinates of the elements of* H*.*

Note that if $s = 2$ then the product of H and $\langle \rho_0 \rangle$ in Equation 5.4 is a direct product and

$$\Gamma(2 \cdot 2^{\mathcal{K}-1}) \cong \mathbb{Z}_2^n \rtimes \Gamma(\mathcal{K}) \tag{5.5}$$

The automorphism group of the polytope $2^{\mathcal{K}}$ is the same as the one in (5.5), and its generators $\rho_1, \dots, \rho_d$ are embedded in the natural way while ρ_0 is assigned to the element $[(0, \dots, 0, 1), \varepsilon]$. Since the polytopes $2^{\mathcal{K}}$ and $2 \cdot 2^{\mathcal{K}-1}$ have the same automorphism group with the same canonical generators, the next corollary follows.

Corollary 5.2.2 *The polytope* $2 \cdot 2^{\mathcal{K}-1}$ *is isomorphic to the polytope* $2^{\mathcal{K}}$*.*

5.3 Main Construction

In Section 5.1 we constructed extensions only for polytopes such that their automorphism groups act faithfully on their vertices. In this section we give a different construction that may be applied to any polytope. It turns out that the polytope obtained by the first construction is the same as the polytope obtained by this one, so we can construct the polytope $2s^{\mathcal{K}-1}$ for any polyope $\mathcal{K}$. We consider the Cayley graphs (see Example 3.1.6 and Appendix B) of the polytopes rather than their vertex CPR graphs, while the set of vertices representing the flags containing the base vertex of the polytope will play the role that played the base vertex x_1 in the first construccion.

Let $\mathcal{K}$ be a regular polytope with vertex set $w_1, \dots, w_n$ and G be its Cayley graph (its CPR graph of the permutations of the flags) with vertex set $\{v_1, \dots, v_m\}$, where $m = nt$ and $v_1, \dots, v_t$ correspond to the flags that include the base vertex w_1 of $\mathcal{K}$. Consider the graph G^{2s} with vertex set

$$\{v_{i,j} \,|\, i \in \{1, \dots, m\}, j \in \mathbb{Z}_{2s}\}$$

and such that two vertices $v_{i,j}, v_{k,l}$ are adjacent if and only if one of the following conditions is satisfied.

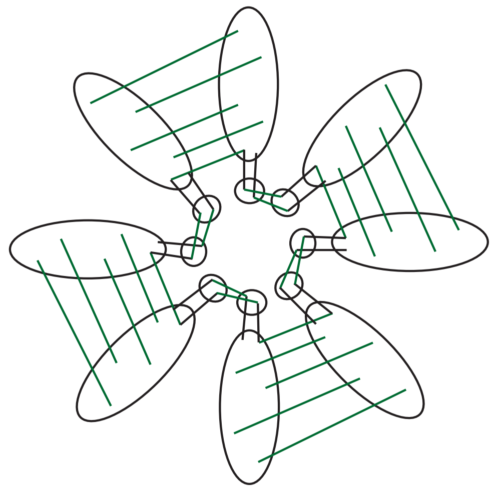


Figure 5.3: G^{2s}

1) $j = l$ and v_i is adjacent to v_k in G. The label of this edge will be 1 plus the label of the edge between v_i and v_k in G.

2) $i = k \in \{1, \dots, t\}$, $j = l + 1$, j even, with label 0.

3) $i = k \notin \{1, \dots, t\}$, $j = l + 1$, j odd, with label 0.

Picture 5.3 shows how the graph G^{2s} looks like. Each elipse altogether with the small circle joined to it by black edges represents a copy of the Cayley CPR graph G of $\mathcal{K}$, each small circle represents the vertices $v_1, \dots, v_t$ of each copy of G (note that they form a connected component of G if we erase the edges of label 1 or color black) and the green edges are again the new edges of label 0.

The following results show that the graph G^{2s} is a CPR graph of an extension of $\mathcal{K}$. Their proofs are analogous to those of Theorem 5.1.1 and Proposition 5.1.2.

Theorem 5.3.1 *For any regular polytope $\mathcal{K}$ of Schläfli type $\{p_1, \dots, p_{d-1}\}$ the graph G^{2s} described above is a CPR graph of a polytope $(2s^{\mathcal{K}-1})'$ with vertex figure isomorphic to $\mathcal{K}$ and Schläfli type $\{2s, p_1, \dots, p_{d-1}\}$.*

Proposition 5.3.2 *The polytope $(2s^{\mathcal{K}-1})'$ has the FAP with respect to its vertex figures.*

To determine the automorphism group of $(2s^{\mathcal{K}-1})'$ we proceed in a similar way to that used for $2s^{\mathcal{K}-1}$.

Propositions 3.5.1 and 5.1.2 also imply that

$$\Gamma((2s^{\mathcal{K}-1})') \cong N_0^-((2s^{\mathcal{K}-1})') \rtimes \langle \rho_1, \dots, \rho_d \rangle.$$

We still have to say which group is $N_0^- = N_0^-((2s^{\mathcal{K}-1})')$ and how does $\langle \rho_1, \dots, \rho_d \rangle$ act on it.

Again we study the group N_0^- by describing first the even subgroup $(N_0^-)^+$.

Every γ in N_0^- still preserves the first subindex of the vertices of G^{2s}, but now (5.2) and (5.3) are changed to

$$\psi^{-1}\rho_0\psi(v_{i,j}) = \begin{cases} v_{i,j-1} & \text{for } j \text{ even, } \psi(v_i) \in \{v_1, \dots, v_t\} \\ & \text{or } j \text{ odd, } \psi(v_i) \notin \{v_1, \dots, v_t\}, \\ v_{i,j+1} & \text{for } j \text{ odd, } \psi(v_i) \in \{v_1, \dots, v_t\} \\ & \text{of } j \text{ even, } \psi(v_i) \notin \{v_1, \dots, v_t\}, \end{cases} \tag{5.6}$$

and (for j even)

$$\psi^{-1}\rho_0\psi\phi^{-1}\rho_0\phi(v_{i,j}) = \begin{cases} v_{i,j} & \text{if } \phi(v_i) \in \{v_1, \dots, v_t\}, \psi(v_i) \in \{v_1, \dots, v_t\}; \\ & \text{or } \phi(v_i) \notin \{v_1, \dots, v_t\}, \psi(v_i) \notin \{v_1, \dots, v_t\}; \\ v_{i,j+2} & \text{if } \phi(v_i) \notin \{v_1, \dots, v_t\}, \psi(v_i) \in \{v_1, \dots, v_t\}; \\ v_{i,j-2} & \text{if } \phi(v_i) \in \{v_1, \dots, v_t\}, \psi(v_i) \notin \{v_1, \dots, v_t\}; \end{cases} \tag{5.7}$$

while for j odd the subindices $j+2$ and $j-2$ in the right side of 5.7 are interchanged.

Now we construct again the mapping Φ from the generators of $(N_0^-)^+$ into $\mathbb{Z}_s^n$ redefining it by $\Phi(\psi^{-1}\rho_0\psi\phi^{-1}\rho_0\phi) = (y_1, \dots, y_n)$, where, for any vertex v of G corresponding to a flag containing the vertex w_i of $\mathcal{K}$

$$y_i = \begin{cases} 0 & \text{if } \phi(v) \in \{v_1, \dots, v_t\}, \psi(v) \in \{v_1, \dots, v_t\}; \\ & \text{or } \phi(v) \notin \{v_1, \dots, v_t\}, \psi(v) \notin \{v_1, \dots, v_t\}; \\ 1 & \text{if } \phi(v) \notin \{v_1, \dots, v_t\}, \psi(v) \in \{v_1, \dots, v_t\}; \\ -1 & \text{if } \phi(v) \in \{v_1, \dots, v_t\}, \psi(v) \notin \{v_1, \dots, v_t\}. \end{cases} \tag{5.8}$$

Note that $\phi(v) \in \{v_1, \dots, v_t\}$ for some v corresponding to a flag containing a given vertex w_i of $\mathcal{K}$ if and only if $\phi(z) \in \{v_1, \dots, v_t\}$ for every z corresponding to a flag containing w_i, so Φ is well defined.

By similar reasons than those explained in Section 5.2 for the first extension, we extend the mapping Φ to the group morphism Ψ from $(N_0^-)^+$ to $\mathbb{Z}_s^n$, implying that

$$(N_0^-)^+ \cong \{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\}.$$

We imply from (5.7) that $\rho_0 \notin (N_0^-)^+$, and we can rewrite

$$N_0^- \cong (N_0^-)^+ \rtimes \langle \rho_0 \rangle \cong \{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\} \rtimes \mathbb{Z}_2.$$

with $\mathbb{Z}_2$ acting on $\{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\}$ by sending each element to its inverse, and $\langle \rho_1, \dots, \rho_d \rangle$ acting on $\{(x_1, \dots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\} \rtimes \mathbb{Z}_2$ by permuting the entries in the first factor.

Since $\Gamma(2s^{\mathcal{K}-1}) = \Gamma((2s^{\mathcal{K}-1})')$, and the string C-group generators of the two polytopes correspond to the same elements in that group, we conclude the following theorem.

Theorem 5.3.3 *For any polytope $\mathcal{K}$ such that $\Gamma(\mathcal{K})$ acts faithfully on its vertices we have that $2s^{\mathcal{K}-1} \cong (2s^{\mathcal{K}-1})'$.*

The construction described in this section along with some results from the following sections will be published in [21].

From now on we refer to $(2s^{\mathcal{K}-1})'$ just by $2s^{\mathcal{K}-1}$.

5.4 Results and Examples

In this section we give some examples and results to illustrate some properties of the polytope $2s^{\mathcal{K}-1}$.

Proposition 5.4.1 *Let $\mathcal{K}$ be a regular d-polytope with even subgroup of index 2 on $\Gamma(\mathcal{K})$, then the even subgroup of $2s^{\mathcal{K}-1}$ has index 2 on $\Gamma(2s^{\mathcal{K}-1})$.*

Proof

Since the even subgroup has index at most 2 it suffices to prove that it is not the whole group $\Gamma(2s^{\mathcal{K}-1})$. We will prove that ρ_d is not an element of the even subgroup.

Let

$$\rho_d = \rho_{i_1}\rho_{i_2}\cdots\rho_{i_m}, \tag{5.9}$$

with $i_k \in \{0, \ldots, d\}$. Proposition 5.3.2 implies that, if we remove all the ρ_0's in (5.9), we get an expression for ρ_d in terms of $\rho_1, \ldots, \rho_d$. Since the even subgroup of $\mathcal{K}$ has index 2 in $\Gamma(\mathcal{K})$, this expression has an odd number of factors.

In the other hand, ρ_0 interchanges the parity of the second subindex of any vertex of G^{2s}, where G^{2s} is the graph constructed in Section 5.3. Since ρ_i preserves that subindex for $i > 0$, there has to be an even number of factors ρ_0's in (5.9). Hence the total number of factors is odd.

□

Corollary 5.4.2 *The polyhedron $2s^{\{n\}-1}$ is an orientable polyhedron with Schläfli type $\{2s, n\}$ with genus $1 + [s^{n-2}(ns - 2s - n)]/2$.*

Proof

To determine the genus of the polyhedron it suffices to notice that the number of vertices is $2s^{n-1}$, the number of edges is ns^{n-1} and the number of facets is ns^{n-2}.

□

The polytope $2^{\mathcal{K}}$ is centrally symmetric if $\mathcal{K}$ has a finite number of vertices (see [19] Chapter 8C). Now we give a sufficient condition for the polytope $2s^{\mathcal{K}-1}$ to be centrally symmetric.

Proposition 5.4.3 *Let $\mathcal{K}$ be a polytope with an even number of vertices. Then, for s even, the polytope $2s^{\mathcal{K}-1}$ is centrally symmetric.*

Proof

The involution ξ given by a half turn of Figure 5.3 corresponds to the element $[\varepsilon, ((s/2, \ldots, s/2), 0)]$ in

$$[\{(x_1, \ldots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\} \rtimes \mathbb{Z}_2] \rtimes \langle \rho_1, \ldots, \rho_d \rangle$$

(here we use that n is even, otherwise the sum of the coordinates would not be 0). Then, ξ is also an element of $\Gamma(2s^{\mathcal{K}-1})$. The commutativity can be checked for each ρ_j either using Proposition 3.2.3 or directly from the group.

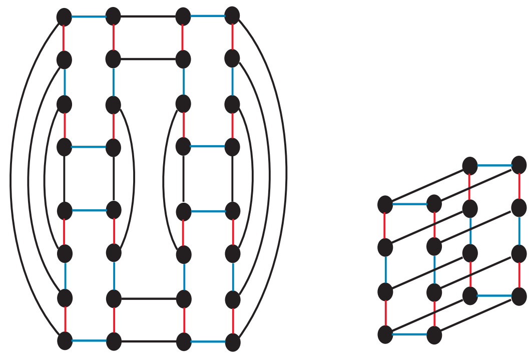


Figure 5.4: CPR graphs of $\{4,4\}_{(4,0)}$ and $\{4,2\}$

□

Unfortunately this conditions are not necessary for the polytope $\mathcal{K}$ to be centrally symmetric. The cube is isomorphic to $2^{\{3\}}$ and is centrally symmetric but the number of vertices of the triangle is odd.

Proposition 5.4.4 *Given a polytope $\mathcal{K}$ of Schläfli type $\{4, p_2, \ldots, p_{d-1}\}$, the polytope $2^{\mathcal{K}}$ is a polytope with 3-faces isomorphic to the toroidal polytope $\{4,4\}_{(4,0)}$.*

Proof

The only connected components of the subgraph of the Cayley graph G of $\mathcal{K}$ induced by edges of labels 0 and 1 are alternating octagons. It is easy to see that the only connected components of the graph G^{2s} (with $s = 2$) defined in Section 5.3 after deleting the edges of labels $3, \ldots, d-1$ are the CPR graphs of $\{4,4\}_{(4,0)}$ and $\{4,2\}$ shown in Figure 5.4. Since $\{4,4\}_{(4,0)} \diamondsuit \{4,2\} = \{4,4\}_{(4,0)}$, there are no other choices for the vertex figure.

□

It was already known that the polytopes $2^{\{4,4\}_{(2,0)}}$ and $2^{\{4,4\}_{(3,0)}}$ are the universal polytopes ${}_2\mathcal{T}^4_{(4,0),(2,0)}$ and ${}_2\mathcal{T}^4_{(4,0),(3,0)}$ respectively (see [19] Chapter 10C). This extensions provide another universal locally toroidal polytope (the facets and vertex figures are toroidal polytopes).

Example 5.4.5 *The polytope $2 \cdot 3^{\{3,3\}-1}$ is the universal polytope ${}_3\mathcal{T}^4_{(3,0)}$ (see [19] Chapter 11B). This follows from the cardinality 1296 of the automor-*

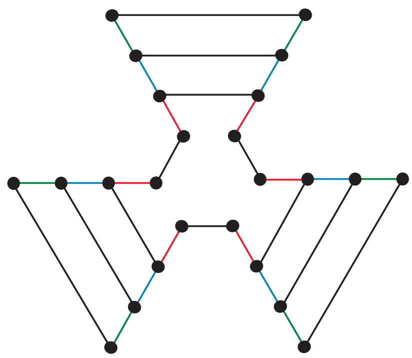


Figure 5.5: CPR graph of $2 \cdot 3^{\{3,3\}-1}$

phism groups of these polytopes. The facet type of $2{\cdot}3^{\{3,3\}-1}$ *is* $6, 3_{(3,0)}$ *and can be derived from the corresponding CPR graph and the relations* $(\rho_0\rho_1\rho_2)^6 = \varepsilon$ *and* $(\rho_2(\rho_1\rho_0)^2)^6 = \varepsilon$ *(see Figure 5.5, black* $= 0$*, red* $= 1$*, blue* $= 2$ *and green* $= 3$*).*

5.5 The Lattice Property

Now we will prove that the polytope $2s^{\mathcal{K}-1}$ satisfies the lattice property with respect to the polytope $\mathcal{K}$. With that in mind we define a polytope $\mathcal{P}_{2s}(\mathcal{K})$, we prove that it is regular and that its dual is an extension of the dual of $\mathcal{K}$ that satisfies the lattice property. Finally we prove that the polytope $\mathcal{P}_{2s}(\mathcal{K})$ is isomorphic to $2s^{\mathcal{K}-1}$.

In [7] they give a purely combinatorial construction of the polytope $2^{\mathcal{K}}$ (see also [29] and [19] Chapter 8D). The construction in this section is somewhat similar to it and provides a combinatorial model for $2s^{\mathcal{K}-1}$.

5.5.1 The Polytope $\mathcal{P}_{2s}(\mathcal{K})$

Let $\mathcal{K}$ be a regular d-polytope with vertex set $\mathcal{K}_0 = \{v_1, \ldots, v_n\}$, with base vertex v_1 and such that each face is determined by its vertex set (it can be easily seen that this is the case if $\mathcal{K}$ is a lattice).

Let ${}_{2s}G(\mathcal{K})$ be the graph with vertex set

$$V({}_{2s}G(\mathcal{K})) = \{\bar{x} = [(x_1, \ldots, x_n), r_{\bar{x}}] \,|\, x_i \in \mathbb{Z}_s,\, \textstyle\sum x_i = 0,\, r_{\bar{x}} \in \mathbb{Z}_2\}$$

such that $\bar{x}$ is adjacent to $\bar{y}$ if and only if $r_{\bar{x}} = 0$, $r_{\bar{y}} = 1$, and one of the following conditions is satisfied

- $x_i = y_i$ for all i, or
- $x_1 = y_1 + 1$, $y_j = x_j + 1$ for some $j \in \{2, \dots, n\}$, and $x_i = y_i$ for $i \in \{2, \dots, n\} \setminus \{j\}$.

Another way of describing the adjacencies of ${}_{2s}G(\mathcal{K})$ is by defining the involutory bijections $\eta_1, \dots, \eta_n$ in $V({}_{2s}G(\mathcal{K}))$, where for $j > 1$

$$\begin{aligned} \eta_j[(x_1, \dots, x_n), 0] &= [(x_1 - 1, x_2, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n), 1], \\ \eta_j[(y_1, \dots, y_n), 1] &= [(y_1 + 1, y_2, \dots, y_{j-1}, y_j - 1, y_{j+1}, \dots, y_n), 0], \end{aligned}$$

and

$$\eta_1[(x_1, \dots, x_n), r_{\bar{x}}] = [(x_1, \dots, x_n), r_{\bar{x}} + 1].$$

Now $\bar{x}$ is adjacent to $\bar{y}$ if and only if $\bar{x} = \eta_j(\bar{y})$ for some j.

Proposition 5.5.1 *The graph ${}_{2s}G(\mathcal{K})$ is connected.*

Proof

Given two vertices $\bar{x} = [(x_1, \dots, x_n), 0]$ and $\bar{y} = [(y_1, \dots, y_n), r_{\bar{y}}]$ it is not hard to see that

$$\bar{y} = \begin{cases} (\eta_1\eta_2)^{y_2 - x_2} \cdots (\eta_1\eta_n)^{y_n - x_n}(\bar{x}) & \text{if } r_{\bar{y}} = 0 \\ \eta_1(\eta_1\eta_2)^{y_2 - x_2}(\eta_1\eta_3)^{y_3 - x_3} \cdots (\eta_1\eta_n)^{y_n - x_n}(\bar{x}) & \text{if } r_{\bar{y}} = 1. \end{cases}$$

Hence all the vertices are in the same connected component as $\bar{x}$ and ${}_{2s}G(\mathcal{K})$ is connected.

□

For any face F of $\mathcal{K}$ of rank at least 1 and any vertex $\bar{x}$ of ${}_{2s}G(\mathcal{K})$ we define $F(\bar{x})$ as the set of vertices

$$\{\bar{z} \mid \bar{z} = \chi(\bar{x}) \text{ for some } \chi \in \langle \eta_j \mid v_j \text{ is a vertex of } F \rangle\}. \tag{5.10}$$

We can extend the definition to a face F of $\mathcal{K}$ of rank 0, that is, a vertex v_j, by taking $v_j(\bar{x})$ to be the edge between $\bar{x}$ and $\eta_j(\bar{x})$. Similarly, for $F = \emptyset$ we define $\emptyset(\bar{x}) := \{\bar{x}\}$. The following remark follows directly from the definition of $F(\bar{x})$.

Remark 5.5.2 *Let F and G be faces of $\mathcal{K}$ such that $F \leq G$, and $\bar{x}$ and $\bar{y}$ vertices of ${}_{2s}G(\mathcal{K})$. If $F(\bar{x}) \subseteq G(\bar{y})$ then $G(\bar{x}) = G(\bar{y})$.*

Now we construct a poset $\mathcal{P}_{2s}(\mathcal{K})$ consisting of the empty set and the set

$$\{F(\bar{z}) \,|\, \bar{z} \in V({}_{2s}G(\mathcal{K})), F \text{ is a face of } \mathcal{K}\}$$

with the order relation given by the set inclusion. Note that for all F and $\bar{x}$, if v_1 is a vertex of F then

$$F(\bar{x}) = \{\bar{z} \in V({}_{2s}G(\mathcal{K})) \,|\, z_i = x_i \text{ if } v_i \text{ is not a vertex of } F\}.$$

First we have to prove that $\mathcal{P}_{2s}(\mathcal{K})$ is an abstract $(d+1)$-polytope.

The empty set is F_{-1} and $V({}_{2s}G(\mathcal{K}))$ is F_{d+1}. Clearly $F(\bar{x}) \subseteq G(\bar{x})$ if $F \leq G$ and we have a proper contention if and only if we $F < G$. As a consequence of this and of Remark 5.5.2 $\mathcal{P}_{2s}(\mathcal{K})$ is a flagged poset where $rank(F(\bar{x})) = rank_{\mathcal{K}}(F) + 1$. Remark 5.5.2 also implies that $\mathcal{P}_{2s}(\mathcal{K})$ satisfies the diamond condition.

To prove the strongly flag connectivity note that if f and g are two flags containing the same vertex, the strongly flag connectivity of $\mathcal{K}$ implies that we can find the desired sequence of flags between them. Since all the edges have exactly two vertices we can also find a 0-adjacent flag to each flag. The connectedness of ${}_{2s}G(\mathcal{K})$ finishes the proof. Hence $\mathcal{P}_{2s}(\mathcal{K})$ is an abstract $(d+1)$-polytope.

Remark 5.5.2 implies that the vertex figure at any vertex is isomorphic to $\mathcal{K}$.

Now that we know that $\mathcal{P}_{2s}(\mathcal{K})$ is an extension (although possibly not regular) of $\mathcal{K}$ we prove the lattice property.

Let $\mathcal{K}$ be a lattice and let $F(\bar{x})$ and $G(\bar{y})$ be two faces of $\mathcal{P}_{2s}(\mathcal{K})$. If they have a vertex $\bar{z}$ in their intersection then $F(\bar{x}) = F(\bar{z})$ and $G(\bar{y}) = G(\bar{z})$. It follows that the meet of $F(\bar{x})$ and $G(\bar{y})$ is $H(\bar{z})$, where H is the meet of F and G. Hence every two elements of $\mathcal{P}_{2s}(\mathcal{K})$ have a meet. Now Proposition B.1.1 implies that $\mathcal{P}_{2s}(\mathcal{K})$ is a lattice.

5.5.2 The Automorphism Group

Now we prove that $\mathcal{P}_{2s}(\mathcal{K}) \cong 2s^{\mathcal{K}-1}$. To do this we prove first that $\mathcal{P}_{2s}(\mathcal{K})$ is regular by giving automorphisms σ_j that send a base flag g to its i-adjacent flags g^i. Some of these automorphisms are defined in terms of automorphisms of $\mathcal{K}$ in the way described next.

Any $\phi \in \Gamma(\mathcal{K})$ determines a permutation τ of the (indices of the) vertices of $\mathcal{K}$ and it induces a bijection ϕ' in $V(\mathcal{P}_{2s}(\mathcal{K}))$ given by the corresponding permutation τ of the coordinates of the vertices

$$\phi'[(x_1, \ldots, x_n), r_{\bar{x}}] = [(x_{\tau(1)}, \ldots, x_{\tau(n)}), r_{\bar{x}}].$$

We choose both a base flag f of $\mathcal{K}$ containing the base vertex v_1 and the edge between the vertices v_1 and v_2, and the base flag g of $\mathcal{P}_{2s}(\mathcal{K})$ containing the vertex $\bar{x}_0 = [(0, \ldots, 0), 0]$ and the faces $F(\bar{x}_0)$ with F a face of f.

Since any face of $\mathcal{P}_{2s}(\mathcal{K})$ is determined by its vertex set it suffices to define the functions $\sigma_0, \ldots, \sigma_d$ on $V(\mathcal{P}_{2s}(\mathcal{K}))$ in the following way.

$$\begin{aligned}
\sigma_i[(x_1, \ldots, x_n), r_{\bar{x}}] &= \rho'_{i-1}[(x_1, \ldots, x_n), r_{\bar{x}}] \quad \text{for } i \geq 2, \\
\sigma_1[(x_1, \ldots, x_n), 0] &= \rho'_0[(x_1, \ldots, x_n), 0], \\
\sigma_1[(x_1, \ldots, x_n), 1] &= \rho'_0[(x_1 + 1, x_2 - 1, x_3, \ldots, x_n), 1], \\
\sigma_0[(x_1, \ldots, x_n), r_{\bar{x}}] &= [(-x_1, \ldots, -x_n), r_{\bar{x}} + 1].
\end{aligned}$$

The proof of the following lemmas is straightforward.

Lemma 5.5.3 *Let $\eta_1, \ldots, \eta_n, \sigma_0$ be as defined above. Then σ_0 commutes with η_j for $j = 1, \ldots, n$.*

Lemma 5.5.4 *The functions $\sigma_1, \ldots, \sigma_d$ and $\eta_1, \ldots, \eta_n$ defined above satisfy*

$$\sigma_i \eta_j = \eta_k \sigma_i$$

where $\rho_{i-1}(v_j) = v_k$.

It follows from 5.10 and Lemmas 5.5.3 and 5.5.4 that the σ_i's map j-faces of $\mathcal{P}_{2s}(\mathcal{K})$ into j-faces of $\mathcal{P}_{2s}(\mathcal{K})$. It is also straightforward that σ_i is an order preserving bijection of the faces of $\mathcal{P}_{2s}(\mathcal{K})$ for $i = 0, \ldots, d$ and that $\sigma_i(g) = g^i$. Proposition 2.1.2 implies that $\mathcal{P}_{2s}(\mathcal{K})$ is a regular polytope with automorphism group $\langle \sigma_0, \ldots, \sigma_d \rangle$.

Now we find explicitely the automorphism group of $\mathcal{P}_{2s}(\mathcal{K})$ in a similar way as for $\Gamma(2s^{\mathcal{K}-1})$. First we prove that $\mathcal{P}_{2s}(\mathcal{K})$ satisfies the FAP with respect to its vertex figures.

Proposition 5.5.5 *Let $\mathcal{K}$ be a regular polytope. Then the polytope $\mathcal{P}_{2s}(\mathcal{K})$ defined above satisfies the FAP with respect to its vertex figures.*

Proof
For any relation

$$\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_t} = \varepsilon$$

we can get another relation of $\Gamma(\mathcal{P}_{2s}(\mathcal{K}))$ by deleting all the factors σ_0's because σ_0 does not involve permutations of the indices of the vertices but the other generators do. Proposition 2.2.4 implies that $\mathcal{P}_{2s}(\mathcal{K})$ satisfies the FAP with respect to its vertex figures.

□

Now note that each generator $\phi^{-1}\sigma_0\phi\psi^{-1}\sigma_0\psi$; $\phi, \psi \in \langle\sigma_1, \ldots, \sigma_d\rangle$, of the even subgroup of the normal closure N_0^- of σ_0 in $\Gamma(\mathcal{P}_{2s}(\mathcal{K}))$ acts on the vertices $\bar{x}$ of $\mathcal{P}_{2s}(\mathcal{K})$ such that $r_{\bar{x}} = 0$ by adding 1 to the j-th entry and -1 to the k-th entry where $\psi(v_j) = \phi(v_k) = v_1$ (we are considering $\phi, \psi \in \langle\sigma_1, \ldots, \sigma_d\rangle \cong \Gamma(\mathcal{K}) = \langle\rho_0, \ldots, \rho_{d-1}\rangle$).

Since σ_0 is not an element of the even subgroup of N_0^- (it changes the entry in $\mathbb{Z}_2$) it follows that

$$[\{(x_1, \ldots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\} \rtimes \mathbb{Z}_2] \rtimes \Gamma(\mathcal{P}_{2s}(\mathcal{K})) \cong \langle\sigma_1, \ldots, \sigma_d\rangle = \Gamma(2s^{\mathcal{K}-1}).$$

Since the identification of the generators $\sigma_0, \ldots, \sigma_d$ of $\Gamma(\mathcal{P}_{2s}(\mathcal{K}))$ is the same as that of the generators $\rho_0, \ldots, \rho_d$ of $\Gamma(2s^{\mathcal{K}-1})$ it follows that the polytopes $\mathcal{P}_{2s}(\mathcal{K})$ and $2s^{\mathcal{K}-1}$ are isomorphic.

5.6 An Alternative Construction

In this section we give a construction of another CPR graph of the polytope $2s^{\mathcal{K}-1}$.

Given a d-polytope $\mathcal{K}$ we construct the graph $(G^{2s})'$ from G^{2s} by deleting the vertices $v_{i,j}$ of G^{2s}, such that $j = 0$ or $j \geq s+1$, while keeping the same rules of adjacency of vertices. We are taking only half of the vertices of G^{2s} (see Figure 5.6).

Note that the graph $(G^{2s})'$ can be obtained from G^{2s} in the way described in Proposition 3.2.5 by taking $\Lambda = \langle\lambda\rangle$ as the group of automorphisms of G^{2s}

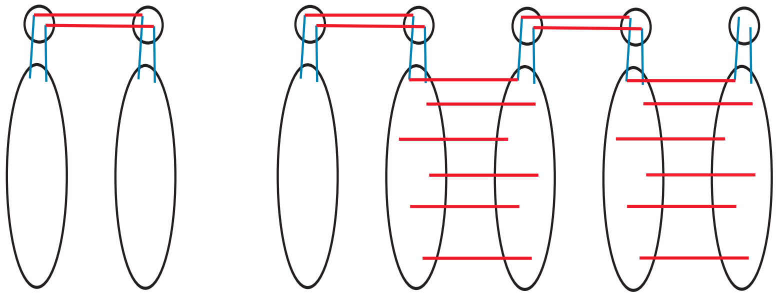


Figure 5.6: The graph $(G^{2s})'$ for $s = 2$ and $s = 5$

as a labeled graph, where λ is the involutory bijection on the vertex set of G^{2s} given by

$$\lambda(v_{i,j}) = v_{i,1-j}.$$

In Figure 5.3, λ is a reflection with respect to a line through the center of the graph that intersects two sets of five red edges that join two pairs of elipses.

It can be proved directly that the graph $(G^{2s})'$ is a CPR graph of a polytope with automorphism group isomorphic to $\Gamma(2s^{\mathcal{K}-1})$ and conclude by studying the generators that $(G^{2s})'$ is another CPR graph of $2s^{\mathcal{K}-1}$, as we did in Section 5.3. This time we will prove it in a different way.

In order to use Proposition 3.2.5 on G^{2s} note that if $\phi \in \Gamma(2s^{\mathcal{K}-1})$ is such that $\phi(v) \in \{v, \lambda(v)\}$ for all vertex v and $\phi(v_0) = \lambda(v_0)$ for some vertex v_0 of G^{2s}, then $\phi(v) = \lambda(v)$ for every vertex v of G^{2s} because λ changes the parity of the second subindex of the vertices. Hence

$$N = \{\phi \in \Gamma(2s^{\mathcal{K}-1}) \,|\, \phi(v) \in \{v, \lambda(v)\}\} \leq \langle \lambda \rangle.$$

We shall prove that N is trivial.

For $s \geq 3$, λ is the action of no automorphism of $\mathcal{K}$ on the vertices of G^{2s} because it acts like ρ_0 on $v_{m,1}$, but not on $v_{m,3}$ and, since there is an automorphism of G^{2s} as a labeled graph that takes $v_{m,1}$ to $v_{m,3}$ (a rotation of $\pi/2$ on Figure 5.3), any automorphism of $\mathcal{K}$ has to act like ρ_0 in both vertices $v_{m,1}$ and $v_{m,3}$, or in none of them.

For $s = 2$, $\rho_0\lambda$ corresponds to the element $[\varepsilon, ((1, 0, \ldots, 0), 0)]$ of

$$[\mathbb{Z}_s^n \rtimes \mathbb{Z}_2] \rtimes \langle \rho_1, \ldots, \rho_d \rangle$$

(see (5.8)) but is not an element of

$$[\{(x_1, \ldots, x_n) \in \mathbb{Z}_s^n \,|\, \textstyle\sum x_i = 0\} \rtimes \mathbb{Z}_2] \rtimes \Gamma(2s^{\mathcal{K}-1}) \cong \langle \rho_1, \ldots, \rho_d \rangle.$$

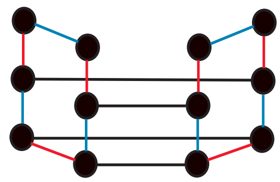


Figure 5.7: CPR graph of the hemicube

Hence λ does not correspond to the action of an element of $\Gamma(2s^{\mathcal{K}-1})$ on the vertex set of G^{2s} for any s.

From the proof of Proposition 3.2.5 we imply that the permutation group generated by the graph $(G^{2s})'$ is isomorphic to $\Gamma(2s^{\mathcal{K}-1})$ with such isomorphism mapping the permutation corresponding to the edges of label i to ρ_i. Hence $(G^{2s})'$ is another CPR graph of the polytope $2s^{\mathcal{K}-1}$.

This construction has the advantage that needs only s copies of G, while graph G^{2s} needs $2s$.

Note that if we change the definition of λ to

$$\lambda'(x_{i,j}) = x_{i,3-j}$$

then, for $s = 2$, we can no longer say that λ' does not corresponds to an action of an element of $\Gamma(2s^{\mathcal{K}-1})$. For example, Figure 5.7 shows the CPR graph of the extension of the triangle $\{3\}$ obtained from λ' with $s = 2$. That graph is a CPR graph of the hemicube (apply the petrie operation to the graph of Figure 3.2C), while $2^{\{3\}} \cong 2 \cdot 2^{\{3\}-1}$ is the cube. In this case λ' corresponds to the central involution of the cube.

5.7 Related Extensions

Given a CPR graph G of a d-polytope $\mathcal{K}$ we can construct several graphs of the type of G_{2s} and G^{2s}. The only important condition of G to construct those graphs is to be connected but to get disconnected if we delete the edges of label 0. In this section we briefly discuss the extensions determined by these constructions of CPR graphs.

For $d \geq 2$ let $G = G_{0,\ldots,d-1}$ be a connected CPR graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ of the d-polytope $\mathcal{K}$ and let C be the union of a nonempty family of connected components of $G_{1,\ldots,d-1}$, but not the whole graph (if we choose G to be the Cayley graph of $\mathcal{K}$ we can choose C as a nonempty proper subset of connected components in $G_{1,\ldots,d-1}$). We construct a new graph $G_{\{2s,C\}}$ in the following way.

$$V(G_{\{2s,C\}}) = \{x_{i,j} : i = 1, \ldots, n; j \in \mathbb{Z}_s\},$$

and there is an edge between the vertices $x_{i,j}$ and $x_{k,l}$ if and only if one of the following conditions is satisfied.

1) $j = l$ and $\{x_i, x_k\} \in E(G)$. The label of $\{x_{i,j}, x_{k,l}\}$ will be 1 plus the label of $\{x_i, x_k\}$.

2) $x_i = x_k \in C$, $j = l + 1$, j even, with label 0.

3) $x_i = x_k \notin C$, $j = l + 1$, j odd, with label 0.

Clearly the graphs $G_{\{2s,C\}}$ and $G_{\{2s,G\setminus C\}}$ are isomorphic.

The proofs of the following results are analogous to those of Theorem 5.1.1 and Proposition 5.1.2.

Theorem 5.7.1 *Let G be a CPR graph of the polytope $\mathcal{K}$. Then the graph $G_{\{2s,C\}}$ described above is a CPR graph of a polytope $2s^{\mathcal{K}-1}_{\{G,C\}}$ with vertex figure type isomorphic to $\mathcal{K}$ and first entry of the Schläfli symbol equal to $2s$.*

Proposition 5.7.2 *The polytope $2s^{\mathcal{K}-1}_{\{G,C\}}$ defined in Theorem 5.7.1 has the FAP with respect to its facets.*

The polytope $2s^{\mathcal{K}-1}$ is a particular case of $2s^{\mathcal{K}-1}_{\{G,C\}}$, taking $G = G_{0,\ldots,d-1}$ to be the Cayley graph of $\mathcal{K}$ and C to be one of the connected components of $G_{1,\ldots,d-1}$.

By similar arguments to those of Sections 5.2 and 5.3, the automorphism group of $2s^{\mathcal{K}-1}_{\{G,C\}}$ is

$$(H \rtimes \mathbb{Z}_2) \rtimes \langle \rho_0, \ldots, \rho_{d-1} \rangle,$$

where the group H is a subgroup of $\{(x_1, \ldots, x_n) \in \mathbb{Z}_s^n \mid \sum x_i = 0\}$ that depends on the graph G and the set C. The actions that determine the semidirect products are the same actions as for the polytope $2s^{\mathcal{K}-1}$.

We finish this section showing an example of these constructions.

The square has only three connected CPR graphs, they are its Cayley graph, its vertex CPR graph, and its edge CPR graph (see Figure 5.8, black = 0, red = 1), and they have $4, 2$ and 3 connected components respectively if we eliminate the edges of label 0. Applying these constructions with $s = 2$, we get the following toroidal polyhedra.

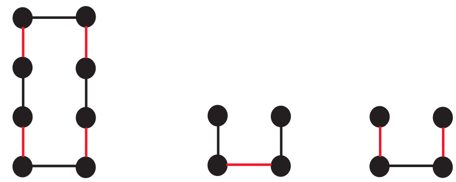


Figure 5.8: Connected CPR graphs of the square

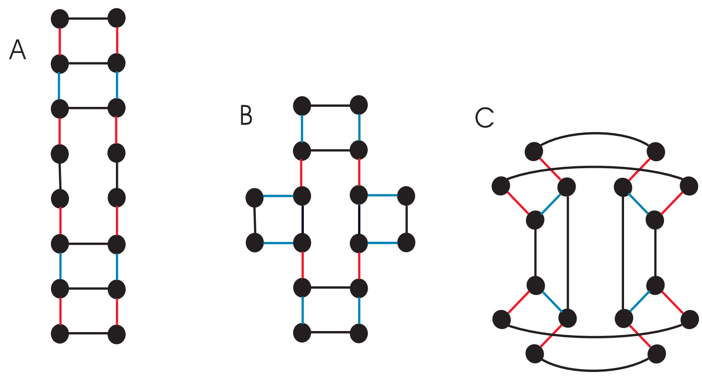


Figure 5.9: CPR graphs of $\{4,4\}_{(4,0)}$, $\{4,4\}_{(2,2)}$ and $\{4,4\}_{(2,0)}$

1) $\{4,4\}_{(4,0)}$ when we take the edge CPR graph with C a component with a single vertex (see Figure 5.9 A, black = 0, red = 1, blue = 2) or the Cayley graph with C a single component.

2) $\{4,4\}_{(2,2)}$ when we take the vertex CPR graph with C any of its two components (see Figure 5.9 B) or the Cayley graph with C consisting of two components that have an edge of label 1 (red) between them.

3) $\{4,4\}_{(2,0)}$, when we take the edge CPR graph with C the component with an edge of label 1 (see Figure 5.9 C) or the Cayley graph with C two opposite components.

5.8 Remarks

The polytope $2s^{\mathcal{K}-1}$ in the way constructed in Section 5.3 gives an affirmative answer to the extension problem for even last entries of the Schläfli symbol.

Additionally it satisfies the FAP with respect to its vertex figures and the finiteness and lattice properties.

Michael Hartley proved in [13] that, for $d \geq 3$, no n-hemicube can be extended with an odd number as last entry of its Schläfli symbol. This gives a negative answer to the extension problem for odd last entries of the Schläfli symbol. It remains open to give sufficient conditions on $\mathcal{K}$ that guarantee the existence of extensions of $\mathcal{K}$ with odd numbers as last entry of the Schläfli symbol.

In [28] Egon Schulte shows that if the vertex figures of a polytope $\mathcal{P}$ are isomorphic to the facets of a polytope $\mathcal{Q}$, and additionally $\mathcal{P}$ has the FAP with respect to its vertex figures and $\mathcal{Q}$ with respect to its facets, then the set of amalgamations of $\mathcal{P}$ and $\mathcal{Q}$ is not empty. Moreover, if $\mathcal{P}$ is the dual of $\mathcal{Q}$ then there is a self-dual amalgamation of $\mathcal{P}$ and $\mathcal{Q}$. In the same paper he states the following conjecture.

Conjecture 5.8.1 *Given a self-dual regular $(d-1)$-polytope $\mathcal{K}$ there is a self-dual regular $(d+1)$-polytope $\mathcal{Q}$ such that its medial section $\{F \,|F_0 \leq F \leq F_d\}$ (F_0 and F_d are incident vertex and facet respectively) is isomorphic to $\mathcal{K}$ and the first and last entries of the Schläfli symbol are equal to a preassigned natural number n.*

The conjecture was already known to be true for $n = 4$. Let $n = 2s$ be even. Since $2s^{\mathcal{K}-1}$ has the FAP with respect to its vertex-figures, we can deduce from [28] that there is a self-dual amalgamation of $2s^{\mathcal{K}-1}$ and its dual. The following result follows.

Theorem 5.8.2 *Given a self-dual regular $(d-1)$-polytope $\mathcal{K}$ there is a self-dual regular $(d+1)$-polytope $\mathcal{Q}$ such that its medial section $\{F \,|F_0 \leq F \leq F_d\}$ is isomorphic to $\mathcal{K}$ and the first and last entries of the Schläfli symbol are equal to a preassigned even natural number n.*

This conjecture is now proved for n even, however it remains open for n odd.

Chapter 6

Extensions of Dually Bipartite Polytopes

In this chapter we construct extensions with an even number as last entry of the Schläfli symbol, for regular polytopes such that the 1-skeleton of their duals are bipartite, and their automorphism groups act faithfully on the facets.

This extension is related to two extensions discovered by Egon Schulte (see [24], [25], [26] and [27]) that can be applied to any polytope such that its automorphism group acts faithfully on its facets. The last entry of the Schläfli symbol of those extensions is 6. One of those extensions is equivalent to the extension described in this chapter for $s = 3$ since for that case the bipartition on the facets of the polytope is not needed. Unfortunately for the remaining values of s the construction does not give as a result an extension of the polytope if its facet graph is not bipartite.

6.1 Previous results

In this chapter we deal mainly with the class of regular polytopes defined next.

Definition 6.1.1 *A* **regular dually bipartite polytope** *is a regular polytope that satisfies that*

- *its automorphism group acts faithfully on its facets and*

- *the* 1*-skeleton (the graph determined by the vertices and edges of the polytope) of its dual is a bipartite graph.*

Before explaining the constructions, some results on the facet CPR graphs of these kind of polytopes are necessary.

Remark 6.1.2 *Let $\mathcal{K}$ be a regular polytope and let $G = G_{0,\dots,d-1}$ be its j-face CPR graph. Then, the vertex of G corresponding to the base j-face of $\mathcal{K}$ has degree* 1*, and the edge incident to it has label j.*

Proposition 6.1.3 *Let $\mathcal{K}$ be a regular dually bipartite polytope and $\mathcal{K}^*$ its dual. Then, the mix $\mathcal{K}^*\Diamond\{\,\}$ is isomorphic to $\mathcal{K}^*$.*

Proof
It follows from Theorem 2.3.7.

□

As a consequence of Proposition 6.1.3 we have the following corollary.

Corollary 6.1.4 *Let G be a CPR graph of the regular dually bipartite polytope $\mathcal{K}$. Then the graph obtained from G by adding two vertices and an edge of label $d-1$ between them is another CPR graph of $\mathcal{K}$.*

The facet CPR graph G of a regular dually bipartite polytope $\mathcal{K}$ behaves well with respect to the bipartition of the vertices of G corresponding to the bipartition of the facets of $\mathcal{K}$. This is described by the following results.

Lemma 6.1.5 *Let $\mathcal{K}$ be a regular dually bipartite d-polytope with bipartition U, V of its set of facets, then ρ_{d-1} induces a perfect matching on the facet CPR graph of $\mathcal{K}$ such that each edge of color $d-1$ is incident to a vertex corresponding to a facet in U and a vertex corresponding to a facet in V.*

Proof
Suppose without loss of generality that the base facet F is an element of V. While ρ_{d-1} moves F to an adjacent facet F', F' has to be in U. In order to preserve the bipartition in the set of facets, the image under ρ_{d-1} of any facet in V has to be in U and vice versa. This induces the perfect matching required on the vertices of the facet CPR graph.

□

Lemma 6.1.6 *Let $\mathcal{K}$ be a regular dually bipartite d-polytope with bipartition U, V of its set of facets, then every edge of labels $0, \dots, d-2$ joins vertices corresponding to two facets in U or two facets in V.*

Proof

Let F be the base facet of $\mathcal{K}$. Then, ρ_k fixes F for $k = 1, \dots, d-2$. In order to preserve the bipartition in the set of facets, the image of any facet in U under ρ_k has to remain in U, and the image of a facet in V under ρ_k has to remain in V.

□

Proposition 6.1.7 *The facet CPR graph of every regular dually bipartite d-polytope has an even number of edges of label k for each $k = 0, \dots, d-3$.*

Proof

Let G be the facet CPR graph of a dually bipartite polytope with bipartition U, V of its facets. Lemma 6.1.6 implies that any edge e of label $k < d-2$ is incident to two vertices corresponding to facets in U or to two vertices corresponding to facets in V. By Lemma 6.1.5 we have that there is an edge of label $d-1$ incident to each vertex of e. Proposition 3.2.1 implies that e belongs to an alternating square with edges of labels k and $d-1$. Since every edge of label $k < d-2$ is in such an alternating square we conclude that the number of edges of label k in G is even.

□

We now know that the facet CPR graph of a regular dually bipartite polytope looks like the one in Figure 6.1. The sets U and V correspond to the sets of the bipartition of the facets of the polytope. The vertex at the right belongs to V and represents the vertex corresponding to the base facet. The edges represent the perfect matching between U and V of edges of label $d-1$ and no other edge joins a vertex on U to a vertex on V.

6.2 The main extension

Now that we have the necessary results we proceed with the construction.

Let $\mathcal{K}$ be a regular dually bipartite d-polytope, $G = G_{0,\dots,d-1}$ its facet CPR graph and let $s \geq 3$. We now construct a new graph $\overline{G}_s(\mathcal{K}) = \overline{G}_s$ by

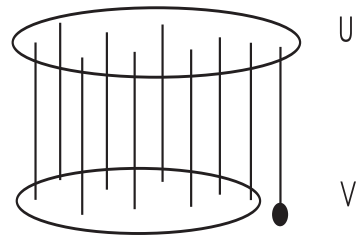


Figure 6.1: Facet CPR graph of a regular dually bipartite polytope

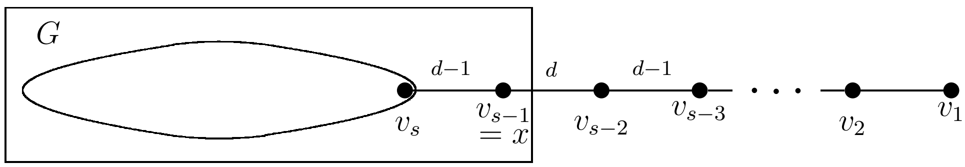


Figure 6.2: Graph $\overline{G}_s$

adding to G an alternating path of length $s-2$ of colors d and $d-1$ to the vertex x correspondent to the base facet. Then, in $\overline{G}_s$, the path P of labels d and $d-1$ of maximal length including x has length $s-1$, and includes the vertices $v_1, \ldots, v_{s-3}, v_{s-2} = \rho_d(x), v_{s-1} = x$ and $v_s = \rho_{d-1}(x)$. (see Figure 6.2).

Proposition 6.1.7 implies the following remark.

Remark 6.2.1 *Let G be the facet CPR graph of a regular dually bipartite polytope $\mathcal{K}$. Then the graph $\overline{G}_s$ has an even number of edges of colors $0, \ldots, d-3$.*

Now we prove that the graph $\overline{G}_s$ is a CPR graph of an extension of $\mathcal{K}$.

Theorem 6.2.2 *Let $\mathcal{K}$ be a regular dually bipartite d-polytope of Schläfli type $\{p_1, \ldots, p_{d-1}\}$ with G its facet CPR graph, and let $d \geq 2$ and $s \geq 3$. Then the graph $\overline{G}_s$ constructed as above is a CPR graph of a $d+1$-polytope $\mathcal{Q}_s(\mathcal{K}) = \mathcal{Q}_s$ with facets isomorphic to $\mathcal{K}$ and Schläfli symbol $\{p_1, \ldots, p_{d-1}, s\}$ if s is even, or $\{p_1, \ldots, p_{d-1}, 2s\}$ if s is odd.*

Proof

We can easily see that this construction with $s = 3$ leads to the extension introduced in [25]. For that particular construction, the intersection property had already been proved in that paper. We use this fact to prove the intersection property in general.

Let $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle$, let ρ_i for $i = 0, \ldots, d$ be determined by the edges of label i in $\overline{G}_s$, and let $\phi \in \langle \rho_0, \ldots, \rho_{d-1} \rangle \cap \langle \rho_j, \ldots, \rho_d \rangle$. We may assume that ϕ fixes v_k for $k = 1, \ldots, s-2$ and thus preserves the sets U and V shown in Figure 6.1 (because an even number of generators ρ_{d-1} are involved in the expression of ϕ in terms of $\rho_0, ..., \rho_{d-1}$); otherwise we take $\rho_{d-1}\phi$ instead of ϕ.

Let ρ'_{d-1} be ρ_{d-1} trivially extended to the vertex v_{s-2} and ρ'_d be ρ_d restricted to the vertex set of the original graph G and v_{s-2}. We can construct an element ϕ' in $\langle \rho_0, \ldots, \rho_{d-2}, \rho'_{d-1} \rangle$ from any expression of ϕ in terms of $\rho_0, \ldots, \rho_{d-1}$ by changing every factor ρ_{d-1} to a factor ρ'_{d-1}. Then the actions of ϕ and ϕ' on $V(G) \cup \{v_{s-2}\}$ are the same. In particular ϕ' also fixes the sets U and V.

Lemma A.2.1 implies that $\langle \rho_j, \ldots, \rho_{d-2}, \rho'_{d-1}, \rho'_d \rangle$ contains a symmetric group on the vertices of the connected component of x on $(G \cup \{v_{s-2}\})_{j,j+1,\ldots,d}$. Since the remaining connected components of $(G \cup \{v_{s-2}\})_{j,j+1,\ldots,d}$ are also connected components of $(\overline{G}_s)_{j,j+1,\ldots,d}$ it is not hard to see that the subgroup of $\langle \rho_j, \ldots, \rho_d \rangle$ that fixes v_k for $k = 1, \ldots, s-2$ is contained (as actions on the vertices) on the subgroup of $\langle \rho_j, \ldots, \rho_{d-2}, \rho'_{d-1}, \rho'_d \rangle$ that fixes v_{s-2}. Hence we can also get an expression for ϕ' in terms of $\rho_j, \ldots, \rho_{d-2}, \rho'_{d-1}, \rho'_d$.

Now we can appeal to the results in [25]. In particular, the above means that $\phi' \in \langle \rho_j, \ldots, \rho'_{d-1} \rangle$ in the construction with $s = 3$, but then we can find an expression for ϕ in terms of $\rho_j, \ldots, \rho_{d-1}$ by changing the factors ρ'_{d-1} to ρ_{d-1} of any expression of ϕ' in terms of $\rho_j, \ldots, \rho_{d-2}, \rho'_{d-1}$ (this can be done because ϕ' fixes the sets U and V), and the intersection property holds for $s \geq 3$.

The facet of the resulting $(d+1)$-polytope $\mathcal{Q}_s$ is isomorphic to $\mathcal{K}$ because of Corollary 6.1.4, and the last entry of the Schläfli sybol can be easily obtained from the polygonal action of $\langle \rho_{d-1}, \rho_d \rangle$ on the connected components of $G_{d-1,d}$ and the fact that G has at least two edges of label $d-1$ (see Lemma 6.1.5).

□

In order to describe the automorphism group for this construction we will discuss the cases s even and s odd separately.

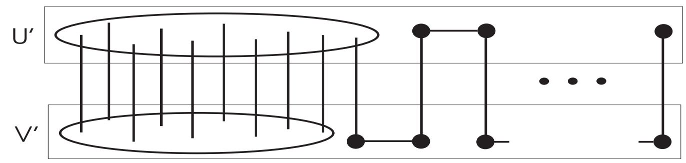


Figure 6.3: Graph $\overline{G}_s$ for s even

6.2.1 s even

For s even, the first and last edges of P will be of color $d-1$, and the graph $\overline{G}_s$ will have a bipartition of the vertices into sets U' and V' of the type described on Lemmas 6.1.5 and 6.1.6 (see Figure 6.3). Proposition 3.5.1 implies that the automorphism group of $\mathcal{Q}_s$ is a semidirect product $\langle \rho_{d-1} \rangle \rtimes H$, where the elements of H are pairs (σ, τ) with $\sigma \in S_{U'}$ and $\tau \in S_{V'}$, and ρ_{d-1} acts on H by interchanging the entries of the elements.

The actions of ρ_d and $\rho_{d-1}\rho_d\rho_{d-1}$ on the vertex set of $\overline{G}_s$ are given by the involutions

$$\begin{aligned} \rho_d &= (xv_{s-2})(v_{s-3}v_{s-4})\dots(v_3v_2) \\ \rho_{d-1}\rho_d\rho_{d-1} &= (v_sv_{s-3})(v_{s-2}v_{s-5})\dots,(v_4v_1). \end{aligned}$$

Now we find certain conjugates of ρ_d in $\Gamma(\mathcal{Q}_s)$ whose action on $V(\overline{G}_s)$ is the same of ρ_d or $\rho_{d-1}\rho_d\rho_{d-1}$ except in one transposition.

Without loss of generality assume $x \in V$ and let $y \in V(\overline{G}_s)\backslash\{v_1, \dots, v_{s-2}\}$ and $\phi \in \langle \rho_0, \dots, \rho_{d-1} \rangle$ such that $\phi(y) = x$. Then

$$\phi^{-1}\rho_d\phi = \begin{cases} \rho_d\,(x\ v_{s-2})(y\ v_{s-2}) & \text{if } y \in V, \\ \rho_{d-1}\rho_d\rho_{d-1}\,(v_s\ v_{s-3})(y\ v_{s-3}) & \text{if } y \in U \end{cases} \tag{6.1}$$

(see Figures 6.2 and 6.3). Now we have

$$\begin{aligned} \rho_d\phi^{-1}\rho_d\phi &= (xv_{s-2}y) \quad \text{for } y \in V, \\ \rho_{d-1}\rho_d\rho_{d-1}\phi^{-1}\rho_d\phi &= (v_sv_{s-3}y) \quad \text{for } y \in U. \end{aligned} \tag{6.2}$$

By Lemma A.2.2 we can obtain any 3-cycle in $A_{V\cup\{v_{s-2}\}}$ and in $A_{U\cup\{v_{s-3}\}}$. Now we consider the element $(\rho_d\rho_{d-1})^2$, that induces two disjoint cycles, one of them on the set $U' \cap P$, and the other on $V' \cap P$. Conjugating a 3-cycle including x (or $\rho_{d-1}(x)$) with this element we obtain all the necessary 3-cycles

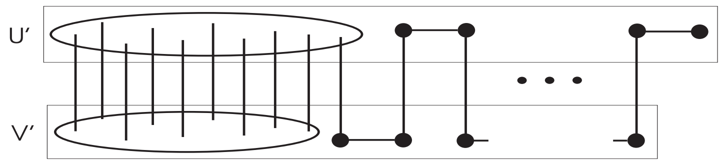


Figure 6.4: Graph $\overline{G}_s$ for s odd

including elements of U' (or V') to use Lemmas A.2.2 and A.2.3 and conclude that

$$A_n \times A_n \leq H \leq S_n \times S_n$$

where $n = f_{d-1}/2 - 1 + s/2$ is the number of vertices in U' (or V').

By Proposition A.3.2, H is isomorphic $A_n \times A_n$, $(S_n \times S_n)](A_n \times A_n)$ or $S_n \times S_n$.

If $\overline{G}_s$ has an odd number of edges of color $d-2$ (or of color d), say an odd number in U' and an even number in V', then, we can multiply ρ_{d-2} (or ρ_d) by $(\rho_{d-2})^{-1}_{|V'}$ (or $(\rho_d)^{-1}_{|V'}$) in order to get an element $(\sigma, \varepsilon) \in \Gamma(\overline{G}_s)$ with σ an odd permutation. In this case,

$$\Gamma(\mathcal{Q}_s) \cong [S_n \times S_n] \rtimes \mathbb{Z}_2. \tag{6.3}$$

Now, if $\overline{G}_s$ has an even number of edges of colors $d-2$ and d, but there is a color $k \in \{1, \ldots, d-2, d\}$ such that U' has an odd number of edges of color k, then

$$\Gamma(\mathcal{Q}_s) \cong \{(S_n \times S_n)](A_n \times A_n)\} \rtimes \mathbb{Z}_2. \tag{6.4}$$

Finally, if both, U' and V' have an even number of edges of colors $1, \ldots, d-2$ and d, then

$$\Gamma(\mathcal{Q}_s) \cong [A_n \times A_n] \rtimes \mathbb{Z}_2. \tag{6.5}$$

6.2.2 s odd

If s is odd then the edge between v_1 and v_2 has label d (see Figure 6.4). In order to find $\Gamma(\mathcal{Q}_s)$ we proceed in a similar way to the case s even.

Assuming that $x \in V'$, the conjugate of ρ_d by an element $\phi \in \langle \rho_0, \ldots, \rho_{d-1} \rangle$ with $\phi(y) = x$ is again

$$\phi\rho_d\phi^{-1} = \begin{cases} \rho_d\,(x\ v_{s-2})(y\ v_{s-2}) & \text{if } y \in V, \\ \rho_{d-1}\rho_d\rho_{d-1}\,(v_s\ v_{s-3})(y\ v_{s-3}) & \text{if } y \in U, \end{cases}$$

and we can also get the 3-cycles

$$\begin{aligned} \rho_d\phi^{-1}\rho_d\phi &= (xv_{s-2}y) \quad \text{for } y \in V, \\ \rho_{d-1}\rho_d\rho_{d-1}\phi^{-1}\rho_d\phi &= (v_sv_{s-3}y) \quad \text{for } y \in U; \end{aligned}$$

but now $v_2 = (\rho_d\rho_{d-1})^2(v_3)$. Since v_2 and v_3 belong neither both to U', nor both to V', we have that $(\rho_d\rho_{d-1})^2$ doesn't fix U' and V' (as sets) any longer. Actually, $(\rho_d\rho_{d-1})^2$ induces a cycle including all the vertices of P while the remaining vertices of $\overline{G}_s$ remain fixed (this can also be seen in the connected components of $(\overline{G}_s)_{d-1,d}$ because P is a path of even length). Lemma A.2.3 allows us to obtain the 3-cycles in $V \cup \{v_{s-2}\}$ and $U \cup \{v_{s-3}\}$ as well as their conjugates by $(\rho_d\rho_{d-1})^2$, so $A_n \leq \Gamma(\mathcal{Q}_s) \leq S_n$, where n is now the number of vertices of $\overline{G}_s$.

If $\overline{G}_s$ has an even number of edges of each label k for $k = d-2, d-1, d$, then

$$\Gamma(\mathcal{Q}_s) \cong A_n; \tag{6.6}$$

and if there exists $k \in \{d-2, d-1, d\}$ such that $\overline{G}_s$ has an odd number of edges of label k then

$$\Gamma(\mathcal{Q}_s) \cong S_n. \tag{6.7}$$

6.3 Results and examples

The result of the construction described above applied to the square with $s = 4$ is the toroidal polyhedron $\{4,4\}_{(3,0)}$. The relation $(\rho_0\rho_1\rho_2\rho_1)^3$ can be derived directly from the graph of the construction, that is shown in Figure 6.5 A.

However the construction applied to any regular dually bipartite polyhedron $\mathcal{K}$ with Schläfli type $\{p,4\}$, except $\pi(\{4,4\}_{3,0})$ (here π means the petrial operation, see Section 2.3, and Figure 6.6 for its facet CPR graph), gives rise to a polytope $\mathcal{Q}_4$ with Schläfli type $\{p,4,4\}$ with vertex figure isomorphic to $\{4,4\}_{(6,0)}$. This is because the CPR graph of the vertex figure of $\mathcal{Q}_4$, obtained from the graph $\overline{G}_s$ of the construction by deleting the edges of label 0,

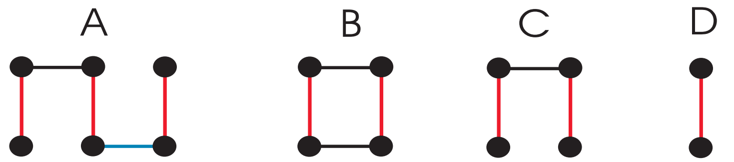


Figure 6.5: Connected components

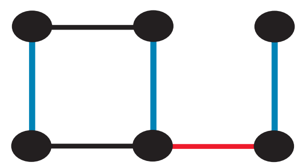


Figure 6.6: Connected components

will have as connected components some copies of the graphs in Figure 6.5. If the graphs B or C of this figure are connected components it is easy to see that the vertex figure of $\mathcal{Q}_4$ is the polyhedron $\{4,4\}_{(3,0)} \Diamond \{\} = \{4,4\}_{(6,0)}$ (see Theorem 2.3.7). The vertex figure of $\mathcal{Q}_4$ will be isomorphic to $\{4,4\}_{(3,0)}$ only if there is only one edge of label 1 in the facet CPR graph of $\mathcal{K}$, but the only regular dually bipartite polyhedron satisfying this extra condition is precisely $\pi(\{4,4\}_{3,0})$.

Since symmetric and alternating groups are involved in the automorphism groups of the extensions described in this chapter we expect no centrally symmetric polytopes as a result of this constructions.

Proposition 6.3.1 *For any polytope $\mathcal{K}$ and $s \geq 3$, the polytope $\mathcal{Q}_s(\mathcal{K})$ described in Section 6.2 is not centrally symmetric.*

Proof
The center of any of the groups A_n, S_n, $A_n \times A_n$, $(S_n \times S_n)](A_n \times A_n)$ and $S_n \times S_n$ is trivial.

$\square$

Now we explore the polyhedra obtained by this construction applied to polygons. Since the automorphism groups involve symmetric and alternating groups, the genus of the polyhedra are expressions involving factorials that are not to hard to obtain but make little contribution to this work, so we

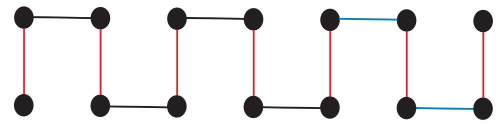


Figure 6.7: CPR graph of $\mathcal{Q}_6(\{10\})$

omit them. Note that the n-gons with n odd are not dually bipartite, so we restrict ourselves to the polyhedra $\mathcal{Q}_s(\{2n\})$.

Proposition 6.3.2 *For all $n \geq 2$ the polyhedron $\mathcal{Q}_s(\{2n\})$ with s odd is non-orientable.*

Proof

Let e_i be the number of edges of label i in $\overline{G}_s$ for $i = 0, 1, 2$. Then e_i is even (odd) if and only if ρ_i induces an even (odd) permutation of the vertices of $\overline{G}_s$.

If e_0, e_1, e_2 are even, then the $\Gamma(\mathcal{Q}_s(\{2n\})) \cong A_n$ and $\mathcal{Q}_s(\{2n\})$ cannot be orientable because A_n has no subgroup of index 2.

Now assume that e_i is odd for some i. From Figure 6.4 we can see that $e_1 = e_0 + e_2$, hence at least one of e_1, e_2, e_3 is even. Then at least one of $\rho_0\rho_1, \rho_0\rho_2, \rho_1\rho_2$ induces an odd permutation on the vertex set of $\overline{G}_s$. Since A_n is the only index 2 subgroup of S_n it follows that the even subgroup is the whole group $\Gamma(\mathcal{Q}_s(\{2n\})) \cong S_n$.

□

For s even we will use Proposition A.3.4 and the action of $\rho_0\rho_1, \rho_0\rho_2$ and $\rho_1\rho_2$ on the vertex set of $\overline{G}_s$ to determine when the polyhedron $\mathcal{Q}_s(\{2n\})$ is orientable.

The action of $\rho_0\rho_1$ on the vertex set of $\overline{G}_s$ is the product of transpositions determined by the edges of label 0 composed with the interchange of the vertices in U' with their correspondents in V'. Analogously the action of $\rho_1\rho_2$ is the product of transpositions determined by the edges of label 2 composed with the interchange of the vertices in U' with their correspondents in V' (see Figure 6.3). Finally the action of $\rho_0\rho_2$ is simply the product of the $n+s/2-1$ transpositions induced by the edges of labels 0 and 2. See Figure 6.7 for the CPR graph of $\mathcal{Q}_6(\{10\})$.

In Table 6.1 we give the automorphism group of $\mathcal{Q}_s(\{2n\})$ for s even and for different cases of values of n and $s/2$. The first two columns indicate that n and $s/2$ are even, odd, or congruent to the number in the table modulo 4. The elements of the even subgroup of $\Gamma(\mathcal{Q}_s(\{2n\}))$ are of type $[(\sigma,\tau),\epsilon]$, where σ and τ are permutations of the vertices in U' and V' respectively. We have that $\epsilon = 0$ if the element fixes the sets U' and V', and $\epsilon = 1$ if the element interchanges these two sets. Note that $\epsilon = 1$ for $\rho_0\rho_1$ and $\rho_1\rho_2$, and $\epsilon = 0$ for $\rho_0\rho_2$. We put "even" ("odd") in the column of $\rho_i\rho_j$ if σ and τ are both even (odd) permutations for that generator of the even subgroup of $\Gamma(\mathcal{Q}_s(\{2n\}))$, in this case the generator will induce an even permutation on the vertices of $\overline{G}_s$. We put "both" in this column if σ is odd and τ is even or vice versa, giving as a result a generator which induces an odd permutation on the vertices of $\overline{G}_s$. Observe that for s even the dual of $\mathcal{Q}_s(\{2n\})$ is the polytope $\mathcal{Q}_s(\{2n\})$ (see Figure 6.7), so there are no more necessary rows in Table 6.1.

n	**s/2**	$\rho_0\rho_1$	$\rho_1\rho_2$	$\rho_0\rho_2$	$\Gamma(\mathcal{Q}_s)$
$\equiv 0$	$\equiv 0$	both	both	odd	$(\mathrm{S}_n \times S_n) \rtimes \mathbb{Z}_2$
$\equiv 0$	$\equiv 2$	both	both	even	$(\mathrm{S}_n \times S_n) \rtimes \mathbb{Z}_2$
$\equiv 2$	$\equiv 2$	both	both	odd	$(\mathrm{S}_n \times S_n) \rtimes \mathbb{Z}_2$
even	$\equiv 1$	both	even	both	$(\mathrm{S}_n \times S_n) \rtimes \mathbb{Z}_2$
even	$\equiv 3$	both	odd	both	$(\mathrm{S}_n \times S_n) \rtimes \mathbb{Z}_2$
$\equiv 1$	$\equiv 1$	even	even	even	$(\mathrm{A}_n \times A_n) \rtimes \mathbb{Z}_2$
$\equiv 1$	$\equiv 3$	even	odd	odd	$(\mathrm{S}_n \times S_n)](A_n \times A_n) \rtimes \mathbb{Z}_2$
$\equiv 3$	$\equiv 3$	odd	odd	even	$(\mathrm{S}_n \times S_n)](A_n \times A_n) \rtimes \mathbb{Z}_2$

Table 6.1: Parameters of $\mathcal{Q}_s(\{2n\})$ for s even

Now it is clear that if n and $s/2$ are even then the three generators of the even subgroup of $\Gamma(\mathcal{Q}_s(\{2n\}))$ are in

$$\Big\{(\tau,\epsilon) \mid \quad \tau \in \quad \{(S_n \times S_n)](A_n \times A_n)\}, \epsilon = 0;$$

$$\text{or} \quad \tau \in \; [(S_n \setminus A_n) \times A_n] \cup [A_n \times (S_n \setminus A_n)], \epsilon = 1\Big\}.$$

Hence the polyhedron $\mathcal{Q}_s(\{2n\})$ is orientable.

If n is even and $s/2$ is odd or n is odd and $s/2$ is even, then $\rho_0\rho_2$ is of type $[(\sigma, \tau), 0]$ with σ even and τ odd or σ odd and τ even. Proposition A.3.4 implies that the even subgroup of $\Gamma(\mathcal{Q}_s(\{2n\}))$ is either $(S_n \times S_n) \rtimes \mathbb{Z}_2$ or $[(S_n \times S_n), 0]$, but since $\rho_0\rho_1$ and $\rho_1\rho_2$ do not belong to the second group we have that the even subgroup of $\Gamma(\mathcal{Q}_s(\{2n\}))$ is of index one, and hence $\mathcal{Q}_s(\{2n\})$ is non-orientable.

Similarly, if n is congruent to 1 and $s/2$ is congruent to 3 modulo 4 or vice versa, then $\rho_0\rho_2$ is not an element of the groups 5) and 6) of Proposition A.3.4. The even subgroup of $\Gamma(\mathcal{Q}_s(\{2n\}))$ cannot be the group 4) of Proposition A.3.4 because $\rho_0\rho_1$ and $\rho_1\rho_2$ interchange U' and V', so it is an index 1 subgroup and $\mathcal{Q}_s(\{2n\})$ is non-orientable.

If n and $s/2$ are congruent to 3 modulo 4 the even subgroup of $\Gamma(\mathcal{Q}_s(\{2n\}))$ is

$$\Big\{(\tau, x) \mid \quad \tau \in \; (A_n \times A_n), x = 0;$$
$$\text{or} \quad \tau \in \; (S_n \setminus A_n) \times (S_n \setminus A_n), x = 1\Big\}.$$

and $\mathcal{Q}_s(\{2n\})$ is orientable.

The only subgroup of index 2 of $(A_m \times A_m) \rtimes C_2$ is $A_m \times A_m$, but $\rho_0\rho_1$ and $\rho_1\rho_2$ interchange the sets U' and V'. Hence the polyhedron $\mathcal{Q}_s(\{2n\})$ is non-orientable if n and s are congruent to 1 modulo 4.

Now we can state the following theorem.

Theorem 6.3.3 *The polytope $\mathcal{Q}_s(\{2n\})$ for s even is orientable if and only if n and $s/2$ are both even or both congruent to 3 modulo 4.*

6.4 Reflection and half turn constructions

Given a regular dually bipartite polytope $\mathcal{K}$ with facet CPR graph G, and the graph $\overline{G}_s(\mathcal{K})$ described in Section 6.2, we can construct a CPR graph $G'_s(\mathcal{K}) = G'_s$ of another extensions $\mathcal{R}_s$ for $\mathcal{K}$ in the following way.

We take two copies of $\overline{G}_s$ and join the last vertices of the corresponding paths P by an edge of label d if s is even, or of label $d-1$ if s is odd (see Figure 6.8). The following result shows that this is a CPR graph of an extension of $\mathcal{K}$.

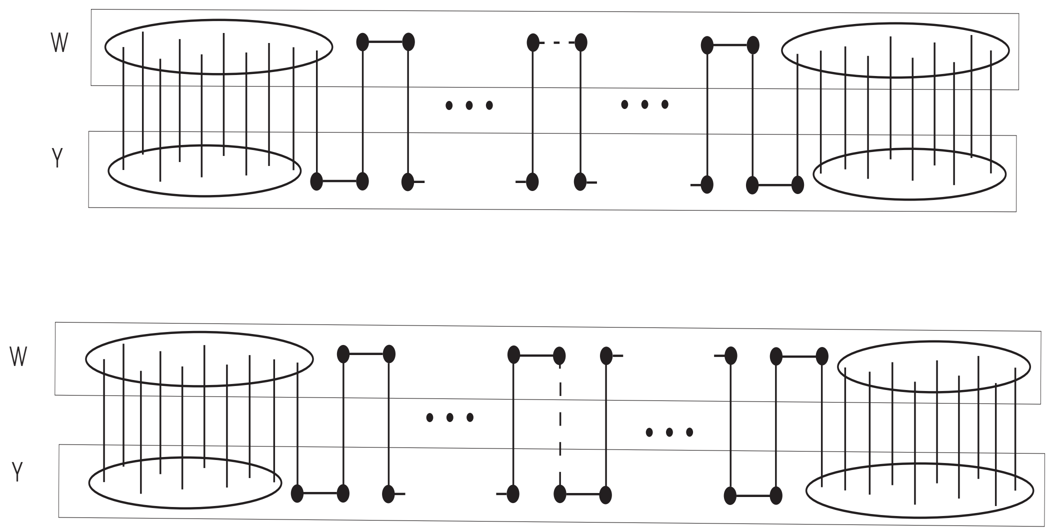


Figure 6.8: CPR graphs G'_s of the reflection and half turn constructions

Theorem 6.4.1 *For $s \geq 3$ and for any regular dually bipartite d-polytope $\mathcal{K}$ with Schläfli symbol $\{p_1, \ldots, p_{d-1}\}$, the graph $G'_s(\mathcal{K})$ described above is a CPR graph of a regular $d+1$-polytope $\mathcal{R}_s$ with Schläfli symbol $\{p_1, \ldots, p_{d-1}, 2s\}$ and facets isomorphic to $\mathcal{K}$.*

Proof
The intersection property can be checked in a similar way than the one in theorem 6.2.2, the last entry of the Schläfli symbol can be obtained from the polygonal action of the corresponding generators on $G'_s(\mathcal{K})$, and the facet type follows from Proposition 6.1.3 and Theorem 2.3.7.

□

It can be seen immediately from Figure 6.8 that this construction has a suitable drawing in the plane that allows a reflection symmetry if s is even, and a half turn symmetry if s is odd. From now on, this construction for s even will be the **reflection construction**, and for s odd will be the **half turn construction**.

Since we have a bipartition of the vertices of each copy of the graph $\overline{G}_s$ constructed in Section 6.2 satisfying the conditions of Lemmas 6.1.5 and 6.1.6, we also have a bypartition of the vertices of the CPR graphs of the reflection and half turn constructions in the sets W and Y shown in Figure 6.8 that also satisfy the conditions of Lemmas 6.1.5 and 6.1.6.

Proposition 3.2.5 implies that $\Gamma(\mathcal{Q}_s(\mathcal{K}))$ is a quotient of $\Gamma(\mathcal{R}_s(\mathcal{K}))$, and

$$N \rtimes \Gamma(\mathcal{R}_s(\mathcal{K})) \cong \Gamma(\mathcal{Q}_s(\mathcal{K})) \qquad (6.8)$$

with N defined as in Lemma 3.2.4.

Now we derive the automorphism groups of the reflection and half turn construction separately. It can be done by different ways but we think this is the simplest.

6.4.1 Half Turn construction

Let η denote the half turn symmetry of G'_s. We recall that

$$N = \{\phi \in \Gamma(\mathcal{R}_s) \,|\, \phi(v) \in \{v, \eta(v)\} \text{ for all } v \in V(G)\}.$$

Since η interchanges the sets W and Y (see Figure 6.8), and a vertex v of G'_s is in W if and only if $\eta(v)$ is in Y, we have that N is either $\{\varepsilon\}$ or $\langle\lambda\rangle$ where λ is an automorphism of $\mathcal{R}_s$ whose action on the vertex set of G'_s is η. Proposition 3.2.3 implies that if λ is an automorphism of $\mathcal{R}_s$ then it is a central involution.

The set W (or Y) contains exactly one element of each orbit of vertices of G'_s under the action of $\langle\eta\rangle$. This implies that

$$\Delta := \{\phi \in \Gamma(\mathcal{R}_s) \,|\, \phi \text{ fixes the sets } W \text{ and } Y\} \cong \Gamma(\mathcal{Q}_s).$$

It follows that the half turn η is an element of $\Gamma(\mathcal{R}_s)$ if and only if Δ contains the product of all the transpositions of type (ab) where $b = \eta\rho_{d-1}(a)$, but this occurs if $\Gamma(\mathcal{Q}_s)$ is symmetric, and if $\Gamma(\mathcal{Q}_s)$ is alternating and the number of vertices in W is congruent to 1 modulo 4 (note that this number cannot be even). Now (6.6), (6.7) and (6.8) imply the following.

Let $\mathcal{K}$ be a regular dually bipartite polytope, k, t and m be the number of edges of labels $d-2$, $d-1$ and d respectively on the graph G'_s of the half turn construction, and let $s \geq 3$. Then if k or m are odd then

$$\Gamma(\mathcal{R}_s(\mathcal{K})) \cong S_n \times \mathbb{Z}_2,$$

if k and m are even and t is congruent to 1 modulo 4 then

$$\Gamma(\mathcal{R}_s(\mathcal{K})) \cong A_n \times \mathbb{Z}_2,$$

and if k and m are even and t is congruent to 3 modulo 4 then

$$\Gamma(\mathcal{R}_s(\mathcal{K})) \cong S_n.$$

The polytope $\mathcal{R}_s(\mathcal{K})$ is centrally symmetric if and only if the half turn is the action of an element $\phi \in \Gamma(\mathcal{R}_s(\mathcal{K}))$. It is clear now that this happens always except if k and m are even and t is congruent to 3 modulo 4.

6.4.2 Reflection Construction

For s even, let ξ denote the reflection symmetry of G'_s. Then

$$N = \{\phi \in \Gamma(\mathcal{R}_s) \,|\, \phi(v) \in \{v, \xi(v)\} \text{ for all } v \in V(G)\}.$$

Now it is possible to have $\phi(u) = \xi(u)$ and $\phi(v) = v$ for two vertices u and v of G'_s and $\phi \in \Gamma(\mathcal{R}_s(\mathcal{K}))$.

Let W be one of the subgraphs of G'_s isomorphic to $\overline{G}_s$. Then W contains exactly one element of each orbit of vertices of G'_s under the action of $\langle\xi\rangle$. We name again $v_1, \ldots, v_s$ the vertices in W corresponding to the path P as in Figure 6.2. Then we have to determine when there exists an automorphism of $\mathcal{R}_s$ whose action on the vertex set of W is the transposition $(v_1, \rho_{d-1}\xi(v_1))$, and by conjugacy, obtain automorphisms whose actions are the remaining transpositions $(v, \rho_{d-1}\xi(v))$. Equivalently, we have to determine when it is possible to obtain an automorphism of $\mathcal{R}_s$ whose action is the permutation

$$\Big(v_2\, v_3\Big) \cdots, \Big(v_{s-2}\, v_{s-1}\Big)\Big(\xi(v_{s-1}) \;\; \xi(v_{s-2})\Big) \cdots \Big(\xi(v_3) \;\; \xi(v_2)\Big). \qquad (6.9)$$

Note that when we use ρ_d we are interchanging a vertex of W with a vertex outside W. In the cases when it is not possible to obtain such automorphism it will remain to determine the permutations induced by the action of automorphisms of $\mathcal{R}_s$ on the vertex set of G'_s that include a single transposition of type $(v, \rho_{d-1}\xi(v))$.

Let n be half of the vertices of W, then equations 6.1 and 6.2 still hold. Similar arguments than those used to derive equations (6.3), (6.4) and (6.5) imply that the permutation in (6.9) is the action of an automorphism of $\mathcal{R}_s$ if

- W has an odd number of edges of label $d-2$,

- W has an even number of edges of label $d-2$ and the numbers of edges of labels d and k are congruent to 2 modulo 4 for some $k \in \{0, \ldots, d-2\}$, and

- The number of edges of every color is a multiple of 4.

In this case

$$\Gamma(\mathcal{R}_s) \cong \mathbb{Z}_2^{2n} \rtimes \Gamma(\mathcal{Q}_s),$$

where $\Gamma(\mathcal{Q}_s)$ acts in $\mathbb{Z}_2^{2n}$ by permuting the coordinates of the elements (note that an element of $\mathbb{Z}_2^{2n}$ has as many coordinates as W has vertices). It follows that the semidirect product is actually a wreath product.

The remaining cases are when W has an odd number of edges of label d and an even number of edges of label $d-2$, and when the number of edges of label d in W is congruent to 2 modulo 4 and the numbers of edges of labels k are multiples of 4 for $k \in \{0, \ldots, d-2\}$. In the first case there is no automorphism of $\mathcal{R}_s$ having the permutation in 6.9 as its action on the vertex set of W, but there is one whose action is

$$\Big(v_2\, v_3\Big) \cdots, \Big(v_{s-4}\, v_{s-3}\Big)\Big(\xi(v_{s-3})\ \ \xi(v_{s-4})\Big) \cdots \Big(\xi(v_3)\ \ \xi(v_2)\Big).$$

This implies that whenever we interchange only one vertex v of W (or an odd number of them) with $\eta(v)$ we get an odd permutation on the vertex set of W.

In a similar way, if the number of edges of label d in W is congruent to 2 modulo 4 and the numbers of edges of labels k are multiples of 4 for $k \in \{0, \ldots, d-2\}$ there is an automorphism whose action on the vertex set of W is

$$\Big(v_2\, v_3\Big) \cdots, \Big(v_{s-6}\, v_{s-5}\Big)\Big(\xi(v_{s-5})\ \ \xi(v_{s-6})\Big) \cdots \Big(\xi(v_3)\ \ \xi(v_2)\Big),$$

implying that whenever we interchange only one vertex v of W (or an odd number of them) with $\eta(v)$ we get a permutation in $(S_n \times S_n)](A_n \times A_n) \rtimes \mathbb{Z}_2$ on the vertex set of W.

Now we describe explicitly the automorphism groups.

Let $\mathcal{K}$ be a regular dually bipartite polytope with G its facet CPR graph. Let G_s be the graph explained in Section 6.2 for s even, let n be half of its number of vertices and let e_i denote the number of edges of label i in G_s for $i = 0, \ldots, d$.

Let N_1 be the subgroup of $\mathbb{Z}_2 \wr [(S_n \times S_n) \rtimes \mathbb{Z}_2]$ consisting of the elements

$$\mathbb{Z}_2^{2n} \rtimes (\tau_1, \tau_2, y, x_1, \ldots, x_{2n}) \in [(S_n \times S_n) \rtimes \mathbb{Z}_2]$$

with $\tau_1\tau_2$ even if $\sum x_i = 0$, and $\tau_1\tau_2$ odd if $\sum x_i = 1$; and N_2 the subgroup where τ_1, τ_2 are even if $\sum x_i = 0$ and τ_1, τ_2 are odd if $\sum x_i = 1$.

If e_{d-2} is odd then

$$\Gamma(\mathcal{R}_s(\mathcal{K})) \cong \mathbb{Z}_2 \wr [(S_n \times S_n) \rtimes \mathbb{Z}_2],$$

if e_{d-2} is even and e_d, e_k are congruent to 2 modulo 4 for some $k \in \{0, \ldots, d-2\}$ then

$$\Gamma(\mathcal{R}_s(\mathcal{K})) \cong \mathbb{Z}_2 \wr [(S_n \times S_n)](A_n \times A_n) \rtimes \mathbb{Z}_2],$$

if e_i is a multiple of four for $i = 0, \ldots, d-2, d$ then

$$\Gamma(\mathcal{R}_s(\mathcal{K})) \cong \mathbb{Z}_2 \wr [(A_n \times A_n) \rtimes \mathbb{Z}_2],$$

if e_d is odd and e_{d-2} is even then

$$\Gamma(\mathcal{R}_s(\mathcal{K})) \cong N_1,$$

and if e_d is congruent to 2 modulo 4 and e_i is a multiple of four for $i = 0, \ldots, d-2$ then

$$\Gamma(\mathcal{R}_s(\mathcal{K})) \cong N_2.$$

In all the cases $\mathbb{Z}_2$ acts on the corresponding group by interchanging the entries of the elements so we actually have a permutation group on $2n$ elements (vertices of W).

Note that the involutions $(a, \xi(a))(b, \xi(b))$ are always actions of automorphisms of $\mathcal{R}_s(\mathcal{K}))$. Since W has an even number of vertices the reflection is always the action of an element $\phi \in \Gamma(\mathcal{R}_s(\mathcal{K}))$. Hence the polytope $\mathcal{R}_s(\mathcal{K}))$ is always centrally symmetric (see Proposition 3.2.3).

The extensions described in this chapter will be published in [22].

Chapter 7

Open questions

In this chapter we pose some open questions originated from the definitions and results of this work.

7.1 CPR Graphs

The main problem to determine whether a d-edge labeled graph is a CPR graph or not is to verify the intersection property.

Problem. Give a characterization of the intersection property for proper d-edge labeled graphs.

It is hard to work with some aspects of the permutation groups, for example to determine the number of elements of the subgroup of S_n generated by certain elements. This makes a hard task to determine most of the properties of a polytope even if we have a CPR graph of it.

Problem. Given two CPR graphs, are there any criteria to determine if they represent the same polytope?

Problem. Given a disconnected CPR graph of a polytope $\mathcal{K}$, how can we find a connected CPR graph for $\mathcal{K}$?

Problem. Given a CPR graph G of a polytope $\mathcal{K}$, is there a procedure to find the Cayley graph of $\mathcal{K}$ from the properties of G as a graph?

7.2 Polyhedra

In the SIGMAC conference, Aveiro, 2006, Roman Nedela talked about *chiral* maps (maps with two orbits of flags under the automorphism group in such a way that adjacent flags belong to different orbits) with alternating automorphism group. He described sufficient conditions for p and q in order to allow infinitely many polytopes with Schläfli symbol $\{p, q\}$ with alternating automorphism group.

Problem. For what p and q are there infinitely many regular polyhedra with Schläfli symbol $\{p, q\}$ and alternating automorphism group?

In Section 4.2 we explain that all the proper 3-labeled graphs with 7 vertices are CPR graphs of polyhedra with automorphism group isomorphic to S_7. It is not hard to see that if we consider all the proper 3-labeled graphs with 5 vertices we get CPR graphs of the seven polyhedra with automorphism group S_5 and the three with automorphism group A_5.

Problem. Is any proper 3-labeled graph with p vertices, a CPR graph of a polyhedron with alternating or symmetric automorphism group, for any prime number p?

7.3 Extensions

For any regular polytope $\mathcal{K}$ and any even number $2s$ we construct in Chapter 5 an extension of $\mathcal{K}$ with $2s$ as last entry of the Schläfli symbol. However there is little information about extensions with odd numbers as last entries of the Schläfli symbol.

Problem. Give necessary conditions for a regular polytope $\mathcal{K}$ in order to admit extensions with any number $m \geq 3$ as last entry of the Schläfli symbol.

And in particular,

Problem. Can the regular dually bipartite polytopes be extended with any number as last entry of the Schläfli symbol?

Conjecture 5.8.1 remains open for odd numbers n. We restate it like the following problem.

Problem. Given a self-dual regular $(d-1)$-polytope $\mathcal{K}$, is there a self-dual regular $(d+1)$-polytope $\mathcal{Q}$ such that its medial section $\{F \,|\, F_0 \leq F \leq F_d\}$ (F_0 and F_d are incident vertex and facet respectively) is isomorphic to $\mathcal{K}$ and the first and last entries of the Schläfli symbol are equal to a preassigned odd natural number $n \geq 3$?.

Appendix A

Algebra

This appendix contains the purely algebraic definitions and results used in the previous chapters, all of them on group theory.

In all the work we may denote the identity element of any group by ε, and we denote the alternating and symmetric groups on any set M by A_M and S_M respectively.

A.1 General Results

Every group acts in a natural way on the left (right) cosets of any of its subgroups. We say that this action is *faithful* if the only element of the group that fixes all the cosets is ε. The following two results give an equivalence for a group acting faithfully on the cosets of one of its subgroup.

Lemma A.1.1 *Let G be a group, $H \leq G$. Then, G acts faithfully on the set of left (right) cosets of H on G if and only if*

$$\bigcap_{g \in G} g^{-1} H g = \varepsilon \tag{A.1}$$

Proof

Let S be the symmetric group on the left cosets of H, and $\tilde{H}$ be the left side of equation A.1. The natural action given by

$$G \to S$$

$$g \mapsto \hat{g},$$

where $\hat{g}(kH) = (gk)H$ acts faithfully if and only if ε is the only element that fixes all the left cosets.

We can easily see that g_0 is such that $g_0kH = kH$ for all $k \in G$ if and only if $g_0 \in \tilde{H}$. This finishes the proof.

□

Lemma A.1.2 *Let G be a group, $H \leq G$. Then, H contains no normal subgroup of G different from $\{\varepsilon\}$ if and only if*

$$\bigcap_{g\in G} g^{-1}Hg = \{\varepsilon\} \tag{A.2}$$

Proof

Let $\tilde{H}$ be the left part of equation A.2.

If H contains a normal subgroup H' of G, then $H' \subseteq \tilde{H}$. In the other hand, $\tilde{H} \trianglelefteq G$. To see this, note that

$$h^{-1}\left(\bigcap_{g\in G} g^{-1}Hg\right)h = \bigcap_{g\in G} h^{-1}g^{-1}Hgh = \bigcap_{g\in G} g^{-1}Hg.$$

□

A.2 Symmetric and Alternating Groups

The following results are a useful tool to determine that a group G is isomorphic to the symmetric group S_M or the alternating group A_M once we have a description of G as a permutation group on the set M. More information about these results can be found in Rotman [23].

Lemma A.2.1 *If a subgroup Γ of S_n contains the transposition $(n-1\ n)$ as well as a subgroup acting transitively on $\{1,\ldots,n-1\}$ while keeping n fixed, then $\Gamma = S_n$.*

Lemma A.2.2 *If a subgroup Γ of S_n contains all the 3-cycles of the form (ijk), $k \in \{1,\ldots,n\} \setminus \{i,j\}$ for any fixed i and j, then $A_n \leq \Gamma \leq S_n$.*

Lemma A.2.3 *If a subgroup Γ of S_n contains the 3-cycle $(n-2\ n-1\ n)$ as well as a subgroup acting transitively on $\{1,\ldots,n-2\}$ while keeping $n-1, n$ fixed, then $A_n \leq \Gamma$.*

A.3 Index 2 Subgroups

Now we present some notation and results involving index 2 subgroups of certain groups.

Given a group Λ and an index 2 subgroup Δ there exists an index 2 subgroup of $\Lambda \times \Lambda$ consisting in the pairs such that both or none of the elements are in Δ.

Notation A.3.1 *Let Δ be a subgroup of index 2 of Λ. The group*

$$[\Delta \times \Delta] \cup [(\Lambda \setminus \Delta) \times (\Lambda \setminus \Delta)]$$

will be denoted by $(\Lambda \times \Lambda)](\Delta \times \Delta)$.

Proposition A.3.2 *Let Δ be a subgroup of index 2 of Λ, and $\alpha \in Aut(\Delta \times \Delta)$ be the automorphism interchanging the entries of the elements. Then the only groups Γ invariant under conjugation by α, and such that $\Delta \times \Delta \leq \Gamma \leq \Lambda \times \Lambda$ are*

- $\Delta \times \Delta$,
- $(\Lambda \times \Lambda)](\Delta \times \Delta)$, *and*
- $\Lambda \times \Lambda$.

Proof

The quotient $(\Lambda \times \Lambda)/(\Delta \times \Delta)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, that has three subgroups of index 2. By the one to one correspondence theorem (mentioned in [14] as the fourth isomorphism theorem), $\Lambda \times \Lambda$ has three subgroups of index 2 containing $\Delta \times \Delta$. The subgroups $\Delta \times \Lambda$ and $\Lambda \times \Delta$ are not invariant under conjugation by α, so the proposition holds.

□

Lemma A.3.3 *The group $A_n \times A_n$ has no index 2 subgroups.*

Proof

Suppose to the contrary that Λ is an index 2 subgroup of $A_n \times A_n$. Consider the normal subgroup Δ of $A_n \times A_n$ with elements (σ, ε) for every $\sigma \in A_n$. Note that $\Delta \cong A_n$. Since $\Lambda \triangleleft A_n \times A_n$, we have that $\Lambda \cap \Delta \triangleleft \Delta$. Then $\Lambda \cap \Delta$ is either trivial or Δ.

If $\Lambda \cap \Delta$ is trivial then we can embed Δ in $(A_n \times A_n)/\Lambda \cong \mathbb{Z}_2$, but that is not possible. On the other hand, if $\Lambda \cap \Delta$ is isomorphic to Δ, then Λ/Δ is an index 2 subgroup of $(A_n \times A_n)/\Delta \cong A_n$, leading again to a contradiction.

□

Proposition A.3.4 *The index 2 subgroups of $(S_n \times S_n) \rtimes \mathbb{Z}_2$ are*

1) $\{(\tau, 0) \mid \tau \in S_n \times S_n\}$

2) $\{(S_n \times S_n)](A_n \times A_n)\} \rtimes \mathbb{Z}_2$

3) $\Big\{(\tau, x) \mid \tau \in \{(S_n \times S_n)](A_n \times A_n)\}, x = 0;$
$or \, \tau \in [(S_n \setminus A_n) \times A_n] \cup [A_n \times (S_n \setminus A_n)], x = 1\Big\};$

and those of $\{(S_n \times S_n)](A_n \times A_n)\} \rtimes \mathbb{Z}_2$ *are*

4) $\{(\tau, 0) \mid \tau \in (S_n \times S_n)](A_n \times A_n)\}$

5) $(A_n \times A_n) \rtimes \mathbb{Z}_2$

6) $\Big\{(\tau, x) \mid \tau \in (A_n \times A_n), x = 0;$
$or \, \tau \in (S_n \setminus A_n) \times (S_n \setminus A_n), x = 1\Big\}$. *In all cases $\mathbb{Z}_2$ acts by interchanging the two coordinates.*

Proof

It follows from Lemma A.3.3 that any index 2 subgroups of $(S_n \times S_n) \rtimes \mathbb{Z}_2$ and $\{(S_n \times S_n)](A_n \times A_n)\} \rtimes \mathbb{Z}_2$ contain the normal subgroup group $\{(\tau, 0) : \tau \in A_n \times A_n\}$.

Note that the quotient

$$(S_n \times S_n) \rtimes \mathbb{Z}_2 \Big/ \{(\tau, 0) : \tau \in A_n \times A_n\} \cong D_4,$$

and

$$\{(S_n \times S_n)](A_n \times A_n)\} \rtimes \mathbb{Z}_2 \Big/ \{(\tau, 0) : \tau \in A_n \times A_n\} \cong \mathbb{Z}_2^2.$$

The one to one correspondence theorem applied to $(S_n \times S_n) \rtimes \mathbb{Z}_2$ and $\{(S_n \times S_n)](A_n \times A_n)\} \rtimes \mathbb{Z}_2$, and their normal subgroup $\{(\tau, 0) : \tau \in A_n \times A_n\}$ implies that there are only three index 2 subgroups of each of these two groups. It is straightforward to check that they are the ones mentioned in the proposition.

□

A.4 Semidirect and Wreath Product

Given a group Λ, if there exist two subgroups Δ and Γ of Λ such that $\Lambda = \Delta\Gamma$, $\Delta \cap \Gamma = \{\varepsilon\}$, and $\Delta \triangleleft \Lambda$, then for any $\gamma \in \Gamma$ there is an automorphism ϕ_γ of Δ given by conjugacy by γ. This induces an action of Γ on Δ that determines the multiplication rule

$$\delta_1\gamma_1 \cdot \delta_2\gamma_2 = \delta_1\phi_{\gamma_1}(\delta_2) \cdot \gamma_1\gamma_2.$$

Conversely, for any morphism $\phi : \Gamma \to Aut(\Delta)$ there is a group Λ with elements (δ, γ), where $\delta \in \Delta$ and $\gamma \in \Gamma$, such that

$$(\delta_1, \gamma_1) \cdot (\delta_2, \gamma_2) = (\delta_1 \cdot \phi(\gamma_1)(\delta_2), \gamma_1\gamma_2).$$

In this case we say that Λ is a *semidirect product* of Δ and Γ and we denote it by

$$\Lambda = \Delta \rtimes \Gamma$$

A particular case of a semidirect product is the wreath product explained next.

Definition A.4.1 *Let Λ be a group and Δ a permutation group on the set $\{1, \ldots, n\}$. The* wreath product *of Λ by Δ, denoted by $\Lambda \wr \Delta$ is the group $\Lambda^n \rtimes \Delta$ where*

$$[(x_1, \ldots, x_n), h][(y_1, \ldots, y_n), k] = [(x_1, \ldots, x_n) \cdot (y_{h(1)}, \ldots, y_{h(n)}), hk].$$

In other words, Δ acts on Λ^n by permuting the coordinates of the elements.

An analogous definition can be seen in M. Hall jr. [9].

Appendix B

Combinatorial Concepts

In this appendix we discuss only the concepts of lattices and Cayley graphs. These are mainly combinatorial concepts.

B.1 Lattices

A *lattice* is a partially ordered set such that for every two elements there exist an unique least upper bound called *join* and an unique greatest lower bound called *meet*.

Since polytopes partially ordered sets with least and greatest elements and has no infinite ascending or descending chain, the following proposition is useful to help determining that a polytope is a lattice (see G. Birkhoff [2], Chapter 2.3 for details).

Proposition B.1.1 *Let* $(P, \leq)$ *be a poset such that*

- *has a least (greatest) element,*
- *has no infinite ascending (descending) chains, and*
- *every two elements have least upper bound (greatest lower bound),*

then $(P, \leq)$ *is a lattice.*

B.2 Cayley Graphs

Any group acts on itself by multiplication by the left (right) side. The *Cayley graph* of a finite group Γ is a representation of this action in a *digraph* (oriented graph) given a generating set of Γ. To construct this graph we consider every element of Γ as a vertex and we add an *arc* (directed edge) from a vertex u to a vertex v whenever $v = gu$ for some g in the generating set.

If the generating set consists only of involutions then the Cayley graph will be *symmetric* (if there is an arc from u to v then there is an arc from v to u). In this case it is enough to add simple edges rather than symmetric arcs.

Let the generating set of the group Γ be $X = \{g_1, \ldots, g_n\}$, then we may label each arc (edge) of the Cayley graph in such a way that the arc (edge) from u to v has label j if $g_j u = v$. In [32] they call these graphs by *Cayley color graphs*. Whenever we refer to "Cayley graphs" on the previous chapters we are talking about the labeled graphs.

For example, Figure 3.2 F shows the Cayley graph of S_4, the automorphism group of the tetrahedron, with the generating set $\{(12), (23), (34)\}$. In this figure the generator (12) induces black edges, (23) induces red edges and (34) induces blue edges.

We mention briefly two properties of the Cayley (color) graphs.

Remark B.2.1 *The Cayley (color) graph of a group Γ with generating set $X = \{g_1, \ldots, g_n\}$ consisting only of involutions is n-regular (every vertex has degree n) with every vertex having an edge of each label incident to it.*

Remark B.2.2 *Let $X = \{g_1, \ldots, g_n\}$ be a generating set of a group Γ such that $X \setminus \{g_n\}$ is no longer a generating set. Then the subgraph induced by the edges of labels $1, \ldots, n-1$ of the Cayley (color) graph of Γ with X as generating set is disconnected.*

For further details about Cayley graphs see [32].

Appendix C

Catalog of CPR graphs

Now we present some CPR graphs of well known polyhedra. Since we can get the vertex CPR graph of a polytope from the face CPR graph of the dual interchanging labels 0 and 2, we do not include CPR graphs of the octahedron, icosahedron, hemioctahedron and hemiicosahedron.

We recall that there are no i-face CPR graphs for polytopes such that their automorphism group does not act faithfully on their i-faces. This is the reason why we do not include a face CPR graph of the hemicube or a vertex (face) CPR graph of the toroid $\{4,4\}_{(2,0)}$.

Color black represents label 0, color red label 1 and color blue label 2.

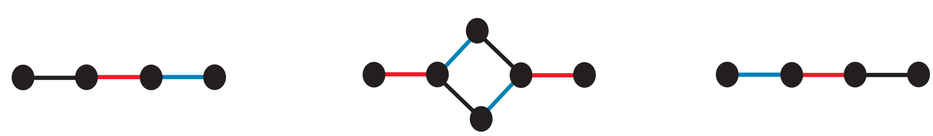


Figure C.1: Vertex, edge and face CPR graphs of the tetrahedron

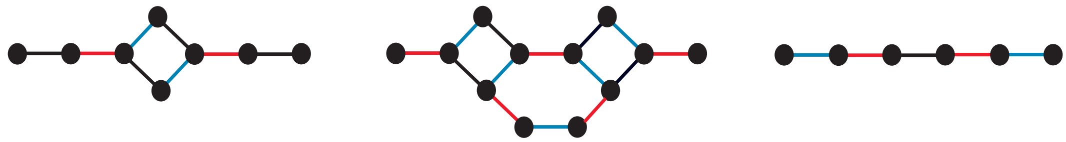


Figure C.2: Vertex, edge and face CPR graphs of the cube

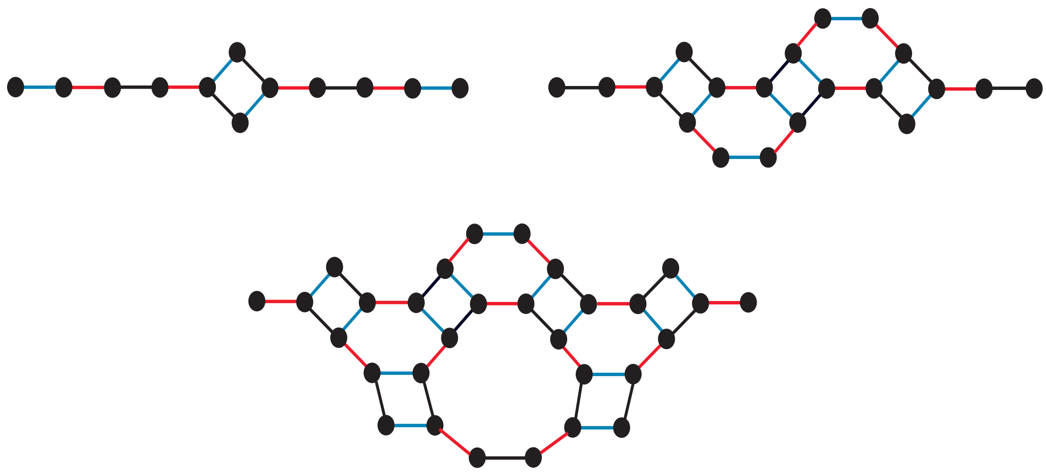


Figure C.3: Face, vertex and edge CPR graphs of the dodecahedron

Figure C.4: Vertex and edge CPR graphs of the hemicube

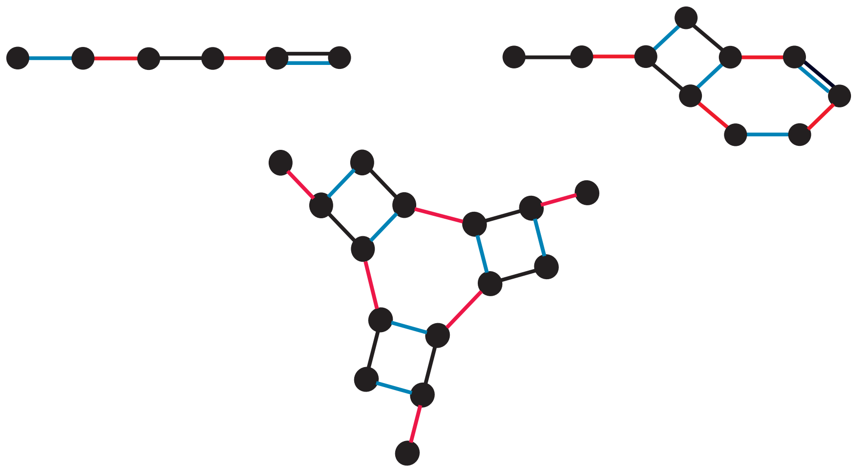


Figure C.5: Face, vertex and edge CPR graphs of the hemidodecahedron

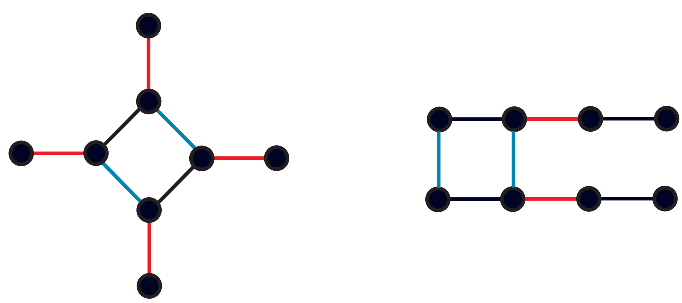


Figure C.6: Edge CPR graphs of $\{4,4\}_{(2,0)}$ and vertex CPR graph of the $\{4,4\}_{(2,2)}$

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